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Anderson localization for a generalized Maryland model with potentials given by skew shifts

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Abstract. In this paper, we proved Anderson localization for the following long-range operator

$$H = \tan \pi \Big(x_0 + m y_0 + \frac{m(m-1)}{2} \omega \Big) \delta_{mn} + \epsilon S_\phi,$$

which generalized the Maryland model to potentials given by skew shifts.

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1. Introduction and main result

Spectral theory of Schrödinger operators is an important topic in both physics and mathematics. Let us begin with the Maryland model

$$H = \lambda \tan \pi (x + n\omega) \delta_{nn'} + \Delta, \tag{1.1}$$

where $\lambda > 0$ is the coupling, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the phase, $\omega \in \mathbb{R} \backslash \mathbb{Q}$ is the frequency, Δ is the lattice Laplacian on $\mathbb{Z}$

$$\Delta(n, n') = \begin{cases} 1 & |n - n'| = 1; \\ 0 & |n - n'| \neq 1. \end{cases}$$

This is an unbounded self adjoint operator on $l^2(\mathbb{Z})$. We assume

$$x + n\omega - \frac{1}{2} \notin \mathbb{Z}, \quad \text{for } n \in \mathbb{Z},$$

to make the operator well defined.

The Maryland model was originally proposed in physics paper [8]. Simon [14] proved Anderson localization for the Maryland model H (1.1) with Diophantine frequencies ω. Anderson localization means that H has pure point spectrum with exponentially decaying eigenfunctions. Recently, using transfer matrix and Lyapounov exponent, Jitomirskaya and Yang [12] developed a constructive method to prove Anderson localization for the Maryland model. More exactly, Jitomirskaya and Liu [10] proved arithmetic spectral transitions for the Maryland model.

Schrödinger operators can be generalized to long range operators. For example, consider the following long-range operator

$$H_\omega(x) = \cos(x + n\omega)\delta_{nn'} + \epsilon S_\phi, \tag{1.2}$$

where S_ϕ is a Toeplitz operator

$$S_\phi(n, n') = \hat{\phi}(n - n')$$

and ϕ is real analytic, $\hat{\phi}(n)$ is the n-th Fourier coefficient of ϕ. Bourgain and Jitomirskaya [5] proved that there is $\epsilon_0 = \epsilon_0(\phi) > 0$, such that if $0 < \epsilon < \epsilon_0$, $H_\omega(x)$, defined in (1.2), satisfies Anderson localization for $(x, \omega) \in \mathbb{T}^2$ in a set of full measure. This result is non-perturbative, since ϵ_0 does not depend on ω. Note that in the long range case, we cannot use the transfer matrix formalism. In this paper, we focus on 1D quasi-periodic operators. For quasi-periodic operators on $\mathbb{Z}^d$, we only mention the recent progress made by Jitomirskaya, Liu and Shi [11].

The Maryland model has many generalizations. Using KAM methods, Bellissard, Lima, and Scoppola [6] generalized localization results of the Maryland model to v-dimensional incommensurate structures. This result is perturbative. Recently, Kachkovskiy [13] established non-perturbative Anderson localization for a wide class of quasi-periodic Schrödinger operators with unbounded monotone potentials, extending the classical localization results of the Maryland model. The authors [15] gave a non-perturbative proof of Anderson localization for the Maryland model with long range interactions

$$H(x) = \tan \pi(x + n\omega)\delta_{nn'} + \epsilon S_\phi. \tag{1.3}$$

Note that for all operators mentioned above, potentials are given by shifts

$$Tx = x + \omega. \tag{1.4}$$

Now, let T be skew shifts

$$T(x_1, x_2) = (x_1 + x_2, x_2 + \omega), \quad (x_1, x_2) \in \mathbb{T}^2. \tag{1.5}$$

In this case, using transfer matrix and Lyapounov exponent, Bourgain, Goldstein, and Schlag [4] proved Anderson localization for

$$H = \lambda v(T^n x) + \Delta. \tag{1.6}$$

For long range case, Bourgain [1] considered the following operator

$$H(x) = v(T^m x)\delta_{mn} + \phi_{m-n}(T^m x) + \overline{\phi_{n-m}(T^n x)}, \tag{1.7}$$

where v is a real, nonconstant, trigonometric polynomial, ϕ_k are trigonometric polynomials and T is the skew shift on $\mathbb{T}^2$. Using multi-scale method, Bourgain proved Anderson localization for the operator (1.7).

In this paper, we will study a generalized Maryland model with potentials given by skew shifts. More precisely, we consider the following operator

$$H(x) = \tan \pi (T^m x)_1 \delta_{mn} + \epsilon S_\phi, \tag{1.8}$$

where T is the skew shift on $\mathbb{T}^2$ and $(T^m x)_1$ refers to the first coordinate of $T^m x$. To make the operator (1.8) well defined, we will always assume

$$(T^m x)_1 - \frac{1}{2} \notin \mathbb{Z}, \quad \text{for all } m \in \mathbb{Z}. \tag{1.9}$$

We will prove the following result.

Theorem 1.1. *Consider a lattice operator $H_\omega(x)$ associated to the skew shift $T = T_\omega$ of the form (1.8). Assume $\omega \in$ DC (diophantine condition)*

$$\|k\omega\| > \gamma |k|^{-2} \quad \textit{for all } k \in \mathbb{Z} \setminus \{0\} \tag{1.10}$$

and ϕ real analytic satisfying

$$|\hat{\phi}(n)| < e^{-\rho|n|}, \quad \textit{for all } n \in \mathbb{Z}, \tag{1.11}$$

for some $\rho > 0$. Fix $x_0 \in \mathbb{T}^2$. Then for almost all $\omega \in DC$ and ϵ taken sufficiently small (depending on γ, ρ), $H_\omega(x_0)$ satisfies Anderson localization.

In the long range case here, the transfer matrix formalism is not applicable. Unlike shifts cases, we cannot obtain non-perturbative results in skew shifts cases. Our basic strategy is the same as that in [1], but the main difficulty is that the potential tan has singularity and the operator H is unbounded.

In order to prove Anderson localization, we need Green's function estimates for

$$G_{[0,N]}(x, E) = (R_{[0,N]}(H(x) - E)R_{[0,N]})^{-1}, \tag{1.12}$$

where R_Λ is the restriction operator to $\Lambda \subset \mathbb{Z}$. Note that

$$R_{[0,N]}(H(x) - E)R_{[0,N]} = D(x)B(x), \tag{1.13}$$

where

$$D(x) = \operatorname{diag}\Big(\frac{1}{\cos \pi x_1}, \dots, \frac{1}{\cos \pi (T^N x)_1}\Big). \tag{1.14}$$

Hence

$$G_{[0,N]}(x, E) = B(x)^{-1} D(x)^{-1}. \tag{1.15}$$

Since in $D(x)^{-1}$, the singularity $\frac{1}{\cos}$ vanishes, we only need Green's function estimates for $B(x)^{-1}$. We need to point out that $B(x)$ is not self-adjoint. Fortunately, we find that multi-scale analysis still applies to this case. Since the operator H is unbounded and the energy E is unbounded, we use the specific property of trigonometric functions to overcome the difficulty of the unboundedness of the energy E.

We summarize the structure of this paper. First, we will prove Green's function estimates in Section 2. Then we recall some facts about semi-algebraic sets in Section 3 and give the proof of Anderson localization in Section 4.

We will use the following notations. For positive numbers a, b, $a \lesssim b$ means $Ca \le b$ for some constant $C > 0$. $a \ll b$ means C is large. $a \sim b$ means $a \lesssim b$ and $b \lesssim a$. N^{1-} means $N^{1-\epsilon}$ with some small $\epsilon > 0$. For $x \in \mathbb{R}$, $\|x\| = \inf_{m \in \mathbb{Z}} |x - m|$, for $x = (x_1, x_2) \in \mathbb{T}^2$, $\|x\| = \|x_1\| + \|x_2\|$.

2. Green's function estimates

In this section, we will prove the Green's function estimates using multi-scale analysis in [1].

We need the following lemma.

Lemma 2.1 (Lemma 3.16 in [1]). *Let $A(x) = \{A_{mn}(x)\}_{1 \le m,n \le N}$ be a matrix-valued function on $\mathbb{T}^d$ such that*

$$A(x) \text{ is self-adjoint for } x \in \mathbb{T}^d, \tag{2.1}$$

$$A_{mn}(x) \text{ is a trigonometric polynomial of degree} < N^{C_1}, \tag{2.2}$$

$$|A_{mn}(x)| < C_2 e^{-c_2|m-n|}, \tag{2.3}$$

where $c_2, C_1, C_2 > 0$ are constants.

Let $0 < \delta < 1$ be sufficiently small, $M = N^{\delta^6}$, $L_0 = N^{\frac{1}{100}\delta^2}$, $0 < c_3 < \frac{1}{10}c_2$.

Assume that for any interval $I \subset [1, N]$ *of size* L_0*, except for* x *in a set of measure* $< e^{-L_0^{\delta^3}}$,

$$\|(R_I A(x) R_I)^{-1}\| < e^{L_0^{1-}}, \tag{2.4}$$

$$|(R_I A(x) R_I)^{-1}(m,n)| < e^{-c_3|m-n|}, \quad m,n \in I, |m-n| > \frac{L_0}{10}. \tag{2.5}$$

For fixed $x \in \mathbb{T}^d$, $n_0 \in [1, N]$ *is called a good site if* $I_0 = \left[n_0 - \frac{M}{2}, n_0 + \frac{M}{2}\right] \subset [1, N]$ *and*

$$\|(R_{I_0} A(x) R_{I_0})^{-1}\| < e^{M^{1-}}, \tag{2.6}$$

$$|(R_{I_0} A(x) R_{I_0})^{-1}(m,n)| < e^{-c_3|m-n|}, \quad m,n \in I_0, |m-n| > \frac{M}{10}. \tag{2.7}$$

Denote $\Omega(x) \subset [1, N]$ *the set of bad sites. Assume that for any interval* $J \subset [1, N]$ *such that* $|J| > N^{\frac{\delta}{5}}$, *we have*

$$|J \cap \Omega(x)| < |J|^{1-\delta}. \tag{2.8}$$

Then

$$\|A(x)^{-1}\| < e^{N^{1-\frac{\delta}{C(d)}}}, \tag{2.9}$$

$$|A(x)^{-1}(m,n)| < e^{-c_3'|m-n|}, \quad |m-n| > \frac{N}{10} \tag{2.10}$$

except for x *in a set of measure* $< e^{-\frac{N^{\delta^2}}{C(d)}}$, *where* $C(d)$ *is a constant depending on* d *and* $c_3' > c_3 - (\log N)^{-8}$.

We also need the following ergodic property of skew shifts on $\mathbb{T}^2$.

Lemma 2.2 (Lemma 15.21 in [2]). *Assume* $\omega \in \mathrm{DC}$, $T = T_\omega$ *is the skew shift on* $\mathbb{T}^2$, $\epsilon > L^{-\frac{1}{10}}$. *Then*

$$\#\{n = 1, \dots, L : \|T^n x - a\| < \epsilon\} < C\epsilon^2 L.$$

Remark 2.3. *In the proof of Lemma 2.2, we only need to assume*

$$\|k\omega\| > \gamma |k|^{-2}, \quad \textit{for all } 0 < |k| \le L.$$

By Lemma 2.1, Lemma 2.2, we can prove the Green's function estimates.

Proposition 2.4. *Let* $T = T_\omega$ *be the skew shift and*

$$H_{mn}(x) = \tan \pi (T^m x)_1 \delta_{mn} + \epsilon S_\phi. \tag{2.11}$$

Assume ϕ real analytic satisfying

$$|\hat{\phi}(n)| < e^{-\rho|n|}, \quad \textit{for all } n \in \mathbb{Z} \tag{2.12}$$

for some $\rho > 0$ and ω satisfying

$$\|k\omega\| > \gamma|k|^{-2}, \quad \textit{for all } 0 < |k| \leq N, \tag{2.13}$$

ϵ is small (depending on γ, ρ). Then for energy E,

$$\|G_{[0,N]}(x, E)\| < e^{N^{1-}}, \tag{2.14}$$

$$|G_{[0,N]}(x, E)(m, n)| < e^{-\frac{\rho}{100}|m-n|}, \quad 0 \leq m, n \leq N, |m-n| > \frac{N}{10} \tag{2.15}$$

for $x \notin \Omega_N(E)$, where

$$\operatorname{mes} \Omega_N(E) < e^{-N^{\sigma}}, \quad \sigma > 0. \tag{2.16}$$

Proof. Write

$$H_{[0,N]}(x) - E = D_{[0,N]}(x) B_{[0,N]}(x), \tag{2.17}$$

where

$$D_{mn}(x) = \frac{\sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}}{\cos \pi (T^m x)_1} \delta_{mn}, \tag{2.18}$$

$$B_{mm}(x) = \frac{1}{\sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}} [\sin \pi (T^m x)_1 + (\epsilon\hat{\phi}(0) - E) \cos \pi (T^m x)_1], \tag{2.19}$$

$$B_{mn}(x) = \frac{\epsilon\hat{\phi}(m-n) \cos \pi (T^m x)_1}{\sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}}, \quad m \neq n. \tag{2.20}$$

We will apply Lemma 2.1 to $B_{[0,N]}(x)$. Note that $B_{[0,N]}(x)$ is not self-adjoint. However, in the proof of Lemma 2.1, we don't need (2.1). Since

$$T^m(x_1, x_2) = \left(x_1 + m x_2 + \frac{m(m-1)}{2}\omega, x_2 + m\omega\right), \tag{2.21}$$

$B_{mn}(x)$ is a trigonometric polynomial of degree $< |m|$. (2.3) holds with $C_2 = 1$, $c_2 = \rho$.

We need to prove

$$\operatorname{mes}\{x \in \mathbb{T}^2 \colon \text{there exist } m \text{ and } n, \text{ with } 0 \leq m, n \leq N, \text{ such that } |B_{[0,N]}(x)^{-1}(m, n)| > e^{N^{1-} - c_3|m-n|\chi_{|m-n| > \frac{N}{10}}}\} < e^{-N^{\delta^3}} \tag{2.22}$$

for some $c_3 > \frac{\rho}{100}$, $0 < \delta < 1$.

By

$$
\begin{aligned}
&|\sin \pi x + (\epsilon\hat{\phi}(0) - E)\cos \pi x| \\
&\quad = \sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}\,|\cos \pi(x - \alpha)|, \quad 0 < \alpha < 1,
\end{aligned}
$$

using the fact

$$
\text{mes}\{x \in [0, 1]\colon |\cos \pi x| < \eta\} < \eta, \quad \text{for all } 0 < \eta < 1,
$$

we have

$$
\text{mes}\left[x \in [0, 1] \,\Big|\, \frac{1}{\sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}}|\sin \pi x + (\epsilon\hat{\phi}(0) - E)\cos \pi x| < \epsilon_0\right] < \epsilon_0. \tag{2.23}
$$

Since T is a measure-preserving transformation,

$$
\begin{aligned}
\text{mes}\Bigg[x \in \mathbb{T}^2 \,\Big|\, &\frac{1}{\sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}}|\sin \pi(T^m x)_1 + (\epsilon\hat{\phi}(0) - E)\cos \pi(T^m x)_1| \\
&< \epsilon_0\Bigg] < \epsilon_0.
\end{aligned} \tag{2.24}
$$

Hence

$$
\text{mes}[x \in \mathbb{T}^2 \mid \min_{0 \le m \le N_0} |B_{mm}(x)| < \epsilon_0] < N_0\epsilon_0. \tag{2.25}
$$

If $\min_{0 \le m \le N_0} |B_{mm}(x)| > \epsilon_0 > \epsilon$, take $\epsilon_0 = e^{-N_0^{\frac{1}{2}}}$, $\epsilon = e^{-N_0}$, by Neumann expansion and (2.25), we have

$$
|B_{[0,N_0]}(x)^{-1}(m, n)| < e^{N_0^{\frac{1}{2}} - \frac{\rho}{2}|m-n|}, \quad m, n \in [0, N_0] \tag{2.26}
$$

except for x in a set of measure $< e^{-cN_0^{\frac{1}{2}}}$. So, (2.22) holds for an initial scale N_0.

Assume (2.22) holds up to scale $L_0 = N^{\frac{1}{100}\delta^2}$, since

$$
B_{m+1,n+1}(x) = B_{mn}(Tx), \tag{2.27}
$$

(2.4) and (2.5) will hold for x outside a set of measure at most $e^{-L_0^{\delta^3}}$. Denote $\Omega(x) \subset [0, N]$ the set of bad sites with respect to scale $M = N^{\delta^6}$. $n_0 \notin \Omega(x)$ means

$$
\begin{aligned}
&|B_{[0,M]}(T^{n_0 - \frac{M}{2}}x)^{-1}(m, n)| \\
&= \left|B_{[n_0 - \frac{M}{2}, n_0 + \frac{M}{2}]}(x)^{-1}\left(m + n_0 - \frac{M}{2}, n + n_0 - \frac{M}{2}\right)\right| \\
&< e^{M^{1-} - c_3|m-n|\chi_{|m-n| > \frac{M}{10}}}, \quad m, n \in [0, M].
\end{aligned} \tag{2.28}
$$

From the inductive hypothesis, we have

$$|B_{[0,M]}(x)^{-1}(m,n)| < e^{M^{1-}-c_3|m-n|\chi_{|m-n|>\frac{M}{10}}}, \quad m,n \in [0,M] \tag{2.29}$$

for $x \notin \Omega_0, \operatorname{mes}\Omega_0 < e^{-M^{\delta^3}}$. By (2.28), (2.29), Lemma 2.1, we only need to show that for any $x \in \mathbb{T}^2$, $N^{\frac{\delta}{5}} < L < N$,

$$\#\{1 \le n \le L \mid T^n x \in \Omega_0\} < L^{1-\delta}. \tag{2.30}$$

Expressing (2.29) as a ratio of determinants and replacing cos, sin by truncated power series, Ω_0 may be viewed as a semi-algebraic set of degree at most M^6. (For properties of semi-algebraic sets, see Section 3.) If $r > e^{-\frac{1}{2}M^{\delta^3}}$, by Proposition 3.2, Ω_0 may be covered by at most $M^C\left(\frac{1}{r}\right)$ r-balls. Choosing $r = L^{-\frac{1}{20}} > N^{-1} > e^{-\frac{1}{2}M^{\delta^3}}$, using Lemma 2.2, Remark 2.3, we have

$$\#\{1 \le n \le L \mid T^n x \in \Omega_0\} < M^C\left(\frac{1}{r}\right)r^2 L < L^{C\delta^5+1-\frac{1}{20}} < L^{1-\delta}.$$

This proves (2.30) and (2.22).

By (2.17),

$$G_{[0,N]}(x,E) = (H_{[0,N]}(x) - E)^{-1} = B_{[0,N]}(x)^{-1} D_{[0,N]}(x)^{-1}, \tag{2.31}$$

hence

$$G_{[0,N]}(x,E)(m,n) = \frac{\cos \pi (T^n x)_1}{\sqrt{1 + (\epsilon\hat{\phi}(0) - E)^2}} B_{[0,N]}(x)^{-1}(m,n), \quad m,n \in [0,N]. \tag{2.32}$$

By (2.31) and (2.32),

$$\|G_{[0,N]}(x,E)\| \le \|B_{[0,N]}(x)^{-1}\|, \tag{2.33}$$

$$|G_{[0,N]}(x,E)(m,n)| \le |B_{[0,N]}(x)^{-1}(m,n)|, \quad m,n \in [0,N]. \tag{2.34}$$

Proposition 2.4 follows from (2.22), (2.33), and (2.34). □

3. Semi-algebraic sets

We recall some basic facts of semi-algebraic sets in this section, which is needed in Section 4. Let $\mathcal{P} = \{P_1, \dots, P_s\} \subset \mathbb{R}[X_1, \dots, X_n]$ be a family of real polynomials whose degrees are bounded by d. A semi-algebraic set is given by

$$S = \bigcup_j \bigcap_{l \in L_j} \{\mathbb{R}^n \mid P_l s_{jl} 0\}, \tag{3.1}$$

where $L_j \subset \{1, \dots, s\}, s_{jl} \in \{\leq, \geq, =\}$ are arbitrary. We say that S has degree at most sd and its degree is the inf of sd over all representations as in (3.1).

We need the following quantitative version of the Tarski-Seidenberg principle.

Proposition 3.1 ([7]). *Let $S \subset \mathbb{R}^n$ be a semi-algebraic set of degree B, then any projection of S is semi-algebraic of degree at most $B^C, C = C(n)$.*

We also need the following fact.

Proposition 3.2 (Corollary 9.6 in [2]). *Let $S \subset [0,1]^n$ be semi-algebraic of degree B. Let $\epsilon > 0$, $\operatorname{mes}_n S < \epsilon^n$. Then S may be covered by at most $B^C (\frac{1}{\epsilon})^{n-1} \epsilon$-balls.*

Finally, we will use the following lemma.

Lemma 3.3 (Lemma 15.26 in [2]). *Let $S \subset \mathbb{T}^3$ be a semi-algebraic set of degree B such that*

$$\operatorname{mes} S < e^{-B^\sigma}, \quad \sigma > 0.$$

Let M be an integer satisfying

$$\log\log M \ll \log B \ll \log M.$$

Then for any fixed $x_0 \in \mathbb{T}^2$,

$$\operatorname{mes}[\omega \in \mathbb{T} \mid \textit{there exists } j \sim M \textit{ such that } (\omega, T_\omega^j x_0) \in S] < M^{-c}, \quad c > 0,$$

where T_ω is the skew shift with frequency ω.

4. Proof of Anderson localization

In this section, we give the proof of Anderson localization as in [3].

By application of the resolvent identity, we have the following

Lemma 4.1. *Let $I \subset \mathbb{Z}$ be an interval of size N and $\{I_\alpha\}$ subintervals of size $M \ll N$, $N = e^{(\log M)^2}$. Assume that, for all $k \in I$, there is some α such that*

$$\left[k - \frac{M}{4}, k + \frac{M}{4}\right] \cap I \subset I_\alpha \tag{4.1}$$

and, for all α,

$$\|G_{I_\alpha}\| < e^{M^{1-}}, \quad |G_{I_\alpha}(n_1, n_2)| < e^{-\frac{\rho}{100}|n_1 - n_2|}, \tag{4.2}$$

with $n_1, n_2 \in I_\alpha$, $|n_1 - n_2| > \frac{M}{10}$.

Then

$$|G_I(n_1,n_2)| < e^{M}, \qquad n_1,n_2 \in I, \tag{4.3}$$

$$|G_I(n_1,n_2)| < e^{-\frac{\rho}{200}|n_1-n_2|}, \quad n_1,n_2 \in I, |n_1-n_2| > \frac{N}{10}. \tag{4.4}$$

Proof. For $m,n \in I$, there is some α such that

$$\left[m - \frac{M}{4}, m + \frac{M}{4}\right] \cap I \subset I_\alpha. \tag{4.5}$$

By resolvent identity,

$$|G_I(m,n)| \leq e^{M^{1-}} + \sum_{n_1 \in I_\alpha, n_2 \notin I_\alpha} |G_{I_\alpha}(m,n_1)| e^{-\rho|n_1-n_2|} |G_I(n_2,n)|. \tag{4.6}$$

If $|m-n_1| \leq \frac{M}{8}$, then $|n_1-n_2| \geq \frac{M}{8}$, hence

$$\sum_{|m-n_1| \leq \frac{M}{8}, m_2 \notin I_\alpha} |G_{I_\alpha}(m,n_1)| e^{-\rho|n_1-n_2|} \leq M e^{M^{1-}} e^{-\rho \frac{M}{8}} < \frac{1}{4}. \tag{4.7}$$

If $|m-n_1| > \frac{M}{8}$, then $|G_{I_\alpha}(m,n_1)| < e^{-\frac{\rho}{100}|m-n_1|}$, hence

$$\sum_{|m-n_1| > \frac{M}{8}, m_2 \notin I_\alpha} |G_{I_\alpha}(m,n_1)| e^{-\rho|n_1-n_2|} < e^{-\frac{\rho}{1000}M} < \frac{1}{4}. \tag{4.8}$$

By (4.6), (4.7), and (4.8),

$$\max_{m,n\in I} |G_I(m,n)| < e^{M^{1-}} + \frac{1}{2} \max_{m,n \in I} |G_I(m,n)|. \tag{4.9}$$

(4.3) follows from (4.9).

Take $m,n \in I$, $|m-n| > \frac{N}{10}$, assume (4.5), by resolvent identity,

$$\begin{aligned} |G_I(m,n)| &\leq \sum_{n_0 \in I_\alpha, n_1 \notin I_\alpha} |G_{I_\alpha}(m,n_0)| e^{-\rho|n_0-n_1|} |G_I(n_1,n)| \\ &\leq M \sum_{|m-n_1| > \frac{M}{4}} e^{-\frac{\rho}{100}|m-n_1|} |G_I(n_1,n)| \\ &\leq M^t \sum_{|m-n_1| > \frac{M}{4}, \ldots, |n_{t-1}-n_t| > \frac{M}{4}} e^{-\frac{\rho}{100}(|m-n_1| + \cdots + |n_{t-1}-n_t|)} |G_I(n_t,n)| \end{aligned} \tag{4.10}$$

where $t \leq 10\frac{N}{M}$.

If $|n-n_t| \leq M$, then by (4.3) and (4.10),

$$|G_I(m,n)| \leq M^t N^t e^{M - \frac{\rho}{100}|m-n_t|} \leq e^{20\frac{N}{M}\log N + 2M - \frac{\rho}{100}|m-n|} < e^{-\frac{\rho}{200}|m-n|}. \tag{4.11}$$

If $t = 10\frac{N}{M}$, then by (4.3) and (4.10),

$$|G_I(m,n)| \le M^t N^t e^{-\frac{\rho}{100}\frac{10N}{M}\frac{M}{4}+M} \le e^{20\frac{N}{M}\log N + M - \frac{\rho}{100}2N} < e^{-\frac{\rho}{100}|m-n|}. \tag{4.12}$$

(4.4) follows from (4.11) and (4.12). This proves Lemma 4.1. □

Now we can prove the main result.

Theorem 4.2. *Consider a lattice operator $H_\omega(x)$ associated to the skew shift $T = T_\omega$ of the form*

$$H_\omega(x) = \tan \pi (T^m x)_1 \delta_{mn} + \epsilon S_\phi. \tag{4.13}$$

Assume $\omega \in$ DC (diophantine condition)

$$\|k\omega\| > \gamma |k|^{-2}, \quad \textit{for all } k \in \mathbb{Z} \setminus \{0\} \tag{4.14}$$

and ϕ real analytic satisfying

$$|\hat{\phi}(n)| < e^{-\rho|n|}, \quad \textit{for all } n \in \mathbb{Z} \tag{4.15}$$

for some $\rho > 0$. Fix $x_0 \in \mathbb{T}^2$. Then for almost all $\omega \in DC$ and ϵ taken sufficiently small(depending on γ, ρ), $H_\omega(x_0)$ satisfies Anderson localization.

Proof. By Shnol's theorem [9], to establish Anderson localization, it suffices to show that if $\xi = (\xi_n)_{n\in\mathbb{Z}}$, $E \in \mathbb{R}$ satisfy

$$\xi_0 = 1, |\xi_n| < C|n|, \quad |n| \to \infty, \tag{4.16}$$

$$H(x_0)\xi = E\xi, \tag{4.17}$$

then

$$|\xi_n| < e^{-c|n|}, \quad |n| \to \infty. \tag{4.18}$$

Denote $\Omega = \Omega(E) \subset \mathbb{T}^2$ the set of x such that

$$|G_{[-N,N]}(x,E)(m,n)| < e^{N^{1-} - \frac{\rho}{100}|m-n|\chi_{|m-n|>\frac{N}{10}}} \tag{4.19}$$

fails for some $|m|, |n| \le N$. Let $N_1 = N^{C_1}$, C_1 is a sufficiently large constant. Then by Proposition 2.4,

$$\operatorname{mes} \Omega(E) < e^{-N^\sigma}, \tag{4.20}$$

$$\#\{|j| \le N_1 | T^j x_0 \in \Omega\} < N_1^{1-\delta}. \tag{4.21}$$

So, we may find an interval $I \subset [0, N_1]$ of size N such that

$$T^j x_0 \notin \Omega, \quad \text{for all } j \in I \cup (-I). \tag{4.22}$$

Hence

$$|G_{[j-N,j+N]}(x_0, E)(m,n)| < e^{N^{1-} - \frac{\rho}{100}|m-n|\chi_{|m-n|>\frac{N}{10}}}, \quad m, n \in [j-N, j+N]. \tag{4.23}$$

By (4.16), (4.17), and (4.23), we have

$$|\xi_j| \le C \sum_{\substack{n_1 \in [j-N, j+N] \\ n_2 \notin [j-N, j+N]}} e^{N^{1-} - \frac{\rho}{100}|j-n_1|\chi_{|j-n_1|>\frac{N}{10}}} e^{-\rho|n_1-n_2|}|n_2| < e^{-\frac{\rho}{200}N}. \tag{4.24}$$

Denoting j_0 the center of I, we have

$$1 = \xi_0 \le \|G_{[-j_0,j_0]}(x_0, E)\| \|R_{[-j_0,j_0]} H(x_0) R_{\mathbb{Z}\setminus[-j_0,j_0]}\xi\|. \tag{4.25}$$

By (4.16) and (4.24), we have for $|n| \le j_0$,

$$\begin{aligned} &|(R_{[-j_0,j_0]} H(x_0) R_{\mathbb{Z}\setminus[-j_0,j_0]}\xi)_n| \\ &\quad \le \sum_{|n_1|>j_0} e^{-\rho|n-n_1|}|\xi_{n_1}| \\ &\quad \le \sum_{j_0<|n_1|\le j_0+\frac{N}{2}} e^{-\rho|n-n_1|} e^{-\frac{\rho}{200}N} + C \sum_{|n_1|>j_0+\frac{N}{2}} e^{-\rho|n-n_1|}|n_1| < e^{-\frac{\rho}{400}N}. \end{aligned} \tag{4.26}$$

By (4.25) and (4.26),

$$\|G_{[-j_0,j_0]}(x_0, E)\| > e^{\frac{\rho}{500}N}, \tag{4.27}$$

hence

$$\operatorname{dist}(E, \operatorname{spec} H_{[-j_0,j_0]}(x_0)) < e^{-\frac{\rho}{500}N}. \tag{4.28}$$

Denote

$$\mathcal{E}_\omega = \bigcup_{|j|\le N_1} \operatorname{spec} H_{[-j,j]}(x_0). \tag{4.29}$$

It follows from (4.28) that if $x \notin \bigcup_{E' \in \mathcal{E}_\omega} \Omega(E')$, then

$$|G_{[-N,N]}(x, E)(m,n)| < e^{N^{1-} - \frac{\rho}{100}|m-n|\chi_{|m-n|>\frac{N}{10}}}, \quad |m|, |n| \le N. \tag{4.30}$$

Consider the set $S \subset \mathbb{T}^3 \times \mathbb{R}$ of (ω, x, E'), where

$$\|k\omega\| > c|k|^{-2}, \quad \text{for all } 0 < |k| \le N, \tag{4.31}$$

$$x \in \Omega(E'), \tag{4.32}$$

$$E' \in \mathcal{E}_\omega. \tag{4.33}$$

By Proposition 3.1,

$$\mathrm{Proj}_{\mathbb{T}^3} S \text{ is a semi-algebraic set of degree } < N^C, \tag{4.34}$$

and by (4.20),

$$\mathrm{mes}(\mathrm{Proj}_{\mathbb{T}^3} S) < e^{-\frac{1}{2}N^\sigma}. \tag{4.35}$$

Let $N_2 = e^{(\log N)^2}$,

$$\mathcal{R}_N = \{\omega \in \mathbb{T} \mid (\omega, T^j x_0) \in \mathrm{Proj}_{\mathbb{T}^3} S,\ \exists |j| \sim N_2\}. \tag{4.36}$$

By (4.34), (4.35), (4.36), using Lemma 3.3, $\mathrm{mes}\,\mathcal{R}_N < N_2^{-c}, c > 0$. Let

$$\mathcal{R} = \bigcap_{N_0 \ge 1} \bigcup_{N \ge N_0} \mathcal{R}_N, \tag{4.37}$$

then, by Borel–Cantelli theorem, $\mathrm{mes}\,\mathcal{R} = 0$. We restrict $\omega \notin \mathcal{R}$.

If $\omega \notin \mathcal{R}_N$, we have for all $|j| \sim N_2$, $(\omega, T^j x_0) \notin \mathrm{Proj}_{\mathbb{T}^3} S$, by (4.30),

$$|G_{[j-N, j+N]}(x_0, E)(m,n)| < e^{N^{1-} - \frac{\rho}{100}|m-n|\chi_{|m-n| > \frac{N}{10}}}, \quad m, n \in [j-N, j+N]. \tag{4.38}$$

Let

$$\Lambda = \bigcup_{\frac{1}{4}N_2 < j < 4N_2} [j-N, j+N] \supset \left[\frac{1}{4}N_2, 4N_2\right],$$

by Lemma 4.1, we deduce from (4.38) that

$$|G_\Lambda(x_0, E)(m,n)| < e^{-\frac{\rho}{200}|m-n|}, \quad |m-n| > \frac{N_2}{10}, \tag{4.39}$$

and therefore

$$|\xi_j| < e^{-\frac{\rho}{4000}|j|}, \quad \frac{1}{2}N_2 \le j \le N_2. \tag{4.40}$$

Since $\omega \notin \mathcal{R}$, by (4.37), there is some $N_0 > 0$, such that for all $N \ge N_0$, $\omega \notin \mathcal{R}_N$. So, (4.40) holds for $j \in \bigcup_{N \ge N_0} [\frac{1}{2}e^{(\log N)^2}, e^{(\log N)^2}] = [\frac{1}{2}e^{(\log N_0)^2}, \infty)$. This proves (4.18) for $j > 0$, similarly for $j < 0$. Hence Theorem 4.2 follows. $\square$

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