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Invertibility issues for a class of Wiener–Hopf plus Hankel operators

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Abstract. The invertibility of Wiener–Hopf plus Hankel operators $W(a) + H(b)$ acting on the spaces $L^p(\mathbb{R}^+)$, $1 \le p < \infty$ is studied. If a and b belong to a subalgebra of $L^\infty(\mathbb{R})$ and satisfy the condition

$$a(t)a(-t) = b(t)b(-t), \quad t \in \mathbb{R},$$

we establish necessary and also sufficient conditions for the operators $W(a) + H(b)$ to be one-sided invertible, invertible or generalized invertible. Besides, efficient representations for the corresponding inverses are given.

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1. Introduction

Let $\mathbb{R}^-$ and $\mathbb{R}^+$ be, respectively, the subsets of all negative and all positive real numbers and χ_E refer to the characteristic function of the subset $E \subset \mathbb{R}$ – i.e.

$$\chi_E(t) := \begin{cases} 1 & \text{if } t \in E, \\ 0 & \text{if } t \in \mathbb{R} \setminus E. \end{cases}$$

In what follows, we often identify the spaces $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R}^-)$, $1 \le p \le \infty$ with the subspaces $\chi_{\mathbb{R}^+} L^p(\mathbb{R})$ and $\chi_{\mathbb{R}^-} L^p(\mathbb{R})$ of the space $L^p(\mathbb{R})$, which consist of the functions vanishing on $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively.

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Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ be the direct and inverse Fourier transforms – i.e.

$$\mathcal{F}\varphi(\xi) := \int_{-\infty}^{\infty} e^{i\xi x}\varphi(x)\,dx, \quad \mathcal{F}^{-1}\psi(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\psi(\xi)\,d\xi, \quad x \in \mathbb{R}.$$

Consider the set $\mathcal{L}$ of functions $c\colon \mathbb{R} \to \mathbb{C}$ such that $c = \mathcal{F}k$ with $k \in L^1(\mathbb{R})$, and let $AP_W(\mathbb{R}) \subset L^\infty(\mathbb{R})$ be the set of functions $a\colon \mathbb{R} \to \mathbb{C}$ having the representation

$$a(t) = \sum_{j\in\mathbb{Z}} a_j e^{i\delta_j t}, \quad t \in \mathbb{R}, \tag{1}$$

with absolutely convergent series (1). It is assumed that $\delta_j \in \mathbb{R}$ for all $j \in \mathbb{Z}$ and $\delta_j \neq \delta_k$ if $j \neq k$. Following [17, Chapter VII], we denote by G the Banach algebra of all functions $g = g(t)$, $t \in \mathbb{R}$, such that

$$\begin{aligned} g &= a + c, \quad a \in AP_W(\mathbb{R}),\ c \in \mathcal{L}, \\ a(t) &= \sum_{j\in\mathbb{Z}} a_j e^{i\delta_j t}, \quad c(t) = (\mathcal{F}k)(t), \end{aligned} \tag{2}$$

equipped with the norm

$$\|g(t)\| = \sum_{j\in\mathbb{Z}} |a_j| + \int_{-\infty}^{\infty} |k(t)|\,dt.$$

We also consider the subalgebra G^+ (G^-) of the algebra G of functions (2) such that all numbers δ_j are non-negative (non-positive) and the functions $c = \mathcal{F}k$ such that $k(t) = 0$ for all $t \le 0$ ($t \ge 0$). The functions from G^+ and G^- admit holomorphic extensions respectively to the upper and lower half-planes and the set $G^+ \cap G^-$ contains constant functions only.

Any function $a \in G$ generates an operator $W^0(a)\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ and operators $W(a), H(a)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ defined by

$$\begin{aligned} W^0(a) &:= \mathcal{F}^{-1} a \mathcal{F}\varphi, \\ W(a) &:= PW^0(a), \\ H(a) &:= PW^0(a)QJ, \end{aligned}$$

where $P\colon f \to \chi_{\mathbb{R}^+} f$ and $Q := I - P$ are the canonical projections on the subspaces $L^p(\mathbb{R}^+)$ and $L^p(\mathbb{R}^-)$, correspondingly, and $J\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is the reflection operator defined by $J\varphi := \tilde{\varphi}$. Here and in what follows, $\tilde{\varphi}(t) := \varphi(-t)$ for any $\varphi \in L^p(\mathbb{R})$, $p \in [1, \infty]$. The operator $W(a)$ is called the convolution on the semi-axis $\mathbb{R}^+$ or the Wiener–Hopf operator, whereas $H(a)$ is referred to as the Hankel operator. It is well known [17] that for $a \in G$ all three operators are bounded on the space L^p for any $p \in [1, \infty)$.

The operators W^0 and $W(a)$ can be also represented as

$$W^0(a)\varphi(t) = \sum_{j=-\infty}^{\infty} a_j \varphi(t - \delta_j) + \int_{-\infty}^{\infty} k(t-s)\varphi(s)\, ds, \quad t \in \mathbb{R},$$

$$W(a)\varphi(t) = \sum_{j=-\infty}^{\infty} a_j B_{\delta_j} \varphi(t) + \int_{0}^{\infty} k(t-s)\varphi(s)\, ds, \quad t \in \mathbb{R}^+,$$

where

$$B_{\delta_j}\varphi(t) = \varphi(t - \delta_j) \qquad \text{if } \delta_j \le 0,$$

$$B_{\delta_j}\varphi(t) = \begin{cases} 0, & 0 \le t \le \delta_j, \\ \varphi(t-\delta_j), & t > \delta_j, \end{cases} \qquad \text{if } \delta_j > 0.$$

Moreover, for $a = \mathcal{F}k$ the operator $H(a)$ acts as

$$H(a)\varphi(t) = \int_{0}^{\infty} k(t+s)\varphi(s)\, ds$$

and for $a = e^{\delta t}$ as

$$H(a)\varphi(t) = \begin{cases} \varphi(\delta - t), & 0 \le t \le \delta, \\ 0, & t > \delta, \end{cases} \qquad \text{if } \delta > 0,$$

$$H(a)\varphi(t) = 0, \qquad t \in \mathbb{R}^+, \qquad \text{if } \delta \le 0.$$

Let us now recall a few useful identities involving the operators mentioned. It is easily seen that if $a, b \in G$, then

$$W^0(ab) = W^0(a)W^0(b).$$

Wiener–Hopf operators $W(a)$ generally do not possess this property, but according to [3, pp. 484, 485] we still have

$$\begin{aligned} W(ab) &= W(a)W(b) + H(a)H(\tilde{b}), \\ H(ab) &= W(a)H(b) + H(a)W(\tilde{b}). \end{aligned} \tag{3}$$

Moreover, if $b \in G$, $c \in G^+$ and $c \in G^-$, then

$$W(abc) = W(a)W(b)W(c).$$

The operators $W(a)$ are well studied. For various classes of generating functions a, the conditions of Fredholmness or semi-Fredholmness of such operators can be efficiently written [3, 4, 6, 14, 15, 16, 17]. Moreover, Fredholm and semi-Fredholm Wiener–Hopf operators are one-sided invertible, the corresponding one-sided inverses are known and there is an efficient description of the kernels and cokernels of $W(a)$, $a \in G$.

Consider now the Wiener–Hopf plus Hankel operators $\mathbb{W}(a,b)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ formally defined by

$$\mathbb{W}(a,b) = W(a) + H(b), \quad a, b \in L^\infty(\mathbb{R}). \tag{4}$$

The study of such operators is much more involved. Nevertheless, Fredholm properties of (4) can be established either directly or by passing to a Wiener–Hopf operator with a matrix symbol. Thus Roch *et al.* [20] studied the Fredholmness of Wiener–Hopf plus Hankel operators with piecewise continuous generating functions, acting on L^p-spaces, $p \in [1,\infty)$. Another approach, called equivalence after extension, has been applied to operators with generating functions from a variety of classes. Nevertheless, in spite of a vast amount of publications, this method is mainly applied to the operators of a special form, namely, to the operators $\mathbb{W}(a,a) = W(a) + H(a)$ acting on the L^2-space. It turns out that the Fredholmness, one-sided invertibility and invertibility of such operators are equivalent to the corresponding properties of the Wiener–Hopf operator $W(a\tilde{a}^{-1})$, so that they can be studied. However, even if an operator $\mathbb{W}(a,a)$ is invertible, the corresponding inverse is not given (see, e.g. [5, Corollary 2.2] for typical results obtained by the method mentioned). If $a \neq b$, then hardly verifiable assumptions concerning the factorization of auxiliary matrix-functions are used. To study the Wiener–Hopf plus Hankel operators of the form $I + H(b)$, another method has been employed in [18, 19], where the essential spectrum and the index of such operators are determined.

On the other hand, recently the Wiener–Hopf plus Hankel operators (4) have been studied under the assumption that the generating functions a and b satisfy the condition

$$a\tilde{a} = b\tilde{b}. \tag{5}$$

In particular, if $a, b \in G$, then the Coburn-Simonenko Theorem for some classes of operators $\mathbb{W}(a,b)$ is established [7], and an efficient description of the space $\ker \mathbb{W}(a,b)$ is obtained [12]. The aim of this work is to find conditions for one-sided invertibility, invertibility and generalized invertibility of the operators $\mathbb{W}(a,b)$ and to provide efficient representations for the corresponding inverses when generating functions a and b satisfy the condition (5). Similar problems for

Toeplitz plus Hankel operators have been recently discussed in [1, 2, 8, 9, 10, 11]. The situation with Wiener–Hopf plus Hankel operators has some special features. The main problem is that the operators here can also be semi-Fredholm – i.e. in general, they may have infinite kernels and co-kernels. This creates additional difficulties. Therefore, in some cases, the results obtained are not as complete as for Fredholm Toeplitz plus Hankel operators.

This paper is organized as follows. Section 2 contains known results on properties of Wiener–Hopf operators, Wiener–Hopf factorization of functions $g \in G$ such that $g(t)g(-t) = 1$, $t \in \mathbb{R}$ and the description of the kernels of Wiener–Hopf plus Hankel operators $W(a) + H(b)$, the generating functions of which satisfy the condition (5). In Section 3, we establish necessary conditions for one-sided invertibility of the operators $\mathbb{W}(a, b)$. Section 4 provides sufficient conditions for one-sided invertibility and presents efficient representations for the corresponding inverses. In Section 5, we construct generalized inverses for Wiener–Hopf plus Hankel operators. The invertibility conditions presented in Section 6 are supported by simple examples.

2. Auxiliary results

Let us recall the properties of Wiener–Hopf and Wiener–Hopf plus Hankel operators with generating functions from the algebra G. It was shown in [17] that for invertible functions g the operators $W(g)$ are one-sided invertible. More precisely, if $a \in AP_W$, $c \in \mathcal{L}$ and $g = a + c$ is invertible in G, then the element a is also invertible in G. Therefore, the numbers

$$\nu(g) := \lim_{l\to\infty} \frac{1}{2l}[\arg a(t)]_{-l}^{l}, \quad n(g) := \frac{1}{2\pi}[\arg(1 + a^{-1}(t)c(t)]_{t=-\infty}^{\infty}, \tag{6}$$

are correctly defined. Moreover, the function g admits a factorization of the form

$$g(t) = g_-(t)e^{i\nu t}\left(\frac{t-i}{t+i}\right)^n g_+(t), \quad -\infty < t < \infty, \tag{7}$$

where $g_+^{\pm 1} \in G^+$, $g_-^{\pm 1} \in G^-$, $\nu = \nu(g)$ and $n = n(g)$.

Let $-\infty < \nu < \infty$ be a real number. On the space $L^p(\mathbb{R}^+)$ we consider the operator U_ν defined by

$$(U_\nu\varphi)(t) := \begin{cases} \varphi(t-\nu) & \text{if } \max(\nu, 0) < t, \\ 0 & \text{if } 0 \le t \le \max(\nu, 0). \end{cases}$$

It is easily seen that for any $\nu \geq 0$, the operator U_ν is left invertible and $U_{-\nu}$ is one of its left-inverses. Moreover, $U_\nu = W(e^{it\nu})$ and $I - U_\nu U_{-\nu}$ is the projection operator,

$$((I - U_\nu U_{-\nu})\varphi)(t) := \begin{cases} \varphi(t) & \text{if } 0 < t < \nu, \\ 0 & \text{if } \nu < t < \infty. \end{cases}$$

We also consider operators V and $V^{(-1)}$ defined by

$$(V\varphi)(t) := \varphi(t) - 2\int_0^t e^{s-t}\varphi(s)\,ds, \quad (V^{(-1)}\varphi)(t) := \varphi(t) - 2\int_t^\infty e^{t-s}\varphi(s)\,ds.$$

Set $V^{(m)} = V^m$ if $m \geq 0$ and $V^{(-m)} = (V^{(-1)})^{-m}$ if $m < 0$. It is known that if $m \in \mathbb{N}$, then $V^{(-m)}V^{(m)} = I$, so that for $m > 0$ the operator $P_m := I - V^{(m)}V^{(-m)}$ is a projection [17, Chapter 7].

The factorization (7) is used to construct one-sided inverses for the Wiener–Hopf operators $W(g)$.

Theorem 2.1 ([17]). *If $g = a + c \in G$, $a \in AP_W$, $c \in \mathcal{L}$, then the operator $W(g)$ is one-sided invertible in $L^p(\mathbb{R}^+)$, $1 \leq p < \infty$ if and only if g is invertible in G. Moreover, if $g \in G$ is invertible in G and $\nu := \nu(g)$, $n := n(g)$, then*

(i) *If $\nu > 0$ and $n \geq 0$, then the operator $W(g)$ is left invertible and*

$$W_l^{-1}(g) = W(g_+^{-1})V^{(-n)}U_{-\nu}W(g_-^{-1}) \tag{8}$$

is one of its left-inverses.

(ii) *If $\nu > 0$ and $n < 0$, then the operator $W(a)$ is left invertible and one of its left-inverses is*

$$W_l^{-1}(g) = W(g_+^{-1})(I - U_{-\nu}P_{-n}U_\nu)^{-1}U_{-\nu}V^{-n}W(g_-^{-1}), \tag{9}$$

where

$$(I - U_{-\nu}P_{-n}U_\nu)^{-1} = \sum_{j=0}^{\infty}(U_{-\nu}P_{-n}U_\nu)^j, \tag{10}$$

and the series in the right-hand side of (10) is uniformly convergent.

(iii) *If $\nu < 0$ and $n \leq 0$, then the operator $W(a)$ is right invertible and*

$$W_r^{-1}(g) = W(g_+^{-1})V^{-n}U_{-\nu}W(g_-^{-1}) \tag{11}$$

is one of its right-inverses.

(iv) *If $\nu < 0$ and $n > 0$, then the operator $W(a)$ is right invertible and one of its right-inverses is*

$$W_r^{-1}(g) = W(g_+^{-1})V^{(-n)}U_{-\nu}(I - U_\nu P_n U_{-\nu})^{-1}W(g_-^{-1}). \tag{12}$$

(v) *If $\nu = 0$ and $n \le 0$ ($n \ge 0$), then the operator $W(g)$ is right (left) invertible and one of the corresponding inverses has the form*

$$W_{r/l}^{-1}(g) = W(g_+^{-1})V^{(-n)}W(g_-^{-1}), \tag{13}$$

Let us point out that there is also an efficient description of the kernels of the operators $W(g)$, but the structure of $\ker W(g)$ depends on the indices $\nu(g)$ and $n(g)$ and will be considered later on.

As far as the Wiener–Hopf plus Hankel operators $\mathbb{W}(a,b) := W(a)+H(b)$ are concerned, here we always assume that the generating functions a, b belong to G and satisfy the matching condition (5). In this case, the duo (a,b) is referred to as a matching pair. Moreover, in what follows, we will only consider the matching pairs (a,b) with the element a invertible in G. Notice that if $\mathbb{W}(a,b)$ is semi-Fredholm, then a is invertible in G and the matching condition yields the invertibility of b in G.

Let us introduce another pair (c,d) with the elements c and d defined by

$$c := ab^{-1} = \tilde{a}^{-1}\tilde{b}, \quad d := a\tilde{b}^{-1} = \tilde{a}^{-1}b.$$

This duo is called the subordinated pair for (a,b). The functions c and d possess a number of remarkable properties – e.g. $c\tilde{c} = 1 = d\tilde{d}$. Following [7], any function $g \in L_\infty(\mathbb{R})$ satisfying the condition $g\tilde{g} = 1$ is called matching function. In passing note that if (c,d) is the subordinated pair for a matching pair (a,b), then $(\bar{d},\bar{c})$ is the subordinated pair for the matching pair $(\bar{a},\bar{\tilde{b}})$, which defines the adjoint operator

$$\mathbb{W}^*(a,b) = W(\bar{a}) + H(\bar{\tilde{b}}) \tag{14}$$

for the operator $\mathbb{W}(a,b)$.

The next proposition comprises results from [7, 12]. For the reader's convenience, they are reformulated in a form suitable for subsequent presentation.

Proposition 2.2. *Assume that $g \in G$ is a matching function – i.e. $g\tilde{g} = 1$. Then*

(i) *Under the condition $g_-(\infty) = 1$, the factors g_+ and g_- in the factorization (7) are uniquely defined – viz. the factorization takes the form*

$$g(t) = (\boldsymbol{\sigma}(g)\,\tilde{g}_+^{-1}(t))e^{i\nu t}\left(\frac{t-i}{t+i}\right)^n g_+(t), \tag{15}$$

where $\nu = \nu(g), n = n(g)$, $\boldsymbol{\sigma}(g) = (-1)^n g(0)$, $\tilde{g}_+^{\pm 1}(t) \in G^-$ *and* $g_-(t) = \boldsymbol{\sigma}(g)\,\tilde{g}_+^{-1}(t)$. *The number* $\boldsymbol{\sigma}(g)$ *takes on only the values* 1 *and* -1 *and is called the factorization signature.*

(ii) *If* $\nu < 0$ *or if* $\nu = 0$ *and* $n < 0$, *then* $W(g)$ *is right-invertible and the operators* $\mathbf{P}_g^{\pm}$,

$$\mathbf{P}_g^{\pm} := (1/2)(I \pm JQBW^0(g)P)\colon \ker W(g) \longrightarrow \ker W(g),$$

considered on the kernel of the operator $W(g)$ *are complementary projections.*

(iii) *If* (c,d) *is the subordinated pair for a matching pair* $(a,b) \in G \times G$ *such that the operator* $W(c)$ *is right-invertible and* $W_r^{-1}(c)$ *is any right-inverse of* $W(c)$, *then*

$$\varphi^+ = \varphi^+(a,b) := \frac{1}{2}(W_r^{-1}(c)W(\tilde{a}^{-1}) - JQW^0(c)PW_r^{-1}(c)W(\tilde{a}^{-1})) + \frac{1}{2}JQW^0(\tilde{a}^{-1}),$$

is an injective operator from $\ker W(d)$ *into* $\ker(W(a) + H(b))$.

(iv) *If* (c,d) *is the subordinated pair for the matching pair* (a,b), *then*

(a) *if the operator* $W(c)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $1 < p < \infty$ *is right-invertible, then*

$$\ker(W(a) + H(b)) = \varphi^+(\operatorname{im} \mathbf{P}_d^+) \dotplus \operatorname{im} \mathbf{P}_c^-; \tag{16}$$

(b) *if the operator* $W(d)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $1 < p < \infty$ *is left-invertible, then*

$$\operatorname{coker}(W(a) + H(b)) = \varphi^+(\operatorname{im} \mathbf{P}_{\bar{c}}^+) \dotplus \operatorname{im} \mathbf{P}_{\bar{d}}^-, \tag{17}$$

where the operator φ^+ *in* (17) *is defined by the matching pair* $(\bar{a}, \bar{\tilde{b}})$.

(v) *Let* Λ_j *be the normalized Laguerre polynomials and the functions* ψ_j, $j \in \mathbb{Z}_+$, *be defined by*

$$\psi_j(t) := \begin{cases} \sqrt{2}e^{-t}\Lambda_j(2t), & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases} \qquad j = 0, 1, \ldots. \tag{18}$$

Then for $\nu = 0$ *and* $n < 0$, *the following systems* $\mathfrak{B}_{\pm}(g)$ *of functions* $W(g_+^{-1})\psi_j$ *form bases in the spaces* $\operatorname{im} \mathbf{P}_g^{\pm}$:

(a) *if* $n = -2m$, $m \in \mathbb{N}$, *then*

$$\mathfrak{B}_{\pm}(g) = \{W(g_{+}^{-1})\,(\psi_{m-k-1} \mp \boldsymbol{\sigma}(g)\psi_{m+k}) : k = 0, 1, \ldots, m-1\},$$

and

$$\dim \operatorname{im} \mathbf{P}_g^{\pm} = m; \tag{19}$$

(b) *if* $n = -2m-1$, $m \in \mathbb{Z}_{+}$, *then*

$$\mathfrak{B}_{\pm}(g) = \{W(g_{+}^{-1})(\psi_{m+k} \mp \boldsymbol{\sigma}(g)\psi_{m-k}) : k = 0, 1, \ldots, m\} \setminus \{0\},$$

and

$$\dim \operatorname{im} \mathbf{P}_g^{\pm} = m + \frac{1 \mp \boldsymbol{\sigma}(g)}{2}. \tag{20}$$

Remark 2.3. If $\nu < 0$, the corresponding spaces $\operatorname{im} \mathbf{P}_g^{\pm}$ are also described in [12]. However, these representations are not used in what follows so that they are not included to the above proposition.

3. Necessary conditions for one-sided invertibility

From now on we always assume without mentioning it specifically that the generating functions a and b constitute a matching pair. Moreover, let us also recall that if an operator $W(a)+H(b)$, $a, b \in G$ acting in the space $L^p(\mathbb{R}^+)$, $p \in (1, \infty)$ is Fredholm or semi-Fredholm, then the generating function a is invertible in G. Therefore, the elements c and d of the subordinated pair (c, d) are also invertible in G and the Wiener–Hopf operators $W(c)$ and $W(d)$ are Fredholm or semi-Fredholm. Let $\nu_1 := \nu(c)$, $n_1 := n(c)$, $\nu_2 := \nu(d)$, and $n_2 := n(d)$ be the corresponding indices (6) of the functions c and d. We start with necessary conditions for one-sided invertibility of the operators $W(a) + H(b)$ in the case where at least one of the indices ν_1, ν_2 is not equal to zero. The situation $\nu_1 = \nu_2 = 0$ will be considered later on.

Theorem 3.1. *Let $a, b \in G$ and the operator $W(a)+H(b)$ be one-sided invertible in $L^p(\mathbb{R}^+)$ and at least one of the indices ν_1 or ν_2 is not equal to zero. Then,*

(i) *either $\nu_1\nu_2 \geq 0$ or $\nu_1 > 0$ and $\nu_2 < 0$;*

(ii) *if $\nu_1 = 0$ and $\nu_2 > 0$, then $n_1 > -1$, or $n_1 = -1$ and $\boldsymbol{\sigma}(c) = -1$;*

(iii) *if $\nu_1 < 0$ and $\nu_2 = 0$, then $n_2 < 1$, or $n_2 = 1$ and $\boldsymbol{\sigma}(d) = -1$.*

Proof. (i) Assume that $\nu_1\nu_2 < 0$. If $\nu_1 < 0$ and $\nu_2 > 0$, then the operator $W(c)$ is right invertible whereas $W(d)$ is left invertible. Moreover, the kernel of the operator $W(c)$ and cokernel of $W(d)$ are infinite-dimensional [17] and so are the spaces $\operatorname{im}\mathbf{P}_c^-$ and $\operatorname{im}\mathbf{P}_{\bar{\tilde{d}}}^-$ [12, Theorems 2.4 and 2.5]. Taking into account Proposition 2.2(iv), we obtain that $\ker(W(a) + H(b)) \neq \{0\}$ and $\operatorname{coker}(W(a) + H(b)) \neq \{0\}$, hence the operator $W(a) + H(b)$ is not one-sided invertible.

(ii) Let $\nu_2 > 0$. By Proposition 2.2(iv), the operator $W(a) + H(b)$ has a non-zero cokernel. If, in addition, $n_1 < -1$ or $n_1 = 1$ and $\boldsymbol{\sigma}(c) = 1$, then (19) and (20) show that in both cases, $\operatorname{im}\mathbf{P}_c^- \neq \{0\}$. Therefore, according to (16), the operator $W(a) + H(b)$ also has a non-trivial kernel and is not one-sided invertible.

Assertion (iii) can be proved analogously. □

Let us briefly discuss the case where $\nu_1 > 0$ and $\nu_2 < 0$. As was mentioned in [12], in this situation it is not clear whether the corresponding Wiener–Hopf operator is even normally solvable. Nevertheless, the kernel and cokernel of $W(a) + H(b)$ can still be described. This allows to establish necessary conditions of one-sided invertibility. However, they are not as transparent as before and, in addition to the relations between the indices ν_1, ν_2, n_1, n_2, the corresponding conditions can include information about the factors in the Wiener–Hopf factorizations of the subordinated functions c and d. We consider one of possible cases.

Theorem 3.2. *Let $\nu_1 > 0$, $\nu_2 < 0$, $n_1 = n_2 = 0$ and let $\mathfrak{N}_\nu^p$, $\nu > 0$ denote the set of functions $f \in L^p(\mathbb{R}^+)$ such that $f(t) = 0$ for $t \in (0, \nu)$.*

(i) *If the operator $W(a) + H(b)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $1 < p < \infty$ is invertible from the left, then*
$$\varphi^+(\mathbf{P}_d^+) \cap \mathfrak{N}_{\nu_1/2}^p = \{0\}, \tag{21}$$
where $\varphi^+ = \varphi^+(ae^{-i\nu_1 t/2}, be^{i\nu_1 t/2})$.

(ii) *If the operator $W(a) + H(b)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $1 < p < \infty$ is invertible from the right, then*
$$\varphi^+(\mathbf{P}_{\bar{c}}^+) \cap \mathfrak{N}_{-\nu_2/2}^p = \{0\}, \tag{22}$$
where $\varphi^+ = \varphi^+(\bar{a}e^{i\nu_2 t/2}, \bar{\tilde{b}}e^{-i\nu_2 t/2})$.

Proof. Let $\nu_1 > 0$, $\nu_2 < 0$, $n_1 = n_2 = 0$ and $W(a) + H(b)$ be a left-invertible operator. It can be represented in the form
$$W(a) + H(b) = (W(ae^{-i\nu_1 t/2}) + H(be^{i\nu_1 t/2}))W(e^{i\nu_1 t/2}). \tag{23}$$

Direct computations show that $(ae^{-i\nu_1 t/2}, be^{i\nu_1 t/2})$ is a matching pair with the subordinated pair $(c_1, d_1) = (ce^{-i\nu_1 t}, d)$. Since $\nu(c_1) = 0$, $n(c_1) = n_1 = 0$, the kernel of the operator $W(c_1)$ is trivial. Consequently, $\ker P^-_{c_1} = \{0\}$ and the relation (16) yields

$$\ker(W(ae^{-i\nu_1 t/2}) + H(be^{i\nu_1 t/2})) = \varphi^+(\operatorname{im}\mathbf{P}^+_d)$$

with the operator $\varphi^+ = \varphi^+(ae^{-i\nu_1 t/2}, be^{i\nu_1 t/2})$. Therefore, taking into account (23), we obtain

$$\ker(W(a) + H(b)) = \{\eta = W(e^{-i\nu_1 t/2})u : u \in \varphi^+(\mathbf{P}^+_d) \cap \operatorname{im} W(e^{i\nu_1 t/2})\}.$$

If the operator $W(a)+H(b)$ is left invertible, its kernel consists of the zero element only. However, since $\operatorname{im} W(e^{i\nu_1 t/2}) = \mathfrak{N}^p_{\nu_1/2}$ and

$$\ker W(e^{-i\nu_1 t/2}) \cap (\varphi^+(\mathbf{P}^+_d) \cap \mathfrak{N}^p_{\nu_1/2}) = \{0\},$$

the assumption

$$\varphi^+(\mathbf{P}^+_d) \cap \mathfrak{N}^p_{\nu_1/2} \neq \{0\}$$

yields the non-triviality of the kernel of $W(a) + H(b)$, so that (21) holds.

The second assertion in Theorem 3.2 comes from the first one by passing to the adjoint operator (see (14)). □

Remark 3.3. Theorem 3.2 raises an interesting question: Do there exist invertible operators $W(a) + H(b)$, such that

$$\dim\operatorname{coker} W(c) = \dim\ker W(d) = \infty?$$

If $\nu(c) = \nu(d) = 0$, we conjecture that for any prescribed natural number N one can find invertible operators $W(a) + H(b)$ for which

$$\operatorname{ind}|W(c)| > N, \quad \operatorname{ind}|W(d)| > N. \tag{24}$$

Note that the set of Toeplitz plus Hankel operators possesses the property (24) – cf. [13], but for Wiener–Hopf plus Hankel operators, this problem requires a separate study.

Remark 3.4. Although the description of the spaces $\operatorname{im}\mathbf{P}^+_d$ and $\operatorname{im}\mathbf{P}^+_{\tilde{c}}$ is available [12], the verification of the conditions (21)–(22) is not trivial. It depends on the properties of Wiener–Hopf operators constituting the operator φ^+ and may require a lot of effort.

Remark 3.5. If $\nu_1 > 0$, $\nu_2 < 0$ but $n_1 \neq 0$ or/and $n_2 \neq 0$, the necessary conditions of one-sided invertibility have the same form (21) and (22) but the representation (23), spaces $\mathfrak{N}_\nu^p$ and operators φ^+ should be redefined accordingly.

We now consider the situation when both indices ν_1 and ν_2 vanish. Let us start with an auxiliary result.

Lemma 3.6. *If $(a, b) \in G \times G$ is a matching pair with the subordinated pair (c, d), then for the factorization signatures of the functions c and d the equation*

$$\boldsymbol{\sigma}(c) = \boldsymbol{\sigma}(d) \tag{25}$$

holds and the indices n_1 and n_2 are simultaneously odd or even.

Proof. Let $n(a)$ and $n(b)$ be the corresponding indices (6) for the functions a and b, respectively. Then

$$n_1 = n(c) = n(ab^{-1}) = n(a) - n(b), \quad n_2 = n(d) = n(a\tilde{b}^{-1}) = n(a) + n(b). \tag{26}$$

Therefore,

$$\begin{aligned} \boldsymbol{\sigma}(c) &= (-1)^{n(a)-n(b)} c(0) = (-1)^{n(a)-n(b)} a(0) b^{-1}(0), \\ \boldsymbol{\sigma}(d) &= (-1)^{n(a)+n(b)} d(0) = (-1)^{n(a)+n(b)} a(0) \tilde{b}^{-1}(0), \end{aligned}$$

and since $b(0) = \tilde{b}(0)$ and the numbers $n(a) - n(b)$ and $n(a) + n(b)$ are simultaneously odd or even, the equation (25) follows.

Moreover, using the relations (26) again, we obtain

$$n_1 + n_2 = 2n(a),$$

so that n_1 has the same evenness as n_2. □

We start with the left invertibility of the operators $\mathbb{W}(a, b)$.

Theorem 3.7. *If $a, b \in G$, $\nu_1 = \nu_2 = 0$, $n_2 \geq n_1$ and the operator $W(a) + H(b)$ is invertible from the left, then the index n_1 satisfies the inequality*

$$n_1 \geq -1$$

and if $n_1 = -1$, then $\boldsymbol{\sigma}(c) = -1$ and $n_2 > n_1$.

Proof. If $n_1 < -1$, the operator $W(c)$ is right invertible. By Proposition 2.2(v), the image of the projection $\mathbf{P}_c^-$ contains non-zero elements, and by (16) so is $\ker(W(a) + H(b))$. This contradicts the left invertibility of the operator $W(a) + H(b)$, hence $n_1 \geq -1$.

Assume now that $n_1 = -1$ and $\boldsymbol{\sigma}(c) = 1$. Using (16) and Proposition 2.2(v) again, we note that $\operatorname{im}\mathbf{P}_c^- \neq \{0\}$, so that the operator $W(a) + H(b)$ has a non-trivial kernel and, therefore, it is not left-invertible. Hence $\boldsymbol{\sigma}(c) = -1$. Assuming, in addition, that $n_2 = -1$ and $\ker(W(a) + H(b)) = \{0\}$, we obtain

$$\boldsymbol{\sigma}(c) = -1, \quad \boldsymbol{\sigma}(d) = 1,$$

which is not possible by Lemma 3.6. Hence, $n_2 > n_1$. □

Theorem 3.8. *If $a, b \in G$, $\nu_1 = \nu_2 = 0$, $n_1 > n_2$ and the operator $W(a) + H(b)$ is invertible from the left, then the inequality*

$$n_1 \geq 1$$

holds. Moreover, the index n_2 is either non-negative or $n_2 < 0$ and $n_1 \geq -n_2$.

Proof. If $n_1 \leq 0$, then $n_2 \leq -2$ – cf. Lemma 3.6, and $W(a) + H(b)$ has a non-trivial kernel, which contradicts the left-invertibility of this operator. On the other hand, if $1 \leq n_1$ and $0 \leq n_2$, then $W(a) + H(b)$ is clearly left-invertible, so we proceed with the case $n_2 < 0$. By Lemma 3.6, both numbers n_1 and n_2 are either even or odd. In both cases the proof of the fact that the indices n_1 and n_2 satisfy the inequality $n_1 \geq n_2$ is similar, but each situation should be examined separately. Here we only analyse the case where n_1 and n_2 are odd numbers. Considering $\operatorname{ind} W(c) := \mathbf{k}_1 = -n_1$ we chose $k_1 \in \mathbb{Z}$ such that

$$2k_1 + \mathbf{k}_1 = 1.$$

Then, according to [12, Theorem 3.2], we have

$$\begin{aligned} &\ker(W(a) + H(b)) \\ &\quad = \left\{ W\left(\left(\frac{t-i}{t+i}\right)^{-k_1}\right)u : \right. \\ &\qquad \left. u \in \left\{ \frac{1+\boldsymbol{\sigma}(c)}{2} W(c_+^{-1})\{\mathbb{C}\psi_0\} \dotplus \varphi^+(\operatorname{im}\mathbf{P}_d^+) \right\} \cap \operatorname{im} W\left(\left(\frac{t-i}{t+i}\right)^{k_1}\right) \right\}, \end{aligned} \tag{27}$$

where the operator $\varphi^+ = \varphi^+(a_1, b_1)$ is defined by the matching pair

$$(a_1, b_1) = \left(a(t)\left(\frac{t-i}{t+i}\right)^{-k_1}, b(t)\left(\frac{t-i}{t+i}\right)^{k_1} \right)$$

and c_+ is the plus factor in the Wiener–Hopf factorization (15) of the function c. The function ψ_0 is defined in (18) and using another representation of the Laguerre polynomials – cf. [12, eq. (2.5)], one can show that

$$\operatorname{im} W\left(\left(\frac{t-i}{t+i}\right)^{k_1}\right) = \operatorname{clos}\operatorname{span}_{L^p(\mathbb{R}^+)}\{\psi_{k_1}, \psi_{k_1+1}, \dots\}.$$

Thus if a function

$$u \in \operatorname{im} W\left(\left(\frac{t-i}{t+i}\right)^{k_1}\right)$$

is expanded in a Fourier series of the Laguerre polynomials ψ_j, $j = 0, 1, \dots,$ its first k_1 Fourier-Laguerre coefficients are equal to zero. If we now assume that the dimension of the subspace

$$\mathfrak{S}(c,d) := \left\{\frac{1+\boldsymbol{\sigma}(c)}{2} W(c_+^{-1})\{\mathbb{C}\psi_0\} \dotplus \varphi^+(\operatorname{im}\mathbf{P}_d^+)\right\}$$

is greater than k_1, then there is a non-zero function $u_0 \in \mathfrak{S}(c,d)$, the first k_1 Fourier-Laguerre coefficients of which vanish. Hence, (27) shows that the kernel of $W(a)+H(b)$ contains a non-zero element. This contradicts the left invertibility of the operator $W(a)+H(b)$. Therefore,

$$k_1 \geq \dim \mathfrak{S}(c,d), \tag{28}$$

and taking into account eq. (20), we rewrite the inequality (28) as

$$k_1 \geq \frac{1+\boldsymbol{\sigma}(c)}{2} + k_2, \tag{29}$$

where

$$k_2 = r + \frac{1-\boldsymbol{\sigma}(d)}{2}$$

and $-n_2 = 2r+1$. Since $k_1 = (1-\mathbf{k}_1)/2 = (1+n_1)/2$, the inequality (29) takes the form

$$\frac{1+n_1}{2} \geq \frac{1+\boldsymbol{\sigma}(c)}{2} + \frac{-n_2-1}{2} + \frac{1-\boldsymbol{\sigma}(d)}{2}$$

or

$$n_1 \geq -n_2 + \boldsymbol{\sigma}(c) - \boldsymbol{\sigma}(d).$$

Since $\boldsymbol{\sigma}(c) = \boldsymbol{\sigma}(d)$ by Lemma 3.6, the proof is completed. □

Theorems 3.7, 3.8 provide necessary conditions for the left invertibility of the operators $\mathbb{W}(a,b) = W(a) + H(b)$. Passing to right-invertible operators, one can recall the simple fact that the operator $\mathbb{W}(a,b)$ is right-invertible if and only if the adjoint operator $\mathbb{W}^*(a,b)$ is left invertible. Relation (14) shows that

$$\mathbb{W}^*(a,b) = \mathbb{W}(\bar{a}, \bar{\tilde{b}}).$$

We note that $(\bar{a}, \bar{\tilde{b}})$ is also a matching pair with the subordinated pair $(c_1, d_1) = (\bar{d}, \bar{c})$, so that

$$\begin{aligned} \nu(c_1) &= \nu(\bar{d}) = -\nu_2, & \nu(d_1) &= \nu(\bar{c}) = -\nu_1, \\ n(c_1) &= n(\bar{d}) = -n_2, & n(d_1) &= n(\bar{c}) = -n_1, \\ \boldsymbol{\sigma}(c_1) &= \boldsymbol{\sigma}(d), & \boldsymbol{\sigma}(d_1) &= \boldsymbol{\sigma}(c). \end{aligned}$$

Now Theorems 3.7 and 3.8 can be used to write the necessary conditions for the right invertibility of the operators $W(a) + H(b)$. Let us just formulate the corresponding results.

Theorem 3.9. *Let $a, b \in G$, $\nu_1 = \nu_2 = 0$, $n_1 \le n_2$ and the operator $W(a) + H(b)$ be invertible from the right. Then*

$$n_2 \le 1$$

and if $n_2 = 1$, then $\boldsymbol{\sigma}(d) = -1$ and $n_1 < n_2$.

Theorem 3.10. *Let $a, b \in G$, $\nu_1 = \nu_2 = 0$, $n_1 > n_2$ and the operator $W(a)+H(b)$ be invertible from the right. Then the inequality*

$$n_2 \le -1$$

holds. Moreover, the index n_1 is either non-positive or $n_1 \le -n_2$.

4. Sufficient conditions of one-sided invertibility and formulas for one-sided inverses

Our next goal is to establish sufficient conditions for one-sided invertibility of the operators $W(a) + H(b)$. In fact, many necessary conditions above are also sufficient ones.

Theorem 4.1. *Let $a, b \in G$ and the indices ν_1, ν_2, n_1 and n_2 satisfy any of the following conditions*:

(i) $\nu_1 < 0$ *and* $\nu_2 < 0$;

(ii) $\nu_1 > 0$, $\nu_2 < 0$, $n_1 = n_2 = 0$, *the operator* $W(a) + H(b)$ *is normally solvable, has a complementable kernel, and satisfies the condition* (22).

(iii) $\nu_1 < 0$, $\nu_2 = 0$ *and* $n_2 < 1$ *or* $n_2 = 1$ *and* $\boldsymbol{\sigma}(d) = -1$;

(iv) $\nu_1 = 0$, $n_1 \leq 0$ *and* $\nu_2 < 0$;

(v) $\nu_1 = 0$ *and* $\nu_2 = 0$

(a) $n_1 \leq 0$, $n_2 \leq 0$,

(b) $n_1 \leq 0$, $n_2 = 1$ *and* $\boldsymbol{\sigma}(d) = -1$.

Then the operator $W(a) + H(b)\colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$, $1 < p < \infty$ *is right invertible.*

Proof. The claim follows from the representation (41) below, the factorization

$$W(V(a,b)) = \begin{pmatrix} -W(d) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & W(\tilde{a}^{-1}) \end{pmatrix} \begin{pmatrix} -W(c) & 0 \\ 0 & I \end{pmatrix},$$

and consequent application of Proposition 2.2. □

Sufficient conditions for the left invertibility of the operators $W(a) + H(b)$ can be obtained from Theorem 4.1 by passing to the adjoint operators and we leave it to the reader.

In the remaining part of this section we deal with the construction of right inverses for the operators $W(a) + H(b)$. Recall that one-sided inverses of Wiener–Hopf operators can be easily determined from the Wiener–Hopf factorizations of the corresponding generating functions – cf. Theorem 2.1. However, finding the inverses for Wiener–Hopf plus Hankel operators is a much more difficult problem and to the best of our knowledge, so far there was no efficient representation of the corresponding inverses even for the simplest pairs of generating functions. Now we want to establish formulas for the left and right inverses of the operators $W(a) + H(b)$ in the case of matching generating functions.

Let us assume that the operators $W(c)$ and $W(d)$ are invertible from the same side. This condition is not necessary for the one-sided invertibility and note that the corresponding inverses can be also constructed even if the condition mentioned is not satisfied.

Theorem 4.2. *Let* $(a,b) \in G \times G$ *be a matching pair such that the operators* $W(c)$ *and* $W(d)$ *are invertible from the right. Then the operator* $W(a) + H(b)$ *is also right invertible and one of its right inverses has the form*

$$B := (I - H(\tilde{c}))\mathbf{A} + H(a^{-1})W_r^{-1}(d), \tag{30}$$

where $\mathbf{A} = W_r^{-1}(c)W(\tilde{a}^{-1})W_r^{-1}(d)$.

Proof. The proof of this result uses equations (3). Consider the product $(W(a) + H(b))B$,

$$\begin{aligned} &(W(a) + H(b))B \\ &\quad = (W(a) + H(b))(I - H(\tilde{c}))\mathbf{A} + (W(a) + H(b))H(a^{-1})W_r^{-1}(d). \end{aligned} \tag{31}$$

It follows from (3) that

$$\begin{aligned} H(b)H(\tilde{c}) &= W(bc) - W(b)W(c) = W(a) - W(b)W(c), \\ W(a)H(\tilde{c}) &= H(a\tilde{c}) - H(a)W(c) = H(b) - H(a)W(c). \end{aligned}$$

Therefore, the first product in the right-hand side of (31) can be rewritten as

$$\begin{aligned} (W(a) + H(b))(I - H(\tilde{c})\mathbf{A} &= (W(b)W(c) + H(a)W(c))\mathbf{A} \\ &= (W(b) + H(a))W(c)\mathbf{A} \\ &= (W(b) + H(a))W(c)W_r^{-1}(c)W(\tilde{a}^{-1})W_r^{-1}(d) \\ &= W(b)W(\tilde{a}^{-1})W_r^{-1}(d) + H(a)W(\tilde{a}^{-1})W_r^{-1}(d). \end{aligned} \tag{32}$$

Analogously,

$$\begin{aligned} W(a)H(a^{-1}) &= H(aa^{-1}) - H(a)W(\tilde{a}^{-1}) = -H(a)W(\tilde{a}^{-1}), \\ H(b)H(a^{-1}) &= W(b\tilde{a}^{-1}) - W(b)W(\tilde{a}^{-1}) = W(d) - W(b)W(\tilde{a}^{-1}), \end{aligned}$$

and the second product in the right-hand side of (31) has the form

$$\begin{aligned} &(W(a) + H(b))H(a^{-1})W_r^{-1}(d) \\ &\quad = -H(a)W(\tilde{a}^{-1})W_r^{-1}(d) + W(d)W_r^{-1}(d) - W(b)W(\tilde{a}^{-1})W_r^{-1}(d) \\ &\quad = I - H(a)W(\tilde{a}^{-1})W_r^{-1}(d) - W(b)W(\tilde{a}^{-1})W_r^{-1}(d). \end{aligned} \tag{33}$$

Combining (32) and (33), one obtains

$$(W(a) + H(b))B = I,$$

hence B is a right inverse for the Wiener–Hopf plus Hankel operator $W(a) + H(b)$. □

Corollary 4.3. *Let $a, b \in G$ and the indices ν_1, ν_2, n_1 and n_2 satisfy one of the following conditions*:

(i) $\nu_1 < 0$ *and* $\nu_2 < 0$;

(ii) $\nu_1 < 0$, $\nu_2 = 0$ *and* $n_2 \leq 0$;

(iii) $\nu_1 = 0$, $n_1 \leq 0$ *and* $\nu_2 < 0$.

Then $W(a) + H(b)$ is invertible from the right and one of its right inverses can be constructed by formula (30).

Example 4.4. Let us consider the operator

$$\mathbb{W}(\nu_1, \nu_2) = W(e^{i\nu_1 t}) + H(e^{i\nu_2 t}), \quad t \in \mathbb{R}, \tag{34}$$

where ν_1 and ν_2 are real numbers such that

$$\nu_1 - \nu_2 \leq 0, \tag{35}$$

$$\nu_1 + \nu_2 \leq 0. \tag{36}$$

In passing note that the conditions (35)–(36) are equivalent to the inequality

$$\nu_1 \leq -|\nu_2|,$$

so that $\nu_1 \leq 0$. Consider now the generating functions $a(t) = e^{i\nu_1 t}$ and $b(t) = e^{i\nu_2 t}$. They satisfy the matching conditions (5), namely,

$$a(t)a(-t) = b(t)b(-t) = 1.$$

The elements c and d of the subordinated pair for the matching pair $(e^{i\nu_1 t}, e^{i\nu_2 t})$ are

$$c(t) = e^{i(\nu_1 - \nu_2)t}, \quad d(t) = e^{i(\nu_1 + \nu_2)t}.$$

Taking into account the conditions (35)–(36), we observe that the corresponding Wiener–Hopf operators $W(c)$, $W(d)$ are right invertible and have infinite dimensional kernels. In order to construct a right inverse of the operator (34) one can use Theorem 4.2. Let us recall some simple properties of Wiener–Hopf and Hankel operators with exponential generating function. Thus for the generating function $a(t) = e^{i\nu t}$ one has:

(i) if $\nu \leq 0$, then the operator $W(e^{i\nu t})$ is right invertible and one of its right inverses is

$$W_r^{-1}(e^{i\nu t}) = W(e^{-i\nu t});$$

(ii) if $\nu \geq 0$, then the operator $W(e^{i\nu t})$ is left invertible and one of its left inverses is

$$W_l^{-1}(e^{i\nu t}) = W(e^{-i\nu t});$$

(iii) if $\nu < 0$, then $H(e^{i\nu t}) = 0$.

Therefore,

$$W_r^{-1}(c) = W(e^{-i(\nu_1-\nu_2)t}), \quad W_r^{-1}(d) = W(e^{-i(\nu_1+\nu_2)t}).$$

Thus the operator (34) is subject to Theorem 4.2. In order to write the corresponding right inverse of $W(a) + H(b)$, we first determine the operator $\mathbf{A}$. Simple computations show that

$$\mathbf{A} = W(e^{-i(\nu_1-\nu_2)t})W(e^{-i\nu_2 t}).$$

Therefore the right inverse (30) for the operator (34) has the form

$$\begin{aligned}(W(e^{i\nu_1 t}) + H(e^{i\nu_2 t}))_r^{-1} = (I - H(e^{-i(\nu_1-\nu_2)t}))W(e^{-i(\nu_1-\nu_2)t})W(e^{-i\nu_2 t}) \\ + H(e^{-i\nu_1 t})W(e^{-i(\nu_1+\nu_2)t}).\end{aligned}$$

Moreover, using formulas (3), one obtains

$$H(e^{-i(\nu_1-\nu_2)t})W(e^{-i(\nu_1-\nu_2)t}) = 0, \quad H(e^{-i\nu_1 t})W(e^{-i(\nu_1+\nu_2)t}) = W(e^{-i\nu_1 t}),$$

and the operator $(W(e^{i\nu_1 t}) + H(e^{i\nu_2 t}))_r^{-1}$ can be finally written as

$$(W(e^{i\nu_1 t}) + H(e^{i\nu_2 t}))_r^{-1} = H(e^{-i\nu_1 t})W(e^{-i(\nu_1+\nu_2)t}) + W(e^{-i\nu_1 t}).$$

We now construct a left inverse for the operator $W(a) + H(b)$.

Theorem 4.5. *Let $(a, b) \in G \times G$ be a matching pair such that the operators $W(c)$ and $W(d)$ are invertible from the left. Then the operator $\mathbb{W}(a, b) = W(a) + H(b)$ is also left-invertible and one of its left-inverses has the form*

$$\mathbb{W}_l(a, b) = \mathbf{C}(I - H(\tilde{d})) + W_l^{-1}(c)H(\tilde{a}^{-1}), \tag{37}$$

where $\mathbf{C} = W_l^{-1}(c)W(\tilde{a}^{-1})W_l^{-1}(d)$.

Proof. Recalling that the adjoint operator $\mathbb{W}^*(a, b)$ to the operator $W(a) + H(b)$ can be identified with the operator

$$\mathbb{W}^*(a, b) = W(a_1) + H(b_1), \quad a_1 = \bar{a}, \quad b_1 = \bar{\tilde{b}},$$

we note that (a_1, b_1) is a matching pair with the subordinated pair $(c_1, d_1) = (\bar{d}, \bar{c})$. Since $W(c_1) = W(\bar{d})$ and $W(d_1) = W(\bar{c})$ are invertible from the right, the operator $\mathbb{W}^*(a, b)$ is also right-invertible by Theorem 4.2 and according to (30), one of its right inverses can be written as

$$\begin{aligned}(\mathbb{W}^*(a,b))_r^{-1} &= (I - H(\tilde{c}_1)\mathbf{A}_1 + H(a_1^{-1})W_r^{-1}(d_1) \\ &= (I - H(\tilde{\bar{d}})\mathbf{A}_1 + H(\bar{a}^{-1})W_r^{-1}(\bar{c}),\end{aligned} \tag{38}$$

where

$$\mathbf{A}_1 = W_r^{-1}(c_1)W(\tilde{a}_1^{-1})W_r^{-1}(d_1) = W_r^{-1}(\bar{d})W(\tilde{\bar{a}}^{-1})W_r^{-1}(\bar{c}). \tag{39}$$

The left inverse to the operator $\mathbb{W}(a, b)$ can be now obtained by computing the adjoint operator for the operator $(\mathbb{W}^*(a, b))_r^{-1}$. Since for any right-invertible operator A one has

$$(A_r^{-1})^* = (A^*)_l^{-1},$$

we can use the relations

$$W^*(g) = W(\bar{g}), \quad H^*(g) = H(\tilde{\bar{g}}), \quad g \in G,$$

to obtain the representation (37) from (38)-(39). □

5. Generalized invertibility of Wiener–Hopf plus Hankel operators

An operator A is called generalized invertible if there exists an operator A_g^{-1}, referred to as a generalized inverse for A, such that

$$AA_g^{-1}A = A.$$

If A_g^{-1} is a generalized inverse for the operator A and the equation

$$Ax = y \tag{40}$$

is solvable, then the element $x_0 = A_g^{-1}y$ is a solution of equation (40).

Our next task is to determine generalized inverses for Wiener–Hopf plus Hankel operators $W(a) + H(b)$ if the generating functions a and b constitute a matching pair. For this we recall some useful formulas connecting Wiener–Hopf plus Hankel operators and matrix Wiener–Hopf operators. According to [7, eq. (2.4)], the diagonal operator $\operatorname{diag}(W(a) + H(b) + Q, W(a) - H(b) + Q)$ can be represented in the form

$$\begin{pmatrix} W(a) + H(b) + Q & 0 \\ 0 & W(a) - H(b) + Q \end{pmatrix} = \mathcal{J} A_1 A_2 (W(V(a,b)) + \mathcal{Q}) C \mathcal{J}^{-1}, \tag{41}$$

where the operators $A_1, A_2, \mathcal{J}, C = C(a,b)$, and $V = V(a,b)$ are defined by

$$A_1 := \operatorname{diag}(I, I) - \operatorname{diag}(P, Q) W^0 \begin{pmatrix} a & b \\ \tilde{b} & \tilde{a} \end{pmatrix} \operatorname{diag}(Q, P),$$

$$A_2 := \operatorname{diag}(I, I) + \mathcal{P} W^0 (V(a,b)) \mathcal{Q},$$

$$\mathcal{J} := \frac{1}{2} \begin{pmatrix} I & J \\ I & -J \end{pmatrix}$$

$$C(a,b) := \begin{pmatrix} I & 0 \\ W^0(\tilde{b}) & W^0(\tilde{a}) \end{pmatrix},$$

$$V(a,b) := \begin{pmatrix} a - b\tilde{b}\tilde{a}^{-1} & b\tilde{a}^{-1} \\ -\tilde{b}\tilde{a}^{-1} & \tilde{a}^{-1} \end{pmatrix},$$

and

$$\mathcal{P} := \operatorname{diag}(P, P), \quad \mathcal{Q} := \operatorname{diag}(Q, Q).$$

Using the notation

$$B := W(V(a,b)) + \mathcal{Q}, \tag{42}$$

$$\mathcal{R} := \operatorname{diag}(W(a) + H(b), W(a) - H(b)), \quad R := \mathcal{R} + \mathcal{Q},$$

we write the equation (41) as

$$R = \mathcal{J} A_1 A_2 B C \mathcal{J}^{-1}. \tag{43}$$

Considering the operator R and taking into account the equation (43) and the invertibility of the operators $\mathcal{J}, C, A_1$ and A_2, we write

$$R_g^{-1} = \mathcal{J} C^{-1} B_g^{-1} A_2^{-1} A_1^{-1} \mathcal{J}^{-1}.$$

Observe that R_g^{-1} is diagonal since so is the operator R. Thus

$$R_g^{-1} = \operatorname{diag}(\mathbf{F}_g^{-1}, \mathbf{K}_g^{-1}),$$

and it is clear that the diagonal elements $\mathbf{F}_g^{-1}$ and $\mathbf{K}_g^{-1}$ have the form

$$\mathbf{F}_g^{-1} = F_g^{-1} + Q, \quad \mathbf{K}_g^{-1} = K_g^{-1} + Q,$$

where $F_g^{-1}, K_g^{-1} \colon L^p(\mathbb{R}^+) \to L^p(\mathbb{R}^+)$ are generalized inverses for the operators $W(a) + H(b)$ and $W(a) - H(b)$, respectively.

In this section we construct a generalized inverse for the operator $W(a) + H(b)$ provided that the operator B is generalized invertible and a generalized inverse of

B can be represented in a special form. The following theorem has been proved in [10] in the case of Toeplitz plus Hankel operators. For Wiener–Hopf plus Hankel operators the proof literally repeats all constructions there and is omitted here.

Theorem 5.1. *Let (a,b) be a matching pair with the subordinated pair (c,d). Assume that the operator B of (42) is generalized invertible and has a generalized inverse B_g^{-1} of the form*

$$B_g^{-1} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{D} & 0 \end{pmatrix} + \mathcal{Q}, \tag{44}$$

where $\mathbf{A}, \mathbf{B}$ and $\mathbf{D}$ are operators acting in the space $L^p(\mathbb{R}^+)$. Then $W(a)+H(b)$ is generalized invertible and the operator G,

$$G := -H(\tilde{c})(\mathbf{A}(I - H(d)) - \mathbf{B}H(\tilde{a}^{-1})) + H(a^{-1})\mathbf{D}(I - H(d)) + W(a^{-1}), \tag{45}$$

is a generalized inverse for the operator $W(a)+H(b)$.

Lemma 5.2. *Let $(a,b) \in G \times G$ be a matching pair such that one of the following conditions holds*:

(i) *the operators $W(c)$ and $W(d)$ are right invertible*;

(ii) *the operators $W(c)$ and $W(d)$ are left invertible*;

(iii) *$W(c)$ and $W(d)$ are, respectively, left and right invertible operators.*

Then the operator B of (42) is generalized invertible and it has a generalized inverse of the form (44).

Proof. For a matching pair (a,b) the operator $W(V(a,b))$ has the form

$$W(V(a,b)) = \begin{pmatrix} 0 & W(d) \\ -W(c) & W(\tilde{a}^{-1}) \end{pmatrix}.$$

Assume for definiteness that both operators $W(c)$ and $W(d)$ are right invertible. Then the operator $W(V(a,b))$ is also right invertible and it is easily seen that one of its right inverses is given by the formula

$$B_g^{-1} = \begin{pmatrix} W_r^{-1}(c)W(\tilde{a}^{-1})W_r^{-1}(d) & -W_r^{-1}(c) \\ W_r^{-1}(d) & 0 \end{pmatrix} + \mathcal{Q},$$

where $W_r^{-1}(c)$ and $W_r^{-1}(d)$ are right-inverses of the operators $W(c)$ and $W(d)$, correspondingly. Thus in this case, condition (44) is satisfied with the operators

$$\mathbf{A} = W_r^{-1}(c)W(\tilde{a}^{-1})W_r^{-1}(d), \quad \mathbf{B} = -W_r^{-1}(c), \quad \mathbf{D} = W_r^{-1}(d). \tag{46}$$

The other cases are considered analogously. Thus if both operators $W(c)$ and $W(d)$ are left invertible, then B is left invertible with a left-inverse having the form (44), where

$$\mathbf{A} = W_l^{-1}(c)W(\tilde{a}^{-1})W_l^{-1}(d), \quad \mathbf{B} = -W_l^{-1}(c), \quad \mathbf{D} = W_l^{-1}(d), \tag{47}$$

and if $W(c)$ is left-invertible and $W(d)$ is right invertible, then the corresponding operators $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{D}$ in (44) are

$$\mathbf{A} = W_r^{-1}(c)W(\tilde{a}^{-1})W_l^{-1}(d), \quad \mathbf{B} = -W_r^{-1}(c), \quad \mathbf{D} = W_l^{-1}(d), \tag{48}$$

which completes the proof. □

Combining Theorem 5.1 and Lemma 5.2 one obtains the following result.

Theorem 5.3. *Let operators $W(c)$ and $W(d)$ satisfy one of the assumptions of Lemma 5.2 and $\mathbf{A}, \mathbf{B}$, and $\mathbf{D}$ be the operators defined by one of the relations (46)–(48). Then the operator $W(a) + H(b)$ is generalized invertible and (45) is one of its generalized inverses.*

Remark 5.4. We note that in cases (i) and (ii), the operator $W(a) + H(b)$ is one-sided invertible and the formulas for the corresponding inverses obtained in Section 4 are simpler than the representation (45).

6. Invertibility of Wiener–Hopf plus Hankel operators

The results of the previous sections can now be used to establish various invertibility conditions for the operators $W(a) + H(b)$ and write down the corresponding inverses. Let us formulate one of such results and provide a few examples.

Corollary 6.1. *Let (a, b), $a, b \in G$ be a matching pair such that the operators $W(c)$ and $W(d)$ are invertible. Then the operator $W(a) + H(b)$ is invertible and*

$$(W(a)+H(b))^{-1} = (I-H(\tilde{c}))W^{-1}(c)W(\tilde{a}^{-1})W^{-1}(d)+H(a^{-1})W^{-1}(d). \tag{49}$$

Proof. If the operators $W(c)$ and $W(d)$ are invertible, then relations (2.7) and (2.4) of [7] show that the operators $W(a)+H(b)$ is invertible and the result follows from Theorem 4.2. □

Let us point out that this is a very surprising result. There is a vast literature devoted to the study of the Fredholmness and one-sided invertibility of Wiener–Hopf plus Hankel operators in the situation where generating functions satisfy the relation $b = a$ or $\tilde{b} = a$. Of course, such generating functions constitute a matching pair. The other case studied is $a(t) = 1$ for all $t \in \mathbb{R}$ and $b = b(t)$ is a specific matching function. However, to the best of our knowledge, so far there are no efficient representations for the inverse operators. On the other hand, for a wide class of generating functions g the inverse operators $W^{-1}(g)$ can be constructed. Therefore, formula (49) is an efficient tool in constructing the inverse operators $(W(a) + H(b))^{-1}$ in the case where a and b constitute a matching generating pair.

Example 6.2. Let us consider the operators $W(a)+H(b)$ in the case where $a = b$. In this situation $c(t) = 1$ and $d(t) = a(t)\tilde{a}^{-1}(t)$. Hence, $H(\tilde{c}) = 0$, $W(c) = I$ and if the operator $W(d)$ is invertible, then the operator $W(a) + H(a)$ is also invertible and

$$(W(a) + H(a))^{-1} = (W(\tilde{a}^{-1}) + H(a^{-1}))W^{-1}(a\tilde{a}^{-1}).$$

Example 6.3. Let $b = \tilde{a}$. Then $c(t) = a(t)\tilde{a}^{-1}(t)$ and $d(t) = 1$. Hence, if the operator $W(c)$ is invertible, then the operator $W(a) + H(a)$ is also invertible and

$$(W(a) + H(\tilde{a}))^{-1} = (I - H(\tilde{a}a^{-1}))W^{-1}(a\tilde{a}^{-1})W(\tilde{a}^{-1}) + H(a^{-1}).$$

Example 6.4. Let $a(t) = 1$ and $b(t)b(-t) = 1$ for all $t \in \mathbb{R}$. In this situation, $c(t) = \tilde{b}(t)$, $d(t) = b(t)$ and if the operator $W(b)$ is invertible, then

$$(I + H(b))^{-1} = (I - H(b))W^{-1}(\tilde{b})W^{-1}(b).$$

Conclusion

For matching generating functions $a, b \in G$, the invertibility of the operators $W(a) + H(b)$ can be described in terms of indices ν and n of the subordinated functions c and d. Moreover, the corresponding inverses can be represented using only auxiliary Wiener–Hopf and Hankel operators along with the corresponding inverses of scalar Wiener–Hopf operators. This approach is efficient and can be realised as soon as the Wiener–Hopf factorization of the functions c and d is available — cf. (8)–(13).

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