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Efficiency and localisation for the first Dirichlet eigenfunction

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Abstract. Bounds are obtained for the efficiency or mean to max ratio $E(\Omega)$ for the first Dirichlet eigenfunction (positive) for open, connected sets Ω with finite measure in Euclidean space $\mathbb{R}^m$. It is shown that (i) localisation implies vanishing efficiency, (ii) a vanishing upper bound for the efficiency implies localisation, (iii) localisation occurs for the first Dirichlet eigenfunctions for a wide class of elongating bounded, open, convex and planar sets, (iv) if Ω_n is any quadrilateral with perpendicular diagonals of lengths 1 and n respectively, then the sequence of first Dirichlet eigenfunctions localises and $E(\Omega_n) = O(n^{-2/3}\log n)$. This disproves some claims in the literature. A key technical tool is the Feynman–Kac formula.

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1. Introduction

Let Ω be a non-empty open set in Euclidean space $\mathbb{R}^m$, $m \geq 2$, with boundary $\partial\Omega$ and finite measure $|\Omega|$. It is well known that the spectrum of the Dirichlet Laplacian acting in $L^2(\Omega)$ is discrete and consists of an increasing sequence of eigenvalues

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots,$$

accumulating at infinity. We denote a corresponding orthonormal basis of eigenfunctions by $\{u_{j,\Omega}, j \in \mathbb{N}\}$,

$$-\Delta u_{j,\Omega} = \lambda_j(\Omega) u_{j,\Omega},\ u_{j,\Omega} \in H_0^1(\Omega).$$

If $\lambda(\Omega) := \lambda_1(\Omega)$ has multiplicity 1, then $u_{1,\Omega}$ is uniquely defined up to a sign. This is the case if Ω is connected, for example. We then write and choose, $u_\Omega := u_{1,\Omega} > 0$.

The Rayleigh–Ritz variational principle asserts that

$$\lambda(\Omega) = \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega \varphi^2}. \tag{1}$$

The *efficiency* or *mean to max ratio* of u_Ω is defined by

$$E(\Omega) = \frac{\|u_\Omega\|_1}{|\Omega| \|u_\Omega\|_\infty}, \tag{2}$$

where $\|\cdot\|_p$, $1 \le p \le \infty$ denotes the standard $L^p(\Omega)$ norm.

The study of $E(\Omega)$ goes back to the pioneering results of [16, 19]. In Theorem 3 of [16], it was shown that if Ω is bounded and convex then

$$E(\Omega) \le \frac{2}{\pi}, \tag{3}$$

with equality in (3) if Ω is a bounded interval in $\mathbb{R}$. A non-linear version has been proved in [9] for the p-Laplacian with $1 < p < \infty$. More general results have been obtained in [6]. It follows from inequality (3) and the main theorem in that paper that if Ω is a bounded region in $\mathbb{R}^m$, then

$$E(\Omega) \ge E(B) \frac{|B|}{|\Omega|} \left(\frac{\lambda(B)}{\lambda(\Omega)} \right)^{m/2},$$

where B is a ball in $\mathbb{R}^m$.

Moreover, it was asserted in Table 1 in [16] that $\frac{2}{\pi}$ is the limit of the efficiency of a thinning annulus in $\mathbb{R}^m$. The proof of this assertion (Theorem 11) will be given in Section 4 below. There we will also compute the efficiency for the equilateral triangle, the square, and the disc. These data support the conjectures that (i) the efficiency of a bounded, convex planar set is maximised by the disc, (ii) if $P_n \subset \mathbb{R}^2$ is a regular n-gon then $n \mapsto E(P_n)$ is increasing. We note that the efficiency for an arbitrarily long rectangle is $(2/\pi)^2 \approx 0.4053$, whereas the efficiency of a disc is approximately 0.4317.

Recently a connection has been established between localisation of eigenfunctions and an effective potential such as the inverse of the torsion function (see [1]). In a similar spirit, it has been shown in certain special cases, such as a bounded interval in $\mathbb{R}$ or a square in $\mathbb{R}^2$, that the eigenfunctions of the Schrödinger operator of Anderson type localise (see [2] and [8]).

The first part of the definition below is very similar to the one in [11] (formula (7.1) for $p = 1$).

Definition 1. Let (Ω_n) be a sequence of non-empty open sets in $\mathbb{R}^m$ with $|\Omega_n| < \infty$.

(i) We say that a sequence (f_n) with $f_n \in L^2(\Omega_n)$, $n \in \mathbb{N}$ and $\|f_n\|_2 = 1$ *localises* if there exists a sequence of measurable sets $A_n \subset \Omega_n$ such that

$$\lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 0, \quad \lim_{n\to\infty} \int_{A_n} f_n^2 = 1. \tag{4}$$

(ii) We say that a sequence (f_n) with $f_n \in L^\infty(\Omega_n)$, $f_n \geq 0$, $\|f_n\|_\infty > 0$, $n \in \mathbb{N}$ *has vanishing efficiency* if

$$\lim_{n\to\infty} \frac{\|f_n\|_1}{|\Omega_n|\|f_n\|_\infty} = 0.$$

We have the following elementary observations.

Lemma 2. *If Ω is a non-empty open set with finite Lebesgue measure and if $\|f\|_2 = 1$, $0 < \|f\|_\infty < \infty$, with $f \geq 0$, then*

(i) $$|\Omega|^{-1}\|f\|_\infty^{-2} \leq \frac{\|f\|_1}{|\Omega|\|f\|_\infty} \leq |\Omega|^{-1/2}\|f\|_\infty^{-1}, \tag{5}$$

(ii) $$\frac{\|f\|_1}{|\Omega|\|f\|_\infty} \leq |\Omega|^{-1}\|f\|_1^2. \tag{6}$$

The proofs of (5) and (6) are immediate, since by Cauchy–Schwarz,

$$1 = \|f\|_2 \leq \|f\|_\infty\|f\|_1 \leq \|f\|_\infty|\Omega|^{1/2}.$$

Lemma 3. *For $n \in \mathbb{N}$, let $f_n \in L^2(\Omega_n)$ with $\|f_n\|_2 = 1$, $f_n \geq 0$, and $|\Omega_n| < \infty$. Then (f_n) localises if and only if*

$$\lim_{n\to\infty} \frac{1}{|\Omega_n|}\|f_n\|_1^2 = 0. \tag{7}$$

By (6) we have that if (f_n) is localising then the mean to max ratio of f_n is vanishing as $n \to \infty$. We were unable to prove that if (u_{Ω_n}) has vanishing efficiency then (u_{Ω_n}) localises.

Denote by $\rho(\Omega) = \sup\{\min\{|x-y|: y \in \partial\Omega\}, x \in \Omega\}$ the inradius of Ω, by $\operatorname{diam}(\Omega) = \sup\{|x-y|: x \in \Omega,\ y \in \Omega\}$ the diameter of Ω, and by $w(\Omega)$ the width of Ω. For a measurable set A in $\mathbb{R}^k$ with $k < m$ we denote its k-dimensional Lebesgue measure by $|A|_k$. The indicator function of a set A is denoted by $\mathbf{1}_A$. We define for $\nu \geq 0$, j_ν to be the first positive zero of the Bessel function J_ν.

Below we show that sets with small $E(\Omega)$ have small inradius and large diameter.

Theorem 4. *For all open, connected $\Omega \subset \mathbb{R}^m$ with $0 < |\Omega| < \infty$,*

$$\frac{\rho(\Omega)}{|\Omega|^{1/m}} \le \left(\frac{ej^2_{(m-2)/2}}{2\pi m}\right)^{1/2} E(\Omega)^{1/m}. \tag{8}$$

If Ω is open, planar, bounded, and convex, then

$$\frac{\mathrm{diam}(\Omega)}{|\Omega|^{1/2}} \ge \left(\frac{\pi}{ej_0^2}\right)^{1/2} E(\Omega)^{-1/2}. \tag{9}$$

It is straightforward to construct sequences (Ω_n) for which (u_{Ω_n}) is localising and, as a consequence of Lemma 3 and (6), have vanishing efficiency. For example, let Ω_n be the disjoint union of one disc B with radius 1 and $4n$ discs with radii $1/2$. All of the L^2 mass of the first eigenfunction of Ω_n is supported on B, with $|B|/|\Omega_n| = \frac{1}{n+1}$, which tends to 0 as $n \to \infty$.

Theorem 6 below together with Lemmas 2 and 3, imply localisation for a wide class of sequences (u_{Ω_n}). We first introduce the necessary notation.

Definition 5. Points in $\mathbb{R}^m$ will be denoted by a Cartesian pair (x_1, x') with $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{m-1}$. If Ω is an open set in $\mathbb{R}^m$, then we define

$$\Omega(x_1) = \{x' \in \mathbb{R}^{m-1} \colon (x_1, x') \in \Omega\}.$$

If $\Omega(x_1)$ is open, bounded, and non-empty in $\mathbb{R}^{m-1}$, then we denote its first $(m-1)$-dimensional Dirichlet eigenvalue by $\mu(\Omega(x_1))$. We also put

$$\Omega' = \bigcup_{x_1 \in \mathbb{R}} \Omega(x_1).$$

A set $\Omega \subset \mathbb{R}^m$ is *horn-shaped* if it is open, connected, $x_1 > x_2 > 0$ implies $\Omega(x_1) \subset \Omega(x_2)$, and $x_1 < x_2 < 0$ implies $\Omega(x_1) \subset \Omega(x_2)$.

Theorem 6. *Let $\Omega \subset \mathbb{R}^m$ be horn-shaped with $|\Omega| < \infty$ and $|\Omega'|_{m-1} < \infty$. If $\lambda \ge \lambda(\Omega)$,*

$$\mu(\Omega') \ge (m-1)(\lambda - \mu(\Omega')), \tag{10}$$

and if

$$\varepsilon \in (0, |\Omega|\mu(\Omega')^{m/2}], \tag{11}$$

then

$$\frac{1}{|\Omega|}\|u_\Omega\|_1^2 \le 2\varepsilon + \frac{2|\Omega'|_{m-1}}{|\Omega|}\Big|\Big\{x_1\in\mathbb{R}: \frac{\mu(\Omega(x_1/2))-\mu(\Omega')}{2(\lambda-\mu(\Omega'))} \le \log(\varepsilon^{-1}|\Omega|\mu(\Omega(x_1/2))^{m/2})\Big\}\Big|_1 + \frac{2^{5/2}|\Omega'|_{m-1}}{|\Omega|}(\lambda-\mu(\Omega'))^{-1/2}(\log(\varepsilon^{-1}|\Omega|\mu(\Omega')^{m/2}))^{1/2}. \tag{12}$$

If $\Omega\subset\mathbb{R}^2$ is open, bounded and convex, then it is always possible to find an isometry of Ω such that this isometric set is horn-shaped: let p and q be points on $\partial\Omega$ such that $|p-q| = w(\Omega)$, and $p-q$ is perpendicular to the pair of straight parallel lines tangent to $\partial\Omega$ at both p and q which define the width $w(\Omega)$. That such a pair p, q exists was shown for example in Theorem 1.5 in [5]. Let $T_{p,q}(\Omega) = \{x-\frac{1}{2}(p+q): x\in\Omega\}$ be the translation of Ω which translates the midpoint of p and q to the origin. Let φ be the angle between the positive x_1 axis and the unit vector $(p-q)/|p-q|$ and let R_φ be rotation over an angle $\frac{\pi}{2}-\varphi$. Then $R_\varphi T_{p,q}(\Omega)$ is isometric with Ω, horn-shaped,

$$R_\varphi T_{p,q}(\Omega)' = (-|p-q|/2, |p-q|/2),$$

and

$$|R_\varphi T_{p,q}(\Omega)'|_1 = w(\Omega). \tag{13}$$

The points p and q need not be unique, and so this isometry need not be unique. However, the construction above always gives (13). If Υ is an ellipse with semi axes a_1 and a_2 with $a_1>a_2$ then $R_\varphi T_{p,q}(\Upsilon) = \big\{(x_1,x_2): \big(\frac{x_1}{a_1}\big)^2 + \big(\frac{x_2}{a_2}\big)^2 < 1\big\}$ and $|R_\varphi T_{p,q}(\Upsilon)'|_1 = w(\Upsilon)$. However, the ellipse $\widetilde{\Upsilon} = \big\{(x_1,x_2): \big(\frac{x_1}{a_2}\big)^2 + \big(\frac{x_2}{a_1}\big)^2 < 1\big\}$ is a horn-shaped isometry of Υ with $|\widetilde{\Upsilon}'|_1 > w(\Upsilon)$.

Corollary 7. *Let $\Omega\subset\mathbb{R}^2$ be a convex horn-shaped set. If $\lambda\ge\lambda(\Omega)$ and $\mu(\Omega')\ge\frac{1}{2}\lambda$, then for $\varepsilon\in(0, |\Omega|\mu(\Omega'))$,*

$$\frac{1}{|\Omega|}\|u_\Omega\|_1^2 \le 2\varepsilon + \frac{2|\Omega'|_1}{|\Omega|}\Big|\Big\{x_1\in\mathbb{R}: \frac{|\Omega'|_1^2\mu(\Omega(x_1/2))-\pi^2}{2(|\Omega'|_1^2\lambda-\pi^2)} \le \log(4\pi^2\varepsilon^{-1}|\Omega'|_1^{-2}|\Omega|)\Big\}\Big|_1 + \frac{2^{5/2}|\Omega'|_1^2}{|\Omega|}(|\Omega'|_1^2\lambda-\pi^2)^{-1/2}(\log(\pi^2\varepsilon^{-1}|\Omega'|_1^{-2}|\Omega|))^{1/2}. \tag{14}$$

Example 8. If $(a_n), (b_n), n \in \mathbb{N}$ are sequences in $\mathbb{R}$ satisfying $a_n \in [0,1], b_n \in [0,n]$, and if Ω_n is the quadrilateral with vertices

$$(0, a_n), \quad (0, -1 + a_n), \quad (b_n, 0), \quad (-n + b_n, 0),$$

then

$$\frac{1}{|\Omega_n|}\|u_{\Omega_n}\|_1^2 = O(n^{-2/3}\log n), \quad n \to \infty, \tag{15}$$

and (u_{Ω_n}) is localising.

Example 9. Let $R_n \subset \mathbb{R}^2$ be the rhombus with vertices

$$\Big(\frac{n}{2}, 0\Big), \quad \Big(-\frac{n}{2}, 0\Big), \quad \Big(0, \frac{1}{2}\Big), \quad \Big(0, -\frac{1}{2}\Big),$$

and let Ω_n be an open subset of R_n which contains the open triangle T_n with vertices

$$\Big(\frac{n}{2}, 0\Big), \quad \Big(0, \frac{1}{2}\Big), \quad \Big(0, -\frac{1}{2}\Big).$$

Then Ω_n satisfies (15) and (u_{Ω_n}) is localising.

It follows by scaling properties of both u_Ω and $|\Omega|$ that if Ω is open and connected with $|\Omega| < \infty$ and if $\alpha > 0$, then

$$E(\alpha\Omega) = E(\Omega),$$

where $\alpha\Omega$ is a homothety of Ω by a factor α. Similarly,

$$\frac{1}{|\alpha\Omega_n|}\|u_{\alpha\Omega_n}\|_1^2 = \frac{1}{|\Omega_n|}\|u_{\Omega_n}\|_1^2.$$

Example 9 then implies that a sequence of suitable translations, rotations and hometheties of sectors $(S_n(r))$, with

$$S_n(r) := \{(\rho, \theta) \colon 0 < \rho < r, 0 < \theta < \pi/n\}$$

satisfies

$$\frac{1}{|S_n(r)|}\|u_{S_n(r)}\|_1^2 = O(n^{-2/3}\log n), \quad n \to \infty,$$

and $(u_{S_n(r)})$ localises as $n \to \infty$. This could have been obtained directly using separation of variables, Kapteyn's inequality, and extensive computations involving Bessel functions. See [14] for similar computations.

Example 10. If $1 \le \alpha < \infty$, $m = 2, 3, \ldots$ and

$$\Omega_{n,\alpha} = \{(x_1, x') \in \mathbb{R}^m : (2n^{-1}|x_1|)^\alpha + |x'|^\alpha < 1\}, \quad n \in \mathbb{N},$$

then

$$\frac{1}{|\Omega_{n,\alpha}|} \|u_{\Omega_{n,\alpha}}\|_1^2 = O(n^{-2/(\alpha+2)} (\log n)^{\max\{1/\alpha, 1/2\}}), \quad n \to \infty, \tag{16}$$

and $(u_{\Omega_{n,\alpha}})$ is localising.

Theorem 11. *If* $R > 0$, $\varepsilon > 0$ *and*

$$\Omega_{R,R+\varepsilon} = \{x \in \mathbb{R}^m : R < |x| < R + \varepsilon\},$$

then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \lambda(\Omega_{R,R+\varepsilon}) = \pi^2, \tag{17}$$

and

$$\lim_{\varepsilon \downarrow 0} E(\Omega_{R,R+\varepsilon}) = \frac{2}{\pi}. \tag{18}$$

If $\triangle \subset \mathbb{R}^2$ *is an equilateral triangle, then*

$$E(\triangle) = \frac{2}{\pi\sqrt{3}}. \tag{19}$$

If $\square \subset \mathbb{R}^2$ *is a rectangle, then*

$$E(\square) = \frac{4}{\pi^2}. \tag{20}$$

If $B \subset \mathbb{R}^2$ *is a disc, then*

$$E(B) \approx 0.6782 \frac{2}{\pi}. \tag{21}$$

Inequalities (6.9) in [11] and (4.7) in [15] state that for Ω open, bounded, planar, and convex,

$$u_\Omega(x) \le \min\{|x - y| : y \in \partial\Omega\} \frac{\lambda(\Omega)^{1/2}}{|\Omega|} \|u_\Omega\|_1, \tag{22}$$

and both papers refer to [16] for details. However, no such inequality can be found in [16]. Inequality (22) would, by first maximising its right-hand side over all $x \in \Omega$, and subsequently its left-hand side over all $x \in \Omega$, imply that

$$\|u_\Omega\|_\infty \le \rho(\Omega) \frac{\lambda(\Omega)^{1/2}}{|\Omega|} \|u_\Omega\|_1. \tag{23}$$

Since the Dirichlet eigenvalues are monotone in the domain and Ω contains a disc of radius $\rho(\Omega)$,

$$\lambda(\Omega) \le \frac{j_0^2}{\rho(\Omega)^2}.$$

This, by (2) and (23), implies that for a bounded, planar convex set Ω,

$$E(\Omega) \ge j_0^{-1}. \tag{24}$$

Inequality (23) was also quoted in formula (2.24) in [10]. However, (23) and (24) cannot hold true. Example 8 above implies that $\lim_{n\to\infty} E(\Omega_n) = 0$ for a collection of sequences of convex quadrilaterals (Ω_n). This collection includes a sequence of rhombi with vertices $\big(\frac{n}{2},0\big), \big(-\frac{n}{2},0\big), \big(0,\frac{1}{2}\big), \big(0,-\frac{1}{2}\big)$. This contradicts (24).

This paper is organised as follows. The proofs of Lemma 3 and Theorem 4 are deferred to Section 2 below. The proofs of Theorem 6, Corollary 7, and Examples 8, 9, and 10 will be given in Section 3. The proof of Theorem 11 will be given in Section 4.

2. Proofs of Lemma 3 and Theorem 4

Proof of Lemma 3. First suppose (7) holds. That is if

$$a_n = \frac{1}{|\Omega_n|}\|f_n\|_1^2, \tag{25}$$

then

$$\lim_{n\to\infty} a_n = 0. \tag{26}$$

Let $\alpha > 0$ and define

$$B_{n,\alpha} = \{x \in \Omega_n : f_n(x) > \alpha\}.$$

It follows that

$$\int_{\Omega_n \setminus B_{n,\alpha}} f_n^2 \le \alpha^2 |\Omega_n \setminus B_{n,\alpha}|,$$

and

$$\int_{B_{n,\alpha}} f_n^2 \ge 1 - \alpha^2 |\Omega_n \setminus B_{n,\alpha}| \ge 1 - \alpha^2 |\Omega_n|.$$

Furthermore,

$$\int_{B_{n,\alpha}} f_n \ge \alpha |B_{n,\alpha}|. \tag{27}$$

It follows by (25) and (27) that

$$|B_{n,\alpha}| \le \alpha^{-1} \int_{B_{n,\alpha}} f_n \le \alpha^{-1} \int_{\Omega_n} f_n \le \alpha^{-1} a_n^{1/2} |\Omega_n|^{1/2}.$$

We now choose

$$\alpha = a_n^{1/4} |\Omega_n|^{-1/2},$$

and conclude that

$$\int_{B_{n, a_n^{1/4}|\Omega_n|^{-1/2}}} f_n^2 \ge 1 - a_n^{1/2}, \quad \frac{|B_{n, a_n^{1/4}|\Omega_n|^{-1/2}}|}{|\Omega_n|} \le a_n^{1/4}.$$

Set $A_n = B_{n, a_n^{1/4}|\Omega_n|^{-1/2}}$. Then A_n satisfies (4) by (26).

Next suppose (4) holds. Let $\varepsilon \in (0, 1)$ be arbitrary. There exists $N_\varepsilon \in \mathbb{N}$ such that both

$$\int_{\Omega_n \setminus A_n} f_n^2 < \varepsilon,$$

and $|A_n|/|\Omega_n| < \varepsilon$. So for $n \ge N_\varepsilon$,

$$\begin{aligned}
\frac{1}{|\Omega_n|} \| f_n \|_1^2 &= \frac{1}{|\Omega_n|} \Big(\int_{A_n} f_n + \int_{\Omega_n \setminus A_n} f_n \Big)^2 \\
&\le \frac{2}{|\Omega_n|} \Big(\Big(\int_{A_n} f_n \Big)^2 + \Big(\int_{\Omega_n \setminus A_n} f_n \Big)^2 \Big) \\
&\le \frac{2}{|\Omega_n|} \Big(|A_n| + |\Omega_n \setminus A_n| \int_{\Omega_n \setminus A_n} f_n^2 \Big) \\
&\le 2 \Big(\frac{|A_n|}{|\Omega_n|} + \varepsilon \Big) \\
&\le 4\varepsilon.
\end{aligned}$$

This concludes the proof since $\varepsilon \in (0, 1)$ was arbitrary. □

Proof of Theorem 4. By Lemma 3.1 in [7] we have, taking into account that the estimates there are for the Dirichlet Laplacian with an extra factor $\frac{1}{2}$, that

$$\| u_\Omega \|_\infty^2 \le \Big(\frac{e}{2\pi m} \Big)^{m/2} \lambda(\Omega)^{m/2}. \tag{28}$$

Since Ω contains a ball with inradius $\rho(\Omega)$, we have by domain monotonicity

$$\lambda(\Omega) \le \frac{j^2_{(m-2)/2}}{\rho(\Omega)^2}. \tag{29}$$

By (28) and (29),

$$\|u_\Omega\|_\infty^{-2} \ge \left(\frac{2\pi m}{e j^2_{(m-2)/2}}\right)^{m/2} \rho(\Omega)^m,$$

and (8) follows by (5). By [13] we have that for planar convex sets, $|\Omega| \le 2\,\mathrm{diam}(\Omega)\rho(\Omega)$. This, together with (8), implies (9). □

3. Proofs of Theorem 6, Corollary 7, and Examples 8, 9, 10

To prove Theorem 6 we proceed via a number of lemmas.

Lemma 12. *If Ω is an open set with $|\Omega| < \infty$ and if $\|u_\Omega\|_2 = 1$, then for any $\varepsilon > 0$,*

$$\frac{1}{|\Omega|}\|u_\Omega\|_1^2 \le 2\varepsilon^2|\Omega| + \frac{2}{|\Omega|}|\{x \in \Omega\colon u_\Omega(x) > \varepsilon\}|. \tag{30}$$

Proof. Let

$$\Omega^\varepsilon = \{x \in \Omega\colon u_\Omega \le \varepsilon\}.$$

We have by Cauchy–Schwarz that

$$\begin{aligned}
\frac{1}{|\Omega|}\|u_\Omega\|_1^2 &= \frac{1}{|\Omega|}\left(\int_{\Omega^\varepsilon} u_\Omega + \int_{\Omega\setminus\Omega^\varepsilon} u_\Omega\right)^2 \\
&\le \frac{2}{|\Omega|}\left(\left(\int_{\Omega^\varepsilon} u_\Omega\right)^2 + \left(\int_{\Omega\setminus\Omega^\varepsilon} u_\Omega\right)^2\right) \\
&\le \frac{2}{|\Omega|}\left(\varepsilon^2|\Omega^\varepsilon|^2 + |\Omega \setminus \Omega^\varepsilon|\int_{\Omega\setminus\Omega^\varepsilon} u_\Omega^2\right) \\
&\le 2\varepsilon^2|\Omega| + \frac{2}{|\Omega|}|\{x \in \Omega\colon u_\Omega(x) > \varepsilon\}|.
\end{aligned}$$

□

For a non-empty open set $\Omega \subset \mathbb{R}^m$, we denote by $p_\Omega(x, y; t)$, $x \in \Omega$, $y \in \Omega$, $t > 0$ its Dirichlet heat kernel.

Lemma 13. *If Ω is an open set in $\mathbb{R}^m$ with $0 < |\Omega| < \infty$, then*

$$p_\Omega(x, x; t) \le \left(\frac{e}{2\pi m}\right)^{m/2} \lambda(\Omega)^{m/2} e^{-t\lambda(\Omega)}, \quad t \ge \frac{m}{2\lambda(\Omega)}. \tag{31}$$

Proof. Since $|\Omega| < \infty$, $p_\Omega(x, y; t)$ has an $L^2(\Omega)$ eigenfunction expansion given by

$$\sum_{j=1}^{\infty} e^{-t\lambda_j(\Omega)} u_{j,\Omega}^2(x) = p_\Omega(x, x; t). \tag{32}$$

It follows from (32) that for $\alpha \in [0, 1)$,

$$\begin{aligned} p_\Omega(x, x; t) &= \sum_{j=1}^{\infty} e^{-(\alpha+1-\alpha)t\lambda_j(\Omega)} u_{j,\Omega}^2(x) \\ &\le e^{-\alpha t\lambda(\Omega)} \sum_{j=1}^{\infty} e^{-(1-\alpha)t\lambda_j(\Omega)} u_{j,\Omega}^2(x) \\ &= e^{-\alpha t\lambda(\Omega)} p_\Omega(x, x; (1-\alpha)t) \\ &\le e^{-\alpha t\lambda(\Omega)} p_{\mathbb{R}^m}(x, x; (1-\alpha)t) \\ &= e^{-\alpha t\lambda(\Omega)} (4\pi(1-\alpha)t)^{-m/2}, \end{aligned} \tag{33}$$

where we have used monotonicity of the Dirichlet heat kernel. For $t \ge m/(2\lambda(\Omega))$ we choose α as to optimise the right-hand side of (33). This yields,

$$\alpha = 1 - \frac{m}{2t\lambda(\Omega)},$$

which in turn gives (31). □

The main idea in the proof of Theorem 6 is to use Brownian motion techniques to achieve an efficient way of separation of variables for horn-shaped domains. These have been used extensively elsewhere. See for example [3] and Lemma 7 in [4]. If $\Omega(x_1)$ is open and non-empty then, following Definition 5, we denote corresponding Dirichlet heat kernel by $\pi_{\Omega(x_1)}(x', y'; t)$, $x' \in \Omega(x_1)$, $y' \in \Omega(x_1)$, $t > 0$. We also put $\mu(\emptyset) = \infty$, $\pi_\emptyset(x', y'; t) = 0$.

Lemma 14. *Let Ω be a horn-shaped set in $\mathbb{R}^m$. If $x_1 \in \mathbb{R}$, $x' \in \Omega(x_1)$, then*

$$p_\Omega(x, x; t) \le (4\pi t)^{-1/2} \pi_{\Omega(x_1/2)}(x', x'; t) + (4\pi t)^{-1/2} e^{-x_1^2/(4t)} \pi_{\Omega'}(x', x'; t). \tag{34}$$

Proof. The proof relies on the Feynman–Kac formula ([17]). We have that for any non-empty open set Ω in $\mathbb{R}^m$,

$$p_\Omega(x, y; t) = (4\pi t)^{-m/2} e^{-|x-y|^2/(4t)} \mathbb{P}\Big(\bigcup_{0 \le \tau \le t} x(\tau) \subset \Omega \colon x(0) = x, x(t) = y \Big), \tag{35}$$

where $\{x(\tau),\, 0 \le \tau \le t\}$ is a Brownian bridge on $\mathbb{R}^m$. The term

$$\mathbb{P}\Big(\bigcup_{0\le\tau\le t} x(\tau) \subset \Omega : x(0)=x,\, x(t)=y\Big)$$

in (35) is the conditional probability that the Brownian bridge stays in Ω, conditioned with $x(0)=x$, $x(t)=y$. We write $x(\tau)=(x_1(\tau), x'(\tau))$ with $x_1(0)=x_1$, $x_1(t)=y_1$, $x'(0)=x'$, $x'(t)=y'$, where $\{x_1(\tau), 0 \le \tau \le t\}$, and $\{x'(\tau), 0\le\tau\le t\}$ are independent Brownian bridges.

For $\xi > 0$ and $x_1 < \xi$, $y_1 < \xi$, we have by the reflection principle,

$$p_{(-\infty,\xi)}(x_1, y_1; t) = \frac{1}{(4\pi t)^{1/2}}\big(e^{-(x_1-y_1)^2/(4t)} - e^{-(2\xi-x_1-y_1)^2/(4t)}\big).$$

By (35), for $x_1 < \xi$, $y_1 < \xi$,

$$\mathbb{P}(\max_{0\le\tau\le t} x_1(\tau) \le \xi : x_1(0) = x_1,\, x_1(t) = y_1) = 1 - e^{-(\xi-x_1)(\xi-y_1)/t}.$$

For $x_1 = y_1 = 0$, $\xi > 0$, we have

$$\mathbb{P}(\max_{0\le\tau\le t} x_1(\tau) \le \xi : x_1(0) = x_1(t) = 0) = 1 - e^{-\xi^2/t}.$$

We arrive at the well-known formula for the density of the maximum of a one-dimensional Brownian bridge,

$$\mathbb{P}(\max_{0\le\tau\le t} x_1(\tau) \in d\xi : x_1(0) = x_1(t) = 0) = \frac{2\xi}{t} e^{-\xi^2/t}\mathbf{1}_{[0,\infty)}(\xi)\, d\xi. \tag{36}$$

We first consider the case $x_1 > 0$. By (35) and (36),

$$\begin{aligned}
p_\Omega(x, x; t) &= (4\pi t)^{-m/2}\mathbb{P}\Big(\bigcup_{0\le\tau\le t} x(\tau) \subset \Omega : x(0) = x(t) = x\Big)\\
&\le (4\pi t)^{-m/2}\int_0^{x_1/2} d\xi \frac{2\xi}{t} e^{-\xi^2/t}\mathbb{P}\Big(\bigcup_{0\le\tau\le t} x'(\tau) \subset \Omega(x_1-\xi) :\\
&\qquad\qquad x'(0) = x'(t) = x'\Big)\\
&\quad + (4\pi t)^{-m/2}\int_{x_1/2}^{\infty} d\xi \frac{2\xi}{t} e^{-\xi^2/t}\mathbb{P}\Big(\bigcup_{0\le\tau\le t} x'(\tau) \subset \Omega' :\\
&\qquad\qquad x'(0) = x'(t) = x'\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le (4\pi t)^{-1/2} \int_0^{x_1/2} d\xi \frac{2\xi}{t} e^{-\xi^2/t} \pi_{\Omega(x_1/2)}(x', x'; t) \\
&\quad + (4\pi t)^{-1/2} \int_{x_1/2}^{\infty} d\xi \frac{2\xi}{t} e^{-\xi^2/t} \pi_{\Omega'}(x', x'; t) \\
&\le (4\pi t)^{-1/2} \pi_{\Omega(x_1/2)}(x', x'; t) + (4\pi t)^{-1/2} e^{-x_1^2/(4t)} \pi_{\Omega'}(x', x'; t),
\end{aligned}
\tag{37}
$$

where we have used that $\Omega(x_1 - \xi) \subset \Omega'$ for $\xi \ge x_1/2$ in the third line, and that $\Omega(x_1 - \xi) \subset \Omega(x_1/2)$ for $\xi \in [0, x_1/2)$ in the fourth line. We next consider the case $x_1 < 0$. By (35) and (36),

$$
\begin{aligned}
p_\Omega(x, x; t) &= (4\pi t)^{-m/2} \mathbb{P}\Big(\bigcup_{0 \le \tau \le t} x(\tau) \subset \Omega \colon x(0) = x(t) = x \Big) \\
&\le (4\pi t)^{-m/2} \int_0^{|x_1|/2} d\xi \frac{2\xi}{t} e^{-\xi^2/t} \mathbb{P}\Big(\bigcup_{0 \le \tau \le t} x'(\tau) \subset \Omega(x_1 + \xi) \colon \\
&\qquad\qquad x'(0) = x'(t) = x' \Big) \\
&\quad + (4\pi t)^{-m/2} \int_{|x_1|/2}^{\infty} d\xi \frac{2\xi}{t} e^{-\xi^2/t} \mathbb{P}\Big(\bigcup_{0 \le \tau \le t} x'(\tau) \subset \Omega' \colon \\
&\qquad\qquad x'(0) = x'(t) = x' \Big) \\
&\le (4\pi t)^{-1/2} \int_0^{|x_1|/2} d\xi \frac{2\xi}{t} e^{-\xi^2/t} \pi_{\Omega(x_1/2)}(x', x'; t) \\
&\quad + (4\pi t)^{-1/2} \int_{|x_1|/2}^{\infty} d\xi \frac{2\xi}{t} e^{-\xi^2/t} \pi_{\Omega'}(x', x'; t) \\
&\le (4\pi t)^{-1/2} \pi_{\Omega(x_1/2)}(x', x'; t) + (4\pi t)^{-1/2} e^{-x_1^2/(4t)} \pi_{\Omega'}(x', x'; t),
\end{aligned}
\tag{38}
$$

where we have used that $\Omega(x_1 + \xi) \subset \Omega'$ for $\xi \ge |x_1|/2$ in the third line, and that $\Omega(x_1 + \xi) \subset \Omega(x_1/2)$ for $\xi \in [0, |x_1|/2)$ in the fourth line. Combining (37) and (38) gives (34). □

Proof of Theorem 6. We apply Lemma 13 to the $(m-1)$-dimensional heat kernels $\pi_{\Omega(x_1/2)}$ and $\pi_{\Omega'}$ respectively, and obtain that for

$$t \geq \frac{m-1}{2\mu(\Omega')} \tag{39}$$

both

$$\pi_{\Omega(x_1/2)}(x', x'; t) \leq \left(\frac{e}{2\pi(m-1)}\right)^{(m-1)/2} \mu(\Omega(x_1/2))^{(m-1)/2} e^{-t\mu(\Omega(x_1/2))}, \tag{40}$$

and

$$\pi_{\Omega'}(x'x'; t) \leq \left(\frac{e}{2\pi(m-1)}\right)^{(m-1)/2} \mu(\Omega')^{(m-1)/2} e^{-t\mu(\Omega')}. \tag{41}$$

Indeed, (39) implies

$$t \geq \frac{m-1}{2\mu(\Omega(x_1/2))}$$

by domain monotonicity. For t satisfying (39),

$$(4\pi t)^{-1/2} \leq (\mu(\Omega')/(2\pi(m-1)))^{1/2}, \tag{42}$$

and we obtain, by Lemma 14, (40), (41), and (42), that for t satisfying (39),

$$\begin{aligned} p_\Omega(x, x; t) \leq e^{-1/2}\left(\frac{e}{2\pi(m-1)}\right)^{m/2} &(\mu(\Omega(x_1/2))^{m/2} e^{-t\mu(\Omega(x_1/2))} \\ &+ \mu(\Omega')^{m/2} e^{-x_1^2/(4t) - t\mu(\Omega')}). \end{aligned} \tag{43}$$

Bounding the left-hand side of (32) from below by $e^{-t\lambda} u_\Omega(x)^2$ we find by (43) that if (39) holds, then

$$\begin{aligned} u_\Omega(x)^2 \leq e^{-1/2}\left(\frac{e}{2\pi(m-1)}\right)^{m/2} &(\mu(\Omega(x_1/2))^{m/2} e^{-t(\mu(\Omega(x_1/2)) - \lambda)} \\ &+ \mu(\Omega')^{m/2} e^{-x_1^2/(4t) - t(\mu(\Omega') - \lambda)}). \end{aligned}$$

It follows that if (39) holds, then

$$\begin{aligned} &\{u_\Omega^2(x) \geq \varepsilon^2\} \\ &\quad \subset \left\{x \in \Omega \colon e^{-1/2}\left(\frac{e}{2\pi(m-1)}\right)^{m/2} \mu(\Omega(x_1/2))^{m/2} e^{-t(\mu(\Omega(x_1/2)) - \lambda)} \geq \frac{\varepsilon^2}{2}\right\} \\ &\quad\quad \cup \left\{x \in \Omega \colon e^{-1/2}\left(\frac{e}{2\pi(m-1)}\right)^{m/2} \mu(\Omega')^{m/2} e^{-x_1^2/(4t) - t(\mu(\Omega') - \lambda)} \geq \frac{\varepsilon^2}{2}\right\} \end{aligned}$$

$$= \left\{ x \in \Omega \colon 2^{1/2} e^{-1/4} \left(\frac{e}{2\pi(m-1)} \right)^{m/4} \mu(\Omega(x_1/2))^{m/4} e^{-t(\mu(\Omega(x_1/2))-\lambda)/2} \geq \varepsilon \right\}$$
$$\cup \left\{ x \in \Omega \colon 2^{1/2} e^{-1/4} \left(\frac{e}{2\pi(m-1)} \right)^{m/4} \mu(\Omega')^{m/4} e^{-x_1^2/(8t)-t(\mu(\Omega')-\lambda)/2} \geq \varepsilon \right\}$$
$$:= A_1 \cup A_2, \tag{44}$$

with obvious notation. We choose

$$t = (2(\lambda - \mu(\Omega')))^{-1},$$

and let

$$\varepsilon \in (0, \mu(\Omega')^{m/4}].$$

Then the constraint on t in (39) is satisfied for all Ω satisfying (10). For the above choice of t we have

$$A_1 \subset \left\{ x \in \Omega \colon \frac{\mu(\Omega(x_1/2)) - \mu(\Omega')}{4(\lambda - \mu(\Omega'))} < \log(\varepsilon^{-1} \mu(\Omega(x_1/2))^{m/4}) \right\},$$

$$|A_1| \leq |\Omega'|_{m-1} \left| \left\{ x_1 \in \mathbb{R} \colon \frac{\mu(\Omega(x_1/2)) - \mu(\Omega')}{4(\lambda - \mu(\Omega'))} < \log(\varepsilon^{-1} \mu(\Omega(x_1/2))^{m/4}) \right\} \right|_1, \tag{45}$$

$$A_2 \subset \{ x \in \Omega \colon x_1^2 (\lambda - \mu(\Omega')) < 4 \log(\varepsilon^{-1} \mu(\Omega')^{m/4}) \},$$

and

$$|A_2| \leq 4 |\Omega'|_{m-1} (\lambda - \mu(\Omega'))^{-1/2} (\log(\varepsilon^{-1} \mu(\Omega')^{m/4}))^{1/2}. \tag{46}$$

By (30), (44), (45), and (46), we obtain

$$\frac{1}{|\Omega|} \|u_\Omega\|_1^2 \leq 2\varepsilon^2 |\Omega| + \frac{2|\Omega'|_{m-1}}{|\Omega|} \left| \left\{ x_1 \in \mathbb{R} \colon \frac{\mu(\Omega(x_1/2)) - \mu(\Omega')}{4(\lambda - \mu(\Omega'))} < \log(\varepsilon^{-1} \mu(\Omega(x_1/2))^{m/4}) \right\} \right|_1 + \frac{8|\Omega'|_{m-1}}{|\Omega|} (\lambda - \mu(\Omega'))^{-1/2} (\log(\varepsilon^{-1} \mu(\Omega')^{m/4}))^{1/2}.$$

Substitution of $\varepsilon^2 |\Omega| = \varepsilon'$ and deleting the $'$ yields (12) for all ε satisfying (11). □

Proof of Corollary 7. Let

$$x_1(\Omega)^+ := \sup\{x_1 \colon \Omega(x_1) \neq \emptyset\} < \infty, \; x_1(\Omega)^- := \inf\{x_1 \colon \Omega(x_1) \neq \emptyset\} > -\infty.$$

Let x_Ω^+, x_Ω^- be points of $\partial\Omega$ with x_1 coordinates $x_1(\Omega)^+$ and $x_1(\Omega)^-$ respectively. By convexity Ω contains triangles with bases Ω' and vertices x_Ω^+ and x_Ω^- respectively. Hence for any $x = (x_1, x') \in \Omega$, $\frac{1}{2}x_1(\Omega)^- \le x_1/2 \le \frac{1}{2}x_1(\Omega)^+$, and $\Omega(x_1/2)$ contains a line segment with length at least $\frac{1}{2}|\Omega'|_1$. So

$$\mu(\Omega(x_1/2)) \le 4\mu(\Omega') = \frac{4\pi^2}{|\Omega'|_1^2}.$$

This, together with (12) for $m = 2$, proves (14). □

P. Kröger observed that one can get upper bounds for the first Dirichlet eigenvalue of the circular sector $S_n(r)$ with radius r and opening angle π/n, which have the correct leading term by choosing an optimal rectangle inside the sector [14]. Similar observations were used in the proof of Theorem 1.5 in [5] and also in the proof of Theorem 1.3 in [12].

Proof of Example 8. Theorem 1.5 in [5] implies the existence of a constant $c_1 < \infty$ such that

$$\lambda(\Omega_n) \le \pi^2 + c_1 n^{-2/3}, \quad n \in \mathbb{N}. \tag{47}$$

We note that Ω_n is horn-shaped with respect to the coordinate system which defines it in Example 8. Note that $|\Omega_n'|_1 = 1$. Straightforward computations show,

$$\mu(\Omega_n(x_1)) = \pi^2\Big(1 - \frac{x_1}{b_n}\Big)^{-2}, \qquad 0 < x_1 < b_n,$$

$$\mu(\Omega_n(x_1)) = \pi^2\Big(1 - \frac{|x_1|}{n - b_n}\Big)^{-2}, \quad b_n - n < x_1 < 0,$$

$$\mu(\Omega_n(x_1/2)) \ge \pi^2\Big(1 + \frac{x_1}{b_n}\Big), \qquad 0 < x_1 < b_n, \tag{48}$$

$$\mu(\Omega_n(x_1/2)) \ge \pi^2\Big(1 + \frac{|x_1|}{n - b_n}\Big), \qquad b_n - n < x_1 < 0, \tag{49}$$

and

$$|\Omega_n| = \frac{n}{2}. \tag{50}$$

By (47) we see that (10) holds for all

$$n \ge N_\Omega := \min\{n \in \mathbb{N} : n^{2/3} \ge \pi^{-2} c_1\}.$$

We obtain by Corollary 7 and (47)–(50) that for

$$\lambda = \pi^2 + c_1 n^{-2/3}, \tag{51}$$

$$\frac{2|\Omega_n'|_1}{|\Omega_n|}\left|\left\{x_1 \in \mathbb{R}: \frac{|\Omega_n'|_1^2\mu(\Omega_n(x_1/2)) - \pi^2}{2(|\Omega_n'|_1^2\lambda - \pi^2)} \le \log(4\pi^2\varepsilon^{-1}|\Omega_n'|_1^{-2}|\Omega_n|)\right\}\right|_1 \le 8\pi^{-2}c_1 n^{-2/3}\log(2\pi^2\varepsilon^{-1}n). \tag{52}$$

The third term in the right-hand side of (12) equals by (51),

$$\frac{2^{7/2}}{n^{2/3}}c_1^{-1/2}(\log(2^{-1}\pi^2\varepsilon^{-1}n))^{1/2}. \tag{53}$$

We find for $n \ge N_\Omega$ and $\varepsilon \in (0, 2^{-1}\pi^2 n]$, by (52), (53), and (12),

$$\frac{1}{|\Omega_n|}\|u_{\Omega_n}\|_1^2 \le 2\varepsilon + 8\pi^{-2}c_1 n^{-2/3}\log(2\pi^2\varepsilon^{-1}n) + 2^{7/2}c_1^{-1/2}n^{-2/3}(\log(2^{-1}\pi^2\varepsilon^{-1}n))^{1/2}. \tag{54}$$

Choosing $\varepsilon = n^{-2/3}$ gives that the right-hand side of (54) is $O(n^{-2/3}\log n)$. This implies localisation by Lemma 3, and (15) follows by (6) and (54) for that choice of ε. □

Proof of Example 9. By choosing an optimal rectangle in T_n one shows, similarly to (47), the existence of $c_3 < \infty$ such that $\lambda(T_n) \le \pi^2 + c_3 n^{-2/3}$. By domain monotonicity of the Dirichlet eigenvalues and (51),

$$\lambda(R_n) \le \lambda(\Omega_n) \le \lambda(T_n) \le \pi^2 + c_3 n^{-2/3}. \tag{55}$$

Furthermore,

$$\frac{n}{4} = |T_n| \le |\Omega_n| \le |R_n| = \frac{n}{2}, \quad \mu(R_n') = \pi^2, \quad |R_n'|_1 = 1.$$

By domain monotonicity of the Dirichlet heat kernels, we have for $\lambda \ge \lambda(\Omega_n)$,

$$\begin{aligned} e^{-t\lambda}u_{\Omega_n}(x)^2 &\le e^{-t\lambda(\Omega_n)}u_{\Omega_n}(x)^2 \\ &\le p_{\Omega_n}(x, x; t) \\ &\le p_{R_n}(x, x; t) \\ &\le (4\pi t)^{-1/2}\pi_{R_n(x_1/2)}(x', x'; t) \\ &\quad + (4\pi t)^{-1/2}e^{-x_1^2/(4t)}\pi_{R_n'}(x', x'; t). \end{aligned}$$

Adapting the proof of Theorem 6 from (40) onwards, and adapting Corollary 7, gives for all n sufficiently large, $\lambda \ge \lambda(\Omega_n)$, and $\varepsilon \le \frac{\pi^2 n}{4}$,

$$\begin{aligned} \frac{1}{|\Omega_n|}\|u_{\Omega_n}\|_1^2 \le 2\varepsilon &+ \frac{2|R_n'|_1}{|\Omega_n|}\left|\left\{x_1 \in \mathbb{R}: \frac{|R_n'|_1^2\mu(R_n(x_1/2)) - \pi^2}{2(|R_n'|_1^2\lambda - \pi^2)} \le \log(4\pi^2\varepsilon^{-1}|R_n'|_1^{-2}|\Omega_n|)\right\}\right|_1 \\ &+ \frac{2^{5/2}|R_n'|_1^2}{|\Omega_n|}(|R_n'|_1^2\lambda - \pi^2)^{-1/2}(\log(\pi^2\varepsilon^{-1}|R_n'|_1^{-2}|\Omega_n|))^{1/2} \end{aligned}$$

$$\leq 2\varepsilon + \frac{8}{n}\left|\left\{x_1 \in \mathbb{R}\colon \frac{\mu(R_n'(x_1/2)) - \pi^2}{2(\lambda - \pi^2)} \leq \log(2\pi^2\varepsilon^{-1}n)\right\}\right|_1 + \frac{2^{9/2}}{n}(\lambda-\pi^2)^{-1/2}(\log(2^{-1}\pi^2\varepsilon^{-1}n))^{1/2},$$

where we have used (55). We now choose $\lambda = \pi^2 + c_3 n^{-2/3}$ and use (48) and (49) with $b_n = \frac{n}{2}$. This gives

$$\frac{1}{|\Omega_n|}\|u_{\Omega_n}\|_1^2 \leq 2\varepsilon + 16\pi^{-2}c_3 n^{-2/3}\log(2\pi^2\varepsilon^{-1}n) + 2^{9/2}c_3^{-1/2}n^{-2/3}(\log(2^{-1}\pi^2\varepsilon^{-1}n))^{1/2}.$$

We choose $\varepsilon = n^{-2/3}$ which gives (15). This proves localisation by Lemma 3. □

Proof of Example 10. Theorem 1.5 in [5] implies the existence of a constant $c(\alpha) \in (1,\infty)$ such that

$$\lambda(\Omega_{n,\alpha}) \leq j^2_{(m-2)/2} + c(\alpha)n^{-2\alpha/(\alpha+2)},\ n \in \mathbb{N}, \tag{56}$$

where $\mu(\{x' \in \mathbb{R}^{m-1}\colon |x'| < 1\}) = j^2_{(m-2)/2}$. For $-\frac{n}{2} < x_1 < \frac{n}{2}$, $\Omega(x_1)$ is an $(m-1)$-dimensional disc with radius $(1-(2|x_1|/n)^\alpha)^{1/\alpha}$. Hence,

$$\begin{aligned}\mu(\Omega(x_1/2)) &= j^2_{(m-3)/2}(1-(n^{-1}|x_1|)^\alpha)^{-2/\alpha}\\ &\geq j^2_{(m-3)/2}(1+2\alpha^{-1}(n^{-1}|x_1|)^\alpha),\end{aligned} \tag{57}$$

and

$$|\Omega'_{n,\alpha}|_1 = |\{x' \in \mathbb{R}^{m-1}\colon |x'| < 1\}|_{m-1} = \omega_{m-1},\ |\Omega_{n,\alpha}| = \omega_m n/2, \tag{58}$$

and ω_m is the measure of the ball with radius 1 in $\mathbb{R}^m$. For $\varepsilon \in (0, 2^{-1}\omega_m j^m_{(m-3)/2}n]$, n sufficiently large, and $\lambda = j^2_{(m-2)/2} + c(\alpha)n^{-2\alpha/(\alpha+2)} \geq \lambda(\Omega_{n,\alpha})$, we have

$$\begin{aligned}&4\frac{\omega_{m-1}}{\omega_m n}\left|\left\{x_1 \in \mathbb{R}\colon \frac{\mu(\Omega_{n,\alpha}(x_1/2)) - j^2_{(m-3)/2}}{2(\lambda - j^2_{(m-3)/2})} \leq \log(2j^m_{(m-3)/2}\omega_m\varepsilon^{-1}n)\right\}\right|_1\\ &\leq \frac{4\omega_{m-1}}{\omega_m}(\alpha c(\alpha)/j^2_{(m-3)/2})^{1/\alpha}n^{-2/(\alpha+2)}(\log(2j^m_{(m-3)/2}\omega_m\varepsilon^{-1}n))^{1/\alpha}.\end{aligned} \tag{59}$$

Similarly we find for $\varepsilon \in (0, 2^{-1}\omega_m j^m_{(m-3)/2}n]$ and all n sufficiently large,

$$\begin{aligned}&2^{7/2}\frac{\omega_{m-1}}{\omega_m n}(\lambda - j^2_{(m-3)/2})^{-1/2}(\log(\pi^2\varepsilon^{-1}n))^{1/2}\\ &\leq 2^{7/2}c(\alpha)^{-1/2}n^{-2/(\alpha+2)}(\log(j^m_{(m-3)/2}\omega_m\varepsilon^{-1}n/2))^{1/2}.\end{aligned} \tag{60}$$

Choosing $\varepsilon = n^{-2/(\alpha+2)}$ gives (16) by Corollary 7, and (56)–(60). Lemma 3 and (16) imply localisation. □

4. Proof of Theorem 11

Proof of Theorem 11. Choosing $\varphi(x) = \sin(\pi(|x|-R)/\varepsilon)$ as a test function in (1), we have that

$$\begin{aligned}
\lambda(\Omega_{R,R+\varepsilon}) &\le \frac{\pi^2}{\varepsilon^2}\frac{\int_R^{R+\varepsilon}\cos^2(\pi(r-R)/\varepsilon)r^{m-1}dr}{\int_R^{R+\varepsilon}\sin^2(\pi(r-R)/\varepsilon)r^{m-1}dr} \\
&\le \frac{\pi^2}{\varepsilon^2}\Big(\frac{R+\varepsilon}{R}\Big)^{m-1}\frac{\int_R^{R+\varepsilon}\cos^2(\pi(r-R)/\varepsilon)dr}{\int_R^{R+\varepsilon}\sin^2(\pi(r-R)/\varepsilon)dr} \\
&= \frac{\pi^2}{\varepsilon^2}\Big(\frac{R+\varepsilon}{R}\Big)^{m-1}.
\end{aligned} \tag{61}$$

On the other hand, since the first Dirichlet eigenfunction of $\Omega_{R,R+\varepsilon}$ is radial, $u_{\Omega_{R,R+\varepsilon}}(x) := u(r)$, we have

$$\begin{aligned}
\lambda(\Omega_{R,R+\varepsilon}) &= \frac{\int_R^{R+\varepsilon}u'(r)^2r^{m-1}dr}{\int_R^{R+\varepsilon}u(r)^2r^{m-1}dr} \\
&\ge \Big(\frac{R}{R+\varepsilon}\Big)^{m-1}\frac{\int_R^{R+\varepsilon}u'(r)^2dr}{\int_R^{R+\varepsilon}u(r)^2dr} \\
&\ge \Big(\frac{R}{R+\varepsilon}\Big)^{m-1}\min_{v\in H_0^1(R,R+\varepsilon)\setminus\{0\}}\frac{\int_R^{R+\varepsilon}v'(r)^2dr}{\int_R^{R+\varepsilon}v(r)^2dr} \\
&= \frac{\pi^2}{\varepsilon^2}\Big(\frac{R}{R+\varepsilon}\Big)^{m-1},
\end{aligned} \tag{62}$$

and (17) follows from (61) and (62).

To prove (18) we consider the radial solution $\psi_\varepsilon(|x|) = u_\varepsilon(x)$ of

$$-\Delta u_{\Omega_{R,R+\varepsilon}} = \lambda(\Omega_{R,R+\varepsilon})u_{\Omega_{R,R+\varepsilon}},$$

with zero boundary condition and $\|\psi_\varepsilon\|_\infty = 1$. The function ψ_ε satisfies

$$\psi_\varepsilon'' + \frac{m-1}{r}\psi_\varepsilon' + \lambda_\varepsilon\psi_\varepsilon = 0 \quad \text{in } (R, R+\varepsilon),$$

with boundary condition $\psi_\varepsilon(R) = \psi_\varepsilon(R+\varepsilon) = 0$ and normalisation $\|\psi_\varepsilon\|_\infty = 1$, where $\lambda_\varepsilon = \lambda(\Omega_{R,R+\varepsilon})$. Define

$$\phi_\varepsilon(t) = \psi_\varepsilon(R+\varepsilon t), \quad t \in (0,1).$$

Then ϕ_ε satisfies

$$\begin{cases} \phi_\varepsilon'' + \dfrac{(m-1)\varepsilon}{R+\varepsilon t}\phi_\varepsilon' + \varepsilon^2\lambda_\varepsilon\phi_\varepsilon = 0 & \text{in } (0,1), \\ \phi_\varepsilon(0) = \phi_\varepsilon(1) = 0, \\ \|\phi_\varepsilon\|_\infty = 1. \end{cases} \tag{63}$$

Integrating between the maximum point t_m of ϕ and $t \in (0,1)$, we get that

$$|\phi_\varepsilon'(t)| = \left| \int_{t_m}^{t} \Big(\frac{(m-1)\varepsilon\phi_\varepsilon'(t)}{R+\varepsilon t} + \varepsilon^2\lambda_\varepsilon\phi_\varepsilon(t) \Big) dt \right| \le (m-1)\Big(\frac{2\varepsilon}{R} + \varepsilon^2\lambda_\varepsilon \Big). \tag{64}$$

Hence $\phi_\varepsilon, \phi_\varepsilon'$ are equibounded in $(0,1)$ and, by the Arzelà–Ascoli theorem, ϕ_ε converges uniformly, as $\varepsilon \to 0^+$, to a continuous function $\phi(t)$ in $(0,1)$. From (63) and (64), we also obtain equiboundedness of the second derivatives ϕ_ε''. Hence ϕ_ε converges uniformly to ϕ in C^1. Moreover, we obtain uniform convergence of the second derivatives ϕ_ε''. Passing to the limit in the equation, we infer that ϕ satisfies

$$\begin{cases} \phi'' + \pi^2\phi = 0 & \text{in } (0,1), \\ \phi(0) = \phi(1) = 0, \\ \|\phi\|_\infty = 1. \end{cases}$$

Hence $\phi(t) = \sin(\pi t)$ and

$$\lim_{\varepsilon \downarrow 0} \int_{[0,1]} \phi_\varepsilon(t)dt = \int_{[0,1]} \phi(t)dt = \frac{2}{\pi}. \tag{65}$$

So we obtain

$$E(\Omega_{R,R+\varepsilon}) = |\Omega_{R,R+\varepsilon}|^{-1} \int_{\Omega_{R,R+\varepsilon}} \psi_\varepsilon \ge \Big(\frac{R}{R+\varepsilon} \Big)^{m-1} \int_{[0,1]} \phi_\varepsilon(t)dt,$$

and, by (65),

$$\liminf_{\varepsilon \downarrow 0} E(\Omega_{R,R+\varepsilon}) \ge \frac{2}{\pi}.$$

Similarly we have

$$E(\Omega_{R,R+\varepsilon}) \le \Big(\frac{R+\varepsilon}{R} \Big)^{m-1} \int_{[0,1]} \phi_\varepsilon(t)dt,$$

and, by (65),

$$\limsup_{\varepsilon \downarrow 0} E(\Omega_{R,R+\varepsilon}) \le \frac{2}{\pi}.$$

To prove (19) we consider an equilateral triangle $\triangle$ with vertices at

$$(0,0), \quad (1,0), \quad \Big(\frac{1}{2}, \frac{1}{2}\sqrt{3}\Big).$$

The first Dirichlet eigenfunction is given by (formula (2.1) in [18]),

$$u_\triangle(x_1,x_2) = \sin\Big(\frac{4\pi x_2}{\sqrt{3}}\Big) - \sin\Big(2\pi\Big(x_1 + \frac{x_2}{\sqrt{3}}\Big)\Big) + \sin\Big(2\pi\Big(x_1 - \frac{x_2}{\sqrt{3}}\Big)\Big).$$

We find that $|\triangle| = \frac{\sqrt{3}}{4}$,

$$\|u_\triangle\|_\infty = u(1/2, \sqrt{3}/6) = \frac{3\sqrt{3}}{2},$$

and

$$\|u_\triangle\|_1 = \int_\triangle u(x_1,x_2) dx_1\, dx_2 = \frac{9}{4\pi\sqrt{3}}.$$

This proves (19).

The efficiency of an interval is given by $\frac{2}{\pi}$. Formula (20) follows by separation of variables. More generally if Ω_1 and Ω_2 are open and connected sets in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively, and with finite measures $|\Omega_1|_{m_1}$ and $|\Omega_2|_{m_2}$ respectively, then

$$E(\Omega_1 \times \Omega_2) = E(\Omega_1)E(\Omega_2),$$

where $\Omega_1 \times \Omega_2$ is the Cartesian product in $\mathbb{R}^{m_1+m_2}$.

To prove (21) we let $B = \{x \in \mathbb{R}^2 \colon |x| < 1\}$. Then

$$u_B(r,\theta) = J_0(j_0 r), 0 \le r < 1,\ 0 < \theta \le 2\pi,$$

and

$$\|u_B\|_1 = \int_{[0,1]} dr\, r \int_{[0,2\pi)} d\theta\, J_0(j_0 r) \approx 0.215882(2\pi).$$

Since $\|u_B\|_\infty = J_0(0) = 1$, we have that

$$E(B) \approx 0.6782\frac{2}{\pi}. \qquad \square$$

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References

[1] D. N. Arnold, G. David, M. Filoche, D. Jerison, and S. Mayboroda, Computing spectra without solving eigenvalue problems. *SIAM J. Sci. Comput.* **41** (2019), no. 1, B69–B92. MR 3904434 Zbl 1420.81009

[2] D. N. Arnold, G. David, M. Filoche, Marcel, D. Jerison, and S. Mayboroda, Localization of eigenfunctions via an effective potential. *Comm. Partial Differential Equations* **44** (2019), no. 11, 1186–1216. MR 3995095 Zbl 1432.35061

[3] R. Bañuelos and M. van den Berg, Dirichlet eigenfunctions for horn-shaped regions and Laplacians on cross sections. *J. London Math. Soc.* (2) **53** (1996), no. 3, 503–511. MR 1396714 Zbl 0863.35070

[4] M. van den Berg, On the asymptotics of the heat equation and bounds on traces associated with the Dirichlet Laplacian. *J. Funct. Anal.* **71** (1987), no. 2, 279–293. MR 0880981 Zbl 0632.35016

[5] M. van den Berg, V. Ferone, C. Nitsch, and C. Trombetti, On Pólya's inequality for torsional rigidity and first Dirichlet eigenvalue. *Integral Equations Operator Theory* **86** (2016), no. 4, 579–600. MR 3578040 Zbl 1388.49012

[6] G. Chiti, A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators. *Z. Angew. Math. Phys.* **33** (1982), no. 1, 143–148. MR 0652928 Zbl 0508.35063

[7] E. B. Davies, Properties of the Green's functions of some Schrödinger operators. *J. London Math. Soc.* (2) **7** (1974), 483–491. MR 0342847 Zbl 0271.47003

[8] G. David, M. Filoche, D. Jerison, and S. Mayboroda, A free boundary problem for the localization of eigenfunctions. *Astérisque* **392** (2017), ii+203 pp. MR 3706139 Zbl 1380.49001

[9] F. Della Pietra, G. di Blasio, and N. Gavitone, Sharp estimates on the first Dirichlet eigenvalue of nonlinear elliptic operators via maximum principle. *Adv. Nonlinear Anal.* **9** (2020), no. 1, 278–291. MR 3935873 Zbl 1417.35080

[10] N. Gamara, A. Hasnaoui, and L. Hermi, Max-to-mean ratio estimates for the fundamental eigenfunction of the Dirichlet Laplacian. In R. Sims and D. Ueltschi (eds.), *Entropy and the quantum* II. Lecture notes from the 2nd Arizona School of Analysis with Applications held at the University of Arizona, Tucson, AZ, March 15–19, 2010. Contemporary Mathematics, 552. American Mathematical Society, Providence, R.I., 2011, 61–70. MR 2868041 Zbl 1269.35023

[11] D. S. Grebenkov and B.-T. Nguyen, Geometrical structure of Laplacian eigenfunctions. *SIAM Rev.* **55** (2013), no. 4, 601–667. MR 3124880 Zbl 1290.35157

[12] D. Grieser and D. Jerison, The size of the first eigenfunction of a convex planar domain. *J. Amer. Math. Soc.* **11** (1998), no. 1, 41–72. MR 1470858 Zbl 0896.35092

[13] M. Henk and G. A. Tsintsifas, Some inequalities for planar convex figures. *Elem. Math.* **49** (1994), no. 3, 120–125. MR 1286601 Zbl 0839.52006

[14] P. Kröger, On the ground state eigenfunction of a convex domain in Euclidean space. *Potential Anal.* **5** (1996), no. 1, 103–108. MR 1373834 Zbl 0842.35065

[15] J. R. Kuttler and V. G. Sigillito, Eigenvalues of the Laplacian in two dimensions. *SIAM Rev.* **26** (1984), no. 2, 163–193. MR 0738929 Zbl 0574.65116

[16] L. E. Payne and I. Stakgold, On the mean value of the fundamental mode in the fixed membrane problem. *Applicable Anal.* **3** (1973), 295–306. Collection of articles dedicated to Alexander Weinstein on the occasion of his 75th birthday. MR 0399633 Zbl 0323.35057

[17] D. B. Ray, On spectra of second-order differential operators. *Trans. Amer. Math. Soc.* **77** (1954), 299–321. MR 0066539 Zbl 0058.32901

[18] B. Siudeja, Sharp bounds for eigenvalues of triangles. *Michigan Math. J.* **55** (2007), no. 2, 243–254. MR 2369934 Zbl 1148.35056

[19] R. P. Sperb, *Maximum principles and applications.* Mathematics in Science and Engineering, 157. Academic Press, New York and London, 1981. MR 0615561 Zbl 0454.35001

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