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Jordan chains of elliptic partial differential operators and Dirichlet-to-Neumann maps

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Abstract. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary Γ. It will be shown that the Jordan chains of m-sectorial second-order elliptic partial differential operators with measurable coefficients and (local or non-local) Robin boundary conditions in $L_2(\Omega)$ can be characterized with the help of Jordan chains of the Dirichlet-to-Neumann map and the boundary operator from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$. This result extends the Birman–Schwinger principle in the framework of elliptic operators for the characterization of eigenvalues, eigenfunctions and geometric eigenspaces to the complete set of all generalized eigenfunctions and algebraic eigenspaces.

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1. Introduction

The Dirichlet-to-Neumann map is an important object in the analysis of elliptic partial differential equations since it can be used to describe the spectra of the associated elliptic operators. The principal strategy and advantage is that a spectral problem for a partial differential operator on a domain Ω is reduced to a spectral problem for an operator function on the boundary Γ of this domain, where, very roughly speaking, the Dirichlet and Neumann data can be *measured*. This type of approach to problems in spectral and scattering theory for elliptic partial differential operators was used in the self-adjoint case in, e.g. [10, 14, 15, 16, 32, 34, 35, 48, 45, 47, 48, 51], for non-self-adjoint situations in, e.g. [13, 19, 37, 42], and we also refer the reader to the more abstract contributions [3, 5, 6, 8, 9, 18, 20, 21, 23, 24, 25, 26, 28, 29, 40, 43, 50].

In the present paper we are interested in a characterization of Jordan chains of eigenvalues of elliptic operators. To motivate our investigations let us consider here in the introduction only the special case of a Schrödinger operator $\mathcal{A} = -\Delta + V$ on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$ and with a complex-valued potential $V \in L_\infty(\Omega)$. Later in this paper much more general second-order partial differential expressions $\mathcal{A}$ with measurable coefficients will be considered; see Section 3 for details. The Dirichlet-to-Neumann map $D(\lambda)$ corresponding to $-\Delta + V$ can be defined as a bounded operator $D(\lambda) \colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ by

$$\operatorname{Tr} f_\lambda \longmapsto \gamma_N f_\lambda,$$

where $f_\lambda \in H^1(\Omega)$ is such that $\mathcal{A} f_\lambda = \lambda f_\lambda$. Here $\operatorname{Tr} f_\lambda \in H^{1/2}(\Gamma)$ and $\gamma_N f_\lambda \in H^{-1/2}(\Gamma)$ denote the Dirichlet and Neumann trace of f_λ, respectively, and $\lambda \in \mathbb{C}$ is not an eigenvalue of the Dirichlet realisation A_D of $-\Delta + V$. Assume for simplicity that $B \colon L_2(\Gamma) \to L_2(\Gamma)$ is a bounded operator and consider the (non-local) Robin realisation of $-\Delta + V$ defined by

$$A_B f = -\Delta f + V f, \tag{1.1a}$$
$$\operatorname{dom} A_B = \{ f \in H^1(\Omega) \colon \gamma_N f = B \operatorname{Tr} f \text{ and } -\Delta f + V f \in L_2(\Omega) \}. \tag{1.1b}$$

Note that the resolvents of A_D and A_B are both compact operators in $L_2(\Omega)$ due to the compactness of the embedding $H^1(\Omega) \hookrightarrow L_2(\Omega)$ and hence the spectra of A_D and A_B are discrete. It is well-known and easy to see that for all $\lambda_0 \notin \sigma_p(A_D)$ one has $\lambda_0 \in \sigma_p(A_B)$ if and only if $\ker(D(\lambda_0) - B) \neq \{0\}$. Sometimes this is referred to as a variant of the Birman–Schwinger principle. In fact, if $\lambda_0 \in \sigma_p(A_B)$ and $f_0 \in \operatorname{dom} A_B$ is a corresponding eigenfunction, then $\operatorname{Tr} f_0 \neq 0$ (as otherwise f_0 would be an eigenfunction for A_D at λ_0) and

$$(D(\lambda_0) - B) \operatorname{Tr} f_0 = D(\lambda_0) \operatorname{Tr} f_0 - B \operatorname{Tr} f_0 = \gamma_N f_0 - B \operatorname{Tr} f_0 = 0,$$

and conversely, if $\varphi \in \ker(D(\lambda_0) - B) \setminus \{0\}$, then the unique solution $f_0 \in H^1(\Omega)$ of the boundary value problem $(-\Delta + V) f_0 = \lambda_0 f_0$ with $\operatorname{Tr} f_0 = \varphi$, satisfies $\gamma_N f_0 - B \operatorname{Tr} f_0 = 0$, so that $f_0 \in \operatorname{dom} A_B$ is an eigenfunction of A_B corresponding to λ_0.

In the situation where the potential V is not real-valued or the Robin boundary operator B is not symmetric the Schrödinger operator A_B in (1.1) is m-sectorial, but not self-adjoint in $L_2(\Omega)$. Therefore, in general, the eigenvalues of A_B are not semisimple and besides an eigenvector f_0 also (finitely many) generalized eigenvectors $f_1, \ldots, f_k$ are associated to an eigenvalue λ_0, which form a so-called Jordan chain. It is the main objective of the present paper to analyse the Jordan chains $f_0, f_1, \ldots, f_k$ corresponding to an eigenvalue λ_0 of A_B with the help of the Dirichlet-to-Neumann operator in a similar form as in the above mentioned Birman–Schwinger principle. In fact, using the notion of Jordan chains for holomorphic operator functions due to M.V. Keldysh [39] (see also [44, §11]), it turns out in our main result Theorem 4.1 that $\{f_0, f_1, \ldots, f_k\}$ form a Jordan chain of A_B at $\lambda_0 \in \sigma_p(A_B) \cap \rho(A_D)$ if and only if the corresponding traces $\varphi_0 = \operatorname{Tr} f_0, \varphi_1 = \operatorname{Tr} f_1, \ldots, \varphi_k = \operatorname{Tr} f_k$ form a Jordan chain for the holomorphic $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$-valued operator function $\lambda \mapsto M(\lambda) = D(\lambda) - B$ at λ_0, that is,

$$\sum_{l=0}^{j} \frac{1}{l!} M^{(l)}(\lambda_0)\varphi_{j-l} = 0 \tag{1.2}$$

for all $j \in \{0, \ldots, k\}$, where $M^{(l)}(\lambda_0)$ denotes the l-th derivative of the function M at λ_0. Note that for $j = 0$ the characterization of the eigenvector f_0 in the Birman–Schwinger principle follows from (1.2); see the above discussion or Corollary 4.2.

The structure of this paper is as follows. In Section 2 we briefly recall the notion of Jordan chains for operators and holomorphic operator functions. In Section 3 we introduce the elliptic differential operators and the corresponding Dirichlet-to-Neumann map that is used for the analysis of the algebraic eigenspaces. Here we treat second-order divergence form elliptic operators with (complex) L_∞-coefficients of the form

$$\mathcal{A} = -\sum_{k,l=1}^{d} \partial_k c_{kl} \partial_l + \sum_{k=1}^{d} c_k \partial_k - \sum_{k=1}^{d} \partial_k b_k + c_0$$

on bounded Lipschitz domains with non-local Robin boundary conditions. In this general situation it is necessary to pay special attention to the definition and properties of the co-normal and adjoint co-normal derivative, and to the properties of the corresponding sesquilinear forms and operators. Furthermore, the unique solvability of the homogeneous and inhomogeneous Dirichlet boundary value problems is discussed. For the convenience of the reader we provide proofs of these preparatory results in Section 3. Our main result on the characterization of Jordan chains of second-order elliptic partial differential operators with local or non-local Robin boundary conditions via Jordan chains of the Dirichlet-to-Neumann map $\lambda \mapsto D(\lambda)$ and the boundary operator B is formulated and proved in Section 4. The proof is technical and requires the preparatory Lemma 4.5. Finally, in Subsection 5.1 we discuss a more regular situation in which the bounded domain Ω is assumed to have a C^2-smooth boundary and the coefficients of the elliptic operator are slightly more regular. In this setting one then obtains a Dirichlet-to-Neumann operator acting from $H^{3/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ and a variant of Theorem 4.1 for $H^2(\Omega)$-smooth Jordan chains. In Subsection 5.2 we reconsider the Dirichlet-to-Neumann operator on a Lipschitz domain, but now we treat the Dirichlet-to-Neumann operator acting from $H^1(\Gamma)$ into $L_2(\Gamma)$. For this we require a smoothness and symmetry condition on the principal coefficients.

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2. Jordan chains of operators and holomorphic operator functions

Throughout this paper the field is the complex numbers. Let A be a linear operator in a Banach space $\mathcal{H}$. Further, let $k \in \mathbb{N}_0$, $f_0, \ldots, f_k \in \mathcal{H}$ and $\lambda_0 \in \mathbb{C}$. Then we say that the vectors $\{f_0, \ldots, f_k\}$ form a *Jordan chain* for A at λ_0 if $f_j \in \operatorname{dom} A$ for all $j \in \{0, \ldots, k\}$ satisfy

$$(A - \lambda_0) f_j = f_{j-1}$$

for all $j \in \{0, \ldots, k\}$ with $f_0 \neq 0$ and we set $f_{-1} = 0$. The vector f_0 is called an *eigenvector* of A at the *eigenvalue* λ_0 and the vectors $f_1, \ldots, f_k$ are said to be *generalized eigenvectors* of A at λ_0. Note that the generalized eigenvectors are all nonzero.

The notion of Jordan chains exists also for holomorphic operator functions and goes back to the work of M.V. Keldysh [39], for more details we also refer the reader to the monograph [44, §11]. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Banach spaces, $\mathcal{O} \subset \mathbb{C}$ an open set and for all $\lambda \in \mathcal{O}$ let $M(\lambda) \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$. Assume, in addition, that the operator function $\lambda \mapsto M(\lambda)$ is holomorphic on $\mathcal{O}$ and denote the l-th derivative of $M(\cdot)$ at $\lambda \in \mathcal{O}$ by $M^{(l)}(\lambda)$. Let $k \in \mathbb{N}_0$ and $\varphi_0, \ldots, \varphi_k \in \mathcal{H}_1$. Then we say that the vectors $\{\varphi_0, \ldots, \varphi_k\}$ form a *Jordan chain* for the function $M(\cdot)$ at $\lambda_0 \in \mathcal{O}$ if

$$\sum_{l=0}^{j} \frac{1}{l!} M^{(l)}(\lambda_0) \varphi_{j-l} = 0$$

for all $j \in \{0, \ldots, k\}$ and $\varphi_0 \neq 0$. The vector φ_0 is called an *eigenvector* of the operator function $M(\cdot)$ at the *eigenvalue* λ_0 and the vectors $\varphi_1, \ldots, \varphi_k$ are said to be *generalized eigenvectors* of $M(\cdot)$ at λ_0.

Observe that in the special case $\mathcal{H}_1 = \mathcal{H}_2$ and $C \in \mathcal{L}(\mathcal{H}_1)$ the notion of Jordan chain for the operator C at $\lambda_0 \in \mathbb{C}$ and the notion of Jordan chain for the function $\lambda \mapsto C - \lambda$ at $\lambda_0 \in \mathbb{C}$ coincide.

3. Elliptic differential operators and Dirichlet-to-Neumann maps

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary Γ. By $H^1(\Omega)$ we denote the L_2-based Sobolev space of order 1 on Ω and $H_0^1(\Omega)$ denotes the closure of the compactly supported $C_c^\infty(\Omega)$-functions in $H^1(\Omega)$. On the Lipschitz boundary Γ the Sobolev space $H^{1/2}(\Gamma)$ of order $1/2$ will play an important role. Its dual is denoted by $H^{-1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle$ stands for the extension of the $L_2(\Gamma)$ inner product onto the pair $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Recall from [46] Theorem 3.37 that there is a continuous *trace map* $\operatorname{Tr}\colon H^1(\Omega) \to H^{1/2}(\Gamma)$ such that $\operatorname{Tr} f = f|_\Gamma$ for all $f \in H^1(\Omega) \cap C^1(\overline{\Omega})$ and it admits a bounded right inverse.

For all $k, l \in \{1, \ldots, d\}$ fix $c_{kl}, b_k, c_k, c_0 \in L_\infty(\Omega)$. We recall that the field is the complex numbers, so we emphasise that all coefficients are complex valued. Assume that there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} c_{kl}(x) \xi_k \overline{\xi_l} \geq \mu \, |\xi|^2$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Define the sesquilinear form $\mathfrak{a}\colon H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(f, g) = \sum_{k,l=1}^{d} \int_\Omega c_{kl} (\partial_l f) \overline{\partial_k g} + \sum_{k=1}^{d} \int_\Omega c_k (\partial_k f) \bar{g} + \sum_{k=1}^{d} \int_\Omega b_k f \overline{\partial_k g} + \int_\Omega c_0 f \bar{g}.$$

The form $\mathfrak{a}$ is continuous in the sense that there exists an $M \geq 0$ such that $|\mathfrak{a}(f, g)| \leq M \, \|f\|_{H^1(\Omega)} \, \|g\|_{H^1(\Omega)}$ for all $f, g \in H^1(\Omega)$. One verifies in the same way as in the proof of [2] Lemma 3.7 that the form is elliptic and hence [4] Lemma 3.1 implies that $\mathfrak{a}$ is a closed sectorial form.

Introduce $\mathcal{A}\colon H^1(\Omega) \to (H_0^1(\Omega))^*$ by

$$\langle \mathcal{A} f, g \rangle_{(H_0^1(\Omega))^* \times H_0^1(\Omega)} = \mathfrak{a}(f, g).$$

In order to introduce the co-normal derivative we need a lemma. Note that the ellipticity condition on the principal coefficients is not needed in the next lemma.

Lemma 3.1. *Let $f \in H^1(\Omega)$ and suppose that $\mathcal{A} f \in L_2(\Omega)$. Then there exists a unique $\psi \in H^{-1/2}(\Gamma)$ such that*

$$\mathfrak{a}(f, g) - (\mathcal{A} f, g)_{L_2(\Omega)} = \langle \psi, \operatorname{Tr} g \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$

for all $g \in H^1(\Omega)$. Moreover, there exists a constant $c > 0$, independent of f, such that $\|\psi\|_{H^{-1/2}(\Gamma)} \leq c(\|f\|_{H^1(\Omega)} + \|\mathcal{A} f\|_{L_2(\Omega)})$.

Proof. Define $F \colon H^1(\Omega) \to \mathbb{C}$ by $F(g) = \mathfrak{a}(f,g) - (\mathcal{A} f, g)_{L_2(\Omega)}$. Then F is anti-linear and bounded. Explicitly, there exists an $M \geq 0$, independent of f, such that

$$\|F\|_{H^1(\Omega)^*} \leq M \, \|f\|_{H^1(\Omega)} + \|\mathcal{A} f\|_{L_2(\Omega)}.$$

Moreover, $F(g) = 0$ for all $g \in H^1_0(\Omega)$. Hence there exists a unique anti-linear $\widetilde{F} \colon H^{1/2}(\Gamma) \to \mathbb{C}$ such that $\widetilde{F}(\operatorname{Tr} g) = F(g)$ for all $g \in H^1(\Omega)$. The map $\widetilde{F}$ is bounded and $\|\widetilde{F}\|_{H^{1/2}(\Gamma)^*} \leq \|F\|_{H^1(\Omega)^*} \, \|Z\|$, where $Z \colon H^{1/2}(\Gamma) \to H^1(\Omega)$ is a bounded right inverse of Tr. Write $\psi = \widetilde{F} \in H^{1/2}(\Gamma)^* = H^{-1/2}(\Gamma)$. Then $\widetilde{F}(\varphi) = \langle \psi, \varphi \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$ for all $\varphi \in H^{1/2}(\Gamma)$ and the lemma follows. □

If $f \in H^1(\Omega)$ with $\mathcal{A} f \in L_2(\Omega)$, then we denote by $\gamma_N f \in H^{-1/2}(\Gamma)$ the function such that

$$\mathfrak{a}(f,g) - (\mathcal{A} f, g)_{L_2(\Omega)} = \langle \gamma_N f, \operatorname{Tr} g \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$

for all $g \in H^1(\Omega)$. We call $\gamma_N f$ the *co-normal derivative* of f.

Denote by $\mathfrak{a}_D$ the restriction of $\mathfrak{a}$ to $H^1_0(\Omega) \times H^1_0(\Omega)$. Then $\mathfrak{a}_D$ is a continuous elliptic form and hence a closed sectorial form (cf. [4] Lemma 3.1.) Denote by A_D the m-sectorial operator associated with the form $\mathfrak{a}_D$. It follows that A_D is the Dirichlet realisation of $\mathcal{A}$ in $L_2(\Omega)$ given by

$$A_D f = \mathcal{A} f, \quad \operatorname{dom} A_D = \{ f \in H^1_0(\Omega) \colon \mathcal{A} f \in L_2(\Omega) \}.$$

Lemma 3.2. *Let $\lambda \in \rho(A_D)$. Then the following assertions hold.*

(a) *For all $\varphi \in H^{1/2}(\Gamma)$ there exists a unique solution $f \in H^1(\Omega)$ of the homogeneous boundary value problem*

$$(\mathcal{A} - \lambda) f = 0 \quad \textit{and} \quad \operatorname{Tr} f = \varphi. \tag{3.1}$$

Moreover, the map $\varphi \mapsto f$ is continuous from $H^{1/2}(\Gamma)$ into $H^1(\Omega)$.

(b) *For all $\varphi \in H^{1/2}(\Gamma)$ and all $h \in L_2(\Omega)$ there exists a unique solution $f \in H^1(\Omega)$ of the inhomogeneous boundary value problem*

$$(\mathcal{A} - \lambda) f = h \quad \textit{and} \quad \operatorname{Tr} f = \varphi. \tag{3.2}$$

Proof. (a) The existence follows as in the proof of [7] Lemma 2.1. For completeness we give the details. There exists a $T \in \mathcal{L}(H^1_0(\Omega))$ such that

$$(Tf, g)_{H^1_0(\Omega)} = \mathfrak{a}_D(f,g) - \lambda (f,g)_{L_2(\Omega)}$$

for all $f, g \in H^1_0(\Omega)$. Further there exists an $\omega > 0$ such that the sesquilinear form $\mathfrak{b} \colon H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ given by $\mathfrak{b}(f,g) = \mathfrak{a}_D(f,g) - \lambda (f,g)_{L_2(\Omega)} + \omega (f,g)_{L_2(\Omega)}$

is coercive. Let $j\colon H_0^1(\Omega) \to L_2(\Omega)$ be the (compact) inclusion map. Then $\mathfrak{b}(f,g) = ((T+K)f,g)_{H_0^1(\Omega)}$ for all $f,g \in H_0^1(\Omega)$, where $K = \omega j^* j$. So $T+K$ is invertible by the Lax–Milgram theorem. Consequently T is a Fredholm operator because K is compact. Now T is injective since $\lambda \in \rho(A_D)$. Hence T is surjective.

There exists an $f_0 \in H^1(\Omega)$ such that $\operatorname{Tr} f_0 = \varphi$. Hence there exists an $h \in H_0^1(\Omega)$ such that $(Th,g)_{H_0^1(\Omega)} = \mathfrak{a}(f_0,g) - \lambda(f_0,g)_{L_2(\Omega)}$ for all $g \in H_0^1(\Omega)$. Then $f = f_0 - h$ satisfies

$$\begin{aligned}&\langle \mathcal{A}f - \lambda f, g\rangle_{(H_0^1(\Omega))^* \times H_0^1(\Omega)}\\ &\qquad = \mathfrak{a}(f_0,g) - \lambda(f_0,g)_{L_2(\Omega)} - \mathfrak{a}_D(h,g) + \lambda(h,g)_{L_2(\Omega)} = 0\end{aligned}$$

and hence $(\mathcal{A} - \lambda)f = 0$. The uniqueness is easy. The continuity of the map follows from the closed graph theorem.

(b) By (a) there exists an $f_0 \in H^1(\Omega)$ such that $(\mathcal{A}-\lambda)f_0 = 0$ and $\operatorname{Tr} f_0 = \varphi$. Then $f_0 + (A_D - \lambda)^{-1}h$ is a solution to the problem (3.2). Again the uniqueness is easy. ☐

Let $\lambda \in \rho(A_D)$. Now we are able to define the *Dirichlet-to-Neumann operator* $D(\lambda)\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. Let $\varphi \in H^{1/2}(\Gamma)$. By Lemma 3.2(a) there exists a unique solution $f \in H^1(\Omega)$ of the homogeneous boundary value problem (3.1). Then $\mathcal{A}f = \lambda f \in L_2(\Omega)$. Hence one can define

$$D(\lambda)\varphi = \gamma_N f.$$

Then $D(\lambda)$ is bounded operator from $H^{1/2}(\Gamma)$ into $H^{-1/2}(\Gamma)$ by the last parts of Lemmas 3.1 and 3.2(a).

We need two holomorphy results.

Lemma 3.3. (a) *Let* $\varphi \in H^{1/2}(\Gamma)$. *For all* $\lambda \in \rho(A_D)$ *let* $g_\lambda \in H^1(\Omega)$ *be the unique element such that* $(\mathcal{A}-\lambda)g_\lambda = 0$ *and* $\operatorname{Tr} g_\lambda = \varphi$. *Then the map* $\lambda \mapsto g_\lambda$ *is holomorphic from* $\rho(A_D)$ *into* $H^1(\Omega)$.

(b) $\lambda \mapsto D(\lambda)$ *is holomorphic from* $\rho(A_D)$ *into* $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$.

Proof. (a) Fix $\lambda_0 \in \rho(A_D)$. By Lemma 3.2(a) there exists a unique $g_{\lambda_0} \in H^1(\Omega)$ such that $(\mathcal{A}-\lambda_0)g_{\lambda_0} = 0$ and $\operatorname{Tr} g_{\lambda_0} = \varphi$. Let $\lambda \in \rho(A_D)$ and consider

$$g = (1 + (\lambda - \lambda_0)(A_D - \lambda)^{-1})g_{\lambda_0} \in H^1(\Omega). \tag{3.3}$$

Then $(\mathcal{A}-\lambda)g = (\mathcal{A}-\lambda)g_{\lambda_0} + (\lambda-\lambda_0)g_{\lambda_0} = 0$ and $\operatorname{Tr} g = \operatorname{Tr} g_{\lambda_0} = \varphi$.

Since the solution of the homogeneous boundary value problem $(\mathcal{A}-\lambda)f = 0$ with $\operatorname{Tr} f = \varphi$, is unique by Lemma 3.2(a) it follows that $g = g_\lambda$. Now the holomorphy of the resolvent $\lambda \mapsto (A_D-\lambda)^{-1}$ in (3.3) implies that the map $\lambda \mapsto g_\lambda$ is holomorphic from $\rho(A_D)$ into $H^1(\Omega)$.

(b) Let $\varphi \in H^{1/2}(\Gamma)$ and $h \in H^1(\Omega)$. For all $\lambda \in \rho(A_D)$ let $g_\lambda \in H^1(\Omega)$ be as in (a). Then

$$\begin{aligned}\langle D(\lambda)\varphi, \operatorname{Tr} h\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} &= \langle \gamma_N g_\lambda, \operatorname{Tr} h\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ &= \mathfrak{a}(g_\lambda, h) - (\mathcal{A} g_\lambda, h)_{L_2(\Omega)}\\ &= \mathfrak{a}(g_\lambda, h) - \lambda (g_\lambda, h)_{L_2(\Omega)}\end{aligned}$$

for all $\lambda \in \rho(A_D)$. Since $\lambda \mapsto g_\lambda$ is holomorphic from $\rho(A_D)$ into $H^1(\Omega)$ by (a), it follows that $\lambda \mapsto D(\lambda)$ is holomorphic with respect to the weak operator topology on $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$, and therefore it is also holomorphic with respect to the uniform operator topology. □

For all $l \in \mathbb{N}$ we denote the l-th derivative of $\lambda \mapsto D(\lambda)$ at $\lambda \in \rho(A_D)$ by $D^{(l)}(\lambda)$. Then according to Lemma 3.3(b) one has

$$D^{(l)}(\lambda) \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$$

for all $\lambda \in \rho(A_D)$.

The *dual form* $\mathfrak{a}^*$ of $\mathfrak{a}$ is defined by $\operatorname{dom}(\mathfrak{a}^*) = H^1(\Omega)$ and $\mathfrak{a}^*(f,g) = \overline{\mathfrak{a}(g,f)}$ for all $f, g \in H^1(\Omega)$. So

$$\mathfrak{a}^*(f,g) = \sum_{k,l=1}^{d} \int_\Omega \overline{c_{lk}} (\partial_l f) \overline{\partial_k g} + \sum_{k=1}^{d} \int_\Omega \overline{b_k} (\partial_k f) \bar{g} + \sum_{k=1}^{d} \int_\Omega \overline{c_k} f \overline{\partial_k g} + \int_\Omega \overline{c_0} f \bar{g}.$$

Obviously $\mathfrak{a}^*$ is of the same type as $\mathfrak{a}$, with c_{kl} replaced by $\overline{c_{lk}}$, etc. Similar to the definition of $\mathcal{A}$ with respect to $\mathfrak{a}$, we can define the operator $\widetilde{\mathcal{A}} \colon H^1(\Omega) \to (H^1_0(\Omega))^*$ by

$$\langle \widetilde{\mathcal{A}} f, g\rangle_{(H^1_0(\Omega))^* \times H^1_0(\Omega)} = \mathfrak{a}^*(f,g).$$

As in Lemma 3.1 it follows that for all $f \in H^1(\Omega)$ with $\widetilde{\mathcal{A}} f \in L_2(\Omega)$, there exists a unique $\tilde{\gamma}_N f \in H^{-1/2}(\Gamma)$ such that

$$\mathfrak{a}^*(f,g) - (\widetilde{\mathcal{A}} f, g)_{L_2(\Omega)} = \langle \tilde{\gamma}_N f, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$$

for all $g \in H^1(\Omega)$. Using all definitions it is easy to prove the following version of Green's second identity.

Lemma 3.4. *Let* $f, g \in H^1(\Omega)$ *and suppose that* $\mathcal{A} f, \widetilde{\mathcal{A}} g \in L_2(\Omega)$. *Then,*

$$\begin{aligned}(\mathcal{A} f, g)_{L_2(\Omega)} &- (f, \widetilde{\mathcal{A}} g)_{L_2(\Omega)} \\ &= \langle \operatorname{Tr} f, \tilde{\gamma}_N g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} - \langle \gamma_N f, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.\end{aligned} \tag{3.4}$$

Denote by $\mathfrak{a}_D^*$ the restriction of the dual form $\mathfrak{a}^*$ to $H_0^1(\Omega) \times H_0^1(\Omega)$. Then $\mathfrak{a}_D^*$ is a closed sectorial form and the m-sectorial operator associated with $\mathfrak{a}_D^*$ is equal to the adjoint A_D^* of A_D, see [38] Theorem VI.2.5. It follows that A_D^* is the Dirichlet realisation of $\widetilde{\mathcal{A}}$ in $L_2(\Omega)$ given by

$$A_D^* f = \widetilde{\mathcal{A}} f, \quad \operatorname{dom} A_D^* = \{f \in H_0^1(\Omega) : \widetilde{\mathcal{A}} f \in L_2(\Omega)\}.$$

Similarly to the Dirichlet-to-Neumann map $D(\lambda) \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ one associates the Dirichlet-to-Neumann map $\widetilde{D}(\lambda) \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ to the adjoint form $\mathfrak{a}^*$ for all $\lambda \in \rho(A_D^*)$. A simple computation based on Greens second identity (3.4) shows

$$\langle D(\lambda)\varphi, \psi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \langle \varphi, \widetilde{D}(\bar{\lambda})\psi\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \tag{3.5}$$

for all $\varphi, \psi \in H^{1/2}(\Gamma)$ and $\lambda \in \rho(A_D)$.

Finally we introduce the Robin operator. Let $B \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. We assume that there is an $\eta > 0$ such that, for all $\varphi \in H^{1/2}(\Gamma)$,

$$\operatorname{Re}\langle B\varphi, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \le \eta \|\varphi\|_{L_2(\Gamma)}^2. \tag{3.6}$$

Note that the restriction to the space $H^{1/2}(\Gamma)$ of every bounded operator B in $L^2(\Gamma)$ can be viewed as an operator in $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ that satisfies (3.6). We also note that the above assumption on $B \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ can be generalized further as in for example [33] Hypothesis 4.1. Next we define the sesquilinear form $\mathfrak{a}_B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_B(f, g) = \mathfrak{a}(f, g) - \langle B \operatorname{Tr} f, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.$$

Proposition 3.5. *The form* $\mathfrak{a}_B$ *is densely defined, closed and sectorial in* $L_2(\Omega)$. *The associated m-sectorial operator*

$$A_B f = \mathcal{A} f, \quad \operatorname{dom} A_B = \{f \in H^1(\Omega) : \mathcal{A} f \in L_2(\Omega) \text{ and } \gamma_N f = B \operatorname{Tr} f\},$$

is the ***Robin realisation*** *of* $\mathcal{A}$ *in* $L_2(\Omega)$.

Proof. We will show first that $\mathfrak{a}_B$ is elliptic, that is, there are $\nu \in \mathbb{R}$ and $\mu > 0$ such that, for all $f \in H^1(\Omega)$,

$$\operatorname{Re} \mathfrak{a}_B(f) + \nu \|f\|_{L_2(\Omega)}^2 \ge \mu \|f\|_{H^1(\Omega)}^2. \tag{3.7}$$

Clearly there are $\mu_1, \omega_1 > 0$ such that $\operatorname{Re} \mathfrak{a}(f) \geq 2\mu_1 \|f\|^2_{H^1(\Omega)} - \omega_1 \|f\|^2_{L_2(\Omega)}$ for all $f \in H^1(\Omega)$ (cf. [2] Lemma 3.7.) Choose $\varepsilon < \frac{\mu_1}{\eta}$, where $\eta > 0$ is as in (3.6). By Ehrling's lemma and the compactness of $\operatorname{Tr}\colon H^1(\Omega) \to L_2(\Gamma)$ there exists a $c > 0$ such that $\| \operatorname{Tr} f\|^2_{L_2(\Gamma)} \leq \varepsilon \|f\|^2_{H^1(\Omega)} + c\|f\|^2_{L_2(\Omega)}$ for all $f \in H^1(\Omega)$. Then

$$\operatorname{Re}\langle B \operatorname{Tr} f, \operatorname{Tr} f\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \leq \eta \| \operatorname{Tr} f\|^2_{L_2(\Gamma)} \leq \mu_1 \|f\|^2_{H^1(\Omega)} + \eta c \|f\|^2_{L_2(\Omega)}$$

and hence

$$\begin{aligned}\operatorname{Re} \mathfrak{a}_B(f) &= \operatorname{Re} \mathfrak{a}(f) - \operatorname{Re}\langle B \operatorname{Tr} f, \operatorname{Tr} f\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &\geq \mu_1 \|f\|^2_{H^1(\Omega)} - (\omega_1 + \eta c)\|f\|^2_{L_2(\Omega)}\end{aligned}$$

for all $f \in H^1(\Omega)$. So (3.7) holds with $\mu = \mu_1$ and $\nu = \omega_1 + \eta c$, therefore $\mathfrak{a}_B$ is elliptic. Hence $\mathfrak{a}_B$ is a densely defined, closed, sectorial form (see [4] Lemma 3.1).

The graph of the m-sectorial operator associated to $\mathfrak{a}_B$ is given by

$$G = \{(f, h) \in H^1(\Omega) \times L_2(\Omega)\colon \mathfrak{a}_B(f, g) = (h, g)_{L_2(\Omega)} \text{ for all } g \in H^1(\Omega)\}$$

and it remains to show that G coincides with the Robin realisation A_B. Now let $f \in \operatorname{dom} G$ and write $h = Gf \in L_2(\Omega)$. Then $f \in H^1(\Omega)$ and

$$\langle \mathcal{A} f, g\rangle_{(H^1_0(\Omega))^* \times H^1_0(\Omega)} = \mathfrak{a}(f, g) = \mathfrak{a}_B(f, g) = (h, g)_{L_2(\Omega)}$$

for all $g \in H^1_0(\Omega)$. So $\mathcal{A} f = h = Gf \in L_2(\Omega)$. If $g \in H^1(\Omega)$, then

$$\begin{aligned}&\mathfrak{a}(f, g) - (\mathcal{A} f, g)_{L_2(\Omega)} \\ &\quad = \mathfrak{a}_B(f, g) + \langle B \operatorname{Tr} f, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} - (h, g)_{L_2(\Omega)} \\ &\quad = \langle B \operatorname{Tr} f, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.\end{aligned}$$

So $\gamma_N f = B \operatorname{Tr} f$ and hence $f \in \operatorname{dom} A_B$. The converse inclusion follows similarly. □

4. Jordan chains of Robin realisations

Adopt the assumptions and notation as in Section 3. In this section we formulate and prove our main result on the characterization of Jordan chains of the m-sectorial Robin realisation A_B of $\mathcal{A}$ via the operator function $\lambda \mapsto D(\lambda) - B$. Our goal is to show the following theorem.

Theorem 4.1. *Let A_B be the Robin realisation of $\mathcal{A}$ in $L_2(\Omega)$ as in Proposition 3.5, let $\lambda_0 \in \rho(A_D)$ and consider the holomorphic function*

$$\lambda \longmapsto D(\lambda) - B \tag{4.1}$$

from $\rho(A_D)$ into $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. Then the following holds.

(a) *Let $\{f_0, \ldots, f_k\}$ be a Jordan chain for A_B at λ_0. For all $m \in \{0, \ldots, k\}$ define $\varphi_m = \operatorname{Tr} f_m$. Then $\{\varphi_0, \ldots, \varphi_k\}$ is a Jordan chain for the function (4.1) at λ_0.*

(b) *Let $\{\varphi_0, \ldots, \varphi_k\}$ be a Jordan chain for the function (4.1) at λ_0. Set $f_{-1} = 0$. For all $m \in \{0, \ldots, k\}$ let $f_m \in H^1(\Omega)$ be the unique solution of the boundary value problem*

$$(\mathcal{A} - \lambda_0) f_m = f_{m-1}, \qquad \operatorname{Tr} f_m = \varphi_m.$$

Then $\{f_0, \ldots, f_k\}$ is a Jordan chain for A_B at λ_0.

For the special case $k = 0$ one obtains the following well-known result.

Corollary 4.2. *Adopt the notation and assumptions as in Theorem 4.1. Then the following holds.*

(a) *If f_0 is an eigenvector of A_B at λ_0, then $D(\lambda_0) \operatorname{Tr} f_0 = B \operatorname{Tr} f_0$ and $\operatorname{Tr} f_0 \neq 0$.*

(b) *If $D(\lambda_0)\varphi_0 = B\varphi_0$ and $\varphi_0 \neq 0$, then the unique solution $f_0 \in H^1(\Omega)$ of the boundary value problem*

$$(\mathcal{A} - \lambda_0) f_0 = 0, \quad \operatorname{Tr} f_0 = \varphi_0,$$

is an eigenvector of A_B at λ_0.

Corollary 4.3. *Adopt the notation and assumptions as in Theorem 4.1. Then*

$$\operatorname{Tr}(\ker(A_B - \lambda_0)) = \ker(D(\lambda_0) - B)$$

and Tr *is a bijection from* $\ker(A_B - \lambda_0)$ *onto* $\ker(D(\lambda_0) - B)$.

Remark 4.4. We can mention here that the assumption $\lambda_0 \in \rho(A_D)$ in Theorem 4.1 and Corollary 4.2 is really needed. In fact, one may define the Dirichlet-to-Neumann graph as a linear relation consisting of the Cauchy data for all $\lambda_0 \in \sigma_p(A_D)$. By [31] Theorem 1 there exist $\mu > 0, \lambda \in \mathbb{R}, u \in C_c^\infty(\mathbb{R}^3) \setminus \{0\}$ and a Hölder continuous function $g: \mathbb{R}^3 \to [\mu, \infty)$ such that $-\operatorname{div} g \nabla u = \lambda u$. Let Ω be a Lipschitz domain with $\operatorname{supp} u \subset \Omega$. Choose $c_{kl} = g|_\Omega\, \delta_{kl}, b_k = c_k = c_0 = 0$ for all $k, l \in \{1, \ldots, d\}$ and $f_0 = u|_\Omega$. Let $B \in \mathcal{L}(L_2(\Gamma))$. Then f_0 is an eigenfunction of A_B at λ. But $\operatorname{Tr} f_0 = 0$. So one cannot drop the assumption $\lambda_0 \in \rho(A_D)$ in Corollary 4.2(a).

Observe that the homogeneous and inhomogeneous boundary value problems in Theorem 4.1(b) and Corollary 4.2(b) admit unique solutions by Lemma 3.2. The proof of Theorem 4.1 requires quite some preparation. The next lemma is particularly useful; its proof is partly based on an argument that was given by V. A. Derkach for symmetric and selfadjoint linear relations in Krein spaces; see also [27] Section 7.4.4.

Lemma 4.5. *Let A_B be the Robin realisation of $\mathcal{A}$ in $L_2(\Omega)$ as in Proposition 3.5 and let $\{f_0, \ldots, f_k\}$ be a Jordan chain of A_B at $\lambda_0 \in \rho(A_D)$. For all $m \in \{0, \ldots, k\}$ define $\varphi_m = \operatorname{Tr} f_m \in H^{1/2}(\Gamma)$. Let $\varphi \in H^{1/2}(\Gamma)$ and let $g \in H^1(\Omega)$ be the unique solution of the adjoint problem $(\widetilde{\mathcal{A}} - \overline{\lambda_0})g = 0$ such that $\operatorname{Tr} g = \varphi$. Then the following holds.*

(a) *If $j \in \{1, \ldots, k\}$, then*

$$(f_{j-1}, g)_{L_2(\Omega)} = \langle D(\lambda_0)\varphi_j - B\varphi_j, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}. \tag{4.2}$$

(b) *If $j \in \{1, \ldots, k+1\}$, then*

$$(f_{j-1}, g)_{L_2(\Omega)} = -\sum_{l=1}^{j} \frac{1}{l!}\langle D^{(l)}(\lambda_0)\varphi_{j-l}, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}. \tag{4.3}$$

Proof. For all $\lambda \in \rho(A_D)$ let $g_{\bar\lambda} \in H^1(\Omega)$ be the unique solution of the adjoint problem $(\widetilde{\mathcal{A}} - \bar\lambda)g_{\bar\lambda} = 0$ such that $\operatorname{Tr} g_{\bar\lambda} = \varphi$; see Lemma 3.2(a). Then $g_{\overline{\lambda_0}} = g$. We set $f_{-1} = 0$.

(a) If $j \in \{0, \ldots, k\}$ and $\lambda \in \rho(A_D)$, then $f_j \in \operatorname{dom} A_B$, so $A_B f_j = \mathcal{A} f_j$ and $\gamma_N f_j = B \operatorname{Tr} f_j$ by Proposition 3.5. Therefore

$$\begin{aligned}
&(A_B f_j, g_{\bar\lambda})_{L_2(\Omega)} - (f_j, \bar\lambda g_{\bar\lambda})_{L_2(\Omega)} \\
&\quad = (\mathcal{A} f_j, g_{\bar\lambda})_{L_2(\Omega)} - (f_j, \widetilde{\mathcal{A}} g_{\bar\lambda})_{L_2(\Omega)} \\
&\quad = \langle \operatorname{Tr} f_j, \tilde\gamma_N g_{\bar\lambda}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} - \langle \gamma_N f_j, \operatorname{Tr} g_{\bar\lambda}\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&\quad = \langle \operatorname{Tr} f_j, \widetilde{D}(\bar\lambda) \operatorname{Tr} g_{\bar\lambda}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} - \langle B \operatorname{Tr} f_j, \operatorname{Tr} g_{\bar\lambda}\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&\quad = \langle D(\lambda)\varphi_j - B\varphi_j, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},
\end{aligned}$$

where we used (3.5) in the last step. Choosing $\lambda = \lambda_0$ gives

$$\begin{aligned}
(f_{j-1}, g)_{L_2(\Omega)} &= ((A_B - \lambda_0) f_j, g_{\overline{\lambda_0}})_{L_2(\Omega)} \\
&= (A_B f_j, g_{\overline{\lambda_0}})_{L_2(\Omega)} - (f_j, \overline{\lambda_0} g_{\overline{\lambda_0}})_{L_2(\Omega)} \\
&= \langle D(\lambda_0)\varphi_j - B\varphi_j, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},
\end{aligned}$$

which proves (4.2). Note that $j = 0$ gives

$$\langle D(\lambda_0)\varphi_0 - B\varphi_0, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = 0$$

and hence

$$D(\lambda_0)\varphi_0 = B\varphi_0. \tag{4.4}$$

(b) We shall show that

$$\begin{aligned} &-(f_{j-1}, g_{\bar\lambda})_{L_2(\Omega)} \\ &= \sum_{l=1}^{j}\Big\langle \frac{1}{(\lambda-\lambda_0)^l}\Big(D(\lambda) - \sum_{s=0}^{l-1}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)\Big)\varphi_{j-l}, \varphi\Big\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \end{aligned} \tag{4.5}$$

for all $j \in \{1, \ldots, k+1\}$ and $\lambda \in \rho(A_D) \setminus \{\lambda_0\}$. Once we have shown this, then the equality (4.3) easily follows by taking the limit $\lambda \to \lambda_0$. In fact, the left hand side of (4.5) tends to $-(f_{j-1}, g_{\bar\lambda_0})_{L_2(\Omega)} = -(f_{j-1}, g)_{L_2(\Omega)}$ by Lemma 3.3(a), and using the Taylor expansion

$$D(\lambda) = \sum_{s=0}^{\infty}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)$$

it is easy to see that for $\lambda \to \lambda_0$ the right hand side in (4.5) tends to

$$\sum_{l=1}^{j}\frac{1}{l!}\langle D^{(l)}(\lambda_0)\varphi_{j-l}, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.$$

We prove formula (4.5) by induction. If $j = 1$ and $\lambda \in \rho(A_D)\setminus\{\lambda_0\}$, then (4.4) gives

$$\begin{aligned} -(\lambda-\lambda_0)(f_0, g_{\bar\lambda})_{L_2(\Omega)} &= (\lambda_0 f_0, g_{\bar\lambda})_{L_2(\Omega)} - (f_0, \bar\lambda g_{\bar\lambda})_{L_2(\Omega)} \\ &= (A_B f_0, g_{\bar\lambda})_{L_2(\Omega)} - (f_0, \bar\lambda g_{\bar\lambda})_{L_2(\Omega)} \\ &= \langle D(\lambda)\varphi_0 - B\varphi_0, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &= \langle (D(\lambda) - D(\lambda_0))\varphi_0, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}, \end{aligned}$$

where we used (4.4) in the last step. So (4.5) is valid if $j = 1$.

Let $m \in \{1, \ldots, k\}$ and suppose that (4.5) is valid for $j = m$. Then by taking the limit $\lambda \to \lambda_0$ one deduces that

$$-(f_{m-1}, g)_{L_2(\Omega)} = \sum_{l=1}^{m}\frac{1}{l!}\langle D^{(l)}(\lambda_0)\varphi_{m-l}, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},$$

and together with (4.2) we conclude

$$\begin{aligned}
&\langle D(\lambda_0)\varphi_m - B\varphi_m, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&\quad = -\sum_{l=1}^{m} \frac{1}{l!}\langle D^{(l)}(\lambda_0)\varphi_{m-l}, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.
\end{aligned} \tag{4.6}$$

Now let us prove the formula (4.5) for $j = m+1$. Let $\lambda \in \rho(A_D)\setminus\{\lambda_0\}$. Then a simple computation shows

$$\begin{aligned}
&\sum_{l=1}^{m+1} \frac{1}{(\lambda-\lambda_0)^l}\Big(D(\lambda) - \sum_{s=0}^{l-1}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)\Big)\varphi_{m+1-l} \\
&\quad = \sum_{l=2}^{m+1} \frac{1}{(\lambda-\lambda_0)^l}\Big(D(\lambda) - \sum_{s=0}^{l-1}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)\Big)\varphi_{m+1-l} \\
&\qquad + \frac{D(\lambda)-D(\lambda_0)}{\lambda-\lambda_0}\varphi_m \\
&\quad = \sum_{l=1}^{m} \frac{1}{(\lambda-\lambda_0)^{l+1}}\Big(D(\lambda) - \sum_{s=0}^{l}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)\Big)\varphi_{m-l} \\
&\qquad + \frac{D(\lambda)-D(\lambda_0)}{\lambda-\lambda_0}\varphi_m \\
&= \frac{1}{\lambda-\lambda_0}\sum_{l=1}^{m} \frac{1}{(\lambda-\lambda_0)^l}\Big(D(\lambda) - \sum_{s=0}^{l-1}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)\Big)\varphi_{m-l} \\
&\qquad - \frac{1}{\lambda-\lambda_0}\sum_{l=1}^{m}\frac{1}{l!}D^{(l)}(\lambda_0)\varphi_{m-l} + \frac{D(\lambda)-D(\lambda_0)}{\lambda-\lambda_0}\varphi_m
\end{aligned}$$

and using (4.5) for $j = m$ for the first term on the right hand side, and (4.6) for the second term on the right hand side gives

$$\begin{aligned}
&\sum_{l=1}^{m+1} \Big\langle \frac{1}{(\lambda-\lambda_0)^l}\Big(D(\lambda) - \sum_{s=0}^{l-1}\frac{1}{s!}(\lambda-\lambda_0)^s D^{(s)}(\lambda_0)\Big)\varphi_{m+1-l}, \varphi\Big\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&\quad = -\frac{1}{\lambda-\lambda_0}(f_{m-1}, g_{\bar\lambda})_{L_2(\Omega)} + \frac{1}{\lambda-\lambda_0}\langle D(\lambda_0)\varphi_m - B\varphi_m, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&\qquad + \frac{1}{\lambda-\lambda_0}\langle D(\lambda)\varphi_m - D(\lambda_0)\varphi_m, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&\quad = \frac{1}{\lambda-\lambda_0}\langle D(\lambda)\varphi_m - B\varphi_m, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} - \frac{1}{\lambda-\lambda_0}(f_{m-1}, g_{\bar\lambda})_{L_2(\Omega)} \\
&\quad = \frac{1}{\lambda-\lambda_0}\big((A_B f_m, g_{\bar\lambda})_{L_2(\Omega)} - (f_m, \bar\lambda g_{\bar\lambda})_{L_2(\Omega)} - (f_{m-1}, g_{\bar\lambda})_{L_2(\Omega)}\big)
\end{aligned}$$

$$= \frac{1}{\lambda - \lambda_0}((f_{m-1} + \lambda_0 f_m, g_{\bar{\lambda}})_{L_2(\Omega)} - (f_m, \bar{\lambda} g_{\bar{\lambda}})_{L_2(\Omega)} - (f_{m-1}, g_{\bar{\lambda}})_{L_2(\Omega)})$$
$$= -(f_m, g_{\bar{\lambda}})_{L_2(\Omega)},$$

where (4.4) was used for $j = m$ in third equality and $(A_B - \lambda_0) f_m = f_{m-1}$ was used in the fourth equality. We have shown (4.5) for $j = m + 1$. The proof of (b) is complete. ☐

Now we are able to prove the main theorem.

Proof of Theorem 4.1. (a) Let $\{f_0, \dots, f_k\}$ form a Jordan chain for A_B at $\lambda_0 \in \rho(A_D)$ and let $\varphi_j = \operatorname{Tr} f_j \in H^{1/2}(\Gamma)$ for all $j \in \{1, \dots, k\}$ be the corresponding traces. We have to prove that

$$\sum_{l=0}^{j} \frac{1}{l!} D^{(l)}(\lambda_0) \varphi_{j-l} = B \varphi_j \tag{4.7}$$

for all $j \in \{0, \dots, k\}$ and that $\varphi_0 \neq 0$.

Using Proposition 3.5 it is easy to see that

$$D(\lambda_0)\varphi_0 - B\varphi_0 = D(\lambda_0) \operatorname{Tr} f_0 - B \operatorname{Tr} f_0 = \gamma_N f_0 - \gamma_N f_0 = 0$$

and hence (4.7) is valid if $j = 0$. Furthermore, $\varphi_0 = \operatorname{Tr} f_0 \neq 0$ as otherwise $f_0 \in \operatorname{dom} A_D$ and therefore $(A_D - \lambda_0) f_0 = (A_B - \lambda_0) f_0 = 0$, which together with $\lambda_0 \in \rho(A_D)$ would imply $f_0 = 0$.

Let $j \in \{1, \dots, k\}$ and let $\varphi \in H^{1/2}(\Gamma)$. Then Lemma 4.5 gives

$$\langle D(\lambda_0)\varphi_j - B\varphi_j, \varphi \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$
$$= -\sum_{l=1}^{j} \frac{1}{l!} \langle D^{(l)}(\lambda_0) \varphi_{j-l}, \varphi \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}.$$

This implies that

$$B\varphi_j = D(\lambda_0)\varphi_j + \sum_{l=1}^{j} \frac{1}{l!} D^{(l)}(\lambda_0) \varphi_{j-l} = \sum_{l=0}^{j} \frac{1}{l!} D^{(l)}(\lambda_0) \varphi_{j-l}$$

as required.

(b) Assume that $\{\varphi_0, \dots, \varphi_k\}$ form a Jordan chain of the function $\lambda \mapsto D(\lambda) - B$ at λ_0, that is, (4.7) is valid for all $j \in \{0, 1, \dots k\}$ and $\varphi_0 \neq 0$. In the following we construct a Jordan chain $\{f_0, \dots, f_k\}$ of A_B at λ_0 such that the corresponding traces are given by the set of vectors $\{\varphi_0, \dots, \varphi_k\}$. We proceed by induction. According to Lemma 3.2(a) there exists a unique $f_0 \in H^1(\Omega)$ such that $(\mathcal{A} - \lambda_0) f_0 = 0$ and $\operatorname{Tr} f_0 = \varphi_0$. Making use of (4.7) for $j = 0$ we obtain

$$\gamma_N f_0 = D(\lambda_0) \operatorname{Tr} f_0 = D(\lambda_0)\varphi_0 = B\varphi_0 = B \operatorname{Tr} f_0$$

and hence $f_0 \in \operatorname{dom} A_B$ with $(A_B - \lambda_0) f_0 = 0$ by Proposition 3.5. Since $\varphi_0 \neq 0$ it is clear that also $f_0 \neq 0$.

Now let $m \in \{1, \dots, k\}$ and assume that there are $f_0, \dots, f_{m-1} \in H^1(\Omega)$ such that $\varphi_j = \operatorname{Tr} f_j$ for all $j \in \{0, \dots, m-1\}$ and the vectors $\{f_0, \dots, f_{m-1}\}$ form a Jordan chain for A_B at λ_0. By Lemma 3.2(b) there exists a unique vector $f_m \in H^1(\Omega)$ such that

$$(\mathcal{A} - \lambda_0) f_m = f_{m-1} \quad \text{and} \quad \operatorname{Tr} f_m = \varphi_m. \tag{4.8}$$

We shall prove that $\gamma_N f_m = B \operatorname{Tr} f_m$. Once we proved that, it follows that $f_m \in \operatorname{dom} A_B$ and $(A_B - \lambda_0) f_m = f_{m-1}$.

By assumption and (4.8) one deduces that

$$\begin{aligned} D(\lambda_0) \operatorname{Tr} f_m &= D(\lambda_0)\varphi_m \\ &= B\varphi_m - \sum_{l=1}^{m} \frac{1}{l!} D^{(l)}(\lambda_0)\varphi_{m-l} \\ &= B \operatorname{Tr} f_m - \sum_{l=1}^{m} \frac{1}{l!} D^{(l)}(\lambda_0)\varphi_{m-l}. \end{aligned}$$

Let $\varphi \in H^{1/2}(\Gamma)$. By Lemma 3.2(a) there exists a unique $g \in H^1(\Omega)$ such that $(\widetilde{\mathcal{A}} - \overline{\lambda_0})g = 0$ and $\operatorname{Tr} g = \varphi$. Then

$$\begin{aligned} &((\mathcal{A} - \lambda_0) f_m, g)_{L_2(\Omega)} \\ &\quad = (\mathcal{A} f_m, g)_{L_2(\Omega)} - (f_m, \overline{\lambda_0} g)_{L_2(\Omega)} \\ &\quad = (\mathcal{A} f_m, g)_{L_2(\Omega)} - (f_m, \widetilde{\mathcal{A}} g)_{L_2(\Omega)} \\ &\quad = \langle \operatorname{Tr} f_m, \widetilde{\gamma}_N g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} - \langle \gamma_N f_m, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &\quad = \langle \operatorname{Tr} f_m, \widetilde{D}(\overline{\lambda_0}) \operatorname{Tr} g\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} - \langle \gamma_N f_m, \operatorname{Tr} g\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &\quad = \langle D(\lambda_0) \operatorname{Tr} f_m - \gamma_N f_m, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &\quad = \Big\langle B \operatorname{Tr} f_m - \gamma_N f_m - \sum_{l=1}^{m} \frac{1}{l!} D^{(l)}(\lambda_0)\varphi_{m-l}, \varphi\Big\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}. \end{aligned}$$

On the other hand, as $\{f_0, \dots, f_{m-1}\}$ is a Jordan chain of A_B at λ_0 we have

$$\begin{aligned}((\mathcal{A} - \lambda_0) f_m, g)_{L_2(\Omega)} &= (f_{m-1}, g)_{L_2(\Omega)} \\ &= -\sum_{l=1}^{m} \frac{1}{l!} \langle D^{(l)}(\lambda_0)\varphi_{m-l}, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\end{aligned}$$

by Lemma 4.5(b). Therefore $\langle B \operatorname{Tr} f_m - \gamma_N f_m, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = 0$ for all $\varphi \in H^{1/2}(\Gamma)$. Thus $\gamma_N f_m = B \operatorname{Tr} f_m$ as required. So $\{f_0, \dots, f_m\}$ is a Jordan chain for A_B at λ_0 with traces $\{\varphi_0, \dots, \varphi_m\}$. ☐

Remark 4.6. Theorem 4.1 may also be interpreted in the abstract context of adjoint pairs of unbounded operators in Hilbert spaces, where A_B can be viewed as an m-sectorial extension of the underlying minimal differential operator and the λ-dependent (minus) Dirichlet-to-Neumann operator is a corresponding Weyl function. In fact, in the abstract setting of (ordinary) boundary triplets and their Weyl functions for adjoint pairs [41, 43, 52] it is known under a natural unique continuation hypothesis that the poles of the Weyl function correspond to the isolated eigenvalues of the fixed extension, see [21, Theorem 4.4]. See also [22, 20, 17] for related results in the context of indefinite inner product spaces.

5. Variations

The aim of this section is to discuss some variations of our main result Theorem 4.1. In the previous section we considered the Dirichlet-to-Neumann operator $D(\lambda)\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and the Jordan chain with respect to the holomorphic operator function $\lambda \mapsto D(\lambda) - B$ from $\rho(A_D)$ into $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$, where $B \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ satisfies (3.6). Except from the obvious ellipticity condition and to have a Lipschitz domain, there were no conditions on the coefficients: merely bounded measurable and complex valued.

There are two other Dirichlet-to-Neumann operators that we consider in this section.

5.1. C^2-domains.

Throughout this subsection we suppose that Ω is a C^2-domain, $c_{kl} \in C^1(\overline{\Omega})$ and $b_k = 0$ for all $k, l \in \{1, \dots, d\}$. We summarise some regularity results that we need in this subsection.

Lemma 5.1. (a) *If* $f \in H^2(\Omega)$, *then* $\operatorname{Tr} f \in H^{3/2}(\Gamma)$, $\mathcal{A} f \in L_2(\Omega)$ *and*

$$\gamma_N f = \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl} \partial_l f) \in H^{1/2}(\Gamma).$$

Moreover, the map $f \mapsto \gamma_N f$ *is continuous from* $H^2(\Omega)$ *into* $H^{1/2}(\Gamma)$.

(b) *Let* $\lambda \in \rho(A_D)$*. For all* $\varphi \in H^{3/2}(\Gamma)$ *there exists a unique* $f \in H^2(\Omega)$ *such that* $(\mathcal{A} - \lambda) f = 0$ *and* $\operatorname{Tr} f = \varphi$*. Moreover, the map* $\varphi \mapsto f$ *is continuous from* $H^{3/2}(\Gamma)$ *into* $H^2(\Omega)$*.*

(c) *Let* $\lambda \in \rho(A_D)$*. For all* $h \in L_2(\Omega)$ *and* $\varphi \in H^{3/2}(\Gamma)$ *there exists a unique* $f \in H^2(\Omega)$ *such that* $(\mathcal{A} - \lambda) f = h$ *and* $\operatorname{Tr} f = \varphi$*.*

Proof. (a) This follows from [36] Theorem 1.5.1.2 and the divergence theorem.

(b) By [36] Theorem 1.5.1.2 there exists an $f_0 \in H^2(\Omega)$ such that $\operatorname{Tr} f_0 = \varphi$. Then it follows from [30] Theorem 6.3.4 that there exists a unique $h \in H^2(\Omega)$ such that $(\mathcal{A} - \lambda) h = (\mathcal{A} - \lambda) f_0$ and $\operatorname{Tr} h = 0$. Therefore $f = f_0 - h$ satisfies the requirements. The uniqueness is easy. The continuity follows from Lemma 3.2(a) and the closed graph theorem.

(c) This can be proved similarly. □

For all $\lambda \in \rho(A_D)$ define the Dirichlet-to-Neumann operator

$$\widehat{D}(\lambda) \colon H^{3/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$$

as follows. Let $\varphi \in H^{3/2}(\Gamma)$. By Lemma 5.1(b) there exists a unique $f \in H^2(\Omega)$ such that $(\mathcal{A} - \lambda) f = 0$ and $\operatorname{Tr} f = \varphi$. Define $\widehat{D}(\lambda)\varphi = \gamma_N f \in H^{1/2}(\Gamma)$ by Lemma 5.1(a). Then $\widehat{D}(\lambda)$ is a bounded operator.

Next we consider holomorphy.

Lemma 5.2. *The map* $\lambda \mapsto \widehat{D}(\lambda)$ *from* $\rho(A_D)$ *into* $\mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$ *is holomorphic.*

Proof. For all $\varphi \in H^{3/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$ define

$$\alpha_{\varphi,\psi} \colon \mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma)) \longrightarrow \mathbb{C}$$

by

$$\alpha_{\varphi,\psi}(F) = (F\varphi, \psi)_{L_2(\Gamma)}.$$

Then $\alpha_{\varphi,\psi} \in \mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))^*$. Let

$$W = \operatorname{span}\{\alpha_{\varphi,\psi} \colon \varphi \in H^{3/2}(\Gamma) \text{ and } \psi \in H^{1/2}(\Gamma)\}.$$

Since $H^{1/2}(\Gamma)$ is dense in $L_2(\Gamma)$, it follows that the space W is separating, that is, if $F \in \mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$ with $\alpha(F) = 0$ for all $\alpha \in W$, then it follows that $F = 0$. If $\varphi \in H^{3/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$, then

$$\alpha_{\varphi,\psi}(\widehat{D}(\lambda)) = (\widehat{D}(\lambda)\varphi, \psi)_{L_2(\Gamma)} = \langle D(\lambda)\varphi, \psi \rangle_{H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)}$$

for all $\lambda \in \rho(A_D)$. Hence the map $\lambda \mapsto \alpha_{\varphi,\psi}(\widehat{D}(\lambda))$ is holomorphic for all $\varphi \in H^{3/2}(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$ by Lemma 3.3(b). Consequently the map $\lambda \mapsto \widehat{D}(\lambda)$ from $\rho(A_D)$ into $\mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$ is holomorphic by [1] Theorem A.7. □

The alluded variation of Theorem 4.1 is as follows.

Theorem 5.3. *Let $B \in \mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$ and suppose there exists an $\eta > 0$ such that*

$$\operatorname{Re}(B\varphi, \varphi)_{L_2(\Gamma)} \leq \eta \|\varphi\|^2_{L_2(\Gamma)}$$

for all $\varphi \in H^{3/2}(\Gamma)$. Let A_B be the Robin realisation of $\mathcal{A}$ in $L_2(\Omega)$ as in Proposition 3.5, let $\lambda_0 \in \rho(A_D)$ and consider the holomorphic function

$$\lambda \longmapsto \widehat{D}(\lambda) - B \tag{5.1}$$

from $\rho(A_D)$ into $\mathcal{L}(H^{3/2}(\Gamma), H^{1/2}(\Gamma))$. Then the following holds.

(a) *Let $f_0, \dots, f_k \in H^2(\Omega)$. Suppose that $\{f_0, \dots, f_k\}$ is a Jordan chain for A_B at λ_0. For all $m \in \{0, \dots, k\}$ define $\varphi_m = \operatorname{Tr} f_m$. Then $\{\varphi_0, \dots, \varphi_k\}$ is a Jordan chain for the function (5.1) at λ_0.*

(b) *Let $\{\varphi_0, \dots, \varphi_k\}$ be a Jordan chain for the function (5.1) at λ_0. Set $f_{-1} = 0$. For all $m \in \{0, \dots, k\}$ let $f_m \in H^2(\Omega)$ be the unique solution of the boundary value problem*

$$(\mathcal{A} - \lambda_0) f_m = f_{m-1} \quad \textit{and} \quad \operatorname{Tr} f_m = \varphi_m.$$

Then $\{f_0, \dots, f_k\}$ is a Jordan chain for A_B at λ_0.

The proof is similar to the proof of Theorem 4.1, with obvious changes.

5.2. m-Sectorial operators. Throughout this subsection we merely assume again that Ω is a Lipschitz domain, but we put conditions on the coefficients of the elliptic operator. We assume that $c_{kl} = c_{lk} \in W^{1,\infty}(\Omega, \mathbb{R})$ is real valued and $b_k = c_k = 0$ for all $k \in \{1, \dots, d\}$. We emphasise that c_0 can be complex valued and merely measurable. An example is the Schrödinger operator with complex potential. The Dirichlet-to-Neumann operator $D(\lambda)\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ has been studied intensively in [11, 13, 32, 34]. Let $\mathcal{D}(\lambda)$ be the part of $D(\lambda)$ in $L_2(\Gamma)$. So $\mathcal{D}(\lambda) \subset D(\lambda)$ and if $\varphi \in L_2(\Gamma)$, then $\varphi \in \operatorname{dom} \mathcal{D}(\lambda)$ if and only if $\varphi \in H^{1/2}(\Gamma)$ and $D(\lambda)\varphi \in L_2(\Gamma)$. The operator $\mathcal{D}(\lambda)$ can be represented by a form.

Lemma 5.4. *Let $\lambda \in \rho(A_D)$. Let $\varphi, \psi \in L_2(\Gamma)$. Then the following are equivalent.*

(i) *$\varphi \in \operatorname{dom} \mathcal{D}(\lambda)$ and $\mathcal{D}(\lambda)\varphi = \psi$.*

(ii) *There exists an* $f \in H^1(\Omega)$ *such that* $\operatorname{Tr} f = \varphi$ *and*

$$\mathfrak{a}(f, g) - \lambda(f, g)_{L_2(\Omega)} = (\psi, \operatorname{Tr} g)_{L_2(\Gamma)}$$

for all $g \in H^1(\Omega)$.

The easy proof is left to the reader.

It seems that the domain of $\mathcal{D}(\lambda)$ depends on λ. This is not the case because of the restriction on the principal part of the elliptic operator. We collect the main properties of the operator $\mathcal{D}(\lambda)$ in the next proposition.

Proposition 5.5. (a) *If* $\lambda \in \rho(A_D)$*, then the operator* $\mathcal{D}(\lambda)$ *is m-sectorial.*

(b) *If* $\lambda \in \rho(A_D)$*, then* $\operatorname{dom} \mathcal{D}(\lambda) = H^1(\Omega)$.

(c) *The map* $\lambda \mapsto \mathcal{D}(\lambda)$ *from* $\rho(A_D)$ *into* $\mathcal{L}(H^1(\Gamma), L_2(\Gamma))$ *is holomorphic.*

Proof. (a) See [49] Corollary 2.3.

(b) Inclusion $\subset$. Let $\varphi \in \operatorname{dom} \mathcal{D}(\lambda)$. Then there exists an $f \in H^1(\Omega)$ such that $\varphi = \operatorname{Tr} f$ and $(\mathcal{A}-\lambda) f = 0$. So $\mathcal{A} f = \lambda f \in L_2(\Omega)$ and $\gamma_N f = \mathcal{D}(\lambda)\varphi \in L_2(\Gamma)$. Therefore [46] Theorem 4.24(ii) implies that $\varphi = \operatorname{Tr} f \in H^1(\Gamma)$.

Inclusion $\supset$. Let $\varphi \in H^1(\Gamma)$. By Lemma 3.2(a) there exists a unique $f \in H^1(\Omega)$ such that $(\mathcal{A} - \lambda) f = 0$ and $\operatorname{Tr} f = \varphi$. Then $\mathcal{A} f = \lambda f \in L_2(\Omega)$. Hence [46] Theorem 4.24(i) gives $\gamma_N f \in L_2(\Gamma)$. So $\varphi \in \operatorname{dom} \mathcal{D}(\lambda)$.

(c) For all $\varphi \in H^1(\Gamma)$ and $\psi \in H^{1/2}(\Gamma)$ define $\alpha_{\varphi,\psi} \colon \mathcal{L}(H^1(\Gamma), L_2(\Gamma)) \to \mathbb{C}$ by

$$\alpha_{\varphi,\psi}(F) = (F\varphi, \psi)_{L_2(\Gamma)}.$$

Then argue as in the proof of Lemma 5.2. □

Now we are able to formulate another version of Theorem 4.1.

Theorem 5.6. *Let* $B \in \mathcal{L}(H^1(\Gamma), L_2(\Gamma))$ *and suppose that there exists an* $\eta > 0$ *such that*

$$\operatorname{Re}(B\varphi, \varphi)_{L_2(\Gamma)} \leq \eta \|\varphi\|^2_{L_2(\Gamma)}$$

for all $\varphi \in H^1(\Gamma)$. *Let* A_B *be the Robin realisation of* $\mathcal{A}$ *in* $L_2(\Omega)$ *as in Proposition* 3.5*, let* $\lambda_0 \in \rho(A_D)$ *and consider the holomorphic function*

$$\lambda \longmapsto \widehat{D}(\lambda) - B$$

from $\rho(A_D)$ *into* $\mathcal{L}(H^1(\Omega), L_2(\Gamma))$. *Then the following holds.*

(a) *Let* $\{f_0, \ldots, f_k\}$ *be a Jordan chain for* A_B *at* λ_0*. For all* $m \in \{0, \ldots, k\}$ *define* $\varphi_m = \operatorname{Tr} f_m$*. Then* $\{\varphi_0, \ldots, \varphi_k\}$ *is a Jordan chain for the function* (5.1) *at* λ_0*.*

(b) *Let* $\{\varphi_0, \ldots, \varphi_k\}$ *be a Jordan chain for the function* (5.1) *at* λ_0*. Set* $f_{-1} = 0$*. For all* $m \in \{0, \ldots, k\}$ *let* $f_m \in H^1(\Omega)$ *be the unique solution of the boundary value problem*

$$(\mathcal{A} - \lambda_0) f_m = f_{m-1} \quad \textit{and} \quad \operatorname{Tr} f_m = \varphi_m.$$

Then $\{f_0, \ldots, f_k\}$ *is a Jordan chain for* A_B *at* λ_0*.*

The proof is similar to the proof of Theorem 4.1, with obvious changes.

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