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Different completions of $A + CX$

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Abstract. In this paper we use Takahashi's idea of using properties of spectral measures in addressing the question of the existence of an operator X such that $A+CX$ is of appropriate types. In particular, we consider the class of right semi-Fredholm operators. Also, in the case of right invertibility we will show how the results of this type can be used to address the appropriate completion problem of the operator matrix $M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix}$.

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1. Introduction and motivations

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For simplicity, we also write $\mathcal{B}(\mathcal{H}, \mathcal{H})$ as $\mathcal{B}(\mathcal{H})$.

For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively. Let $n(A) = \dim \mathcal{N}(A)$, $\beta(A) = \operatorname{codim} \mathcal{R}(A)$ and $d(A) = \dim \mathcal{R}(A)^{\perp}$. For subspaces $K, L, M \subseteq \mathcal{H}$, by $K \oplus L = M$ we will denote the fact that $K + L = M$ and $K \cap L = \{0\}$, i.e. that the sum is direct.

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is such that $\mathcal{R}(A)$ is closed and $n(A) < \infty$, then A is said to be a left semi-Fredholm operator. If $\beta(A) < \infty$, then A is called a right semi-Fredholm operator. A semi-Fredholm operator is one which is either left semi-Fredholm or right semi-Fredholm. An operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called Fredholm if it is both left semi-Fredholm and right semi-Fredholm. The subset of $\mathcal{B}(\mathcal{H}, \mathcal{K})$

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consisting of all Fredholm operators is denoted by $\Phi(\mathcal{H},\mathcal{K})$. If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a semi-Fredholm operator, the index of A is defined by $\mathrm{ind}(A) = n(A) - d(A)$. If $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is a semi-Fredholm operator with $\mathrm{ind}(A) = 0$, then A is a Weyl operator.

For $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and subspaces $\mathcal{S}$ of $\mathcal{H}$ and $\mathcal{T}$ of $\mathcal{K}$ we denote

$$A[\mathcal{S}] = \{As : s \in \mathcal{S}\}$$

and

$$A^{-1}[\mathcal{T}] = \{x \in \mathcal{H} : Ax \in \mathcal{T}\}.$$

For $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ define

$$\mathcal{N}(A|C) = \{G \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathcal{R}(AG) \subseteq \mathcal{R}(C)\}.$$

All Hilbert spaces under consideration in this paper are assumed to be infinitedimensional and separable.

The initial motivation for the results presented in this paper was the paper of Takahashi [6] in which he characterizes the pairs of operators (S, R) for which the operator $S + RX$ is invertible for some $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Beside the role this problem plays in the spectrum assignment problems in systems theory, this also allowed him to completely solve the problem of completion of

$$\begin{bmatrix} A & C \\ ? & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}, \tag{1}$$

to invertibility, in the case when $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ are given operators.

The problem of completion of the operator matrix

$$M_X = \begin{bmatrix} A & C \\ X & B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

to an operator of a certain type T, in the case when $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ are given operators, is closely related with the problem of existence of $X \in \mathcal{B}(\mathcal{H},\mathcal{L})$ such that $S + RX$ is of type T, where the operators S, R are appropriately linked with A, B, C. One connection between these two problems follows directly from the equality:

$$\begin{bmatrix} I & -C \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ X & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} = \begin{bmatrix} A - CX & 0 \\ 0 & I \end{bmatrix}.$$

It directly links the existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\left[\begin{smallmatrix} A & C \\ X & I \end{smallmatrix}\right]$ is of an appropriate type to the equivalent problem of existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $A - CX$ is of an appropriate type (see [3]). However it should be noted that the argument will work only in the case when pre and post multiplying with an invertible operator does not the change of type of operators we are interested in. On the other hand, there is a second approach when we first solve the problem of existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $S + RX$ is of a certain type and then, using this result and some more work, succeed to solve a completion problem of M_X to the same type of operators. This usually leads to more elegant and simpler proofs. This is the way that Takahashi proceeded in the case of invertibility. In this paper we will show that Takahashi's technique that uses properties of spectral measures can also be employed for some other completion problems. Using this technique of Takahashi, we characterize all operators $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $R \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ for which there exists an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $S + RX$ is a right semi-Fredholm operator. We will show the advantages that Takahashi's idea has. We will also point out that this technique can only be used for all those classes of operators to which the operator matrix is to be completed that are closed sets with respect to the operator norm. Notice that completing $S + RX$ and M_X to right semi-Fredholmness has already been proved in [3]. There the first approach was used. Namely first the completion problem of M_X to right semi-Fredholm operator was solved and later, as a consequence, the result on the existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $S + RX$ is a right semi-Fredholm operator followed. It is clear that from the solution of the problem of the existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $S + RX$ is of type T the solution of the problem of completion of M_X to an operator of type T will not directly follow, and we will need to do some more work to make a link between these two problems. How we can switch from one problem to the other will be described in the case of right invertible operators.

The analogous problem in the case of injectivity and surjectivity was considered in [2] and [5], respectively.

2. Existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is right semi-Fredholm operator

In this section we characterize all operators $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $R \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ for which there exists an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $S + RX$ is a right semi-Fredholm operator. Notice that the right semi-Fredholmness of $[S \;\; R]: \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ is clearly a necessary condition for the existence of X such that $S + RX$ is right semi-Fredholm.

Theorem 2.1. *Let $S \in \mathcal{B}(\mathcal{H})$ and $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The following hold*:

(1) *if R is compact, then there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is right semi-Fredholm if and only if S is right semi-Fredholm*;

(2) *if R is non-compact, then there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is right semi-Fredholm if and only if $[S \ \ R] \colon \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ is right semi-Fredholm and $\mathcal{N}(S|R)$ contains a non-compact operator.*

Proof. (1) If R is compact and there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is right semi-Fredholm, then S is right semi-Fredholm. If S is right semi-Fredholm, take $X = 0$.

(2) Suppose that R is non-compact and that there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is right semi-Fredholm. Since

$$S + RX = [S \ \ R]\begin{bmatrix} I \\ X \end{bmatrix},$$

it follows that $[S \ \ R]$ is right semi-Fredholm and that $\mathcal{R}\left(\left[\begin{smallmatrix} I \\ X \end{smallmatrix}\right]\right) + \mathcal{N}([S \ \ R])$ is a subspace of finite codimension in $\mathcal{H} \oplus \mathcal{K}$. Now, let $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right] \colon \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ be such that

$$\mathcal{R}\left(\begin{bmatrix} G \\ H \end{bmatrix}\right) = \mathcal{N}([S \ \ R]). \tag{2}$$

Now, we have that $W = \left[\begin{smallmatrix} I & G \\ X & H \end{smallmatrix}\right]$ is right semi-Fredholm. If $\mathcal{N}(S|R)$ contains only compact operators, then G is a compact operator so $W = \left[\begin{smallmatrix} I & 0 \\ X & H \end{smallmatrix}\right]$ is right semi-Fredholm. This implies that H is right semi-Fredholm so there exists $T \in \mathcal{B}(H)$ such that $HT = I + K$ for some compact operator $K \in \mathcal{B}(\mathcal{H})$. Since $SG + RH = 0$, we have that $SGT + R + RK = 0$ i.e. R is a compact operator. Thus we get a contradiction, so $\mathcal{N}(S|R)$ contains a non-compact operator.

For the converse implication, if $[S \ \ R]$ is right semi-Fredholm and $\mathcal{N}(S|R)$ contains a non-compact operator the proof will follow Takahashi's proof for the case of invertibility but we will present it anyway for the sake of completeness. First remark that if $\mathcal{N}(S|R)$ contains a non-compact operator, then we have $\dim \mathcal{N}([S \ \ R]) = \infty$. Let $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right] \colon \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ be left invertible such that

$$\mathcal{R}\left(\begin{bmatrix} G \\ H \end{bmatrix}\right) = \mathcal{N}([S \ \ R]). \tag{3}$$

We will consider three cases.

Case 1: R is left invertible. Evidently, since $R_l^{-1}SG + H = 0$, we get that G is left invertible. Now, there exists an operator $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $H - XG$ is invertible (for example let $X = G_l^{-1} - R_l^{-1}S$). Now, from

$$\begin{bmatrix} I & G \\ X & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} I & G \\ 0 & H - XG \end{bmatrix}$$

we have that $\left[\begin{smallmatrix} I & G \\ X & H \end{smallmatrix}\right]$ is invertible. From

$$[S \ \ R] \begin{bmatrix} I & G \\ X & H \end{bmatrix} = [S + RX \ \ 0],$$

we get that $S + RX$ is right semi-Fredholm. Hence the case when R is left invertible is solved.

Case 2: S is not left semi-Fredholm. Let E be the spectral measure of R^*R and for $x \in \mathcal{K}$ let E_x be the positive measure given by $E_x(S) = (E(S)x, x)$ for a Borel subset S of $\mathbb{C}$. First we will prove that there exists $\epsilon > 0$ such that $\mathcal{K}_\epsilon = \mathcal{R}(E(\epsilon, \infty))$ is infinitedimensional. Let $R_\epsilon = RE(\epsilon, \infty)$. Then

$$\|(R - R_\epsilon)x\|^2 = \int \lambda \chi_{[0,\epsilon]} dE_{x_0}(\lambda) \le \epsilon \|x\|^2,$$

where $x_0 = E[0, \epsilon]x$. If for each $\epsilon > 0$, $\mathcal{K}_\epsilon$ is finitedimensional then we will have that R is a compact as a limit of finitedimensional operators R_ϵ. Hence there exists $\epsilon_0 > 0$ such that $\mathcal{K}_{\epsilon_0} = \mathcal{R}(E(\epsilon_0, \infty))$ is infinitedimensional. Since the set of right semi-Fredholm operators is open in the operator norm topology, we get that for small enough $\epsilon > 0$ such that $\epsilon < \epsilon_0$, the operator $[S \ \ R_\epsilon]$ is right semi-Fredholm i.e. $[S \ \ R|_{\mathcal{K}_\epsilon}]$ is right semi-Fredholm, where $\mathcal{K}_\epsilon$ is an infinitedimensional closed subspace since $\mathcal{K}_{\epsilon_0} \subseteq \mathcal{K}_\epsilon$. Also, we have that

$$\|R_\epsilon x\|^2 = (R^*Rx_0, x_0) = \int \lambda \chi_{(\epsilon,\infty)} dE_{x_0}(\lambda) \ge \epsilon \|x_0\|^2,$$

where $x_0 = E(\epsilon, \infty)x$, implying that $R|_{\mathcal{K}_\epsilon}$ is left invertible. Furthermore, we have $\dim \mathcal{N}([S \ \ R|_{\mathcal{K}_\epsilon}]) = \infty$. In fact, if $\dim \mathcal{N}([S \ \ R|_{\mathcal{K}_\epsilon}]) < \infty$, then $[S \ \ R|_{\mathcal{K}_\epsilon}]$ is left semi-Fredholm, which contradicts the assumption that S is not left semi-Fredholm. Indeed, from $\dim \mathcal{N}([S \ \ R|_{\mathcal{K}_\epsilon}]) < \infty$ we can conclude that $\dim \mathcal{N}(S) < \infty$ and that $\dim \mathcal{R}(S) \cap \mathcal{R}(R|_{\mathcal{K}_\epsilon}) < \infty$ which, using the fact that $[S \ \ R|_{\mathcal{K}_\epsilon}]$ is right semi-Fredholm, implies that $\mathcal{R}(S)$ is closed. Hence S is left semi-Fredholm. Now, by case 1 of this proof there exists $F_1 \in \mathcal{B}(\mathcal{L}, \mathcal{K}_\epsilon)$ such that $S + (R|_{\mathcal{K}_\epsilon})F_1$ is invertible. Define $F = JF_1$ where $J : \mathcal{K}_\epsilon \to \mathcal{K}$ is the inclusion mapping. Then $S + RF$ is right semi-Fredholm.

Case 3: S is left semi-Fredholm. It is sufficient to show that there exists a closed infinitedimensional subspace $\mathcal{K}_1$ of $\mathcal{K}$ such that $R_1 = R|_{\mathcal{K}_1}$ is left invertible and $[S\ R_1]$ is a right semi-Fredholm operator with $\dim \mathcal{N}([S\ \ R_1]) = \infty$. By assumption there exists non-compact $D \in \mathcal{N}(S|R)$ i.e. $SD + RT = 0$, for some $T \in \mathcal{B}(\mathcal{K})$. Thus $\mathcal{R}\left(\left[\begin{smallmatrix} D \\ T \end{smallmatrix}\right]\right) \subseteq \mathcal{N}([S\ \ R])$ which by (3) implies that $\mathcal{R}(D) \subseteq \mathcal{R}(G)$, so G is non-compact. Also, from (3) we have that $SG + RH = 0$ which, together with the fact that S is left semi-Fredholm, implies that RH is non-compact. Now similarly as in the previous case using the spectral measure of $(RH)^*(RH)$, there exists an infinitedimensional subspace $\mathcal{K}_0$ of $\mathcal{K}$ such that $RH|_{\mathcal{K}_0}$ is left invertible. Let $\mathcal{K}_1 = H[\mathcal{K}_0] \oplus \mathcal{R}(H)^{\perp}$. To prove that $\mathcal{K}_1$ is closed it is sufficient to prove that $H[\mathcal{K}_0]$ is closed. Since $RH[K_0]$ is closed it follows that $R^{-1}[RH[\mathcal{K}_0]] = H[\mathcal{K}_0] \oplus \mathcal{N}(R)$ is closed as an inverse image of a closed subspace. Hence $H[\mathcal{K}_0]$ is a closed subspace. Now we have that $\mathcal{K}_1$ is closed. Let us show that it satisfies the required conditions. First, we will prove that $P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}} : \mathcal{R}(H)^{\perp} \to \mathcal{R}(S)^{\perp}$ is injective and right semi-Fredholm. Suppose that for $h \in \mathcal{R}(H)^{\perp}$, $P_{\mathcal{R}(S)^{\perp}} Rh = 0$. Then $Rh \in \mathcal{R}(S)$, so there exists $t \in \mathcal{K}$ such that $Rh = St$. Now $\left[\begin{smallmatrix} t \\ -h \end{smallmatrix}\right] \in \mathcal{N}([S\ \ R])$ implying by (3) that $h \in \mathcal{R}(H)$. Hence, $h = 0$ and we have that $P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}} : \mathcal{R}(H)^{\perp} \to \mathcal{R}(S)^{\perp}$ is injective. Right semi-Fredholmness of $P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}} : \mathcal{R}(H)^{\perp} \to \mathcal{R}(S)^{\perp}$ is established as follows.

Since $[S\ \ R]$ is right semi-Fredholm we have that $P_{\mathcal{R}(S)^{\perp}} R$ is of finite codimension in $\mathcal{R}(S)^{\perp}$. Since $\mathcal{R}(R|_{\mathcal{R}(H)}) \subseteq \mathcal{R}(S)$ (follows by (3)) we have that $\mathcal{R}(P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}}) = \mathcal{R}(P_{\mathcal{R}(S)^{\perp}} R)$ implying that $P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}} : \mathcal{R}(H)^{\perp} \to \mathcal{R}(S)^{\perp}$ is right semi-Fredholm.

Now, by the right semi-Fredholmness of $P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}} : \mathcal{R}(H)^{\perp} \to \mathcal{R}(S)^{\perp}$, having in mind that $\mathcal{R}(R|_{\mathcal{R}(H)^{\perp}}) \subseteq \mathcal{R}(R_1)$ and that $[S\ \ R]$ is right semi-Fredholm, we get that $[S\ \ R_1]$ is right semi-Fredholm, where $R_1 = R|_{\mathcal{K}_1}$.

Left invertibility of $RH|_{\mathcal{K}_0}$ implies that of $R|_{H\mathcal{K}_0} : \mathcal{R}(H\mathcal{K}_0) \to \mathcal{R}(S)$, which together with left invertibility of $P_{\mathcal{R}(S)^{\perp}} R|_{\mathcal{R}(H)^{\perp}}$ implies that $\mathcal{R}_1$ is left invertible.

Since

$$\mathcal{R}\left(\left.\begin{bmatrix} G \\ H \end{bmatrix}\right|_{\mathcal{K}_0}\right) \subseteq \mathcal{N}([S\ \ R_1])$$

and since $\mathcal{K}_0$ is infinitedimensional and $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right]$ is left invertible, it follows that $\dim \mathcal{N}([S\ \ R_1]) = \infty$. □

Notice that the case of left semi-Fredholmness is a much simpler one.

Theorem 2.2. *Let $S \in \mathcal{B}(\mathcal{H})$ and $R \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The following hold:*

(1) *if R is compact, then there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is left semi-Fredholm if and only if S is left semi-Fredholm;*

(2) *if R is non-compact, then there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is left semi-Fredholm.*

Proof. (1) If R is compact and there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is left semi-Fredholm, then S is left semi-Fredholm. If S is left semi-Fredholm, take $X = 0$.

(2) If R is non-compact, then by Theorem 3.2 [1], there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $S + RX$ is left invertible i.e. it is left semi-Fredholm. □

Now we will describe how the result on the existence of $X \in \mathcal{B}(\mathcal{H}, \mathcal{L})$ such that $S + RX$ is of a certain type can be used to solve a completion problem of M_X to the same type of operators. Here we will present such a technique in the case of right invertibility. Namely, this problem was already considered in [4]. Here we will present a much shorter proof with the intention of pointing out the connection between the completion problem for this type of operator matrices and the existence of an operator X for which $A + CX$ is an operator of the appropriate type. The corresponding result on right invertibility is given in [1]. Notice that right invertibility of $[S \;\; R]: \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ is clearly a necessary condition for the existence of X such that $S + RX$ is right invertible.

Theorem 2.3 ([1]). *Let $S, R \in \mathcal{B}(\mathcal{H})$ be such that $[S \;\; R]: \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$ is right invertible. The following hold:*

(1) *if R is compact, there exists $X \in \mathcal{B}(\mathcal{H})$ such that $S + RX$ is right invertible if and only if S is right semi-Fredholm operator and* $\mathrm{ind}(S) \geq 0$;

(2) *if R is non-compact, there exists $X \in \mathcal{B}(\mathcal{H})$ such that $S + RX$ is right invertible if and only if $\mathcal{N}(S|R)$ contains a non-compact operator.*

First we will state an auxiliary result:

Theorem 2.4. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then M_X is right invertible for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ if and only if $[A \;\; C]: \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ is right invertible,* $\dim \mathcal{N}([A \;\; C]) = \infty$ *and $BH + XG$ is a right invertible operator, where $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right]: \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ is a left invertible operator such that*

$$\mathcal{R}\left(\begin{bmatrix} G \\ H \end{bmatrix}\right) = \mathcal{N}([A \;\; C]). \tag{4}$$

Proof. If M_X is right invertible for some $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then there exists $W = \left[\begin{smallmatrix} Y & G \\ Z & H \end{smallmatrix}\right]: \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ such that

$$M_X W = \begin{bmatrix} A & C \\ X & B \end{bmatrix} \begin{bmatrix} Y & G \\ Z & H \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{5}$$

Evidently, we have that $[A \;\; C]$ is right invertible and that $\dim \mathcal{N}([A \;\; C]) = \infty$, since $\mathcal{R}\left(\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right]\right) \subseteq \mathcal{N}([A \;\; C])$ and $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right]$ is left invertible.

Now, let $[G \;\; H]: \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ be a left invertible operator such that

$$\mathcal{R}\left(\begin{bmatrix} G \\ H \end{bmatrix}\right) = \mathcal{N}([A \;\; C]). \tag{6}$$

Since $[A \;\; C]$ is right invertible, there exists an (left invertible) operator

$$\begin{bmatrix} E \\ F \end{bmatrix}: \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{K}$$

such that $AE + CF = I$. Let

$$W = \begin{bmatrix} E & G \\ F & H \end{bmatrix}: \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{H} \oplus \mathcal{K}. \tag{7}$$

Clearly W is invertible and satisfies

$$M_X W = \begin{bmatrix} I & 0 \\ XE + BF & XG + BH \end{bmatrix}. \tag{8}$$

Hence $XG + BH$ is right invertible. The reverse implication follows similarly. □

Theorem 2.5. *Let* $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ *and* $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. *Then there exists* $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *such that* M_X *is right invertible if and only if* $[A \;\; C]: \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ *is right invertible,* $\dim \mathcal{N}([A \;\; C]) = \infty$ *and one of the following conditions is satisfied*:

(i) $\mathcal{N}(A|C)$ *contains a non-compact operator*;

(ii) *the operator* M_0 *is right semi-Fredholm and*

$$d(M_0) \leq n(A) + \dim(\mathcal{R}(A) \cap \mathcal{R}(C|_{\mathcal{N}(B)})).$$

Proof. If M_X is right invertible for some $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then evidently $[A \;\; C]$ is right invertible and $\dim \mathcal{N}([A \;\; C]) = \infty$. Let W be defined by (7). If G is non-compact then evidently (i) holds. So, let us suppose that G is a compact operator.

Since $XG + BH$ is right invertible we have that BH is right semi-Fredholm and that $\mathrm{ind}(()BH) \geq 0$. Since

$$M_0 W = \begin{bmatrix} I & 0 \\ BF & BH \end{bmatrix} \tag{9}$$

it follows that M_0 is right semi-Fredholm and $\mathrm{ind}(M_0) = \mathrm{ind}(BH) \geq 0$. Hence, $d(M_0) \leq n(M_0)$. Also, we can suppose that $\mathcal{N}(C) \cap \mathcal{N}(B) = \{0\}$ since if this is not the case we will consider

$$M'_X = \begin{bmatrix} A & C' \\ X & B' \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K}' \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix},$$

where $\mathcal{K}' = (\mathcal{N}(B) \cap \mathcal{N}(C))^{\perp}$, $B' = B|_{\mathcal{K}'}$ and $C' = C|_{\mathcal{K}'}$ with the remark that M_X is right invertible if and only if M'_X is right invertible. Hence,

$$d(M_0) \leq n(M_0) \leq n(A) + \dim(\mathcal{R}(A) \cap \mathcal{R}(C|_{\mathcal{N}(B)})).$$

For the converse implication, if (i) holds, then there exists $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right] : \mathcal{K} \to \mathcal{H} \oplus \mathcal{K}$ such that $AG + CH = 0$ and G is non-compact. Since G^* is non-compact, then by Theorem 3.2 of [1], there exists $X_0 \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $(BH)^* + G^* X_0^*$ is left invertible i.e. $X_0 G + BH$ is right invertible. Define

$$W = \begin{bmatrix} E & G \\ F & H \end{bmatrix} : \mathcal{H} \oplus \mathcal{K} \longrightarrow \mathcal{H} \oplus \mathcal{K}.$$

where $\left[\begin{smallmatrix} E \\ F \end{smallmatrix}\right] : \mathcal{H} \to \mathcal{H} \oplus \mathcal{K}$ is a right inverse of $[A \;\; C]$. Then

$$M_{X_0} W = \begin{bmatrix} I & 0 \\ X_0 E + BF & X_0 G + BH \end{bmatrix}$$

is a right invertible operator, so M_{X_0} is right invertible.

If (ii) holds, then let W be defined by (7). By (9) we have that BH is right semi-Fredholm and $d(M_0) = d(BH)$, $n(M_0) = n(BH)$. □

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