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Lieb–Thirring inequalities for an effective Hamiltonian of bilayer graphene

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Abstract. Combining the methods of Cuenin [7] and Borichev, Golinskii, and Kupin [4] and [5], we obtain the so-called Lieb–Thirring inequalities for non-selfadjoint perturbations of an effective Hamiltonian for bilayer graphene.

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Contents

Introduction and main results

Since the early 2000s, a certain amount of attention of the mathematical community has been attracted by the spectral properties of complex (non-selfadjoint) perturbations of model operators from mathematical physics. Among relatively recent papers in this direction, we quote articles by Demuth, Hansmann, and Katriel [13], Frank [19] and [20], Frank and Simon [22], Frank and Sabin [21], Frank,

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Laptev, and Safronov [23], Fanelli, Krejčiřík, and Vega [15] and [16], Mizutani [40], Fanelli and Krejčiřík [17], Cuenin and Kenig [10], and Lee and Seo [38], dealing with spectral properties of complex Schrödinger operators. Similar problems for Dirac, fractional Schrödinger and other types of operators were treated in Cuenin, Laptev, and Tretter [8], Cuenin and Seigl [9], Dubuisson [14], Cuenin [6] and [11], Cossetti [12], Ibrogimov, Krejčiřík, and Laptev [34], and Hulko [30] and [31]. A series of results on spectral analysis of Jacobi matrices can be found in Borichev, Golinskii, and Kupin [4] and [5] and Golinskii and Kupin [26]–[29].

In the present article, we are interested in the study of perturbations of bilayer graphene Hamiltonian given by

$$D_{\mathrm{bg},m} := \begin{bmatrix} m & 4\partial_z^2 \\ 4\partial_{\bar z}^2 & -m \end{bmatrix}, \tag{1}$$

where $m \geq 0$ and

$$\partial_z := \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}), \quad \partial_{\bar z} := \frac{1}{2}(\partial_{x_1} - i\partial_{x_2}).$$

As usual, we let

$$L^2(\mathbb{R}^2;\mathbb{C}^2) := \left\{ f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : \|f\|_2^2 = \int\limits_{\mathbb{R}^2} |f(x)|^2\, dx < \infty \right\}$$

to be the standard space of measurable vector-valued functions; here

$$|f(x)| = (|f_1(x)|^2 + |f_2(x)|^2)^{1/2}.$$

Furthermore, let

$$H^2(\mathbb{R}^2;\mathbb{C}^2) := \left\{ f \in L^2(\mathbb{R}^2;\mathbb{C}^2) : \|f\|_{H^2}^2 = \int\limits_{\mathbb{R}^2} (1+|\xi|^2)^2 |\hat f(\xi)|^2\, d\xi < \infty \right\}$$

be the corresponding second order Sobolev space, where $\hat f$ denotes the Fourier transform of a function f, see Section 1.1 for more notation. It is not difficult to see that

$$D_{\mathrm{bg},m} : H^2(\mathbb{R}^2;\mathbb{C}^2) \longrightarrow L^2(\mathbb{R}^2;\mathbb{C}^2)$$

is a selfadjoint operator. Since

$$D_{\mathrm{bg},m}^2 = (\Delta^2 + m^2) I_2,$$

the spectral mapping theorem yields $\sigma(D_{\mathrm{bg},m}) := (-\infty, -m] \cup [m, +\infty)$. The resolvent set of $D_{\mathrm{bg},m}$ is denoted by $\rho(D_{\mathrm{bg},m}) := \mathbb{C} \backslash \sigma(D_{\mathrm{bg},m})$.

Detailed discussion of this and other similar operators from the physical point of view can be found in the book of Katznelson [36].

We consider the perturbed operator

$$D_{\mathrm{bg}} := D_{\mathrm{bg},m} + V \tag{2}$$

with $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, $q \geq 1$. Since the perturbation V is not assumed to be selfadjoint, the operator D_{bg} may be non-selfadjoint as well. For the formal definition of $D_{\mathrm{bg},m} + V$ for the class of potentials considered here we allude to the "factorization method" of Kato [35]; see also Gesztesy, Latushkin, et al. [24]. A version of Weyl's theorem [24, Theorem 4.5] asserts that

$$\sigma_{\mathrm{ess}}(D_{\mathrm{bg}}) = \sigma_{\mathrm{ess}}(D_{\mathrm{bg},m}) = (-\infty, -m] \cup [m, +\infty), \tag{3}$$

where we adopt the convention that $\sigma_{\mathrm{ess}}(D_{\mathrm{bg}}) := \sigma(D_{\mathrm{bg}}) \backslash \sigma_d(D_{\mathrm{bg}})$ and the discrete spectrum $\sigma_d(D)$ is the set of isolated eigenvalues of D of finite multiplicity.

We shall be interested in distribution properties of the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$ of the perturbed operator D_{bg}. Note that $\sigma_d(D_{\mathrm{bg}})$ can only accumulate to $\sigma_{\mathrm{ess}}(D_{\mathrm{bg}})$, and we want to find some quantitative characteristics of the rate of accumulation.

The first step in this direction is to understand better the localization of the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$. The well-established Birman–Schwinger operator

$$\mathrm{BS}_z := |V|^{1/2}(D_{\mathrm{bg},m} - z)^{-1} V^{1/2}, \quad z \in \rho(D_{\mathrm{bg},m}), \tag{4}$$

plays a key role in this problem, see original references by Birman [3] and Schwinger [43]. Here, $V(x) = |V(x)|U(x)$ is the polar decomposition of the matrix $V(x)$, $|V(x)| := (V(x)^* V(x))^{1/2}$ and $U(x)$ is the corresponding partial isometry. So, $V^{1/2}(x) := |V(x)|^{1/2} U(x)$ for a. e. $x \in \mathbb{R}^2$. The Birman–Schwinger principle [24, Theorem 3.2] says that $z \in \rho(D_{\mathrm{bg},m})$ is an eigenvalue of D_{bg} if and only if -1 is an eigenvalue of the operator BS_z. In particular, we have the inclusion

$$\sigma_d(D_{\mathrm{bg}}) \subset \{z \in \rho(D_{\mathrm{bg},m}) \colon \| \mathrm{BS}_z \| \geq 1\}.$$

Laptev, Ferrulli, and Safronov [18, Theorem 1.1] obtain the following interesting result.

Theorem 0.1 ([18]). *Let $D_{\mathrm{bg},m}, D_{\mathrm{bg}}$ be as above and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, $1 < q < 4/3$. Then,*

(1) *for $z \in \rho(D_{\mathrm{bg},m})$,*

$$\| \mathrm{BS}_z \|^q = \| |V|^{1/2}(D_{\mathrm{bg},m} - z)^{-1} V^{1/2} \|^q \leq C_q \|V\|_q^q \frac{(|z-m| + |z+m|)^q}{|z^2 - m^2|^{q-1/2}}; \tag{5}$$

(2) *in particular,*

$$\sigma_d(D_{\rm bg}) \subset \left\{z\colon C_q \|V\|_q^q \frac{(|z-m|+|z+m|)^q}{|z^2-m^2|^{q-1/2}} \ge 1\right\}.$$

Slightly later, the Cuenin [7, Theorem 1.1 and Proposition 2.4] improved the resolvent bound in several respects. First, he showed that the norm of the Birman–Schwinger operator BS_z in the left hand side of (5) can be taken in an appropriate Schatten–von Neumann class $\mathcal{S}_p$, $p = p(q)$; second, the range of parameter q is extended to $1 \le q \le 3/2$. It was observed that these results were optimal in a certain sense. We mention also that [7, Proposition A.5] addresses more general situations as compared to [18, Theorem 1.1]; in particular, the former is valid for more general differential operators than the bilayer graphene Hamiltonian.

The key to the Lieb–Thirring type inequalities obtained in this article is a claim similar to [7, Proposition 2.4]. We feel that it is appropriate to give a detailed and a self-contained proof of this result, see Theorem 0.2 below. As compared to [7, Proposition 2.4], we extend the range of parameter q to $1 \le q < \infty$.

Theorem 0.2. *Let $D_{{\rm bg},m}$, $D_{\rm bg}$ be defined in (1), (2), and $m > 0$. For $q \ge 1$ and $\varepsilon > 0$, set*

$$p = p(q,\varepsilon) := \begin{cases} \dfrac{q}{2-q} + \varepsilon, & 1 \le q < 4/3, \\ \dfrac{q}{2-q}, & 4/3 \le q \le 3/2, \\ 2q, & q > 3/2. \end{cases} \tag{6}$$

(I) *Let $1 \le q \le 3/2$. There exists a $C_3 > 0$ such that, for any $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, one has*

$$\|A(D_{{\rm bg},m} - z)^{-1} B\|_{\mathcal{S}_p} \le C_3 \Phi(z) \|A\|_{2q} \|B\|_{2q}, \tag{7}$$

where

$$\Phi(z) = \Phi_q(z) := \frac{|z+m|+|z-m|}{|z^2-m^2|^{q_1}},$$

$z \in \rho(D_{{\rm bg},m})$ and $q_1 := 1 - 1/(2q)$.

(II) *Let $q > 3/2$. There exists a $C_4 > 0$ such that, for any $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, one has*

$$\|A(D_{{\rm bg},m} - z)^{-1} B\|_{\mathcal{S}_p} \le C_4 \Psi(z) \, \|A\|_{2q} \|B\|_{2q}, \tag{8}$$

where

$$\Psi(z) = \Psi_q(z) := \frac{(|z+m|+|z-m|)^{q_2}}{|z^2-m^2|^{1/q}} \, \frac{1}{d^{1-q_2}(z, \sigma(D_{{\rm bg},m}))},$$

$z \in \rho(D_{\mathrm{bg},m})$ and $q_2 := 3/(2q) < 1$. Here, $d(z, \sigma(D_{\mathrm{bg},m}))$ is the distance from z to $\sigma(D_{\mathrm{bg},m})$. The constants C_3, C_4 depend on m, q, ε, but not on $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$.

The above result along with discussion on Birman–Schwinger operators preceding Theorem 0.1 provides the following corollary.

Corollary 0.3. (1) *For $1 \le q \le 3/2$ and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$,*

$$\sigma_d(D_{\mathrm{bg}}) \subset \{z \colon C_3 \Phi(z) \|V\|_q \ge 1\}.$$

In particular, the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$ is bounded.

(2) *For $q > 3/2$ and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$,*

$$\sigma_d(D_{\mathrm{bg}}) \subset \{z \colon C_4 \Psi(z) \|V\|_q \ge 1\}.$$

Theorem 0.2 combined with techniques developed in Borichev, Golinskii, and Kupin [4] and [5] implies the following result.

Theorem 0.4. *Let $D_{\mathrm{bg},m}, D_{\mathrm{bg}}$ be defined in (1), (2), and $m > 0$. For $q > 1$ and $\varepsilon > 0$, set*

$$\beta = \beta(q, \varepsilon) := \begin{cases} \dfrac{4q-5}{2(2-q)} + \dfrac{2q-1}{2q}\varepsilon, & 1 < q < \dfrac{4}{3}, \\ \dfrac{4q-5}{2(2-q)}, & \dfrac{4}{3} \le q \le \dfrac{3}{2}. \end{cases} \tag{9}$$

Assume that $\|V\|_q \le 1$. Then the Lieb–Thirring inequalities for the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$ hold:

(I) *for $1 \le q \le 3/2$,*

$$\sum_{\zeta \in \sigma_d(D_{\mathrm{bg}})} d^{1+\varepsilon}(\zeta, \sigma(D_{\mathrm{bg},m}))\, |\zeta^2 - m^2|^\beta \le C_5 \|V\|_q; \tag{10}$$

(II) *for $q > 3/2$,*

$$\sum_{\zeta \in \sigma_d(D_{\mathrm{bg}})} \frac{|\zeta|^{2q+1+\varepsilon} d^{2q-2+\varepsilon}(\zeta, \sigma(D_{\mathrm{bg},m}))\, |\zeta^2 - m^2|}{(1 + |\zeta|)^{2q+1+\varepsilon}} \le C_6 \|V\|_q. \tag{11}$$

The constants C_5 and C_6 depend on m, q, ε, but not on $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$.

The counterparts of the above theorems for the case $m = 0$ are given below. Their proofs are similar to Theorems 0.2, 0.4, and therefore they are omitted.

Theorem 0.5. *Let $D_{\mathrm{bg},0}, D_{\mathrm{bg}}$ be given by* (1), (2), *and* $z \in \rho(D_{\mathrm{bg},0}) := \mathbb{C}\backslash\mathbb{R}$. *Take an $\varepsilon > 0$ and put $p = p(q,\varepsilon)$ as in* (6).

(I) *Let $1 \le q \le 3/2$. There exists a $C_3' > 0$ such that, for any $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, one has*

$$\|A(D_{\mathrm{bg},0} - z)^{-1}B\|_{\mathcal{S}_p} \le C_3'|z|^{-(1-\frac{1}{q})}\|A\|_{2q}\|B\|_{2q}. \tag{12}$$

(II) *Let $q > 3/2$. There exists a $C_4' > 0$ such that, for any $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, one has*

$$\|A(D_{\mathrm{bg},0} - z)^{-1}B\|_{\mathcal{S}_p} \le C_4'|z|^{-\frac{1}{2q}}|\mathrm{Im}\, z|^{-(1-\frac{3}{2q})}\|A\|_{2q}\|B\|_{2q}, \tag{13}$$

Above, $|\mathrm{Im}\, z| = d(z,\mathbb{R})$ is the distance from z to the real line $\mathbb{R}$. The constants C_3' and C_4' depend on q, ε, but not on $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$.

Similarly to Corollary 0.3, we can describe the regions containing the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$ for $m = 0$. In particular, the set is bounded for $1 \le q \le 3/2$ and $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$.

Theorem 0.6. *Let $D_{\mathrm{bg},0}, D_{\mathrm{bg}}$ be defined as above. Let $q > 1$ and $\varepsilon > 0$ be small enough. Assume that $\|V\|_q \le 1$. Then the Lieb–Thirring inequalities for the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$ hold*:

(I) *for $1 \le q \le 3/2$,*

$$\sum_{\zeta \in \sigma_d(D_{\mathrm{bg}})} |\mathrm{Im}\, \zeta|^{1+\varepsilon} \le C_5'\|V\|_q; \tag{14}$$

(II) *for $q > 3/2$,*

$$\sum_{\zeta \in \sigma_d(D_{\mathrm{bg}})} \frac{|\mathrm{Im}\, \zeta|^{2-\frac{3}{2q}+\varepsilon}}{(1+|\zeta|)^{1-\frac{3}{2q}+2\varepsilon}} \le C_6'\|V\|_q. \tag{15}$$

The constants C_5' and C_6' depend on q, ε, but not on $V \in L^q(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$.

Remark 0.7. (1) In order to prove the above theorems we need the $\mathcal{S}_p$-norm of the Birman–Schwinger operator $\|V_2(D_{\mathrm{bg},m} - z(iy))^{-1}V_1\|_{\mathcal{S}_p}$ to go to zero when $y \to +\infty$, see (45). For this reason inequality (10) is obtained for $1 < q \le 3/2$, even though the case $q = 1$ is treated in Theorem 0.2.

(2) The assumption $\|V\|_q \le 1$ does not mean that the perturbation is small. Theorem 0.4 holds uniformly over any bounded in L^q set of potentials V, i.e., 1 can be replaced with a constant $C(q, m, \varepsilon)$.

The paper is organized in the following manner. We start Section 1 recalling some basic facts and notation on differential operators. The second part of Section 1 is devoted to the proof of Theorem 0.2. The proof of Theorem 0.4 is in Section 2. Section 3 is an appendix containing results on interpolation between $\mathcal{S}_p$-spaces and the Kato–Selier–Simon lemma.

The space of infinitely differentiable functions on $\mathbb{R}^2$ is denoted by $C^\infty(\mathbb{R}^2)$; $C_0^\infty(\mathbb{R}^2)$ are infinitely differentiable functions with compact support. The notation $L^p(\mathbb{R}^2)$, $1 \le p \le \infty$, stays for the familiar space of p-summable measurable functions. $L_0^\infty(\mathbb{R}^2)$ refers also to functions from $L^\infty(\mathbb{R}^2)$ with compact support. Meaningful constants are written as C_j, C_j', $j = 0, 1, \ldots$; technical constants are denoted by c, C, and they change from one relation to another.

1. Resolvent bounds for the bilayer graphene Hamiltonian

1.1. Fourier transforms. The purpose of this subsection is to fix some notation and recall some basic properties of the Fourier transformation. For this purpose we temporarily consider the case of arbitrary dimension n. At the end of the subsection we will compute Fourier transforms of some tempered distributions (homogeneous distributions and surface-carried measures) that will play an important role in the next subsection. We refer to Hörmander [32] and Sogge [45] for more details on the subject.

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined as

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

Let $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space, *i.e.,* the space of rapidly decreasing smooth functions on $\mathbb{R}^n$. The Fourier transformation is an isomorphism $\mathcal{F}\colon \mathcal{S} \to \mathcal{S}$, and its inverse is furnished by the Fourier inversion formula,

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi}\, dx.$$

We use the standard notation $\check{f} := \mathcal{F}^{-1} f$. Hence, $\mathcal{F}$ may be extended to the dual space $\mathcal{S}'$, the space of tempered distributions, by setting $\hat{u}(\phi) = u(\hat{\phi})$ for $u \in \mathcal{S}'$, $\phi \in \mathcal{S}$. Moreover, Plancherel's formula,

$$\|\hat{f}\|_2 = (2\pi)^{n/2} \|f\|_2, \quad f \in \mathcal{S}, \tag{16}$$

gives rise to a continuous extension $\mathcal{F}\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.

Let $D = \nabla$ be a formal differential operator. The Fourier multiplier

$$m(D)\colon \mathcal{S} \longrightarrow \mathcal{S}'$$

associated to a tempered distribution $m \in \mathcal{S}'$ is the operator

$$m(D)f := \mathcal{F}^{-1}(m\hat{f}), \quad f \in \mathcal{S},$$

and (16) shows that m is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$, and $\|m(D)\| = \|m\|_\infty$. We also have

$$(m(D)\varphi)(x) = \check{m} * \varphi = \int_{\mathbb{R}^n} \check{m}(x-y)\varphi(y)\,dy, \quad \varphi \in \mathcal{S}, \tag{17}$$

with the understanding that $*\colon \mathcal{S}' \times \mathcal{S} \to \mathcal{S}'$ is the convolution between a Schwartz function and a tempered distribution. The second identity in (17) is in general only formal, but it is rigorous if $\check{m}$ is a regular tempered distribution. To simplify notation, the expression $(m(D))(x)$, refers to the convolution kernel $\check{m}(x)$ of the integral operator in (17).

Consider now a smooth real-valued function ρ which we think of as (a normalized power of) a Hamiltonian. Then, for $\lambda \in \mathbb{R}$, we define the level sets of ρ (*i.e.,* the sets of constant energy) as

$$S_\lambda := \rho^{-1}(\lambda) = \{\xi \in \mathbb{R}^n\colon \rho(\xi) = \lambda\}. \tag{18}$$

These sets play a crucial role in scattering theory, see e.g. Hörmander [33, Chapter XIV]. In the present paper the main feature of S_λ is its nowhere vanishing Gaussian curvature. To ensure that S_λ is in fact a manifold (a curve) we make the assumption that ρ is normalized such that $|\nabla\rho| = 1$ on S_λ. In the following we will only deal with[1] $\rho(\xi) = |\xi|$, in which case S_λ is just the sphere of radius λ. Let $d\sigma_{S_\lambda}$ be the canonical surface measure on S_λ. As usual, $L^2(d\sigma_{S_\lambda})$ is the space of measurable square-summable functions on S_λ. The Fourier restriction operator for S_λ is defined by

$$R(\lambda)\varphi := \hat{\varphi}\big|_{S_\lambda}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

Its formal adjoint (the Fourier extension operator) is given by

$$R(\lambda)^*\varphi = \widehat{\varphi\, d\sigma_{S_\lambda}}, \quad \varphi \in \mathcal{S}(\mathbb{R}^n).$$

[1] The fact that $\xi \mapsto |\xi|$ is not smooth at $\xi = 0$ is irrelevant for our purposes since (by homogeneity) we will only need smoothness in a neighborhood of the unit sphere $S_1 = \{\xi\colon |\xi| = 1\}$.

Here, the Fourier transform of the measure $\varphi\, d\sigma_{S_\lambda}$ is defined as

$$\widehat{\varphi\, d\sigma_{S_\lambda}}(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(\xi) d\sigma_{S_\lambda}(\xi).$$

The multiplier corresponding to the function $\xi \mapsto |\xi|$ is denoted by $\sqrt{-\Delta}$. Denote by $E_{\sqrt{-\Delta}}(\lambda)$ the (operator-valued) spectral measure associated to this operator, viewed as an unbounded selfadjoint operator on $L^2(\mathbb{R}^n)$. Since its spectrum is absolutely continuous we may write

$$dE_{\sqrt{-\Delta}}(\lambda) = \frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda} d\lambda,$$

where the convolution kernel of the density is given by

$$\frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda}(x-y) = (2\pi)^{-n} \int_{|\xi|=\lambda} e^{i(x-y)\cdot\xi} d\sigma_{S_\lambda}(\xi).$$

By a change of variables $\xi = \lambda\xi'$, $|\xi'| = 1$, we see that

$$\frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda} = \frac{\lambda^{n-1}}{(2\pi)^n} R(\lambda)^* R(\lambda), \tag{19}$$

where $R(\lambda)$ is the restriction operator discussed above. It is also plain that

$$R(\lambda) f = \lambda^{-n} R(1)(f(\lambda^{-1}\cdot)).$$

Define

$$\chi_+^w(\tau) := \mathbf{1}_{[0,\infty)}(\tau)\tau^w/\Gamma(w+1), \quad w \in \mathbb{C},$$

where Γ is the usual Gamma function.

Lemma 1.1. *Let $z, \zeta \in \mathbb{C}$, $\operatorname{Im} z > 0$. The one-dimensional inverse Fourier transform of the function*

$$\eta_{z,\zeta}(x) := (x-z)^{-\zeta}, \quad x \in \mathbb{R},$$

is given by

$$\check{\eta_{z,\zeta}}(\tau) = e^{i(\pi\zeta/2 + z\tau)} \chi_+^{\zeta-1}(\tau). \tag{20}$$

Proof. After a change of variables, this follows immediately by applying the inverse Fourier transformation to the following identity (see [32], specifically the explanation after Example 7.1.17)

$$\mathcal{F}(x \mapsto e^{-\epsilon x}\chi_+^z(x))(\xi) = e^{-i\pi(z+1)/2}(\xi - i\epsilon)^{-z-1}, \quad \epsilon > 0,\ z \in \mathbb{C}. \qquad \square$$

Lemma 1.2. *Let $\beta \in C_0^\infty(\mathbb{R}^n)$ and let S_1 be the unit sphere in $\mathbb{R}^n$. Then the inverse Fourier transform of the surface measure $d\mu := \beta\, d\sigma_{S_1}$ admits the representation*

$$\check{d\mu}(x) = \sum_{\pm} e^{\pm i|x|} a_\pm(|x|) := e^{i|x|} a_+(|x|) - e^{-i|x|} a_-(|x|),$$

where $a_\pm \in C^\infty(\mathbb{R}_+)$ satisfy the symbol bounds

$$|\partial^k a_\pm(s)| \le C_{k\pm}(1+|s|)^{-\frac{n-1}{2}-k}. \tag{21}$$

Proof. This is a special case of [45, Theorem 1.2.1]. □

Lemma 1.3. *Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be supported in the annulus $\{1/2 \le |\xi| \le 3/2\}$, and $S = \{\zeta : a \le \operatorname{Re}\zeta \le b\}$ be a vertical strip in $\mathbb{C}$. Then*

$$\left| \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\chi(\xi)}{(|\xi|-z)^\zeta}\, d\xi \right| \le C e^{\pi^2|\operatorname{Im}\zeta|^2} (1+|x|)^{-\frac{n+1}{2}+\operatorname{Re}\zeta}, \quad \zeta \in S,\ |z| = 1,$$

where the constant depends on a, b and finitely many derivatives of χ, but is independent of ζ, z.

Proof. It suffices to prove this for $|x| > 1$ since the case $|x| \le 1$ is trivial. Writing the integral in polar coordinates and using Lemma 1.2 we find that

$$\int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{\chi(\xi)}{(|\xi|-z)^\zeta}\, d\xi = \sum_{\pm} \int_{-\infty}^{\infty} e^{\pm ir|x|} \frac{r^{n-1} a_\pm(r|x|)}{(r-z)^\zeta}\, dr,$$

where the function $r \mapsto r^{n-1} a_\pm(r|x|)$ is supported in a neighborhood of $r = 1$ and it satisfies

$$|r^{n-1} a_\pm(r|x|)| \le C(1+|x|)^{-\frac{n-1}{2}}$$

for any fixed Schwartz norm $|\cdot|$. Hence, by Lemma 1.2 again, its inverse Fourier transform is bounded by

$$|\mathcal{F}^{-1}(r \longmapsto r^{n-1} a_\pm(r|x|))(\tau)| \le C_N (1+|\tau|)^{-N} (1+|x|)^{-\frac{n-1}{2}}$$

for any $N > 0$. The convolution theorem and Lemma 1.1 yield

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} e^{\pm ir|x|} \frac{r^{n-1} a_\pm(r|x|)}{(r-z)^\zeta}\, dr \right| \\ &\quad \le C_N e^{\pi|\operatorname{Im}\zeta|} (1+|x|)^{-\frac{n-1}{2}} \int_{-\infty}^{\infty} (1+|\tau-|x||)^{-N} \chi_+^{\operatorname{Re}\zeta-1}(\tau)\, d\tau \\ &\quad \le C e^{\pi|\operatorname{Im}\zeta|} |\Gamma(\zeta)^{-1}| (1+|x|)^{-\frac{n+1}{2}+\operatorname{Re}\zeta}. \end{aligned}$$

The claim now follows from the estimate $|\Gamma(\zeta)^{-1}| \le Ce^{\pi^2|\zeta|^2}$; see e.g. formula (11.21) in Muscalu and Schlag [41]. $\square$

1.2. Resolvent bounds in $\mathcal{S}_p$-norm for bilayer graphene. We now return to the case $n = 2$ and the bilayer Hamiltonian. The coming bound is a special case of [7, Lemma A.6]. It is crucial for coming resolvent estimates.

In the following, we fix a function $\chi \in C_0^\infty(\mathbb{R}^2)$ supported in the annulus $\{1/2 \le |\xi| \le 3/2\}$ such that, in addition, $\chi(\xi) = 1$ for $3/4 \le |\xi| \le 5/4$.

Proposition 1.4. *Let* $1 \le a \le 3/2, t \in \mathbb{R}$, *and* $z \notin \mathbb{R}_+$. *There exists a constant* $C_1' > 0$ (*depending on* χ *only*) *such that*

$$|\chi(D)(\Delta^2 - z)^{-(a+it)}(x)| \le \frac{C_1' e^{\pi^2 t^2}}{(1+|x|)^{3/2-a}}, \quad x \in \mathbb{R}^2, \ |z| = 1. \tag{22}$$

Proof. Set $z^{1/4} = |z|^{1/4} e^{(i \operatorname{Arg} z)/4}$. Clearly the 4-th power complex roots of z are given by $\{i^m z^{1/4}\}$, $m = 0, 1, 2, 3$. Without loss of generality, we suppose that $m = 0$ and $|\operatorname{Arg} z| \le \pi$, or $|\operatorname{Arg} z^{1/4}| \le \pi/4$, the other cases being analogous. Writing

$$(|\xi|^4 - z) = (|\xi| - z^{1/4})\Big(\prod_{k=1}^{3} (|\xi| - i^k z^{1/4})\Big)$$

and absorbing the second factor into χ, we see that it suffices to prove

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\tilde{\chi}(\xi; a; t)}{(|\xi| - z^{1/4})^{a+it}} \, d\xi \le \frac{Ce^{\pi^2 t^2}}{(1+|x|)^{3/2-a}},$$

whenever $\tilde{\chi}(\xi; a, t)$ satisfies the bounds

$$\sum_{|\alpha| \le N} \|\partial_\xi^\alpha \tilde{\chi}(\cdot; a, t)\|_\infty \le C_N e^{2\pi|t|}$$

for a fixed, sufficiently large $N > 0$. This follows directly from Lemma 1.3. $\square$

Remark 1.5. In view of the identity

$$\frac{1}{|\xi|^2 - z^{1/2}} - \frac{1}{|\xi|^2 + z^{1/2}} = \frac{2z^{1/2}}{|\xi|^4 - z},$$

inequality (22) also follows from a two-dimensional version of estimates (2.23) and (2.25) in Kenig, Ruiz, and Sogge [37]; see also (44) in Frank and Sabin [21]. To keep the article self-contained, we provided the above proof which rests only on the stationary phase method (Lemma 1.2) and formula (20).

Proposition 1.6. *Fix an* $\varepsilon > 0$ *and set the function* χ *as above. For* $q \geq 1$, *let*

$$p = p(q,\varepsilon) := \begin{cases} \dfrac{q}{2-q} + \varepsilon, & 1 \leq q < 4/3, \\ \dfrac{q}{2-q}, & 4/3 \leq q \leq 3/2, \\ 2q, & q > 3/2. \end{cases} \tag{23}$$

For $A, B \in L^{2q}(\mathbb{R}^2)$, *the following bounds hold true*:

(I) *for* $1 \leq q \leq 3/2$,

$$\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_p} \leq C_7 \, \|A\|_{2q}\|B\|_{2q}, \quad |z| = 1; \tag{24}$$

(II) *for* $q > 3/2$

$$\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_p} \leq \frac{C_8}{d(z,\mathbb{R}_+)^{1-3/(2q)}} \, \|A\|_{2q}\|B\|_{2q}, \quad |z| = 1. \tag{25}$$

Here, $C_j = C_j(q,\varepsilon)$, $j = 7, 8$, *are independent of* A, B, *and* z.

Proof. The proof relies heavily on interpolation between Schatten–von Neumann classes $\mathcal{S}_p$, $p \geq 1$, presented in Section 3. It is convenient to separate part (I) of the proposition in two cases: Case I.1 for $1 \leq q < 4/3$ and Case I.2 for $4/3 \leq q \leq 3/2$. We begin with the proof of Case I.2.

Case I.2: $4/3 \leq q \leq 3/2$. Without loss of generality we may assume that $A > 0$ and $B > 0$. At the moment, we suppose also that $A, B \in L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$. We wish to apply Corollary 3.4 to the analytic family of operators given by

$$T_\zeta := A^\zeta \chi(D)(\Delta^2 - z)^{-\zeta} B^\zeta$$

on the strip

$$S = S_{0,a_0} := \{\zeta : 0 \leq \operatorname{Re} z \leq a_0\}, \quad \text{with } 1 \leq a_0 \leq 3/2.$$

Here, $\zeta = a + it$, $0 \leq a \leq a_0$, and $t \in \mathbb{R}$.

We start by checking assumptions of Corollary 3.4, see also Theorem 3.3. For arbitrary $f, g \in L^2(\mathbb{R}^2)$ we have, by Plancherel's identity,

$$(T_\zeta f, g) = \int_{\mathbb{R}^2} \chi(\xi)(|\xi|^4 - z)^{-\zeta} \widehat{B^\zeta f}(\xi) \overline{\widehat{A^\zeta g}(\xi)} d\xi,$$

which shows that $\zeta \mapsto (T_\zeta f, g)$ is analytic in S. By the Cauchy–Schwarz inequality,

$$|(T_\zeta f, g)| \leq \|\chi\|_\infty \|(|\cdot|^4 - z)^{-\zeta}\|_\infty \|B^\zeta f\|_2 \, \|A^\zeta g\|_2.$$

Since $|\arg(|\xi|^4 - z)| \leq 2\pi$, we have that

$$\begin{aligned} |(|\xi|^4 - z)^{-\zeta}| &= |\exp(-(a+it)(\log \||\xi|^4 - z| + i \arg(|\xi|^4 - z))| \\ &\leq |(|\xi|^4 - z)|^{-a} \exp(2\pi|t|). \end{aligned}$$

Observe that a varies over a compact interval and z is fixed. Putting all this together, we obtain that

$$|(T_\zeta f, g)| \leq Ce^{2\pi|t|} \|\chi\|_\infty \|A\|_\infty^a \|B\|_\infty^a \|f\|_2 \|g\|_2, \quad \zeta = a + it,$$

showing that (61) is satisfied. It also yields that

$$\|T_\zeta\|_{\mathfrak{S}_\infty} \leq Ce^{2\pi|\operatorname{Im}\zeta|} \tag{26}$$

for $\operatorname{Re}\zeta = 0$. Note that T_ζ is compact since we have the Hilbert-Schmidt bound

$$\begin{aligned} \|T_\zeta\|_{\mathfrak{S}_2}^2 &= \int_{\mathbb{R}_x^2}\int_{\mathbb{R}_y^2} |A^\zeta(x)|^2 |\mathcal{F}(\chi(|\cdot|^4 - z)^{-\zeta})(x-y)|^2 |A^\zeta(x)|^2 dxdy \\ &\leq e^{4\pi|\operatorname{Im}\zeta|} \|\chi(|\cdot|^4 - z)^{-\operatorname{Re}\zeta}\|_1^2 \|A\|_2^{2\operatorname{Re}\zeta} \|B\|_2^{2\operatorname{Re}\zeta}, \end{aligned}$$

and the right hand side is finite by the assumption that $A, B \in L_0^\infty(\mathbb{R}^2)$.

On the vertical line $\{\zeta : \operatorname{Re}\zeta = a_0\}$, Proposition 1.4 and Hardy–Littlewood–Sobolev inequality (see Lieb and Loss [39, Section 4.3]) yield that

$$\begin{aligned} \|T_{a_0+it}\|_{\mathfrak{S}_2}^2 &\leq \int_{\mathbb{R}_x^2}\int_{\mathbb{R}_y^2} |\chi(D)(\Delta^2 - z)^{-(a_0+it)}(x-y)|^2 |A(x)|^{2a_0} |B(y)|^{2a_0}\, dxdy \\ &\leq Ce^{2\pi^2t^2} \int_{\mathbb{R}_x^2}\int_{\mathbb{R}_y^2} \frac{1}{|x-y|^{3-2a_0}} |A(x)|^{2a_0} |B(y)|^{2a_0}\, dxdy \\ &\leq Ce^{2\pi^2t^2} \||A|^{2a_0}\|_s \||B|^{2a_0}\|_s, \end{aligned}$$

where $2/s + (3-2a_0)/2 = 2$, or $s = 4/(1+2a_0)$. In particular,

$$\||A|^{2a_0}\|_s = \|A\|_{8a_0/(1+2a_0)}^{2a_0},$$

the same equality holding for $\||B|^{2a_0}\|_s$. Hence, gathering the above computations, we arrive at the bound

$$\|T_\zeta\|_{\mathfrak{S}_2} \leq Ce^{\pi^2|\operatorname{Im}\zeta|^2} \|A\|_{8a_0/(1+2a_0)}^{a_0} \|B\|_{8a_0/(1+2a_0)}^{a_0} \quad \text{for } \operatorname{Re}\zeta = a_0. \tag{27}$$

We recall now Corollary 3.4 (see also Theorem 3.3) with parameters chosen as

$$\zeta := 1, \quad 1 = \gamma \cdot a_0 + (1-\gamma) \cdot 0, \quad \frac{1}{s_\gamma} = \frac{\gamma}{2} + \frac{(1-\gamma)}{\infty} = \frac{\gamma}{2},$$

to interpolate between (26) and (27). Solving first for γ and then for s_γ yields $\gamma = 1/a_0$ and $s_\gamma = 2a_0$. Corollary 3.4 then implies that

$$\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_{2a_0}} \le C_7 \, \|A\|_{8a_0/(1+2a_0)} \|B\|_{8a_0/(1+2a_0)},$$

which is exactly (24) with $4/3 \le q \le 3/2$ if one puts $2q = 8a_0/(1+2a_0)$.

To sum up, we proved (24) for $4/3 \le q \le 3/2$ and $A, B \in L^{2q}(\mathbb{R}^2) \cap L^\infty_0(\mathbb{R}^2)$. It remains to get rid of the assumption that $A, B \in L^\infty_0(\mathbb{R}^2)$. The proof relies essentially on the fact that the constant C_7 from (24) *does not depend* on A, B. We proceed by a limiting argument. Let $A, B \in L^{2q}(\mathbb{R}^2)$. For $n \in \mathbb{N}$, define

$$E_n = \{x \in \mathbb{R}^2 \colon |x| + |A(x)| + |B(x)| \le n\}$$

and set the "truncations" of A, B to be

$$A_n = A\mathbf{1}_{E_n}, \quad B_n = B\mathbf{1}_{E_n}.$$

Let $P_n \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be the corresponding orthogonal projection

$$P_n f = \mathbf{1}_{E_n} f, \quad f \in L^2(\mathbb{R}^2).$$

The elementary properties of L^{2q}-integrable functions yield that

$$\lim_{n\to+\infty} \|P_n f - f\|_2 = 0 \quad \text{for any } f \in L^2(\mathbb{R}^2).$$

Recalling [25, Theorem 5.2] and inequality (24) for functions from $L^{2q}(\mathbb{R}^2) \cap L^\infty_0(\mathbb{R}^2)$, we obtain

$$\begin{aligned}
\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathcal{S}_p} &= \sup_n \|P_n(A\chi(D)(\Delta^2 - z)^{-1}B)P_n\|_{\mathcal{S}_p} \\
&= \sup_n \|A_n\chi(D)(\Delta^2 - z)^{-1}B_n\|_{\mathcal{S}_p} \\
&\le C_7 \|A_n\|_{2q} \|B_n\|_{2q} \\
&\le C_7 \|A\|_{2q} \|B\|_{2q}.
\end{aligned}$$

Case I.2 follows.

Case II: $q > 3/2$. As before, we may assume without loss of generality that $A, B \in L^{2q}(\mathbb{R}^2) \cap L^\infty_0(\mathbb{R}^2)$, and that $A, B > 0$.

Let $S := S_{0,a_0} := \{a + it : 0 \le a \le a_0 = 2q/3,\ t \in \mathbb{R}\}$. Notice that $q > 3/2$ implies that $a_0 = 2q/3 > 1$. Consider the analytic family of operators

$$T_\zeta = A^\zeta \chi(D)(\Delta^2 - z)^{-1} B^\zeta,$$

defined on S. For $\operatorname{Re}\zeta = a_0$, inequality (24) applied with $p_0 = 3, q_0 = 3/2$ instead of p, q yields

$$\|T_\zeta\|_{\mathfrak{S}_3} \le C_3 \|A^{2q/3}\|_3 \|B^{2q/3}\|_3 = C_3 \|A\|_{2q}^{2q/3} \|B\|_{2q}^{2q/3} \tag{28}$$

for $\operatorname{Re}\zeta = a_0$. On the other hand, since for $\operatorname{Re}\zeta = 0$ we have $|A^\zeta| = |B^\zeta| = 1$ a.e. on $\mathbb{R}^2$, we also see that

$$\|T_\zeta\|_{\mathfrak{S}_\infty} \le \frac{\|\chi\|_\infty}{d(z, \mathbb{R}_+)}. \tag{29}$$

by the spectral theorem for Δ^2. Compactness of T_ζ follows by the same argument as in Case I.1. Interpolating in between (28) and (29), with

$$\zeta := 1, \quad 1 = \frac{2q}{3} \cdot \gamma + 0 \cdot (1 - \gamma) = \frac{2q}{3}\gamma,$$

we get $\gamma = 3/(2q) \in (0, 1)$ and consequently

$$\frac{1}{p_{0\gamma}} = \frac{\gamma}{3} + \frac{(1-\gamma)}{\infty} = \frac{\gamma}{3},$$

which means that $p_{0\gamma} = 2q$. That is,

$$\|A\chi(D)(\Delta^2 - z)^{-1} B\|_{\mathfrak{S}_{2q}} \le \frac{C_8}{d(z, \mathbb{R}_+)^{1-\gamma}} \|A\|_{2q} \|B\|_{2q},$$

By the same limiting argument as before, we get relation (25).

Case I.1: $1 \le q \le 4/3$. Let $\tilde\chi$ be a cutoff function with the same support properties as χ and such that $\tilde\chi = 1$ on the support of χ; in particular, $\tilde\chi\chi = \chi$.

Let $A, B \in L^2(\mathbb{R}^2)$. We start by proving that

$$\|A\chi(D) \frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda} \tilde\chi(D) B\|_{\mathfrak{S}_1} \le C\, \|A\|_2 \|B\|_2. \tag{30}$$

Indeed, using (19), we re-write the operator on the left hand side of (30) as

$$A\chi(D) \frac{dE_{\sqrt{-\Delta}}(\lambda)}{d\lambda} \tilde\chi(D) B = \frac{\lambda^{n-1}}{(2\pi)^n} (R(\lambda)\chi(D)A)^* (R(\lambda)\tilde\chi(D)B). \tag{31}$$

The kernel of the operator $R(\lambda)\chi(D)A\colon L^2(\mathbb{R}^2) \to L^2(S_\lambda)$ is given by

$$(R(\lambda)\chi(D)A)(\xi, x) = \chi(\xi)e^{ix\xi}A(x), \quad x \in \mathbb{R}^2, \xi \in S_\lambda,$$

and thus

$$\|R(\lambda)\chi(D)A\|_{\mathbb{S}_2}^2 = \int_{\mathbb{R}^2_x}\int_{S_{\lambda,\xi}} |\chi(\xi)A(x)|^2\, dx d\sigma_{S_\lambda}(\xi) = \|\chi\|^2_{L^2(S_\lambda)}\|A\|_2^2 \le C\|A\|_2^2.$$

Since the same bound holds for $R(\lambda)\tilde{\chi}(D)B$, Hölder's inequality for $\mathbb{S}_p$-classes yields (30).

Set $0 < a_0 < 1$. Using the formula

$$(\Delta^2 - z)^{-(a_0+it)} = \int_{\mathbb{R}} (\lambda^4 - z)^{-(a_0+it)}\, dE_{\sqrt{-\Delta}}(\lambda).$$

inequality (30) and the fact that the functions $\|\chi_j\|_{S_\lambda}$ are supported on the set where $1/2 \le \lambda \le 3/2$, we get the bound

$$\|A\chi(D)(\Delta^2 - z)^{-(a_0+it)}\chi(D)B\|_{\mathbb{S}_1} \le C\,\frac{e^{2\pi|t|}}{(1-a_0)}\|A\|_2\|B\|_2. \tag{32}$$

On the other hand, from (22), we see that

$$|\chi(D)(\Delta^2 - z)^{-3/2+it}(x)| \le C_1' e^{\pi^2 t^2},$$

that is, the kernel of $\chi(D)(\Delta^2 - z)^{-3/2+it}(x)$ is uniformly bounded with respect to the "space variable" $x \in \mathbb{R}^2$. The Hilbert-Schmidt bound for integral operators implies immediately

$$\|A\chi(D)(\Delta^2 - z)^{-3/2+it}B\|_{\mathbb{S}_2} \le C\,\|A\|_2\|B\|_2. \tag{33}$$

Let $0 < \varepsilon < 1/2$ be fixed. Suppose, as in Cases I.2 and II, that one has $A, B \in L^2(\mathbb{R}^d) \cap L_0^\infty(\mathbb{R}^2)$. Furthermore, set

$$T_\zeta := A\chi(D)^2(\Delta^2 - z)^{-\zeta}B$$

and $S = S_{a_0,b_0} := \{\zeta\colon a_0 \le \operatorname{Re}\zeta \le b_0\}$ to be the vertical strip with

$$a_0 = \frac{(1-2\varepsilon)}{(1-\varepsilon)} < 1, \quad b_0 = 3/2 > 1.$$

As previously, the family (T_ζ) on S_{a_0,b_0} satisfies the assumptions of Theorem 3.3 and we can interpolate between (32) and (33). More precisely, for the parameters of the corollary we take $\zeta := 1$ and

$$1 = \frac{1-2\varepsilon}{1-\varepsilon}\gamma + \frac{3}{2}(1-\gamma),$$

i.e., $\gamma = (1-\varepsilon)/(1+\varepsilon)$. Hence the relation

$$\frac{1}{s_\gamma} = \frac{\gamma}{1} + \frac{(1-\gamma)}{2}$$

gives $s_\gamma = 1+\varepsilon$. To sum up, we arrive at

$$\|A\chi(D)(\Delta^2 - z)^{-1}B\|_{\mathfrak{S}_{1+\varepsilon}} \le C\varepsilon^{-(1-\varepsilon)/(1+\varepsilon)}\|A\|_2\|B\|_2. \tag{34}$$

We interpolate once again in between (34) and (24) for $q = 4/3$ to obtain (24) for $1 \le q < 4/3$. Passing from $A, B \in L^{2q}(\mathbb{R}^2) \cap L_0^\infty(\mathbb{R}^2)$ to general $A, B \in L^{2q}(\mathbb{R}^2)$ is carried out as in the previous cases. □

We introduce some notation before going to the proof of Theorem 0.2. Let

$$k(u)^4 := (u^2 - m^2),$$

where we use the principal branch of 4-th complex root, so that one has $k(u) = (u^2 - m^2)^{1/4} \in \mathbb{R}_+$ for $u = x \in \mathbb{R}$, $x > m$. Furthermore,

$$\zeta(u) := \frac{u+m}{k(u)^2} = \left(\frac{u+m}{u-m}\right)^{1/2}, \quad u \neq \pm m$$

with the standard choice of the branch of the square complex root.

1.3. Proof of Theorem 0.2. In order to distinguish the variable referred to in operators $\partial_z, \partial_{\bar z}$ and the spectral parameter of the operator $D_{\mathrm{bg},m}$, the latter will be denoted by $u \in \rho(D_{\mathrm{bg},m})$ in this subsection.

We consider first Case I of the theorem, *i.e.*, $1 \le q \le 3/2$. Let $A, B \in L^{2q}(\mathbb{R}^2; \mathrm{Mat}_{2,2}(\mathbb{C}))$, that is

$$A(x) = [A_{jl}(x)]_{j,l=1,2}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

and $A_{jl}(x) \in L^{2q}(\mathbb{R}^2)$. Recalling the identities

$$4\partial_z\partial_{\bar z} = 4\partial_{\bar z}\partial_z = (\partial_{x_1}^2 + \partial_{x_2}^2)^2 = \Delta^2,$$

we readily see

$$\begin{aligned} D_{\mathrm{bg},m}^2 - u^2 &= \begin{bmatrix} m & 4\partial_{\bar z}^2 \\ 4\partial_z^2 & -m \end{bmatrix}^2 - u^2 \\ &= \begin{bmatrix} \Delta^2 + (m^2 - u^2) & 0 \\ 0 & \Delta^2 + (m^2 - u^2) \end{bmatrix} \\ &= (\Delta^2 - k(u)^4)I_2. \end{aligned}$$

For $k(u)^4 \in \mathbb{C}\backslash\mathbb{R}_+$, we have

$$(D_{\mathrm{bg},m} - u)^{-1} = (\Delta^2 - k(u)^4)^{-1}(D_{\mathrm{bg},m} + u).$$

We are interested in Schatten–von Neumann properties of Birman–Schwinger operator of the bilayer Hamiltonian, *i.e.,*

$$\mathrm{BS}_u := [\mathrm{BS}_{u,jl}]_{j,l=1,2} = A(D_{\mathrm{bg},m} - u)^{-1}B = A(\Delta^2 - k(u)^4)^{-1}(D_{\mathrm{bg},m} + u)B.$$

Of course, a bound of the form

$$\| \mathrm{BS}_u \|_{\mathcal{S}_p} \le C(u)\|A\|_{2q}\|B\|_{2q},$$

see (7) and (8), will follow if we prove it "entry-by-entry", that is

$$\| \mathrm{BS}_{u,jl} \|_{\mathcal{S}_p} \le C(u)\|A\|_{2q}\|B\|_{2q}, \quad j,l = 1,2.$$

We shall do the computation for the entry $\mathrm{BS}_{u,11}$; the bounds for other entries of the operator BS_u are obtained in a similar way. We have

$$\begin{aligned} \mathrm{BS}_{u,11} = (m+u)&A_{11}(\Delta^2 - k(u)^4)^{-1}B_{11} \\ &+ 4A_{11}(\Delta^2 - k(u)^4)^{-1}\partial_{\bar z}^2 B_{21} \\ &+ 4A_{12}(\Delta^2 - k(u)^4)^{-1}\partial_z^2 B_{11} \\ &+ (m-u)A_{12}(\Delta^2 - k(u)^4)^{-1}B_{21}. \end{aligned} \tag{35}$$

To simplify the following computations, we use the following homogeneity argument. Let $f \in L^s(\mathbb{R}^2)$, $s > 0$, $f = f(x)$, $x \in \mathbb{R}^2$. Set $x = ay, a > 0, y \in \mathbb{R}^2$. We write $g(y) = f(ay)$; to make the writing of differential operators more precise, we write x- or y-subindex to indicate the variable the differential operator is computed with. For instance Δ_x and Δ_y are the Laplacians computed with respect to x and y, respectively.

It is plain that for $j = 1,2$

$$\begin{aligned} \partial_{y_j} g(y) &= a\partial_{x_j} f(ay) = a\partial_{x_j} f(x), \\ \partial^2_{y_j^2} g(y) &= a^2\partial^2_{x_j^2} f(ay) = a^2\partial^2_{x_j^2} f(x). \end{aligned}$$

In particular, $\partial_{z,y} g = a\partial_{z,x} f$, $\partial^2_{z,y} g = a^2\partial^2_{z,x} f$, $\Delta^2_y g = a^4\Delta^2_x f$, etc.

Furthermore, one has

$$\|g\|_s^s = \int_{\mathbb{R}^2_y} |g(y)|^s\,dy = \int_{\mathbb{R}^2_y} |f(ay)|^s\,dy = a^{-2}\int_{\mathbb{R}^2_x} |f(x)|^s\,dx = a^{-2}\|f\|_s^s, \tag{36}$$

or $\|g\|_s = a^{-2/s}\|f\|_s$.

Suppose that $k(u) \neq 0$ and write $k(u)^4$ as $k(u)^4 = |k(u)|^4 e^{i\varphi}$. We assume also that $e^{i\varphi} \neq 1$; the case $e^{i\varphi} = 1$ can be obtained by a standard argument passing to the limit in relations (24), (25). So, putting $a = 1/|k(u)|$,

$$\begin{aligned}(\Delta_x^2 - k(u)^4) f(x) &= |k(u)|^4 (|k(u)|^{-4} \Delta_x^2 - e^{i\varphi}) f(x) \\ &= |k(u)|^4 (\Delta_y^2 - e^{i\varphi}) g(y),\end{aligned}$$

where $g(y) = f(ay)$, $x = ay$. In the same way,

$$\partial_{z,x}^2 f(x) = |k(u)|^2 \partial_{z,y}^2 g(y), \quad \partial_{\bar{z},x}^2 f(x) = |k(u)|^2 \partial_{\bar{z},y}^2 g(y).$$

Set $\widetilde{A}_{jl}(y) = A_{jl}(ay)$ and $\widetilde{B}_{jl}(y) = B_{jl}(ay)$ for $j, l = 1, 2$. Turning back to (35), we rewrite it as

$$\begin{aligned}\mathrm{BS}_{u,11} = \frac{1}{|k(u)|^2} \Big(&\frac{(m+u)}{|k(u)|^2} \widetilde{A}_{11}(y)(\Delta_y^2 - e^{i\varphi})^{-1} \widetilde{B}_{11}(y) \\ &+ 4\widetilde{A}_{11}(y)(\Delta_y^2 - e^{i\varphi})^{-1} \partial_{\bar{z},y}^2 \widetilde{B}_{21}(y) \\ &+ 4\widetilde{A}_{12}(y)(\Delta_y^2 - e^{i\varphi})^{-1} \partial_{z,y}^2 \widetilde{B}_{11}(y) \\ &+ \frac{(m-u)}{|k(u)|^2} \widetilde{A}_{12}(y)(\Delta_y^2 - e^{i\varphi})^{-1} \widetilde{B}_{21}(y) \Big).\end{aligned} \tag{37}$$

Suppose momentarily that we could prove the following estimates,

$$\begin{aligned}\|\widetilde{A}_{11}(\Delta_y^2 - e^{i\varphi})^{-1} \widetilde{B}_{11}\|_{\mathcal{S}_p} &\le C \|\widetilde{A}_{11}\|_{2q} \|\widetilde{B}_{11}\|_{2q}, \\ \|\widetilde{A}_{11}(\Delta_y^2 - e^{i\varphi})^{-1} \partial_{\bar{z},y}^2 \widetilde{B}_{21}\|_{\mathcal{S}_p} &\le C \|\widetilde{A}_{11}\|_{2q} \|\widetilde{B}_{21}\|_{2q}, \\ \|\widetilde{A}_{12}(\Delta_y^2 - e^{i\varphi})^{-1} \partial_{z,y}^2 \widetilde{B}_{11}\|_{\mathcal{S}_p} &\le C \|\widetilde{A}_{12}\|_{2q} \|\widetilde{B}_{11}\|_{2q}, \\ \|\widetilde{A}_{12}(\Delta_y^2 - e^{i\varphi})^{-1} \widetilde{B}_{21}\|_{\mathcal{S}_p} &\le C \|\widetilde{A}_{12}\|_{2q} \|\widetilde{B}_{21}\|_{2q},\end{aligned} \tag{38}$$

Recall that $|(m+u)/|k(u)|^2| = |\zeta(u)|$ and $|(m-u)/|k(u)|^2| = |\zeta(u)|^{-1}$, while

$$1 \le C(|\zeta(u)| + |\zeta(u)|^{-1}), \quad u \in \mathbb{C}.$$

Plugging these bounds in (37) implies

$$\begin{aligned}\| \mathrm{BS}_{u,11} \|_{\mathcal{S}_p} &\le \frac{C}{|k(u)|^2} (1 + |\zeta(u)| + |\zeta(u)|^{-1}) \|\widetilde{A}\|_{2q} \|\widetilde{B}\|_{2q} \\ &\le \frac{C}{|k(u)|^2} (|\zeta(u)| + |\zeta(u)|^{-1}) \|\widetilde{A}\|_{2q} \|\widetilde{B}\|_{2q} \\ &= C(|\zeta(u)| + |\zeta(u)|^{-1}) |k(u)|^{2/q-2} \|A\|_{2q} \|B\|_{2q},\end{aligned} \tag{39}$$

where we used the rescaling (36) in the last line. We notice that

$$(|\zeta(u)| + |\zeta(u)|^{-1}) |k(u)|^{2/q-2} \le C \Phi_q(u), \quad u \in \rho(D_{\mathrm{bg},m}).$$

Hence (39) is exactly the formula claimed in (7).

Consequently, it remains to prove (38). Set

$$m_1(\xi) := \frac{1}{(|\xi|^4 - e^{i\varphi})}, \quad m_2(\xi) := \frac{(\xi_1 \pm i\xi_2)^2}{(|\xi|^4 - e^{i\varphi})}.$$

Furthermore, take $\chi_1 \in C_0^\infty(\mathbb{R}^2)$ with the properties: $0 \le \chi_1(x) \le 1$ for all $x \in \mathbb{R}^2$, χ_1 is supported in $\{x \in \mathbb{R}^2 : 1/2 \le |x| \le 3/2\}$ and $\chi_1(x) = 1$ for $x \in \{x \in \mathbb{R}^2 : 3/4 \le |x| \le 5/4\}$. Let $\chi_2 := 1 - \chi_1$; by definition $\chi_1 + \chi_2 = 1$ is a smooth partition of unity. Rewriting (38) in terms of symbols of differential operators, we shall show that

$$\|\widetilde{A}\chi_l(D)m_j(D)\widetilde{B}\|_{\mathcal{S}_p} \le C\|\widetilde{A}\|_{2q}\|\widetilde{B}\|_{2q}, \quad l, j = 1, 2.$$

For $1 \le q \le 3/2$, the bound for $l = 1$ is exactly Case I of Proposition 1.6.

Consider the case $l = 2$ now. Notice that for the range of q's we are interested in, one can always choose $\varepsilon > 0$ small enough so that $p = p(q, \varepsilon) \ge q$. Thus we shall prove the bound

$$\|\widetilde{A}\chi_2(D)m_j(D)\widetilde{B}\|_{\mathcal{S}_q} \le C\|\widetilde{A}\|_{2q}\|\widetilde{B}\|_{2q}, \quad j = 1, 2,$$

which is stronger than (38). Notice that

$$|\chi_2(\xi)m_1(\chi)| = \left|\frac{\chi_2(\xi)}{|\xi|^4 - e^{i\varphi}}\right| \le \frac{C}{(1 + |\xi|^2)},$$

$$|\chi_2(\xi)m_2(\chi)| = \left|\frac{\chi_2(\xi)(\xi_1 \pm i\xi_2)^2}{|\xi|^4 - e^{i\varphi}}\right| \le \frac{C}{(1 + |\xi|^2)}.$$

Lemma 3.1 applied to the operator $\widetilde{A}\chi_2(D)m_j(D)\widetilde{B}$ gives

$$\|\widetilde{A}\chi_2(D)m_j(D)\widetilde{B}\|_{\mathcal{S}_q} \le \|(1 + |\xi|^2)^{-1}\|_q \,\|\widetilde{A}\|_{2q}\|B\|_{2q}, \quad j = 1, 2,$$

as needed.

Let us turn to Case II, $q > 3/2$. The proof closely follows the proof of Proposition 1.6, Case II. It consists in interpolation in between bounds for parameters $q = 3/2$ (*i.e.*, Case I), and $q = \infty$.

Assume that $A > 0$ and $B > 0$. Fix $q > 3/2$ and let $p = p(q) := 2q$. This choice implies in particular that $2q/3 > 1$. Set $a_0 = 0, b_0 = 2q/3$ and consider the strip

$$S := \{\zeta = a + it : a_0 \le a \le b_0,\ t \in \mathbb{R}\}.$$

The family of operators

$$T_\zeta = A^\zeta (D_{\mathrm{bg},m} - u)^{-1} B^\zeta,$$

is analytic on S. Apply (7) with $q_0 = 3/2$ in place of q to the family T_ζ on $\operatorname{Re}\zeta = b_0 = 2q/3$; that is

$$\|A^{2q/3+it}(D_{\mathrm{bg},m} - u)^{-1} B^{2q/3+it}\|_{\mathfrak{S}_3} \le C\Phi(u)\|A\|_{2q}^{2q/3}\|B\|_{2q}^{2q/3}, \tag{40}$$

where we used that $\|A^{2q/3+it}\|_3 = \|A\|_{2q}^{2q/3}$, and the same relation holds for B. Notice that $q_{01} = 1 - 1/(2q_0) = 2/3$. For $\operatorname{Re}\zeta = a_0 = 0$, we have the trivial bound

$$\|A^{it}(D_{\mathrm{bg},m} - u)^{-1} B^{it}\|_{\mathfrak{S}_\infty} \le \frac{1}{d(u, \sigma(D_{\mathrm{bg},m}))}. \tag{41}$$

As in Proposition 1.6, we interpolate between (40) and (41) using Theorem 3.3 with parameters $\zeta := 1$ and

$$1 = \frac{2q}{3}\gamma + (1-\gamma)0, \quad \frac{1}{p_\gamma} = \frac{3}{\gamma} + \frac{(1-\gamma)}{\infty} = \frac{1}{2q}.$$

Hence, $\gamma = 3/(2q)$ and $p_\gamma = 2q$. Claim (8) follows, and this finishes the proof of the theorem. □

2. Lieb–Thirring inequalities for bilayer graphene

In what follows we always assume that $m > 0$. We begin with the standard Zhukovsky transform

$$z = z(w) = \frac{m}{2}\left(w + \frac{1}{w}\right), \tag{42}$$

which maps the upper half-plane $\mathbb{C}_+$ onto the domain $\rho(D_{\mathrm{bg},m})$. Since

$$|z(w) \pm m| = \frac{m}{2|w|}|w \pm 1|^2,$$

we have

$$\begin{aligned} |z+m| + |z-m| &= \frac{m}{2|w|}(|w+1|^2 + |w-1|^2) = \frac{m}{|w|}(1+|w|^2), \\ |z^2 - m^2|^{\frac{1}{2}} &= \frac{m}{2|w|}|w^2 - 1|. \end{aligned} \tag{43}$$

The distortion [42, Corollary 1.4] for the Zhukovsky transform reads as

$$\frac{d(z, \sigma(D_{\mathrm{bg},m}))}{\operatorname{Im} w} \asymp |z'(w)| = \frac{m|w^2-1|}{2|w|^2} = \frac{|z^2-m^2|^{1/2}}{|w|}, \quad w \in \mathbb{C}_+. \tag{44}$$

2.1. Proof of Theorem 0.4, Case I: $1 < q \le 3/2$. We have, by (43),

$$\Phi(z(w)) = C(1+|w|^2)\,\frac{|w|^{p_1}}{|w^2-1|^{2q_1}}, \qquad p_1 := 2q_1 - 1 = 1 - \frac{1}{q} > 0.$$

The bound (7) in the variable w reads

$$\|V_2(D_{\mathrm{bg},m} - z(w))^{-1}V_1\|_{\mathcal{S}_p} \le C_9(1+|w|^2)\,\frac{|w|^{p_1}}{|w^2-1|^{2q_1}}\,\|V\|_q, \qquad w \in \mathbb{C}_+, \tag{45}$$

where $V_2 = A := |V|^{1/2}$ and $V_1 = B := V^{1/2}$, see the discussion preceding (4). For $w = iy$, $y > 0$,

$$\|V_2(D_{\mathrm{bg},m} - z(iy))^{-1}V_1\|_{\mathcal{S}_p} \le C_9\Big(\frac{y}{1+y^2}\Big)^{p_1}\|V\|_q < \frac{C_9}{y^{p_1}}\|V\|_q. \tag{46}$$

We proceed with the *regularized perturbation determinant*

$$H(w) := \det_p(I + V_2(D_{\mathrm{bg},m} - z(w))^{-1}V_1), \qquad w \in \mathbb{C}_+,$$

which admits the bounds, see [44, Theorem 9.2]

$$\log|H(w)| \le \Gamma_p\,\|V_2(D_{\mathrm{bg},m} - z(w))^{-1}V_1\|_{\mathcal{S}_p}^p \tag{47}$$

and

$$|H(w) - 1| \le \varphi(\|V_2(D_{\mathrm{bg},m} - z(w))^{-1}V_1\|_{\mathcal{S}_p}), \tag{48}$$

where

$$\varphi(x) := x\exp\{\Gamma_p(x+1)^p\}, \qquad x \ge 0.$$

Denote

$$h(w) = h_y(w) := \frac{H(yw)}{H(iy)}, \qquad h(i) = 1, \tag{49}$$

$y \ge 1$ is chosen later on.

Proposition 2.1. *Assume that*

$$\|V\|_q \le 1. \tag{50}$$

Then there is a constant $C_{10} = C_{10}(m, q, \varepsilon)$ so that for $y = C_{10}$ the following holds

$$\log|h(w)| \le C_{11}\,\frac{(1+|w|)^{4pq_1}}{|w^2 - y^{-2}|^{2pq_1}}\,\|V\|_q, \qquad w \in \mathbb{C}_+.$$

Proof. Without loss of generality we assume that $C_9 > 1$. If $y^{p_1} \ge C_9 \ge C_9\|V\|_q$, we have, by (46),

$$\|V_2(D_{\mathrm{bg},m} - z(iy))^{-1}V_1\|_{\mathcal{S}_p} \le \frac{C_9}{y^{p_1}}\|V\|_q \le \|V\|_q \le 1. \tag{51}$$

An obvious bound $\varphi(x) \le \exp\{2^p\Gamma_p\}\, x$, $0 \le x \le 1$, implies, in view of (51),

$$\varphi(\|V_2(D_{\mathrm{bg},m} - z(iy))^{-1}V_1\|_{\mathcal{S}_p}) \le e^{2^p\Gamma_p}\,\|V_2(D_{\mathrm{bg},m} - z(iy))^{-1}V_1\|_{\mathcal{S}_p},$$

and so, by (48),

$$1 - |H(iy)| \le |H(iy) - 1| \le \frac{C_9 e^{2^p\Gamma_p}}{y^{p_1}}\,\|V\|_q \le \frac{1}{2},$$

as soon as $y^{p_1} \ge 2C_9 \exp\{2^p\Gamma_p\} =: C_{12}$. The case $|H(iy)| > 1$ being trivial, we continue with the case $\frac{1}{2} \le |H(iy)| \le 1$. Hence,

$$|H(iy)| \ge \frac{1}{2}, \quad \log|H(iy)| \ge -(1 - |H(iy)|) \ge -C_{12}\,\frac{\|V\|_q}{y^{p_1}}. \tag{52}$$

A combination of (47), (45), and (52) leads to the bound

$$\begin{aligned}\log|h(w)| &= \log|H(yw)| - \log|H(iy)| \\ &\le C(1 + y|w|)^{2p}\,\frac{(y|w|)^{pp_1}}{|y^2w^2 - 1|^{2pq_1}}\,\|V\|_q^p + C_{12}\,\frac{\|V\|_q}{y^{p_1}} \\ &\le C_{13}\Big[\frac{(1 + |w|)^{2p}|w|^{pp_1}}{|w^2 - y^{-2}|^{2pq_1}}\,\frac{\|V\|_q^p}{y^{pp_1}} + \frac{\|V\|_q}{y^{p_1}}\Big] \\ &\le C_{13}\frac{\|V\|_q}{y^{p_1}}\Big[\frac{(1 + |w|)^{2p}|w|^{pp_1}}{|w^2 - y^{-2}|^{2pq_1}} + 1\Big].\end{aligned}$$

As $2p + pp_1 - 4pq_1 = -pp_1 < 0$, we have for $y \ge 1$

$$\begin{aligned}(1 + |w|)^{2p}|w|^{pp_1} + |w^2 - y^{-2}|^{2pq_1} &\le (1 + |w|)^{2p+pp_1} + (1 + |w|)^{4pq_1} \\ &< 2(1 + |w|)^{4pq_1}.\end{aligned}$$

The result follows with $y = C_{10} = C_{12}^{1/p_1}$, $C_{11} = 2C_{13}$. □

It is well known that the Lieb–Thirring inequalities agree with the Blaschke type conditions for the zeros of the corresponding perturbation determinants. So, the next step is an application of [5, Theorem 4.4] to the above function h. The input parameters are

$$a = 0, \quad b = 2pq_1, \quad c_j = 0; \quad x_1' = y^{-1},\ x_2' = -y^{-1}, \quad K = C\|V\|_q,$$

$$d_1 = d_2 = d = 2pq_1 = \begin{cases} \dfrac{2q-1}{2-q} + \Big(2 - \dfrac{1}{q}\Big)\varepsilon, & 1 < q < \dfrac{4}{3}, \\ \dfrac{2q-1}{2-q}, & \dfrac{4}{3} \le q \le \dfrac{3}{2}. \end{cases}$$

The output parameters in [5, Theorem 4.4] are

$$l = \{l\}_{a,\varepsilon} = 0, \quad (d-1+\varepsilon)_+ = \frac{3q-3}{2-q} + \omega_q \varepsilon, \quad l_1 = \frac{4q-2}{2-q} + \tau_q \varepsilon,$$

with

$$\omega_q = \begin{cases} \dfrac{3q-1}{q}, & 1 < q < \dfrac{4}{3}, \\ 1, & \dfrac{4}{3} \le q \le \dfrac{3}{2}; \end{cases} \qquad \tau_q = \begin{cases} \dfrac{6q-1}{q}, & 1 < q < \dfrac{4}{3}, \\ 1, & \dfrac{4}{3} \le q \le \dfrac{3}{2}. \end{cases}$$

So, the Blaschke type condition of [5, Theorem 4.4] takes the form

$$\sum_{\xi \in Z(h)} \frac{(\operatorname{Im} \xi)^{1+\varepsilon}}{(1+|\xi|)^{l_1}} |\xi^2 - y^{-2}|^{(d-1+\varepsilon)_+} \le C_{14} \|V\|_q, \tag{53}$$

and, since the "test point" y in Proposition 2.1 does not depend on V, the constant $C_{14}(m, q, \varepsilon)$ does not depend on V either.

In terms of the zeros of H we have

$$\xi \in Z(h) \iff y\xi = \lambda \in Z(H), \quad \xi = \frac{\lambda}{y},$$

and as $y = C_{10}$ is a constant, condition (53) does not alter

$$\sum_{\lambda \in Z(H)} \frac{(\operatorname{Im} \lambda)^{1+\varepsilon}}{(1+|\lambda|)^{l_1}} |\lambda^2 - 1|^{(d-1+\varepsilon)_+} \le C_{15} \|V\|_q. \tag{54}$$

It remains to get back to the spectral variable $z \in \rho(D_{\mathrm{bg},m})$, keeping in mind that for the discrete spectrum of D_{bg} the equivalence holds

$$\zeta \in \sigma_d(D_{\mathrm{bg}}) \iff \lambda \in Z(H).$$

To make the final result transparent, we invoke the main result [7, Theorem 1.1], which claims, in particular, that the discrete spectrum $\sigma_d(D_{\mathrm{bg}})$ is bounded, that is, $|\zeta| \le C_{16}$, for all $\zeta \in \sigma_d(D_{\mathrm{bg}})$. In the Zhukovsky variable the latter means

$$0 < c \le |\lambda| \le C < \infty, \quad \text{for all } \lambda \in Z(H). \tag{55}$$

So we can neglect the term $1 + |\lambda|$ in (53). Next, as in (43),

$$|\zeta^2 - m^2| = \frac{m^2}{4} \frac{|\lambda^2 - 1|^2}{|\lambda|^2} \implies c|\lambda^2 - 1| \le |\zeta^2 - m^2|^{1/2} \le C|\lambda^2 - 1|.$$

Finally, the distortions (44) and (55) imply

$$c \implies \lambda \le \frac{d(\zeta, \sigma(D_{\mathrm{bg},m}))}{|\zeta^2 - m^2|^{1/2}} \le C \operatorname{Im} \lambda.$$

Case I of Theorem 0.4 is proved. □

2.2. Proof of Theorem 0.4, Case II: $q > 3/2$. We use the distortion (44) to obtain the bound similar to (45)

$$\|V_2(D_{\mathrm{bg},m} - z(w))^{-1}V_1\|_{\mathcal{S}_p} \le C_9 \frac{(1+|w|)^{2q_2}}{(\operatorname{Im} w)^{p_2}} \frac{|w|^{p_3}}{|w^2-1|^{p_4}} \|V\|_q, \quad w \in \mathbb{C}_+, \tag{56}$$

where

$$p = 2q, \quad p_2 := 1 - q_2 = 1 - \frac{3}{2q} > 0, \quad p_3 := 2 - \frac{5}{2q}, \quad p_4 := 1 + \frac{1}{2q}.$$

Note that $p_3 - p_2 = p_1$. For $w = iy$, $y > 0$, the bound is exactly the same as (46)

$$\|V_2(D_{\mathrm{bg},m} - z(iy))^{-1}V_1\|_{\mathcal{S}_p} < \frac{C_9}{y^{p_1}} \|V\|_q. \tag{57}$$

We argue as in the proof of Proposition 2.1 to obtain the bound for h (49)

$$\log|h(w)| \le C_{11} \frac{|w|^{pp_2}(1+|w|)^{2pp_4}}{(\operatorname{Im} w)^{pp_2}\,|w^2 - y^{-2}|^{pp_4}} \|V\|_q. \tag{58}$$

Indeed,

$$\begin{aligned}
\log|h(w)| &= \log|H(yw)| - \log|H(iy)| \\
&\le C \frac{(1+y|w|)^{2pq_2}(y|w|)^{pp_3}}{(\operatorname{Im} yw)^{pp_2}|y^2w^2-1|^{pp_4}} \|V\|_q^p + C_{12} \frac{\|V\|_q}{y^{p_1}} \\
&\le C_{13}\Big[\frac{(1+|w|)^{2pq_2}|w|^{pp_3}}{(\operatorname{Im} w)^{pp_2}|w^2-y^{-2}|^{pp_4}} \frac{\|V\|_q^p}{y^{pp_1}} + \frac{\|V\|_q}{y^{p_1}}\Big] \\
&\le C_{13} \frac{\|V\|_q}{y^{p_1}}\Big[\frac{(1+|w|)^{2pq_2}|w|^{pp_3}}{(\operatorname{Im} w)^{pp_2}|w^2-y^{-2}|^{pp_4}} + 1\Big].
\end{aligned}$$

Next,

$$\begin{aligned}
&(1+|w|)^{2pq_2}|w|^{pp_3} + (\operatorname{Im} w)^{pp_2}|w^2-y^{-2}|^{pp_4} \\
&\quad \le (1+|w|)^{2pq_2}|w|^{pp_3} + |w|^{pp_2}(1+|w|^2)^{pp_4} \\
&\quad \le |w|^{pp_2}((1+|w|)^{2pq_2}|w|^{pp_1} + (1+|w|)^{2pp_4}) \\
&\quad \le 2|w|^{pp_2}(1+|w|)^{2pp_4},
\end{aligned}$$

and (58) follows.

The computation with [5, Theorem 4.4] is a bit more complicated now. The input parameters are

$$\begin{gathered} a = pp_2 = 2q - 3 > 0, \quad b = pp_4 = 2q + 1, \\ x_1' = y^{-1}, \quad x_2' = -y^{-1}, \quad x_1 = 0, \\ c_1 = pp_2 = a, \quad c_j = 0, \ j \geq 2, \\ d_1 = d_2 = d = pp_4 = b, \\ K = C\|V\|_q. \end{gathered}$$

The output parameters in [5, Theorem 4.4] are

$$l = a, \quad \{l\}_{a,\varepsilon} = -a, \quad (d - 1 + \varepsilon)_+ = 2q + \varepsilon, \quad l_1 = 2 + 4q + 4\varepsilon,$$

so the Blaschke type condition takes the form

$$\sum_{\xi \in Z(h)} \frac{(\operatorname{Im} \xi)^{a+1+\varepsilon}}{(1 + |\xi|)^{2+4q+4\varepsilon}} \frac{|\xi^2 - y^{-2}|^{2q+\varepsilon}}{|\xi|^a} \leq C_{14}\|V\|_q.$$

After the change of variable $\lambda = y\,\xi = C_{10}\xi$, we come to

$$\sum_{\lambda \in Z(H)} \frac{(\operatorname{Im} \lambda)^{a+1+\varepsilon}}{(1 + |\lambda|)^{2+4q+4\varepsilon}} \frac{|\lambda^2 - 1|^{2q+\varepsilon}}{|\lambda|^a} \leq C_{15}\|V\|_q. \tag{59}$$

As before, the final step relies on the distortion relations for the Zhukovsky transform. Indeed, separate the upper-half plane $\mathbb{C}_+$ in three regions

$$\begin{aligned} \Omega_1 &:= \{\lambda \in \mathbb{C}_+ : c \leq |\lambda| \leq C\}, \\ \Omega_2 &:= \{\lambda \in \mathbb{C}_+ : |\lambda| \geq C\}, \\ \Omega_3 &:= \{\lambda \in \mathbb{C}_+ : |\lambda| \leq c\}, \end{aligned}$$

with constants c, C chosen as $0 < c < 1 < C < +\infty$. It is clear that

$$\sum_{\lambda \in Z(H) \cap \Omega_1} (\operatorname{Im} \lambda)^{a+1+\varepsilon} |\lambda^2 - 1|^{2q+\varepsilon} \leq C \sum_{\lambda \in Z(H) \cap \Omega_1} \frac{(\operatorname{Im} \lambda)^{a+1+\varepsilon}}{(1 + |\lambda|)^{2+4q+4\varepsilon}} \frac{|\lambda^2 - 1|^{2q+\varepsilon}}{|\lambda|^a}.$$

On the other hand, one has $|\zeta(\lambda)| \asymp |\lambda|$ for $\lambda \in \Omega_2$, and $|\zeta(\lambda)| \asymp |\lambda|^{-1}$ for $\lambda \in \Omega_3$. Using these relations along with inequalities given next to (55), we cut the sum (59) in parts corresponding to domains Ω_i, $i = 1, 2, 3$, and rewrite these partial sums in terms of ζ-variable.

Case II of Theorem 0.4 is proved as well. □

3. Some technical tools: interpolation theorems and Kato–Selier–Simon lemma

3.1. Kato–Selier–Simon lemma. Recall the notation introduced in Section 1.1. We have the following proposition usually called *Kato–Selier–Simon lemma.*

Proposition 3.1 ([44, Theorem 4.1]). (1) *Let* $f, g \in L^q(\mathbb{R}^d)$, $d \geq 1$. *Then, for* $2 \leq q < \infty$, $f(x)g(D) \in \mathcal{S}_q$, *and*

$$\|f(x)g(D)\|_{\mathcal{S}_q} \leq (2\pi)^{-d}\|f\|_q\|g\|_q.$$

(2) *Let* $f \in L^q(\mathbb{R}^d)$, $d \geq 1$, *and* $A, B \in L^{2q}(\mathbb{R}^d)$. *For* $2 \leq q < \infty$,

$$\|A(x)f(D)B(y)\|_{\mathcal{S}_q} \leq (2\pi)^{-d}\|f\|_q\,\|A\|_{2q}\|B\|_{2q}.$$

The first claim of the above proposition is in Simon [44, Theorem 4.1]; the second claim is a "symmetrized" version of the first one and it is proved similarly.

3.2. Interpolation theorem for bounded analytic families. In this subsection, we follow mainly the presentation of Zhu [47, Chapter 2].

Let X_0, X_1 be two Banach spaces. We say that the pair X_0, X_1 is compatible, if there is a topological Hausdorff space X containing both X_0 and X_1. We have the following theorem.

Theorem 3.2 ([47, Theorem 2.4]). *Let* X_0, X_1 *be a pair of compatible Banach spaces, idem for* Y_0, Y_1. *For a* γ, $0 < \gamma < 1$, *there are Banach spaces* X_γ, Y_γ,

$$X_\gamma = [X_0, X_1]_\gamma, \quad Y_\gamma = [Y_0, Y_1]_\gamma,$$

interpolating in between X_0 *and* X_1 *and* Y_0 *and* Y_1, *respectively, in the following sense.*

Let $T : X_0 + X_1 \to Y_0 + Y_1$ *be a* ***bounded*** *linear map such that*

$$\|Tx\|_{Y_0} \leq C_0\|x\|_{X_0}, \quad x \in X_0,$$
$$\|Tx\|_{Y_1} \leq C_1\|x\|_{X_1}, \quad x \in X_1.$$

Then T *induces a linear map* $T_\gamma \colon X_\gamma \to Y_\gamma$ *with the property*

$$\|T_\gamma\| \leq C_0^\gamma C_1^{1-\gamma}.$$

Saying "interpolation" we mean "complex interpolation" throughout the article. For instance, we have

$$[L^{p_0}(\mathbb{R}^d), L^{p_1}(\mathbb{R}^d)]_\gamma = L^{p_\gamma}(\mathbb{R}^d), \tag{60}$$

where $1 \le p_0, p_1 \le \infty$, $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$, and $d \ge 1$, see [47, Theorem 2.5].

It is important that a similar construction holds for "non-commutative" L^p-spaces as well. That is, denoting by $\mathcal{S}_p$ the Schatten–von Neumann classes of compact operators, we have

$$[\mathcal{S}_{p_0}, \mathcal{S}_{p_1}]_\gamma = \mathcal{S}_{p_\gamma},$$

where $1 \le p_0, p_1 \le \infty$ and $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$. A proof of this result is in [47, Theorem 2.6]. Much more information and further references on the interpolation theory of Banach spaces are in the monographs by Bennett and Sharpley [1] and by Bergh and Löfström [2].

For $1 \le p_{01}, p_{02} \le +\infty$, it is plain to see that

$$L^{p_{01}}(\mathbb{R}^d_x) \times L^{p_{02}}(\mathbb{R}^d_y) \simeq L^{p_{01}}(\mathbb{R}^d_x) \dotplus L^{p_{02}}(\mathbb{R}^d_y), \quad x, y \in \mathbb{R}^d,$$

and so interpolation (60) holds for these spaces as well. This observation is often applied to an operator $\mathcal{A}$ of the form

$$\mathcal{A}\colon L^{p_{01}}(\mathbb{R}^d) \times L^{p_{02}}(\mathbb{R}^d) \longrightarrow \mathcal{S}_{q_{01}}, \quad 1 \le q_{01} \le +\infty,$$

see Section 1.

3.3. Interpolation theorem for general analytic families. Following Gohberg and Krein [25, Chapter III.13], we present a generalized version of interpolation in between $\mathcal{S}_p$-spaces.

Let $a, b \in \mathbb{R}$, $a < b$, and

$$S = \{\zeta\colon a \le \operatorname{Re}\zeta \le b\}$$

be a vertical strip in the complex plane. For a Hilbert space H, we say that a family of bounded operators $(T_\zeta)_{\zeta\in S}$, $T_\zeta\colon H \to H$ is is *analytic* on S, if $(T_\zeta f, g)$ is analytic on an open neighborhood of S for any fixed $f, g \in H$.

Theorem 3.3 ([25, Theorem 13.1]). *Let $(T_\zeta)_{\zeta\in S}$ be an analytic family of operators. Assume that for any $f, g \in H$*

$$\log|(T_\zeta f, g)| \le C_{1;f,g}\, e^{C_{2;f,g}|\operatorname{Im}\zeta|}, \quad \zeta \in S, \tag{61}$$

where the constants $C_{j;f,g}$, $j = 1, 2$ depend on f, g, but not on $\zeta \in S$, and

$$0 \le C_{2;f,g} < \frac{\pi}{b-a}.$$

Furthermore, suppose that

(1) *for* $\operatorname{Re}\zeta = a$, $T_\zeta \in \mathcal{S}_{p_0}$, *with* $1 \le p_0 < \infty$ *and*

$$\|T_\zeta\|_{\mathcal{S}_{p_0}} \le C_0;$$

(2) *for* $\operatorname{Re}\zeta = b$, $T_\zeta \in \mathcal{S}_{p_1}$, *with* $p_0 < p_1 \le \infty$ *and*

$$\|T_\zeta\|_{\mathcal{S}_{p_1}} \le C_1.$$

Take an $x \in (a,b)$ *and write it as* $x = \gamma\, a + (1-\gamma)\, b$, $\gamma \in (0,1)$. *For* $\zeta \in S, \operatorname{Re}\zeta = x$ *we have that* $T_\zeta \in \mathcal{S}_{p_\gamma}$, *and moreover*

$$\|T_\zeta\|_{\mathcal{S}_{p_\gamma}} \le C_0^\gamma C_1^{1-\gamma},$$

where $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$.

We often use the following corollary of the above theorem.

Corollary 3.4. *Let* $(T_\zeta)_{\zeta\in S}$ *be an analytic family of operators satisfying the assumption of Theorem* 3.3 *with conditions* (1) *and* (2) *replaced by the following assumptions*:

(1′) *for* $\operatorname{Re}\zeta = a$, $T_\zeta \in \mathcal{S}_{p_0}$, *with* $1 \le p_0 < \infty$ *and*

$$\|T_\zeta\|_{\mathcal{S}_{p_0}} \le C_0 e^{A_0|\operatorname{Im}\zeta|^2}.$$

(2′) *for* $\operatorname{Re}\zeta = b$, $T_\zeta \in \mathcal{S}_{p_1}$, *with* $p_0 < p_1 \le \infty$ *and*

$$\|T_\zeta\|_{\mathcal{S}_{p_1}} \le C_1 e^{A_1|\operatorname{Im}\zeta|^2},$$

for some constants $A_0, A_1 \ge 0$.

As above, for an $x = \gamma\, a + (1-\gamma)\, b \in (a,b)$, $\gamma \in (0,1)$ *and* $\zeta \in S, \operatorname{Re}\zeta = x$ *we have that* $T_\zeta \in \mathcal{S}_{p_\gamma}$, *and moreover*

$$\|T_x\|_{\mathcal{S}_{p_\gamma}} \le C''\, C_0^\gamma\, C_1^{1-\gamma},$$

where $1/p_\gamma = \gamma/p_0 + (1-\gamma)/p_1$. *The constant* C'' *depends on* a, b, C_0, C_1, A_0 *and* A_1.

The corollary follows immediately by applying Theorem 3.3 to the analytic family of operators $\widetilde{T}_\zeta = e^{\max(A_0,A_1)\zeta^2}\, T_\zeta$, $\zeta \in S$.

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