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Partial retraction of "Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian"

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We regret that we have to retract portions of the article "Two-term, asymptotically sharp estimates for eigenvalue means of the Laplacian" [J. Spectral Theory 8 (2018), 1529–1550] due to an essential error in the proof of Theorem 1.2, which is used in other places in the paper.

An error in the proof of Theorem 1.2 was pointed out to us by S. Larson. The proof relies on an average over certain translations, but the parameter L there cannot be chosen independently of the spectral parameter z in order to eliminate the remainder term called $G(z)$ in the proof. Since we have been unable to remedy the error and Theorem 1.2 is used throughout, we retract Theorem 1.2 and all claims depending on it.

Several salient claims of the paper do not depend on the erroneous averaging and remain unaffected. Before listing them we recall some definitions for the reader's convenience:

The eigenvalues of the Neumann Laplacian on a bounded domain Ω are denoted

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots, \tag{1.2}$$

and some related quantities that will appear are

$$m_k := C_d \left(\frac{k}{|\Omega|}\right)^{\frac{2}{d}}, \quad S_k := \frac{\frac{d+2}{d}\frac{1}{k}\sum_{j=1}^{k} \mu_j}{m_k}.$$

The "classical constant" is written $C_d = (2\pi)^2 B_d^{-\frac{2}{d}}$, where $B_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}$ is the volume of the d-dimensional unit ball. Pólya's conjecture for Neumann domains reads

$$\mu_j \le C_d |\Omega|^{-\frac{2}{d}} (j-1)^{-\frac{2}{d}}. \tag{1.5}$$

Claims that remain valid include the following:

Theorem 1.1 (a sharpening of Kröger's inequality). *Let $d \geq 2$. Then for all $k \geq 0$ the Neumann eigenvalue μ_{k+1} satisfies*

$$m_k^2(1 - S_k) \geq (\mu_{k+1} - m_k)^2, \tag{1.13}$$

i.e.,

$$m_k(1 - \sqrt{1 - S_k}) \leq \mu_{k+1} \leq m_k(1 + \sqrt{1 - S_k}). \tag{1.14}$$

Corollary 1.3. *Let $d \geq 2$ and $\Omega = \Omega' \times [0, \delta]$ be a bounded domain. Then for all $z \geq 0$,*

$$\begin{aligned} &\sum_{j=1} (z - \mu_j)_+ \\ &\geq L_{1,d}^{cl} |\Omega| z^{\frac{d}{2}+1} + \frac{1}{2} L_{1,d-1}^{cl} \frac{|\Omega|}{\delta} z^{\frac{d}{2}+\frac{1}{2}} - \frac{1}{24} (2\pi)^{2-d} B_d \frac{|\Omega|}{\delta^2} z^{\frac{d}{2}}. \end{aligned} \tag{1.20}$$

The statement of Corollary 1.4 needs to drop a lower-order nonnegative contribution derived from Theorem 1.2. After correction, it reads:

Corollary 1.4 (Pólya's conjecture for Cartesian products). *Suppose that $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^d$ where $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ are two bounded domains with spectra consisting of increasing eigenvalues satisfying eq.* (1.2)*, and where Pólya's conjecture* (1.5) *holds for Ω_1. Then*

$$\mathcal{N}(z) \geq 1 + |\Omega| L_{0,d}^{cl} z^{\frac{d}{2}}. \tag{1.22}$$

This implies Pólya's conjecture for Ω of the form $\Omega_1 \times \Omega_2$.

Section 3, containing detailed calculations for rectangles, and the Appendix, discussing refinements of Young's and Hölder's inequalities, are entirely independent of Theorem 1.2 and hence unaffected by the error.

Other parts of Section 1 aside from those listed above and Lemma 1.5 (which is from an earlier work), as well as Section 2 and Section 4, can be disregarded.

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