Existence of metrics maximizing the first eigenvalue on non-orientable surfaces

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Abstract. We prove the existence of metrics maximizing the first eigenvalue normalized by area on closed, non-orientable surfaces assuming two spectral gap conditions. These spectral gap conditions are proved by the authors in [21].

1. Introduction

For a closed Riemannian surface (Σ, g) the spectrum of the Laplace operator acting on smooth functions, is purely discrete and can be written as

$$0 = \lambda_0 < \lambda_1(\Sigma, g) \leq \lambda_2(\Sigma, g) \leq \lambda_3(\Sigma, g) \leq \cdots \longrightarrow \infty,$$

where we repeat an eigenvalue as often as its multiplicity requires. Via the variational characterization of eigenvalues this extends to a much larger class of metrics with lower regularity properties. For details we refer to [18], which in particular covers the metrics considered here.

The pioneering work of Hersch [13] and Yang and Yau [36] raised the natural question, whether there are metrics g that maximize the scale-invariant quantities

$$\bar{\lambda}_1(\Sigma) := \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g)$$

if Σ is a closed surface of fixed topological type (see also [15, 19] for the case of non-orientable surfaces). Such maximizers have remarkable properties. In fact, they always arise as immersed minimal surfaces (of possibly high codimension) in a sphere [9] and are unique in their conformal class unless they are branched

immersions into the two sphere [6, 23, 31]. By a slight abuse of notation, we also call Σ, endowed with a maximizing metric, a 'maximizer'.

For the statement of our results and related work, we need to introduce some notation. We write Σ_γ for a closed orientable surface of genus γ. Similarly, Σ_δ^K denotes a closed non-orientable surface of non-orientable genus δ. Here, K stands for Klein. We briefly elaborate on these notions in Appendix A. Furthermore, we use the common notation

$$\Lambda_1(\gamma) = \sup_g \lambda_1(\Sigma_\gamma, g)\, \mathrm{area}(\Sigma_\gamma, g),$$

and similarly,

$$\Lambda_1^K(\delta) = \sup_g \lambda_1(\Sigma_\delta^K, g)\, \mathrm{area}(\Sigma_\delta^K, g),$$

with the supremum taken over all smooth metrics on Σ_γ, respectively Σ_δ^K.

Explicit values for $\Lambda_1(\gamma)$ or $\Lambda_1^K(\delta)$ are only known in very few cases. However, in all of these cases not only the values but also explicit maximizing metrics are known.

The case of the sphere is due to Hersch. We have $\Lambda_1(\mathbb{S}^2) = 8\pi$ with unique maximizer the round metric [13]. His arguments are very elegant and a cornerstone in the development of the subject. For the real projective plane, we have $\Lambda_1(\mathbb{RP}^2) = 12\pi$ with unique maximizer the round metric [19]. The proof extends the ideas from [13] in a conceptually very nice way.

The first result for higher genus surfaces is due to Nadirashvili, namely $\Lambda_1(T^2) = 8\pi^2/\sqrt{3}$ with unique maximizer the flat equilateral torus [25]. Nadirashvili's arguments are very different from the previously employed methods. The crucial step in his proof is to obtain the existence of a maximizer. Using [23] (see also [6]) it follows that such a maximizer necessarily has to be flat. The sharp bound follows then from earlier work of Berger [2].

For the Klein bottle, $\Lambda_1(K) = 12\pi E(2\sqrt{2}/3)$ with unique maximizer a metric of revolution [6, 8, 25]. Here E is the complete elliptic integral of the second kind.

There is also a conjecture concerning the sharp bound on genus 2 surfaces [14], a proof of which has very recently been given by Nayatani and Shoda in [31].

Let us also mention that there are quite some results concerning similar questions for higher order eigenvalues, see e.g. [29, 33] and [26, 30, 17] for the case of $\mathbb{S}^2$ and [27, 16] for the case of $\mathbb{R}P^2$.

The growing interest in finding maximizers for eigenvalue functionals on surfaces starting from Nadirashvili's paper [25] is certainly connected to the

connection of the problem to minimal surfaces in spheres. Similarly, for the Steklov eigenvalue problem, there is a connection to free boundary minimal surfaces in Euclidean balls. In pioneering work Fraser and Schoen showed the existence of maximizers for the first Steklov eigenvalue on surfaces with boundary of genus 0 [11].

Recently, Petrides used many of the ideas in [11] to prove the following beautiful result concerning metrics realizing $\Lambda_1(\gamma)$.

Theorem 1.1 (Theorem 2 in [32]). *If $\Lambda_1(\gamma - 1) < \Lambda_1(\gamma)$, there is a metric g on $\Sigma = \Sigma_\gamma$, which is smooth away from finitely many conical singularities, such that*

$$\lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g) = \Lambda_1(\gamma).$$

We extend this to non-orientable surfaces. Since non-orientable surfaces can degenerate to non-orientable surfaces as well as orientable ones, we need to make two instead of only a single spectral assumption.

Theorem 1.2. *If $\Lambda_1^K(\delta - 1) < \Lambda_1^K(\delta)$ and $\Lambda_1(\lfloor (\delta - 1)/2 \rfloor) < \Lambda_1^K(\delta)$, there is a metric g on $\Sigma = \Sigma_\delta^K$, which is smooth away from at most finitely many conical singularities, such that*

$$\lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g) = \Lambda_1^K(\delta).$$

Our methods are very similar to those in [32]. In addition to the cases already handled by Petrides, we also need to take care of degenerating one-sided geodesics.

The non-strict inequality $\Lambda_1(\gamma - 1) \leq \Lambda_1(\gamma)$ was proved by Colbois and El Soufi in [7] using a result of Anné [1]. It is easy to obtain the non-strict versions of the inequalities assumed in Theorem 1.2 along the same lines, see also [20]. In [20] we prove the monotonicity $\Lambda_1(\gamma - 1) < \Lambda_1(\gamma)$ under some extra assumptions on the maximizing metric using a relatively simple glueing construction. In [21], by means of a much more complicated glueing construction and using Theorem 1.1 and Theorem 1.2, we prove all of the spectral gap conditions assumed in Theorem 1.1 and Theorem 1.2 above. In particular, this implies the existence of maximizing metrics on closed surfaces of any topological type.

Theorem 1.3 (Theorem 1.3 in [21] using Theorem 1.1 and Theorem 1.2). *Let Σ be a closed surface. Then there is a metric g on Σ, which is smooth away from at most finitely many conical singularities, such that*

$$\lambda_1(\Sigma, h) \operatorname{area}(\Sigma, h) \leq \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g)$$

for any smooth metric h on Σ.

Outline.   In Section 2 we prove that the set of orientable, hyperbolic surfaces
with injectivity radius bounded below is a compact subset of the moduli space.
This is a version of the Mumford compactness criterion for non-orientable sur-
faces. We use this result in the main section, namely Section 3, in which we prove
Theorem 1.2.

2. Compactness for non-orientable surfaces

The Mumford compactness criterion [24] states that the set of orientable, hyper-
bolic surfaces with injectivity radius bounded below is a compact subset of the
moduli space. In this section we show that the Mumford argument for orientable
surfaces holds for the non-orientable case as well.

Given any Riemannian metric g_0 on $\Sigma = \Sigma_\delta^K$, the Poincaré uniformization
theorem asserts that we can find a new metric on Σ which is conformal to g_0 and
has constant curvature $+1, 0$, or -1, depending on the sign of $\chi(\Sigma)$. Assuming
$\delta \geq 3$, these metrics have curvature -1. Let h_k be a sequence of such metrics on
Σ with injectivity radius bounded uniformly from below, $\mathrm{inj}(\Sigma, h_k) \geq c > 0$. The
goal is to prove that there exist diffeomorphisms σ_k of Σ and a hyperbolic metric
h of Σ, such that $\sigma_k^* h_k$ converges smoothly to h as $k \to \infty$. Our strategy is to
apply the Mumford compactness criterion to the orientation double covers of the
surfaces (Σ, h_k).

Consider the orientation double cover $\hat{\Sigma} = \Sigma_{\delta-1}$ of Σ endowed with the
pullback metrics of h_k, denoted by $\hat{h}_k$. Since $\delta \geq 3$, these are orientable hyperbolic
surface of genus $\delta - 1$ and may thus be regarded (if we also fix a marking) as
elements in Teichmüller space $\mathcal{T}_{\delta-1}$, which in addition admit fixed point free,
isometric, orientation reversing involutions ι_k.

We have the following lemma.

Lemma 2.1. *Assume that* $\inf_k \mathrm{inj}(\Sigma_\delta^K, h_k) > 0$. *Then there exists a sequence
of diffeomorphisms* $\tau_k \colon \Sigma_{\delta-1} \to \Sigma_{\delta-1}$, *such that, up to taking a subsequence,*
$\tau_k^* \hat{h}_k \to \hat{h}$ *in* C^∞. *Moreover,* $(\Sigma_{\delta-1}, \hat{h})$ *admits a fixed point free, isometric, ori-
entation reversing involution* ι, *which is obtained as a* C^1*-limit of the involutions*
$\tau_k^{-1} \circ \iota_k \circ \tau_k$.

Proof. As above, we simply write Σ instead of Σ_δ^K, and $\widehat{\Sigma}$ instead of $\Sigma_{\delta-1}$. It is elementary to see that $\text{inj}(\widehat{\Sigma}, \hat{h}_k) \geq \text{inj}(\Sigma, h_k)$. Therefore, we can apply the Mumford compactness criterion [24] and find diffeomorphisms τ_k and a limit metric $\hat{h}$ as asserted.

It remains to show that we can find the involution ι. Since $\tau_k^* \hat{h}_k \to \hat{h}$ in C^∞, we have the uniform Lipschitz bound

$$
\begin{aligned}
d_{\hat{h}}((\tau_k^{-1} &\circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\
&\leq C d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\
&= C d_{\tau_k^* \hat{h}_k}(p, q) \\
&\leq C d_{\hat{h}}(p, q).
\end{aligned}
$$

Similarly, we obtain a uniform bound on $\|D\iota\|_{C^{0,1}(\widehat{\Sigma}, \hat{h})}$. Since $\widehat{\Sigma}$ is compact, it follows from Arzelà–Ascoli, that, up to taking a subsequence, $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in $C^1(\widehat{\Sigma}, \hat{h})$. We have

$$
\begin{aligned}
d_{\hat{h}}(\iota(p), \iota(q)) &\leq \lim_{k \to \infty} d_{\tau_k^* \hat{h}_k}(\iota(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(p)) \\
&\quad + \lim_{k \to \infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\
&\quad + \lim_{k \to \infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(q), \iota(q)) \\
&\leq C \lim_{k \to \infty} d_{C^0(\widehat{\Sigma}, \hat{h})}(\tau_k^{-1} \circ \iota_k \circ \tau_k, \iota) \\
&\quad + \lim_{k \to \infty} d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), (\tau_k^{-1} \circ \iota_k \circ \tau_k)(q)) \\
&= d_{\hat{h}}(p, q),
\end{aligned}
\tag{2.2}
$$

using that $\tau_k^* \hat{h}_k \to \hat{h}$ in C^∞, and $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in $C^0(\widehat{\Sigma}, \hat{h})$. Observe that ι is an involution again, hence (2.2) implies that actually

$$
d_{\hat{h}}(\iota(p), \iota(q)) = d_{\hat{h}}(p, q).
$$

By the Myers–Steenrod theorem it thus follows that ι is a smooth, isometric involution.

We need to show that ι does not have any fixed points. But this is a consequence of the general bound $d_{\tau_k^* \hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), p) \geq c > 0$ for some uniform c. To prove this let $c > 0$ be such that $B_{\hat{h}}(x, 2c) \subset \widehat{\Sigma}$ is strictly geodesically convex for any $x \in \widehat{\Sigma}$. Then $B_{\tau_k^* \hat{h}_k}(x, c)$ is strictly geodesically convex for $k \geq K$ sufficiently large. Assume now that there is $k \geq K$, such that $d_{\hat{h}_k}((\tau_k^{-1} \circ \iota_k \circ \tau_k)(p), p) < c$.

Let γ be the unique minimizing geodesic connecting p to $(\tau_k^{-1} \circ \iota_k \circ \tau_k)(p)$. Since $\tau_k^{-1} \circ \iota_k \circ \tau_k$ is an isometry, we need to have $\text{im}(\tau_k^{-1} \circ \iota_k \circ \tau_k \circ \gamma) = \text{im}\,\gamma$. Since ι_k is fixed point free, γ is non-constant. Therefore, $\tau_k^{-1} \circ \iota_k \circ \tau_k$ restricted to $\text{im}\,\gamma$ induces an involution of the interval $[0, 1]$ mapping 0 to 1 and vice versa. But such an involution needs to have a fixed point. It follows that ι_k has a fixed point for large k, which is a contradiction.

Finally, note that ι is orientation reversing by C^0-convergence. $\square$

Since $\tau_k^{-1} \circ \iota_k \circ \tau_k \to \iota$ in C^1 it follows that the metric $\hat{h}$ on $\hat{\Sigma}$ is ι-invariant. Therefore, it induces a smooth hyperbolic metric h on Σ. Moreover, the hyperbolic metrics on Σ induced from $\tau_k^* h_k$ and $\tau_k^{-1} \circ \iota_k \circ \tau_k$ converge smoothly to h on Σ. Finally, observe that the diffeomorphisms τ_k induce diffeomorphisms σ_k of Σ, such that $\sigma_k^* h_k$ are the metrics described above and converge smoothly to h.

Thus we have proved the following proposition.

Proposition 2.3. *Let (h_k) be a sequence of hyperbolic metrics on Σ_δ^K such that their injectivity radius is uniformly bounded from below $\text{inj}(\Sigma_\delta^K, h_k) \geq c > 0$. Then there are diffeomorphisms σ_k of Σ_δ^K and a hyperbolic metric h, such that $\sigma_k^* h_k \to h$ smoothly.*

3. Maximizing the first eigenvalue

In this section we extend [32, Theorem 2] to the non-orientable case. The strategy is the same as in [32]. That is, we first use that we can maximize the first eigenvalue in each conformal class. We then pick a maximizing sequence, consisting of maximizers in their own conformal class. This has the advantage, that these metrics can be studied in terms of sphere-valued harmonic maps. Using these harmonic maps it is possible to estimate the first eigenvalue along the maximizing sequence in case that the conformal class degenerates. To do so, we extend the results from [37] to non-orientable surfaces.

For fixed non-orientable genus $\delta \geq 3$, let c_k be a sequence of conformal classes on $\Sigma = \Sigma_\delta^K$ represented by hyperbolic metrics h_k, such that

$$\lim_{k \to \infty} \sup_{g \in c_k} \lambda_1(\Sigma, g)\,\text{area}(\Sigma, g) = \Lambda_1^K(\delta).$$

We will now use the following result due to Nadirashvili–Sire (with an extra assumption not relevant for our purposes) and, independently, Petrides.

Theorem 3.1 ([28, Theorem 2.1] or [32, Theorem 1]). *For each conformal class c_k as above, there is a metric g_k, which is smooth away from finitely many conical singularities, such that*

$$\lambda_1(\Sigma, g_k) \operatorname{area}(\Sigma, g_k) = \sup_{g \in c_k} \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g).$$

From now on we assume that $g_k \in c_k$ is picked as in the preceding theorem. Moreover, we assume that they are normalized to have

$$\operatorname{area}(\Sigma, g_k) = 1.$$

Since these metrics are maximizers, there is a family of first eigenfunctions $u_1^k, \ldots, u_{\ell(k)+1}^k$, such that $\Phi_k = (u_1^k, \ldots, u_{\ell(k)+1}^k) \colon (\Sigma, h_k) \to \mathbb{S}^{\ell(k)}$ is a harmonic map [10]. Since the multiplicity of λ_1 is uniformly bounded in terms of the topology of Σ [3, 5], we may pass to a subsequence, such that $\ell(k)$ is some constant number l. Moreover, in this situation the maximizing metrics can be recovered by

$$g_k = \frac{|\nabla \Phi_k|_{h_k}^2}{\lambda_1(\Sigma, g_k)} h_k.$$

In view of Proposition 2.3, we want to show the following proposition.

Proposition 3.2. *The injectivity radius of h_k is uniformly bounded from below, provided that $\Lambda_1^K(\delta) > \Lambda_1(\delta - 1)$, and $\Lambda_1^K(\delta) > \Lambda_1^K(\delta - 1)$.*

We will argue by contradiction and assume $\operatorname{inj}(\Sigma, h_k) \to 0$. The Margulis lemma implies that we can find closed geodesics $\gamma_k^1, \ldots, \gamma_k^s$ in (Σ, h_k), such that their lengths go to zero, i.e. $l_{h_k}(\gamma_i^k) \to 0$, as $k \to \infty$. We assume that s is chosen maximal with this property.

Each of these geodesics is either one-sided or two-sided. If a such a geodesic is two-sided, tubular neighborhoods are described by the classical collar lemma for hyperbolic surfaces [4]. In the second case we may apply the collar lemma to the orientation double cover as follows.

Let c be a one-sided closed geodesic in Σ. We write $\hat{\Sigma}$ for the orientation double cover and τ for the non-trivial deck transformation. The lifts of c to $\hat{\Sigma}$ can not be closed, since in this case they would be disjoint and it would follow that c is two-sided. Thus the lifts c_1 and c_2 are geodesic segments with $\tau \circ c_1 = c_2$. Let $\mathcal{C}$ be a collar around the closed geodesic $c_2 * c_1$. It is not very difficult to see that the action of τ near $c_2 * c_1$ is just given by rotation about π and reflection at $c_2 * c_1$. Therefore, τ maps $\mathcal{C}$ to itself (by the explicit construction of $\mathcal{C}$), so that we can use $\mathcal{C}/\tau$ as a tubular neighborhood of c.

Our first goal is to prove that for the situation at hand the volume, measured with respect to g_k, either concentrates in the neighborhood of a pinching geodesic, or in one connected component of the complement of these neighborhoods. Before stating and proving this result we need to introduce some notation, which we borrow from Section 4 in [32].

We write s_1 for the number of one-sided closed geodesics with length going to 0. Moreover, we denote by s_2 the number of such geodesics that are two-sided. Clearly, $s = s_1 + s_2$ and $0 \leq s_1, s_2 \leq s$. From now on we assume that the closed geodesics γ_k^i are ordered such that the first s_1 geodesics are one-sided. Moreover, we write $l_k^i = l_{h_k}(\gamma_k^i)$ for the hyperbolic length of the short geodesics.

For all $s_1 + 1 \leq i \leq s$ the collar theorem [4] asserts the existence of an open neighborhood P_k^i of γ_k^i isometric to the following truncated hyperbolic cylinder

$$\mathcal{C}_k^i = \left\{ (t, \theta) \mid -w_k^i < t < w_k^i, \, 0 \leq \theta < 2\pi \right\}$$

with

$$w_k^i = \frac{\pi}{l_k^i} \left(\pi - 2 \arctan \left(\sinh \frac{l_k^i}{2} \right) \right)$$

endowed with the metric

$$\left(\frac{l_k^i}{2\pi \cos \left(\frac{l_k^i}{2\pi} t \right)} \right)^2 (dt^2 + d\theta^2).$$

Below we identify $(\theta, t) = (0, t)$ with $(\theta, t) = (2\pi, t)$. Thus the closed geodesic γ_k^i corresponds to $\{t = 0\}$.

By the discussion above and the collar theorem again, we get that for all $1 \leq i \leq s_1$, there exists an open neighborhood P_k^i of γ_k^i isometric to the following truncated Möbius strip

$$\mathcal{M}_k^i = \{ (t, \theta) \mid -w_k^i < t < w_k^i, \, 0 \leq \theta < 2\pi \} / \sim$$

with

$$w_k^i = \frac{\pi}{2l_k^i} (\pi - 2 \arctan(\sinh l_k^i))$$

endowed with the metric

$$\left(\frac{2l_k^i}{2\pi \cos \left(\frac{2l_k^i}{2\pi} t \right)} \right)^2 (dt^2 + d\theta^2).$$

Moreover, the equivalence relation $\sim$ is given by identifying $(t, \theta,) \sim (-t, \theta + \pi)$, where $\theta + \pi \in \mathbb{R}/2\pi\mathbb{R}$. Hence, the closed geodesic γ_k^i corresponds to $\{t = 0\}$.

We denote by $\Sigma_k^1, \ldots, \Sigma_k^r$ the connected components of $\Sigma \setminus \bigcup_{i=1}^s P_k^i$. Consequently, Σ can be written as the disjoint union

$$\Sigma = \Big(\bigcup_{i=1}^s P_k^i \Big) \cup \Big(\bigcup_{j=1}^r \Sigma_k^j \Big).$$

For $s_1 + 1 \leq i \leq s$ and $0 < b < w_k^i$ we denote by $P_k^i(b)$ the truncated hyperbolic cylinder whose length, compared to P_k^i, is reduced by b, i.e.,

$$P_k^i(b) = \{(t, \theta), -w_k^i + b < t < w_k^i - b\}.$$

Analogously, for $1 \leq i \leq s_1$ and $0 < b < w_k^i$, we introduce

$$P_k^i(b) = \{(t, \theta), -w_k^i + b < t < w_k^i - b\}/ \sim .$$

Finally, we denote by $\Sigma_k^j(b)$ the connected components of $\Sigma \setminus \bigcup_{i=1}^s P_k^i(b)$ which contains Σ_k^j.

We are now ready to prove the above mentioned result, namely, that the volume either concentrates in the neighborhood of a pinching geodesic P_k^i, or in one connected component Σ_k^j of the complement of these neighborhoods.

Lemma 3.3. *There exists $D > 0$ such that one of the two following assertions is true*:

(1) *there exists an $i \in \{1, \ldots, s\}$ such that*

$$\mathrm{area}_{g_k}(P_k^i(a_k)) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ with $\frac{a_k}{w_k^i} \to 0$ as $k \to +\infty$ for all $1 \leq i \leq s$;

(2) *there exists a $j \in \{1, \ldots, r\}$ such that*

$$\mathrm{area}_{g_k}(\Sigma_k^j(9a_k)) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ with $\frac{a_k}{w_k^i} \to 0$ as $k \to +\infty$ for all $1 \leq i \leq s$.

Proof. The proof of Claim 11 in [32] can easily be adapted to the present situation. First recall the rough strategy of the proof: construct suitable test functions for $\lambda_1(\Sigma, g_k)$ in the P_k^i and the Σ_k^j's, apply the min-max formula for the first eigenvalue and prove the claim by contradiction. More precisely, on $\hat{\Sigma}$, the test functions are constructed with linear decay in the t variable in neck regions of

the type $\widehat{P}_k^i(2a_k) \setminus \widehat{P}_k^i(3a_k)$ and $\widehat{P}_k^i(1a_k) \setminus \widehat{P}_k^i(2a_k)$, respectively, where the hat indicates that we consider the preimages under the covering map $\widehat{\Sigma} \to \Sigma$. By conformal invariance, the Dirichlet energy of these can be estimated using the hyperbolic metric and decays like a_k^{-1}. From the construction it is clear that these functions are invariant under the relevant involutions. From this point on, one can just follow the arguments in [32]. □

Below we consider the two possible cases of the preceding lemma separately. The following lemma deals with the first case, i.e. when the volume concentrates in one of the P_k^i. We show that in this case we would have $\Lambda_1^K(\delta) \leq 8\pi$ if γ_k^i is 2-sided; and $\Lambda_1^K(\delta) \leq 12\pi$ if γ_k^i is 1-sided.

Lemma 3.4. *Suppose that there exists an $i \in \{1, \dots, s\}$ such that*

$$\operatorname{area}_{g_k}(P_k^i(a_k)) \geq 1 - \frac{D}{a_k}$$

for all sequences $a_k \to +\infty$ with $\frac{a_k}{w_k^i} \to 0$ as $k \to +\infty$ for all $1 \leq i \leq s$.

(1) *If γ_k^i is 2-sided, then $\Lambda_1^K(\delta) \leq 8\pi$.*
(2) *If γ_k^i is 1-sided, then $\Lambda_1^K(\delta) \leq 12\pi$.*

Proof. In [32], Petrides proved the first statement by following ideas of Girouard (see [12]). The proof of the second statement is carried out analogously.

By assumption, there exists an $i \in \{1, \dots, s\}$, such that the volume concentrates on $P_k := P_k^i$. On P_k we have coordinates (t, θ) as above (on $\mathcal{M}_k$). By the assumptions on the volume and a_k, we can find cut-off functions η_k which are 1 on $P_k(a_k)$ and 0 outside P_k, and satisfy

$$\int_\Sigma |\nabla \eta_k|^2 dv_{g_k} \to 0. \tag{3.5}$$

We denote by $\mathcal{C} = (-\infty, \infty) \times \mathbb{S}^1$ the infinite cylinder with its canonical coordinates $(t, \theta) \in (-\infty, \infty) \times [0, 2\pi)$. Let $\phi: \mathcal{C} \to \mathbb{S}^2 \subset \mathbb{R}^3$ be given by

$$\phi(t, \theta) = \frac{1}{e^{2t} + 1} (2e^t \cos(\theta), 2e^t \sin(\theta), e^{2t} - 1).$$

Observe that this induces a map $\psi: \mathcal{M} \to \mathbb{RP}^2(\sqrt{3})$ if we divide by the $\mathbb{Z}/2$-actions that we have on both sides. More precisely, $\mathcal{M} = \mathcal{C}/\sim$, where $(t, \theta) \sim (-t, \theta + \pi)$ as above, and on $\mathbb{S}^2$ we simply take the antipodal map. If we denote by $v: \mathbb{RP}^2(\sqrt{3}) \to \mathbb{S}^4$ the Veronese map, the concatenation $v \circ \phi: \mathcal{M} \to \mathbb{S}^4$

is a conformal map [12]. We may regard $\mathcal{M}_k \subset \mathcal{M}$ using Fermi coordinates as introduced above.

By a theorem of Hersch [13], there exists a conformal diffeomorphism τ_k of $\mathbb{S}^4$, such that

$$\int_{P_k} (\pi \circ \tau_k \circ v \circ \phi)\eta_k dv_{g_k} = 0,$$

where $\pi \colon \mathbb{S}^4 \hookrightarrow \mathbb{R}^5$ is the standard embedding. Set $u_k^i = (\pi_i \circ \tau_k \circ v \circ \phi)\eta_k$. By construction, we have

$$\sum_{i=1}^{5} \int_{\mathcal{M}_k} (u_k^i)^2 dv_{g_k} \geq 1 - \frac{D}{a_k},$$

since $\text{area}_{g_k}(P_\alpha^i(a_k)) \geq 1 - \frac{D}{a_k}$. Using conformal invariance and (3.5), one easily finds that

$$\int_{\Sigma} |\nabla u_k|_{g_k}^2 dv_{g_k} \leq 12\pi + o(1).$$

For details we refer to [12]. Consequently, there is $i = i(k) \in \{1, \ldots, 5\}$, such that

$$\lambda_1(\Sigma, g_k) \leq \frac{\int_M |\nabla u_k^i|_{g_k}^2 dv_{g_k}}{\int_M (u_k^i)^2 dv_{g_k}} \leq 12\pi + o(1).$$

This finally implies

$$\Lambda_1^K(\delta) \leq \limsup_{k \to \infty} \lambda_1(\Sigma, g_k) \leq 12\pi,$$

which establishes the claim. $\square$

We are thus left with the case second case from Lemma 3.3. In this case, we have the following lemma, which concludes the proof of Proposition 3.2.

Lemma 3.6. *Suppose that the second alternative from* 3.3 *holds, then either*

(i) $\Lambda_1^K(\delta) \leq \Lambda_1^K(\delta - 1)$, *or*

(ii) $\Lambda_1^K(\delta) \leq \Lambda_1(\gamma)$,

where $\gamma = \lfloor (\delta - 1)/2 \rfloor$.

Proof. Again, we apply the machinery from [32] to the orientation cover. The essential point is to keep track of the geometry of the corresponding involutions. Denote by $(\hat{\Sigma}, \hat{h}_k)$ the orientation covers of (Σ, h_k), and by ι_k the corresponding deck transformations.

We can then identify the spectrum of the Laplacian for any metric g in $[h_k]$ with the spectrum of the Laplacian acting only on the even functions on $(\hat{\Sigma}, \hat{g})$. We consider the associated harmonic maps $\Phi_k : (\Sigma, g_k) \rightarrow \mathbb{S}^l$. By conformal invariance, we can also view these as harmonic maps from (Σ, h_k) to $\mathbb{S}^l$. In this situation, the metric can be recovered by

$$g_k = \frac{|\nabla \Phi_k|^2_{h_k}}{\lambda_1(\Sigma, g_k)} h_k,$$

see [32, proof of Theorem 1]. By pulling back the Φ_k's to $\hat{\Sigma}$, we obtain even harmonic maps $\hat{\Phi}_k : (\hat{\Sigma}, \hat{h}_k) \rightarrow \mathbb{S}^l$, such that

$$\hat{g}_k = \frac{|\nabla \hat{\Phi}_k|^2_{\hat{h}_k}}{\lambda_1(\Sigma, g_k)} \hat{h}_k.$$

With out loss of generality, we may assume that the volume (with respect to g_k) concentrates in $\Sigma^1_k(9a_k)$. Denote by $\hat{\Sigma}^1_k(9a_k)$ its preimage under the covering projection. Note that this preimage might be disconnected. As in [32, Section 4], there are a compact Riemann surface $\bar{\Sigma}$ and diffeomorphisms $\tau_k : \bar{\Sigma} \setminus \{p_1, \ldots, p_r\} \rightarrow \hat{\Sigma}^1_k(9a_k)$. Moreover, the hyperbolic metrics $\bar{h}_k = \tau_k^* \hat{h}_k$ converge in $C^\infty_{loc}(\bar{\Sigma} \setminus \{p_1, \ldots, p_r\})$ to a hyperbolic metric $\bar{h}$.

Observe, that we can restrict and pullback the involutions ι_k to get involutions $\bar{\iota}_k$ of $\bar{\Sigma} \setminus \{p_1, \ldots, p_r\}$. Clearly, these involutions are isometric with respect to the hyperbolic metrics $\bar{h}_k$.

In a next step, we construct a fixed point free limit involution on $\bar{\Sigma}$. For the compact subsets $\bar{\Sigma}_c := \{x \in \bar{\Sigma} \mid \text{inj}_x(\bar{\Sigma}, \bar{h}) \geq c\}$, we can argue exactly as in the proof of Lemma 2.1 to get limit involutions $\bar{\sigma}_n$ on $\bar{\Sigma}_{1/n}$. Since any isometric involution must map $\bar{\Sigma}_c$ to itself, we may take subsequences, such that for $m \geq n$, we have $\bar{\sigma}_m|_{\bar{\Sigma}_{1/n}} = \bar{\sigma}_n$. Using a standard diagonal argument, we find a limit involution on $\bar{\Sigma} \setminus \{p_1, \ldots, p_r\}$. Clearly, this involution extends to an involution $\bar{\iota}$ on all of $\bar{\Sigma}$. Moreover, $\bar{\iota}$ is fixed point free: Arguing again as in Lemma 2.1, we can not have fixed points different from the p_i's. If say p_1 is fixed under $\bar{\iota}$, the involution is just rotation by π in a disc centered at p_1. By C^0-convergence away from p_1, we see that the involutions $\hat{\iota}_k$ act just via rotation on the collars around the degenerating geodesic. But this is impossible, since this implies that $\hat{\iota}_k$ is orientation preserving.

By [37], the pullbacks $\bar{\Phi}_k$ of the harmonic maps $\hat{\Phi}_k$ along the diffeomorphisms τ_k are then harmonic maps that converge in $C^\infty_{\text{loc}}(\bar{\Sigma} \setminus \{p_1, \ldots, p_r, x_1, \ldots, x_s\})$ to a limit harmonic map $\bar{\Phi}$. Clearly, $\bar{\Phi}$ is invariant under $\bar{\iota}$. Note, that no energy can be lost at the points x_i or p_i. By construction, no volume concentrates near the closed geodesics bounding $\hat{\Sigma}^1_k(9a_k) \subset \hat{\Sigma}$, which implies that no energy is lost at the points p_i. Observe next, that the points x_i always come in pairs by the invariance of the harmonic maps. Moreover, from the construction of the limit involution, it is clear, that two such points are bounded away from each other. Therefore, energy concentration of the harmonic maps in a point x_i implies that the volume with respect to the metric g_k concentrates at a point in Σ. But by [18, Lemma 2.1 and 3.1] this implies

$$\Lambda_1^K(\delta) = \lim_{k \to \infty}(\Sigma, g_k) \le 8\pi.$$

Finally, the energy identity from [37] implies that there is also no energy lost in the necks. Let $\bar{h}_0$ be the hyperbolic metric in the conformal class of the cusp compactification of $(\bar{\Sigma} \setminus \{p_1, \ldots, p_r\}, [\bar{h}])$. Since $\bar{\Phi}: (\bar{\Sigma} \setminus \{p_1, \ldots, p_s\}, [\bar{h}_0]) \to \mathbb{S}^l$ has finite energy, $\bar{\Phi}$ extends to a harmonic map $(\bar{\Sigma}, [\bar{h}_0]) \to \mathbb{S}^l$ [34, Theorem 3.6]. Moreover, this extension is certainly invariant under $\bar{\iota}$. In conclusion, $\bar{\Phi}: (\bar{\Sigma}, \bar{h}_0) \to \mathbb{S}^l$ is an $\bar{\iota}$ invariant harmonic map with energy

$$\int_{\bar{\Sigma}} |\nabla \bar{\Phi}|^2 dv_{\bar{h}_0} = \lim_{k \to \infty} \int_{\hat{\Sigma}} |\nabla \hat{\Phi}_k| dv_{\hat{h}_k}.$$

We consider the metric

$$\bar{g} = \frac{|\nabla \bar{\Phi}|^2_{\bar{h}_0}}{\Lambda_1^K(\delta)} \bar{h}_0$$

and observe that it is invariant under the involution $\bar{\iota}$, so that it descends to a metric g on $\bar{\Sigma}/\bar{\iota}$. Since there is no energy lost along the sequence $\bar{\Phi}_k$ of harmonic maps, we have

$$\text{area}(\bar{\Sigma}/\bar{\iota}, g) = 1.$$

Using that the capacity of a point relative to any ball is 0 [22, Chapter 2.2.4], it is easy to construct $\bar{\iota}_k$-invariant cut-offs $\eta_{\varepsilon,k}$ on $\bar{\Sigma}$ with the following two properties. For ε small, there are neighborhoods $U_\varepsilon \subset V_\varepsilon$ of $\{p_1, \ldots, p_r\}$ such that

- $\bigcap_{\varepsilon > 0} V_\varepsilon = \{p_1, \ldots, p_r\}$,
- $\eta_{\varepsilon,k} = 0$ in U_ε,
- $\eta_{\varepsilon,k} = 1$ outside V_ε, and
- $\int_{\bar{\Sigma}} |\nabla \eta_{\varepsilon,k}|^2 dv_{\bar{g}} \le \varepsilon^2$.

We write $\bar{g}_k = \tau_k^*(\hat{g}_k)$. Let u be the lift of a first eigenfunction of $(\bar{\Sigma}/\bar{\iota}, g)$ to $\bar{\Sigma}$. Using $\eta_{\varepsilon,k} u$ as a test function on $\hat{\Sigma}_k(9a_k)$ for k large enough, we find with the help of the dominated convergence theorem that

$$
\begin{aligned}
\Lambda_1^K(\delta) &= \lim_{k \to \infty} \lambda_1(\Sigma, g_k) \\
&\leq \limsup_{\varepsilon \to 0} \lim_{k \to \infty} \frac{\int_{\bar{\Sigma}} |\nabla(\eta_{\varepsilon,k} u)|^2 dv_{\hat{g}_k}}{\int_{\bar{\Sigma}} |\eta_{\varepsilon,k} u|^2 dv_{\bar{g}_k} - \left(\int_{\bar{\Sigma}} \eta_{\varepsilon,k} u dv_{\bar{g}_k}\right)^2} \\
&\leq \limsup_{\varepsilon \to 0} \frac{\int_{\bar{\Sigma}} |\nabla u|^2 dv_{\bar{g}} + C\varepsilon}{\int_{\bar{\Sigma} \backslash V_\varepsilon} |u|^2 dv_{\bar{g}} - \left(\int_{\bar{\Sigma}} u dv_{\bar{g}}\right)^2} \\
&\leq \frac{\int_{\hat{\Sigma}} |\nabla u|^2 dv_{\bar{g}}}{\int_{\bar{\Sigma}} |u|^2 dv_{\bar{g}}} \\
&\leq \lambda_1(\bar{\Sigma}/\bar{\iota}, g).
\end{aligned}
$$

If $\bar{\Sigma}$ is disconnected, it has two connected components and the genus of each component is at most $\lfloor (\delta - 1)/2 \rfloor$. Therefore, the quotient $\bar{\Sigma}/\bar{\iota}$ is an orientable surface of genus at most $\lfloor (\delta - 1)/2 \rfloor$ in this case and thus

$$
\Lambda_1^K(\delta) \leq \Lambda_1(\lfloor (\delta - 1)/2 \rfloor)
$$

thanks to [7]. In case $\bar{\Sigma}$ is connected, the quotient is non-orientable of non-orientable genus at most $\delta - 1$ and we have that

$$
\Lambda_1^K(\delta) \leq \Lambda_1^K(\delta - 1)
$$

again thanks to [7].[1]   □

Thanks to the weak inequality $\Lambda_1^K(\delta + 1) \geq \Lambda_1^K(\delta - 1)$ and the fact that $\Lambda_1^K(2) > 12\pi$, we can always rule out the first scenario from Lemma 3.3. Thanks to [19] and [6, 8, 25] our main result Theorem 1.2 follows from the theorem below.

Theorem 3.7. *Let $\delta \geq 3$. If $\Lambda_1^K(\delta) > \max\{\Lambda_1^K(\delta-1), \Lambda_1(\lfloor(\delta-1)/2\rfloor)\}$, there is a metric smooth away from finitely many singularities on Σ_δ^K that achieves $\Lambda_1^K(\delta)$.*

Proof. By the assumptions, Proposition 2.3, and Proposition 3.2, we can take hyperbolic metrics $h_k \to h$ in C^∞, such that

$$
\lim_{k \to \infty} \sup_{g \in [h_k]} \lambda_1(\Sigma, g) \, \mathrm{area}(\Sigma, g) = \Lambda_1^K(\delta).
$$

---

[1] More precisely, the non-orientable version of the result.

As above, we take unit volume metrics $g_k \in [h_k]$, such that

$$\lambda_1(\Sigma, g_k) = \sup_{g \in [h_k]} \lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g).$$

For the corresponding sequence of harmonic maps $\Phi_k \colon (\Sigma, h_k) \to \mathbb{S}^l$ no bubbling can occur since this would imply $\Lambda_1^K(\delta) \leq 8\pi$, by the same argument as above. Therefore, we can take a subsequence such that $\Phi_k \to \Phi$ in C^∞, which implies that $g_k \to g = \frac{|\nabla \Phi|_h^2}{\Lambda_1^K(\delta)} h$ in C^∞. In particular,

$$\lambda_1(\Sigma, g) \operatorname{area}(\Sigma, g) = \Lambda_1^K(\delta)$$

and g is smooth away from the branch points of Φ. The number of branch points is finite and the branch points correspond to conical singularities of g [35]. $\qquad \square$

A. Topology of surfaces

For convenience of the reader and the authors, we review here the notion of non-orientable genus.

Recall the classification of closed surfaces. The classes of closed orientable and non-orientable surfaces are both uniquely described up to diffeomorphism by the Euler characteristic. More precisely, any closed orientable surface is diffeomorphic to a surface of the form

$$\Sigma_\gamma = \mathbb{S}^2 \# \underbrace{T^2 \# \ldots \# T^2}_{\gamma \text{ times}},$$

and any closed non-orientable surface is diffeomorphic to a surface of the form

$$\Sigma_\delta^K = \mathbb{S}^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\delta \text{ times}}.$$

These two families provide – up to diffeomorphism – a complete list of all orientable respectively non-orientable closed surfaces. We call γ the *genus* of Σ_γ and δ the *non-orientable genus* of Σ_δ^K. Note that with this convention, the real projective plane has non-orientable genus 1. We have $\chi(\Sigma_\gamma) = 2 - 2\gamma$ and $\chi(\Sigma_\delta^K) = 2 - \delta$, so that the orientation cover of Σ_δ^K is given by $\Sigma_{\delta-1}$. Some authors prefer to refer to the genus of the orientation cover as the non-orientable genus. As explained above these two definitions differ. Moreover, recall that we have the relation

$$\mathbb{S}^2 \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{\delta \text{ times}} \cong \mathbb{S}^2 \# \underbrace{T^2 \# \ldots \# T^2}_{k \text{ times}} \# \underbrace{\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2}_{(\delta - 2k) \text{ times}},$$

if $2k < \delta$.

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