Quantum evolution and sub-Laplacian operators on groups of Heisenberg type

Clotilde Fermanian-Kammerer and Véronique Fischer

Abstract. In this paper we analyze the evolution of the time averaged energy densities associated with a family of solutions to a Schrödinger equation on a Lie group of Heisenberg type. We use a semi-classical approach adapted to the stratified structure of the group and describe the semi-classical measures (also called *quantum limits*) that are associated with this family. This allows us to prove an Egorov's type Theorem describing the quantum evolution of a pseudodifferential semi-classical operator through the semi-group generated by a sub-Laplacian.

Mathematics Subject Classification (2020). 58J45, 43A80, 35Q40.

Keywords. Analysis on nilpotent Lie groups, evolution of solutions to the Schrodinger equation, semi-classical analysis for sub-elliptic operators, abstract harmonic analysis, C*-algebra theory.

Contents

1. Introduction

We consider groups of Heisenberg type, or H-type groups G, which are a special case of simply connected Lie groups stratified of step 2 as described more precisely later. As a step 2 stratified group, its Lie algebra $\mathfrak{g}$ is equipped with a vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z},$$

such that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z} \neq \{0\}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Choosing a basis V_j of $\mathfrak{v}$ and identifying $\mathfrak{g}$ with the Lie algebra of left-invariant vector fields on G, one defines the sublaplacian

$$\mathbb{L}_G = \sum_{1 \leq j \leq \dim \mathfrak{v}} V_j^2$$

together with the associated Schrödinger propagator $\mathrm{e}^{it\mathbb{L}_G}$. We are interested in the asymptotic analysis as ε goes to 0 of quantities of the form

$$\frac{1}{T} \int\limits_0^T \int\limits_G \phi(x) |\, \mathrm{e}^{i\frac{t}{2\varepsilon^{\aleph}}\mathbb{L}_G}\, \psi_0^\varepsilon(x)|^2 \, dx \, dt \tag{1.1}$$

for $\phi \in \mathcal{C}_c^\infty(G)$, $T \in \mathbb{R}$, $\aleph \in \mathbb{R}$ and $(\psi_0^\varepsilon)_{\varepsilon > 0}$ a bounded family in $L^2(G)$ which satisfies

$$
\begin{aligned}
&\text{there exist } s, C_s > 0 \text{ such that for all } \varepsilon > 0,\\
&\varepsilon^s \|(-\mathbb{L}_G)^{\frac{s}{2}} \psi_0^\varepsilon\|_{L^2(G)} + \varepsilon^{-s} \|(-\mathbb{L}_G)^{-\frac{s}{2}} \psi_0^\varepsilon\|_{L^2(G)} \leq C_s,
\end{aligned}
\tag{1.2}
$$

so that the oscillations of the initial data are exactly of size $1/\varepsilon$. Taking into account that the operator $\mathbb{L}_G$ is homogeneous of degree 2 and writing

$$\frac{t}{\varepsilon^{\aleph}} \mathbb{L}_G = \frac{t}{\varepsilon^{\aleph+2}} \varepsilon^2 \mathbb{L}_G,$$

we choose $\aleph > -2$. Considering the asymptotics $\varepsilon \to 0$ then consists in doing an analysis in large times (times of sizes $O(\varepsilon^{-\aleph-2})$) simultaneously with the study of the dispersion of the concentration or oscillation effects that are present in the initial data. A consequence of our main results is the next theorem where we denote by $\mathcal{M}^+(Z)$ the subset of positive elements of the set $\mathcal{M}(Z)$ of finite Radon measures on a locally compact Hausdorff set Z (see Definition 2.7 for more precision about $L^\infty(\mathbb{R}, \mathcal{M}^+(Z))$).

Theorem 1.1. *Let G be a H-type group and $(\psi_0^\varepsilon)_{\varepsilon>0}$ a bounded family in $L^2(G)$ satisfying* (1.2). *Any weak limit of the measure $|\operatorname{e}^{i\frac{t}{2\varepsilon^\aleph}\mathbb{L}_G}\psi_0^\varepsilon(x)|^2 dx\, dt$ is of the form $d\varrho_t(x)\otimes dt$ where $t\mapsto\varrho_t$ is a map in $L^\infty(\mathbb{R},\mathcal{M}^+(G))$. Moreover, for almost all $t\in\mathbb{R}$, the measure ϱ_t writes*

$$\varrho_t = \varrho_t^{\mathfrak{v}^*} + \varrho_t^{\mathfrak{z}^*}, \qquad \text{with } t\longmapsto \varrho_t^{\mathfrak{z}^*} \text{ and } t\longmapsto \varrho_t^{\mathfrak{v}^*} \text{ in } L^\infty(\mathbb{R},\mathcal{M}^+(G)),$$

and has the following properties.

- *If $\aleph \in (-2,-1)$, then, for all $t \in \mathbb{R}$, $\varrho_t^{\mathfrak{v}^*} = \varrho_0^{\mathfrak{v}^*}$ and $\varrho_t^{\mathfrak{z}^*} = \varrho_0^{\mathfrak{z}^*}$ are independent of the time $t \in \mathbb{R}$, and $\varrho_0^{\mathfrak{v}^*} + \varrho_0^{\mathfrak{z}^*}$ is equal to a weak limit of $|\psi_0^\varepsilon(x)|^2 dx$.*

- *If $\aleph = -1$, then $\varrho_t^{\mathfrak{z}^*} = \varrho_0^{\mathfrak{z}^*}$, where $\varrho_0^{\mathfrak{z}^*} \in \mathcal{M}^+(G)$ depends only on (ψ_0^ε), and*

$$\varrho_t^{\mathfrak{v}^*}(x) = \int_{\mathfrak{v}^*} \varsigma_0(x\operatorname{Exp}(t\omega\cdot V), d\omega). \tag{1.3}$$

  for $\varsigma_0 \in \mathcal{M}^+(G\times\mathfrak{v}^*)$ which depends only on (ψ_0^ε).

- *If $\aleph \in (-1,0)$, then $\varrho_t^{\mathfrak{v}^*} = 0$ and $\partial_t\varrho_t^{\mathfrak{z}^*} = 0$ holds in the sense of distributions on $\mathbb{R}\times G$.*

- *If $\aleph = 0$, then $\varrho_t^{\mathfrak{v}^*} = 0$ and $\varrho_t^{\mathfrak{z}^*}$ decomposes into*

$$\varrho_t^{\mathfrak{z}^*} = \sum_{n\in\mathbb{N}} \int_{\mathfrak{z}^*\setminus\{0\}} \gamma_{n,t}(x,d\lambda)$$

  where $t\mapsto\gamma_{n,t}$ is in $L^\infty(\mathbb{R},\mathcal{M}^+(G\times(\mathfrak{z}^*\setminus\{0\})))$. Furthermore we have in the sense of distributions on $\mathbb{R}\times G\times(\mathfrak{z}^*\setminus\{0\})$,

$$\partial_t\gamma_{n,t} - \left(n+\frac{d}{2}\right)\mathcal{Z}^{(\lambda)}\gamma_{n,t} = 0, \tag{1.4}$$

  where $\mathcal{Z}(\lambda)$ is the left invariant vector field corresponding to $|\lambda|^{-1}\lambda \in \mathfrak{z}^*$.

- *If $\aleph > 0$, then $\varrho_t = 0$ for all $t \in \mathbb{R}$.*

We point out that the dependence of the measures mentioned in the statement above on the initial data will be made precise in the article. Several aspects are interesting to notice. Firstly, there exists a threshold, $\aleph = 0$, above which the weak limits of the time-averaged energy density is 0; this means that for sufficiently large scale of times, all the concentrations and oscillations effects have disappeared: the dispersion is complete. A similar picture holds in the Euclidean setting, however

the threshold occurs at $\aleph = -1$ (see [2] and Appendix A in this article). This illustrates the fact that the dispersion is slower in sub-Riemannian geometries than in Euclidean ones, as already noticed in [6, 14, 5]. Secondly, one observes a decomposition of these weak limits into two parts $\varrho_t = \varrho_t^{\mathfrak{v}^*} + \varrho_t^{\mathfrak{z}^*}$ which turn out to have different transitional regimes: $\aleph = -1$ for $\varrho_t^{\mathfrak{v}^*}$ and $\aleph = 0$ for $\varrho_t^{\mathfrak{z}^*}$. This splitting is also present in the works [8] about the Grushin–Schrödinger equation and [43, 12] about sublaplacians on contact manifolds. The part $\varrho_t^{\mathfrak{v}^*}$ behaves like in the Euclidean setting and equation (1.3) also presents Euclidean features. However, the other part $\varrho_t^{\mathfrak{z}^*}$ looks completely different and is specific to the nilpotent Lie group context, showing that the structure of the limiting objects is more complex than in the Euclidean case.

Similar questions have been addressed for the Laplace operator in different geometries, including compact ones: in the torus and for integrable systems [3, 2], in Zoll manifolds (see [33, 30] and the review [31]), or on manifolds such as the sphere [32]. In contrast with the non-compact case (which is ours here), the compactness of the manifold implies that the complete dispersion of the energy is not possible; furthermore, the weak limits of the energy densities possess structural properties due to the geometry of the manifold, such as invariance by some flows, that may allow for their determination. For example, on compact Riemannian manifolds, such a measure belongs to the set of measures which are invariant under the geodesic flow, and this property is at the root of quantum ergodicity theorem [39, 11, 42] (see the introductory survey [1] and the articles [44, 16] for more recent developments in the topic). The question of quantum ergodicity also arises in sub-Riemannian geometries and have been addressed for contact [43, 12] and quasi-contact [38] manifolds. As will be made precise in the next sections, we observe invariance properties by a flow that turns out to coincide with the Reeb flow used in [43, 12] when G is the Heisenberg group.

Theorem 1.1 is a consequence of the main results of this paper which use the semi-classical approach introduced in [17] for H-type Lie groups and are in the spirit of the article [30] for the treatment of the large time evolution together with the oscillations. They are as follows.

(1) The first result is an Egorov's type theorem on H-type groups (see Theorem 2.5), which describes as ε goes to 0 the asymptotics of quantities of the form

$$\int_{\mathbb{R}} \theta(t)(e^{-i\frac{t}{2\varepsilon^\aleph}\mathbb{L}_G} \mathrm{Op}_\varepsilon(\sigma)\, e^{i\frac{t}{2\varepsilon^\aleph}\mathbb{L}_G}\, f, f)_{L^2(G)}\, dt \qquad (1.5)$$

for $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$, $f \in L^2(G)$ and where the operator $\mathrm{Op}_\varepsilon(\sigma)$ is the semi-classical operator of symbol σ as introduced in [17] (see also [18, 21, 4, 41]) All these elements are carefully explained in Section 2.

(2) The second result concerns the structure of the limiting objects when passing to the limit in (1.5), and for $f = \psi_0^\varepsilon$, a bounded family in $L^2(G)$. We extend the notion of semi-classical measure introduced in [17] to a time-dependent context (Theorem 2.8) and analyze the properties of the semi-classical measures associated in that manner with the family $(e^{i\frac{t}{2\varepsilon\aleph}\mathbb{L}_G} \psi_0^\varepsilon)_{\varepsilon>0}$, depending on the value of $\aleph$. We give a complete description of these limiting objects in Theorem 2.10 below.

The proof of Theorem 1.1 is based on the fact that, under certain hypothesis on the size of the oscillations, the analysis of the weak limits of the energy density can be deduced from those of its semi-classical measures, which are also called *quantum limits* in some geometric contexts. This idea was introduced in the 90's in the Euclidean case (see [26, 23, 29]), and adapted for H-type groups in [17]. The hypothesis on the size of the oscillations of $(\psi_0^\varepsilon)_{\varepsilon>0}$ is a *uniform strict ε-oscillation* property (see Section 5.2) which guarantees that the oscillations are of sizes ε^{-1} and is implied by the condition in (1.2). Then, using the semi-classical pseudodifferential operators constructed in [17], we determine the semi-classical measures that are associated with the family $e^{i\frac{t}{2\varepsilon\aleph}\mathbb{L}_G} \psi_0^\varepsilon$ and prove Theorem 1.1.

A straightforward generalization of our result would consist in adding a scalar potential $\varepsilon^\theta V(x)$ for a smooth function V defined on G and a parameter $\theta \in \mathbb{R}^+$. Then, one could exhibit regimes depending on the position of θ with respect to $\aleph$ and the vector fields to consider should be modified in a non-trivial manner. One should then consider operations on symbols $\sigma(x, \lambda)$ that involve differentials of the potential $V(x)$ and difference operators acting on the operator part of $\sigma(x, \lambda)$. A second generalization would be to consider more general stratified and graded groups. This would require to obtain in this more general setting similar results to those of Appendix B which at the moment heavily rely on the special case of H-type groups. However, the authors think this is doable and they have this generalization in mind. They also think that this approach can be adapted to homogeneous spaces.

In the next section, we recall the definition of H-type groups and present our two main results shortly described above, the Egorov theorem 2.5 and the analysis of the semi-classical measures associated with a family of solutions to the Schrödinger equation in 2.10. After some preliminary results on semi-classical symbols in Section 3, we prove both theorems in Section 4. Theorem 1.1 is a

consequence of this analysis and is proved in Section 5. Appendix A is devoted to a short description of the Euclidean case and Appendix B to some technical auxiliary results.

2. Main results

2.1. H-type groups, notations and definitions. A simply connected Lie group G is said *stratified of step* 2 if its Lie algebra $\mathfrak{g}$ is equipped with a vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z},$$

such that $[\mathfrak{v}, \mathfrak{v}] = \mathfrak{z} \neq \{0\}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}$. Via the exponential map

$$\mathrm{Exp}\colon \mathfrak{g} \longrightarrow G$$

which is a diffeomorphism from $\mathfrak{g}$ to G, one identifies G and $\mathfrak{g}$ as a set and a manifold. Under this identification, the group law on G (which is generally not commutative) is provided by the Campbell–Baker–Hausdorff formula, and $(x, y) \mapsto xy$ is a polynomial map. More precisely, if $x = \mathrm{Exp}(v_x + z_x)$ and $y = \mathrm{Exp}(v_y + z_y)$ then

$$xy = \mathrm{Exp}(v + z), \quad v = v_x + v_y \in \mathfrak{v}, \quad z = z_x + z_y + \frac{1}{2}[v_x, v_y] \in \mathfrak{z}.$$

If $x = \mathrm{Exp}(v)$ then $x^{-1} = \mathrm{Exp}(-v)$. We may identify $\mathfrak{g}$ with the space of left-invariant vector fields via

$$Xf(x) = \frac{d}{dt} f(x\,\mathrm{Exp}(tX))\Big|_{t=0}, \quad x \in G.$$

For any $\lambda \in \mathfrak{z}^\star$ (the dual of the center $\mathfrak{z}$) we define a skew-symmetric bilinear form on $\mathfrak{v}$ by

$$B(\lambda)(U, V) := \lambda([U, V]), \quad \text{for all } U, V \in \mathfrak{v}. \tag{2.1}$$

Following [28] (see also [13] in which a classification of all the H-type algebras is given), we say that G is *of H-type* (or *of Heisenberg type*) when, once the inner products on $\mathfrak{v}$ and on $\mathfrak{z}$ are fixed, the endomorphism of this skew symmetric form (that we still denote by $B(\lambda)$) satisfies

$$B(\lambda)^2 = -|\lambda|^2\,\mathrm{Id}_{\mathfrak{v}}, \quad \text{for all } \lambda \in \mathfrak{z}^*. \tag{2.2}$$

This implies in particular that the dimension of $\mathfrak{v}$ is even. We set

$$\dim \mathfrak{v} = 2d, \quad \dim \mathfrak{z} = p.$$

2.1.1. Orthonormal basis of $\mathfrak{g}$. For each $\lambda \in \mathfrak{z}^* \setminus \{0\}$, one can find an orthonormal basis

$$(P_1, \ldots, P_d, Q_1, \ldots, Q_d) = (P_1^{(\lambda)}, \ldots, P_d^{(\lambda)}, Q_1^{(\lambda)}, \ldots, Q_d^{(\lambda)}),$$

where $B(\lambda)$ is represented by

$$B(\lambda)(U, V) = |\lambda| U^t J V, \quad \text{with } J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}. \tag{2.3}$$

for two vectors $U, V \in \mathfrak{v}$ written in the $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$-basis. We decompose $\mathfrak{v}$ in a λ-depending way as $\mathfrak{v} = \mathfrak{p}_\lambda + \mathfrak{q}_\lambda$ with

$$\mathfrak{p} := \mathfrak{p}_\lambda := \mathrm{Span}(P_1, \ldots, P_d), \quad \mathfrak{q} := \mathfrak{q}_\lambda := \mathrm{Span}(Q_1, \ldots, Q_d).$$

The fundamental property (2.2) satisfied by $B(\lambda)$ considered as an endomorphism on $\mathfrak{v}$ implies that for all $V \in \mathfrak{v}$, $|B(\lambda)V|_\mathfrak{v}^2 = |\lambda|^2 |V|_\mathfrak{v}^2$, and, by linearization, we deduce for all $U, U' \in \mathfrak{v}$ and $\lambda, \lambda' \in \mathfrak{z}^*$,

$$(B(\lambda)U, B(\lambda')U')_\mathfrak{v} + (B(\lambda')U, B(\lambda)U')_\mathfrak{v} = 2(\lambda, \lambda')_{\mathfrak{z}^*}(U, U')_\mathfrak{v}. \tag{2.4}$$

Finally, by the definition of $B(\lambda)$, we have

$$(B(\lambda)U, B(\lambda')U')_\mathfrak{v} = \lambda([U, B(\lambda')U']) = \lambda'([B(\lambda)U, U']). \tag{2.5}$$

Applying (2.4) and (2.5) to $U = U' = P_j^{(\lambda)}$, since $Q_j = -|\lambda|^{-1} B(\lambda) P_j^{(\lambda)}$ by (2.3), one obtains

$$\begin{aligned} \lambda'([P_j, Q_j]) &= -|\lambda|^{-1} \lambda'([P_j, B(\lambda)P_j]) \\ &= |\lambda|^{-1} (B(\lambda)P_j, B(\lambda')P_j)_\mathfrak{v} \\ &= |\lambda|^{-1} (\lambda, \lambda')_{\mathfrak{z}^*} \|P_j\|_\mathfrak{v}^2 \\ &= (|\lambda|^{-1}\lambda, \lambda')_{\mathfrak{z}^*}. \end{aligned}$$

Denoting by $\mathcal{Z}^{(\lambda)}$ the unique vector of $\mathfrak{z}$ equal to $|\lambda|^{-1}\lambda$ through the identification of $\mathfrak{z}^*$ to $\mathfrak{z}$ via the inner product,

$$(\mathcal{Z}^{(\lambda)}, Z)_\mathfrak{z} = |\lambda|^{-1}\lambda(Z), \quad \text{for all } Z \in \mathfrak{z},$$

we deduce that

$$[P_j, Q_j] = \mathcal{Z}^{(\lambda)}, \quad j = 1, \ldots, d. \tag{2.6}$$

One proves similarly that, for all $1 \le j_1, j_2 \le d$,

$$j_1 \ne j_2 \implies [P_{j_1}, P_{j_2}] \in \ker \lambda, \quad [Q_{j_1}, Q_{j_2}] \in \ker \lambda, \quad [P_{j_1}, Q_{j_2}] \in \ker \lambda. \tag{2.7}$$

2.1.2. Realisation of the elements in G. Denoting by $z = (z_1, \ldots, z_p)$ the coordinate of Z in a fixed orthonormal basis $(Z_1, \ldots, Z_p)$ of $\mathfrak{z}$, and once given $\lambda \in \mathfrak{z}^*$, we will often use the writing of an element $x \in G$ or $X \in \mathfrak{g}$ as

$$
\begin{aligned}
x &= \mathrm{Exp}(X), \\
X &= p_1 P_1 + \cdots + p_d P_d + q_1 Q_1 + \cdots + q_d Q_d + z_1 Z_1 + \cdots + z_p Z_p,
\end{aligned}
\tag{2.8}
$$

where $p = (p_1, \ldots, p_d)$ are the λ-dependent coordinates of P on the vector basis $(P_1, \ldots, P_d)$, by $q = (q_1, \ldots, q_d)$ those of Q on $(Q_1, \ldots, Q_d)$, while the coordinates $z = (z_1, \ldots, z_p)$ of Z are independent of λ. We will also fix an orthonormal basis $(V_1, \ldots, V_{2d})$ of $\mathfrak{v}$ to write the coordinates

$$
v = (v_1, \ldots, v_{2d}), \quad \text{of an element } V = v_1 V_1 + \cdots + v_{2d} V_{2d}
$$

of $\mathfrak{v}$; both this orthonormal basis and the coordinates are independent of λ.

2.1.3. Functional spaces on G. The inner products on $\mathfrak{v}$ and $\mathfrak{z}$ allow us to consider the Lebesgue measure $dv\,dz$ on $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$. Via the identification of G with $\mathfrak{g}$ by the exponential map, this induces a Haar measure dx on G. This measure is invariant under left and right translations:

$$
\int_G f(y)\,dy = \int_G f(xy)\,dy = \int_G f(yx)\,dy \quad \text{for all } f \in L^1(G), x \in G.
$$

Note that the convolution of two functions f and g on G is given by

$$
f * g(x) := \int_G f(xy^{-1})g(y)\,dy = \int_G f(y)g(y^{-1}x)\,dy
\tag{2.9}
$$

and as in the Euclidean case we define Lebesgue spaces by

$$
\|f\|_{L^q(G)} := \left(\int_G |f(y)|^q\,dy \right)^{\frac{1}{q}},
$$

for $q \in [1, \infty)$, with the standard modification when $q = \infty$.

We define the Schwartz space $\mathcal{S}(G)$ as the set of smooth functions on G such that for all α, β in $\mathbb{N}^{2d+p}$, the function $x \mapsto x^\beta \mathcal{X}^\alpha f(x)$ belongs to $L^\infty(G)$, where $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left invariant vector fields forming a basis of $\mathfrak{g}$ and x^β a product of $|\beta|$ coordinate functions on $G \sim \mathfrak{v} \times \mathfrak{z}$. The Schwartz space $\mathcal{S}(G)$ can be naturally identified with the Schwartz space $\mathcal{S}(\mathbb{R}^{2d+p})$; in particular, it is dense in Lebesgue spaces.

2.1.4. Dilations. Since G is stratified, there is a natural family of dilations on $\mathfrak{g}$ defined for $t > 0$ as follows: if X belongs to $\mathfrak{g}$, we decompose X as $X = V + Z$ with $V \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ and we set

$$\delta_t X := tV + t^2 Z.$$

This allows us to define the dilation on the Lie group G via the identification by the exponential map:

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\ \delta_t\ } & \mathfrak{g} \\
{\scriptstyle \mathrm{Exp}}\downarrow & & \downarrow{\scriptstyle \mathrm{Exp}} \\
G & \xrightarrow{\ \mathrm{Exp}\,\circ\,\delta_t\,\circ\,\mathrm{Exp}^{-1}\ } & G
\end{array}
$$

To simplify the notation, we shall still denote by δ_t the map $\mathrm{Exp} \circ \delta_t \circ \mathrm{Exp}^{-1}$. The dilations δ_t, $t > 0$, on $\mathfrak{g}$ and G form a one-parameter group of automorphisms of the Lie algebra $\mathfrak{g}$ and of the group G. The Jacobian of the dilation δ_t is t^Q where

$$Q := \dim \mathfrak{v} + 2\dim \mathfrak{z} = 2d + 2p$$

is called the *homogeneous dimension* of G. A differential operator T on G (and more generally any operator T defined on $C_c^\infty(G)$ and valued in the distributions of $G \sim \mathbb{R}^{2d+p}$) is said *homogeneous of degree v* (or *v-homogeneous*) when $T(f \circ \delta_t) = t^v (Tf) \circ \delta_t$.

2.2. The irreducible unitary representations and the Fourier transform

2.2.1. Irreducible unitary representations. For $\lambda \in \mathfrak{z}^* \setminus \{0\}$, the irreducible unitary representation π_x^λ of G on $L^2(\mathfrak{p}_\lambda)$ is defined by

$$\pi_x^\lambda \Phi(\xi) = e^{i\lambda(z) + \frac{i}{2}|\lambda| pq + i \sqrt{|\lambda|} \xi q} \, \Phi(\xi + \sqrt{|\lambda|} p),$$

where x has been written as in (2.8). The representations π^λ, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, are infinite dimensional. The other unitary irreducible representations of G are given by the characters of the first stratum in the following way: for every $\omega \in \mathfrak{v}^*$, we set

$$\pi_x^{0,\omega} = e^{i\omega(V)}, \quad x = \mathrm{Exp}(V + Z) \in G, \quad \text{with } V \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}.$$

The set $\widehat{G}$ of all unitary irreducible representations modulo unitary equivalence is then parameterized by $(\mathfrak{z}^* \setminus \{0\}) \sqcup \mathfrak{v}^*$:

$$\widehat{G} = \{\text{class of } \pi^\lambda \colon \lambda \in \mathfrak{z}^* \setminus \{0\}\} \sqcup \{\text{class of } \pi^{0,\omega} \colon \omega \in \mathfrak{v}^*\}. \tag{2.10}$$

We will often identify each representation π^λ with its equivalence class; in this case, we may write $\mathcal{H}_\lambda$ for the Hilbert space of the representation instead of $L^2(\mathfrak{p}_\lambda) \sim L^2(\mathbb{R}^d)$; we also set $\mathcal{H}_{(0,\mu)} = \mathbb{C}$. Note that the trivial representation $1_{\widehat{G}}$ corresponds to the class of $\pi^{(0,\omega)}$ with $\omega = 0$, i.e. $1_{\widehat{G}} := \pi^{(0,0)}$.

2.2.2. The Fourier transform. In contrast with the Euclidean case, the Fourier transform is defined on $\widehat{G}$ and is valued in the space of bounded operators on $L^2(\mathfrak{p}_\lambda)$. More precisely, the Fourier transform of a function f in $L^1(G)$ is defined as follows: for any $\lambda \in \mathfrak{z}^*$, $\lambda \neq 0$,

$$\hat{f}(\lambda) := \mathcal{F}f(\lambda) := \int_G f(x)(\pi_x^\lambda)^* \, dx,$$

Note that for any $\lambda \in \mathfrak{z}^*$, $\lambda \neq 0$, we have $(\pi_x^\lambda)^* = \pi_{x^{-1}}^\lambda$ and the map π_x^λ is a group homomorphism from G into the group $U(L^2(\mathfrak{p}_\lambda))$ of unitary operators of $L^2(\mathfrak{p}_\lambda)$, so functions f of $L^1(G)$ have a Fourier transform $(\mathcal{F}(f)(\lambda))_\lambda$ which is a bounded family of bounded operators on $L^2(\mathfrak{p}_\lambda)$ with uniform bound:

$$\|\mathcal{F}f(\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} \leq \int_G |f(x)| \|(\pi_x^\lambda)^*\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} \, dx = \|f\|_{L^1(G)}. \qquad (2.11)$$

since the unitarity of π^λ implies $\|(\pi_x^\lambda)^*\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} = 1$.

2.2.3. Plancherel formula. The Fourier transform can be extended to an isometry from $L^2(G)$ onto the Hilbert space of measurable families $A = \{A(\lambda)\}_{(\lambda)\in\mathfrak{z}^*\backslash\{0\}}$ of operators on $L^2(\mathfrak{p}_\lambda)$ which are Hilbert–Schmidt for almost every $\lambda \in \mathfrak{z}^* \backslash \{0\}$, with norm

$$\|A\| := \left(c_0 \int_{\mathfrak{z}^*\backslash\{0\}} \|A(\lambda)\|_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))}^2 |\lambda|^d \, d\lambda \right)^{\frac{1}{2}} < \infty,$$

where $c_0 > 0$ is a computable constant. We have the following Fourier–Plancherel formula:

$$\int_G |f(x)|^2 \, dx = c_0 \int_{\mathfrak{z}^*\backslash\{0\}} \|\mathcal{F}f(\lambda)\|_{\mathrm{HS}(L^2(\mathfrak{p}_\lambda))}^2 |\lambda|^d \, d\lambda. \qquad (2.12)$$

This yields an inversion formula for any $f \in \mathcal{S}(G)$ and $x \in G$:

$$f(x) = c_0 \int_{\mathfrak{z}^*\backslash\{0\}} \mathrm{Tr}(\pi_x^\lambda \mathcal{F}f(\lambda)) |\lambda|^d \, d\lambda, \qquad (2.13)$$

where Tr denotes the trace of operators of $\mathcal{L}(L^2(\mathfrak{p}_\lambda))$. This formula makes sense since for $f \in \mathcal{S}(G)$, the operators $\mathcal{F}f(\lambda)$, $\lambda \in \mathfrak{z}^* \backslash \{0\}$, are trace-class and $\int_{\mathfrak{z}^*\backslash\{0\}} \mathrm{Tr} |\mathcal{F}f(\lambda)| |\lambda|^d \, d\lambda$ is finite.

2.2.4. Fourier transform and finite-dimensional representations. Usually, the Fourier transform of a locally compact group G would be defined on $\widehat{G}$, the set of unitary irreducible representations of G modulo equivalence, via

$$\hat{f}(\pi) = \mathcal{F}f(\pi) = \int_G f(x)\pi(x)^* \, dx,$$

for a representation π of G, and then considering the unitary equivalence we obtain a measurable field of operators $\mathcal{F}f(\pi)$, $\pi \in \widehat{G}$. Here, the Plancherel measure is supported in the subset {class of $\pi^\lambda: \lambda \in \mathfrak{z}^* \setminus \{0\}$} of $\widehat{G}$ (see (2.10)) since it is $c_0|\lambda|^d \, d\lambda$. This allows us to identify $\widehat{G}$ and $\mathfrak{z}^* \setminus \{0\}$ when considering measurable objects up to null sets for the Plancherel measure. However, our semiclassical analysis will lead us to consider objects which are also supported in the other part of $\widehat{G}$. For this reason, we also set for $\omega \in \mathfrak{v}^*$ and $f \in L^1(G)$:

$$\hat{f}(0, \omega) = \mathcal{F}f(0, \omega) := \int_G f(x)(\pi_x^{(0,\omega)})^* \, dx = \int_{\mathfrak{v} \times \mathfrak{z}} f(\mathrm{Exp}(V+Z))e^{-i\omega(V)} \, dV \, dZ.$$

2.2.5. Convolution and Fourier operators. The Fourier transform sends the convolution, whose definition is recalled in (2.9), to composition in the following way:

$$\mathcal{F}(f \star g)(\lambda) = \mathcal{F}g(\lambda) \, \mathcal{F}f(\lambda). \tag{2.14}$$

We recall that a convolution operator T with integrable convolution kernel $\kappa \in L^1(G)$ is defined by $Tf = f * \kappa$ and we have $\mathcal{F}(Tf) = \mathcal{F}\kappa\mathcal{F}f$ by (2.14); hence, T appears as a Fourier multiplier with Fourier symbol $\mathcal{F}\kappa$ acting on the left of $\mathcal{F}f$. Consequently, T is invariant under left-translation and bounded on $L^2(G)$ with operator norm

$$\|T\|_{\mathcal{L}(L^2(G))} \le \sup_{\lambda \in \widehat{G}} \|\mathcal{F}\kappa(\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))}.$$

In other words, T is in the space $\mathcal{L}(L^2(G))^G$ of the left-invariant bounded operators on $L^2(G)$.

2.2.6. The von Neumann algebra of the group. Let us denote by $L^\infty(\widehat{G})$ the space of bounded symbols, that is, here, measurable fields of operators $\sigma = \{\sigma(\lambda): \lambda \in \widehat{G}\}$ which are bounded in the sense that the essential supremum for the Plancherel measure $c_0|\lambda|^d \, d\lambda$

$$\|\sigma\|_{L^\infty(\widehat{G})} := \mathrm{ess} \sup_{\lambda \in \widehat{G}} \|\sigma(\lambda)\|_{\mathcal{L}(\mathcal{H}_\lambda)}$$

is finite. The space $L^\infty(\widehat{G})$ is naturally equipped with a von Neumann algebra structure, and is called the *von Neumann algebra of the group*. As explained above, we already know $L^\infty(\widehat{G}) \supset \mathcal{F}L^1(G)$ by (2.11), but this inclusion is strict.

The full Plancherel theorem [15] implies that the von Neumann algebras $L^\infty(\widehat{G})$ and the space $\mathcal{L}(L^2(G))^G$ of left-invariant bounded operators on $L^2(G)$ introduced above are isomorphic via the mapping $\sigma \mapsto \mathrm{Op}_1(\sigma)$ where $\mathrm{Op}_1(\sigma)$ is the operator with Fourier operator symbol σ, i.e.,

$$\mathcal{F}(\mathrm{Op}_1(\sigma)f) = \sigma \mathcal{F}f, \quad f \in L^2(G).$$

The isomorphism between $L^\infty(\widehat{G})$ and $\mathcal{L}(L^2(G))^G$ allows us to naturally extend the group Fourier transform to distributions $\kappa \in \mathcal{S}'(G)$ such that the convolution operator $f \mapsto f * \kappa$ is bounded on $L^2(G)$ by setting that $\mathcal{F}(\kappa)$ is the symbol of the corresponding operator in $\mathcal{L}(L^2(G))^G$.

2.2.7. Infinitesimal representations and Fourier transforms of left-invariant vector fields.
The group Fourier transform can also be extended to certain classes of distributions whose convolution operators yield left-invariant operators. Indeed, denoting by $\pi(X)$ the infinitesimal representation of π at $X \in \mathfrak{g}$, i.e.

$$\pi(X) = \frac{d}{dt}\pi(\mathrm{Exp}(tX))\Big|_{t=0},$$

we have

$$\mathcal{F}(Xf)(\pi) = \pi(X)\mathcal{F}f(\pi);$$

here (the class of) π is equal to (the class of) π^λ or $\pi^{(0,\omega)}$ identified with λ or ω respectively. For instance, if $Z \in \mathfrak{z}$,

$$\pi^\lambda(Z) = i\lambda(Z). \tag{2.15}$$

We also compute for any $\lambda \in \mathfrak{z}^* \setminus \{0\}$

$$\pi^\lambda(P_j) = \sqrt{|\lambda|}\partial_{\xi_j}, \quad \pi^\lambda(Q_j) = i\sqrt{|\lambda|}\xi_j, \quad \pi^\lambda(\mathcal{Z}^{(\lambda)}) = i|\lambda|, \tag{2.16}$$

and for $\omega \in \mathfrak{v}^\star$ and $j \in \{1, \ldots, p\}$,

$$\pi^{0,\omega}(V_j) = i\omega_j \quad \text{and} \quad \pi^{0,\omega}(\mathcal{Z}^{(\lambda)}) = 0. \tag{2.17}$$

The infinitesimal representation of π extends to the universal enveloping Lie algebra of $\mathfrak{g}$ that we identify with the left invariant differential operators on G. Then for such a differential operator T we have $\mathcal{F}(Tf)(\pi) = \pi(T)\mathcal{F}f(\pi)$ and we may write $\pi(T) = \mathcal{F}(T)$. For instance, if as before $\mathcal{X}^\alpha$ denotes a product of $|\alpha|$ left invariant vector fields forming a basis of $\mathfrak{g}$, then

$$\mathcal{F}(\mathcal{X}^\alpha f)(\pi) = \pi(\mathcal{X})^\alpha \mathcal{F}f(\pi) \quad \text{and} \quad \mathcal{F}(\mathcal{X}^\alpha) = \mathcal{F}(\mathcal{X})^\alpha.$$

Note that $\pi(\mathcal{X})^\alpha$ may be considered as a field of unbounded operators on $\widehat{G}$ defined on the smooth vectors of the representations [21].

2.3. The sublaplacian. The sublaplacian on G is defined by

$$\mathbb{L}_G := \sum_{j=1}^{2d} V_j^2.$$

One checks easily that $\mathbb{L}_G$ is a differential operator which is left invariant and homogeneous of degree 2. In this paper, we shall consider its associated *Schrödinger equation*

$$i\,\partial_t\psi = -\frac{1}{2}\mathbb{L}_G\psi, \quad \psi_{t=0} = \psi_0.$$

The operator $\mathbb{L}_G$ is essentially self-adjoint on $C_c^\infty(G)$ (see [21, Section 4.1.3] or [40, Proof of Lemma 12.1], or the older contributions [27]), so the Schrödinger equation has a unique solution for any data $\psi_0 \in L^2(G)$ by Stone's theorem. We keep the same notation for its unique self-adjoint (unbounded) extension to $L^2(G)$. More precisely, to deal with high-frequencies data, we shall be concerned with the *semi-classical Schrödinger equation*

$$i\varepsilon^\tau\partial_t\psi^\varepsilon = -\frac{\varepsilon^2}{2}\mathbb{L}_G\psi^\varepsilon, \quad (\psi^\varepsilon)_{|t=0} = \psi_0^\varepsilon, \tag{2.18}$$

where $\varepsilon > 0$ is a small parameter taking into account the size of the oscillations of the initial data and $\tau > 0$ is a parameter allowing us to consider large time behaviour, and as the same time the asymptotics $\varepsilon \to 0$.

The definition of $\mathbb{L}_G$ is independent of the chosen orthonormal basis for $\mathfrak{v}$, although it depends on the scalar product that we have fixed at the very beginning on $\mathfrak{v}$. In particular, choosing the basis fixed in Section 2.1 for any $\lambda \in \mathfrak{z}^* \setminus \{0\}$ we have

$$\mathbb{L}_G = \sum_{j=1}^{d}(P_j^2 + Q_j^2). \tag{2.19}$$

The infinitesimal representation (or Fourier transform) of $\mathbb{L}_G$ can be computed thanks to the equalities in (2.16), (2.17), and (2.19): at $\pi^{(0,\omega)}$, $\omega \in \mathfrak{v}^*$, it is the number

$$\mathcal{F}(-\mathbb{L}_G)(0,\omega) = |\omega|^2,$$

and at π^λ, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, it is the operator

$$\mathcal{F}(-\mathbb{L}_G)(\lambda) = H(\lambda), \tag{2.20}$$

where $H(\lambda)$ is defined on $L^2(\mathbb{R}^d)$ through the identification $\mathfrak{p}_\lambda \sim \mathbb{R}^d$, by

$$H(\lambda) = |\lambda|\sum_{1\le j\le d}(-\partial_{\xi_j}^2 + \xi_j^2). \tag{2.21}$$

Up to a constant, this is the quantum harmonic oscillator. We observe that the spectrum $\{|\lambda|(2n + d), n \in \mathbb{N}\}$ of $H(\lambda)$ is discrete and the eigenspaces are finite dimensional. To each eigenvalue $|\lambda|(2n + d)$, we denote by $\Pi_n^{(\lambda)}$ and $\mathcal{V}_n^{(\lambda)}$ the corresponding spectral orthogonal projection and eigenspace. The well-known description of the eigenspaces in terms of Hermite functions is recalled in Appendix B. In particular, for each $n \in \mathbb{N}$, all the eigenspaces $\mathcal{V}_n^{(\lambda)}$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, are isomorphic, and may be denoted by $\mathcal{V}_n$.

2.4. Semi-classical pseudodifferential operators

2.4.1. The space $\mathcal{A}_0$ of semi-classical symbols.
We denote by $\mathcal{A}_0$ the space of symbols $\sigma = \{\sigma(x, \pi) \colon (x, \pi) \in G \times \widehat{G}\}$ of the form

$$\sigma(x, \lambda) = \mathcal{F}\kappa_x(\lambda) = \int\limits_G \kappa_x(z)(\pi_z^\lambda)^* \, dz,$$

where $x \mapsto \kappa_x(\cdot)$ is a smooth and compactly supported function from G to $\mathcal{S}(G)$. Being compactly supported means that $\kappa_x(z) = 0$ for x outside a compact of G and any $z \in G$. The definition of pseudodifferential operators on Lie groups has a long history starting from the pioneer works of [41, 7, 22, 10, 35, 34, 36, 37]. We follow here the recent developments performed in [4, 21] where the reader will also find an overview on the subject.

Remark 2.1. The algebra $\mathcal{A}_0$ is the space of smoothing symbols with compact support in x. We will recall the definition of the space $S^{-\infty}$ of smoothing symbols introduced in [21] at the beginning of Section 3.4 below. Examples of smoothing symbols include the spectrally defined symbols $f(H(\lambda))$ for any $f \in \mathcal{S}(\mathbb{R})$ [21, Chapter 4].

As the Fourier transform is injective, it yields a one-to-one correspondence between the symbol σ and the function κ: we have $\sigma(x, \lambda) = \mathcal{F}\kappa_x(\lambda)$ and conversely the Fourier inversion formula (2.13) yields

$$\kappa_x(z) = c_0 \int\limits_{\widehat{G}} \text{Tr}(\pi_z^\lambda \sigma(x, \lambda))|\lambda|^d \, d\lambda, \quad \text{for all } x, z \in G.$$

The set $\mathcal{A}_0$ is an algebra for the composition of symbols since if $\sigma_1(x, \lambda) = \mathcal{F}\kappa_{1,x}(\lambda)$ and $\sigma_2(x, \lambda) = \mathcal{F}\kappa_{2,x}(\lambda)$ are in $\mathcal{A}_0$, then so is $\sigma_1(x, \lambda)\sigma_2(x, \lambda) = \mathcal{F}(\kappa_{2,x} * \kappa_{1,x})(\lambda)$ by (2.14).

In the case of representations of finite dimension, we distinguish between all the finite-dimensional representations by replacing $\lambda = 0$ with the parameters

$(0, \omega)$, $\omega \in \mathfrak{v}^*$. The operator $\mathcal{F}\kappa_x(0, \omega) = \sigma(x, (0, \omega))$ then reduces to a complex number since $\mathcal{H}_{(0,\mu)} = \mathbb{C}$.

2.4.2. Semi-classical pseudodifferential operators. Given $\varepsilon > 0$, the *semi-classical parameter*, that we use to weigh the oscillations of the functions that we shall consider, we quantify the symbols of $\mathcal{A}_0$ by setting as in [18] (see also [21, 4, 41])

$$\mathrm{Op}_\varepsilon(\sigma)f(x) = c_0 \int_{\widehat{G}} \mathrm{Tr}(\pi_x^\lambda \sigma(x, \varepsilon^2\lambda)\mathcal{F}f(\lambda))|\lambda|^d \, d\lambda, \quad f \in \mathcal{S}(G). \quad (2.22)$$

The kernel of the operator $\mathrm{Op}_\varepsilon(\sigma)$ is the function

$$G \times G \ni (x, y) \longmapsto \kappa_x^\varepsilon(y^{-1}x)$$

where $\kappa_x^\varepsilon(z) = \varepsilon^{-Q}\kappa_x(\delta_{\varepsilon^{-1}}z)$ and κ_x, which is such that $\mathcal{F}(\kappa_x)(\lambda) = \sigma(x, \lambda)$, is called the *convolution kernel* of σ. Note that $\varepsilon^2\lambda$ can be understood as the action on λ of the dilation induced on $\widehat{G}$ by the dilation δ_ε of G (see Remark 3.3 in [18]).

Following [18], the action of the symbols in $\mathcal{A}_0$ on $L^2(G)$ is bounded:

there exists $C > 0$ such that for all $\sigma \in \mathcal{A}_0$ and $\varepsilon > 0$,

$$\| \mathrm{Op}_\varepsilon(\sigma)\|_{\mathcal{L}(L^2(G))} \leq C \int_G \sup_{x \in G} |\kappa_x(z)| \, dz.$$

One also has to mention that there exists a symbolic calculus for these operators (see [17]). In this paper, we will mainly use the description of the commutator between the sub-Laplacian and a semi-classical pseudodifferential operator, which comes from the explicit computation and writes: for all $\sigma \in \mathcal{A}_0$,

$$[-\varepsilon^2\mathbb{L}_G, \mathrm{Op}_\varepsilon(\sigma)] = \mathrm{Op}_\varepsilon([H(\lambda), \sigma]) - 2\varepsilon\,\mathrm{Op}_\varepsilon(\pi^\lambda(V)\cdot V\sigma) - \varepsilon^2\,\mathrm{Op}_\varepsilon(\mathbb{L}_G\sigma), \quad (2.23)$$

where $H(\lambda) = \mathcal{F}(-\mathbb{L}_G)$ has been defined in (2.20) and (2.21). Above, $\pi^\lambda(V)\cdot V\sigma$ denotes the symbol obtained by letting $\sum_{1 \leq j \leq 2d} \pi(V_j)V_j$ act on the symbol σ; more precisely the differential operator V_j acts on $x \mapsto \sigma(x, \lambda)$ while the operator $\pi(V_j)$ acts by multiplication on the left on the symbol; recall that the infinitesimal representation of the group representation π is also denoted by π, see Section 2.2.7.

2.4.3. The subspace $\mathcal{A}_H$ of $\mathcal{A}_0$. The Egorov theorem that we are going to state in the next section is valid for symbols compactly supported with respect to both the Fourier transform and the spectral decomposition of $H(\lambda)$:

Definition 2.2. A symbol $\sigma \in \mathcal{A}_0$ is in $\mathcal{A}_H$ when it vanishes for λ in a neighbourhood of $\{\lambda = 0\}$ and when its kernel and image contain a finite number of $\mathcal{V}_n$, in the sense that

$$\Pi_n^{(\lambda)}\sigma(x, \lambda)\Pi_{n'}^{(\lambda)} = 0, \quad \text{for all } (x, \lambda) \in G \times (\mathfrak{z}^* \setminus \{0\}),$$

for all but a finite number of integers $n, n' \in \mathbb{N}$.

One checks readily that $\mathcal{A}_H$ is a subalgebra of $\mathcal{A}_0$. It is non-trivial since it contains for instance all the symbols of the form $a(x)f(H(\lambda))b(\lambda)$ where $a \in \mathcal{C}_c^\infty(G)$, $f \in \mathcal{C}_c^\infty(\mathbb{R})$ and where $b \in \mathcal{S}(\mathfrak{z}^*)$ vanish in a neighbourhood of 0 (see Remark 2.1); for other symbols in $\mathcal{A}_H$, see Remark 3.13. Although $\mathcal{A}_H$ cannot be dense in $\mathcal{A}_0$ for the Fréchet topology of $\mathcal{A}_0$, we will see in Corollary 3.11 that it satisfies a property of weak density. Besides, symbols $\sigma \in \mathcal{A}_H$ can be decomposed in commuting and non-commuting symbols according to the following definition.

Definition 2.3. With the setting of Definition 2.2, the symbol $\sigma \in \mathcal{A}_H$ is called *H-diagonal* when $\sigma(x, \lambda) = \sum_{n \in \mathbb{N}} \Pi_n^{(\lambda)}\sigma(x, \lambda)\Pi_n^{(\lambda)}$ and *anti-H-diagonal* when $\sigma(x, \lambda) = \sum_{n \neq n'} \Pi_n^{(\lambda)}\sigma(x, \lambda)\Pi_{n'}^{(\lambda)}$. We denote by $\mathcal{A}_H^{(d)}$ the space of *H-diagonal* symbols.

Lemma 2.4. *A symbol $\sigma \in \mathcal{A}_H$ is uniquely decomposed as the sum*

$$\sigma = \sigma^{(d)} + \sigma^{(a)}$$

of an H-diagonal symbol $\sigma^{(d)} \in \mathcal{A}_H$ with an anti-H-diagonal symbol $\sigma^{(a)} \in \mathcal{A}_H$. Furthermore, for each $n, n' \in \mathbb{N}$, the symbol $\Pi_n \sigma \Pi_{n'}$ given by

$$(\Pi_n \sigma \Pi_{n'})(x, \lambda) = \begin{cases} \Pi_n^{(\lambda)}\sigma(x, \lambda)\Pi_{n'}^{(\lambda)} & \text{for } (x, \lambda) \in G \times \mathfrak{z}^* \setminus \{0\}, \\ 0 & \text{for } x \in G \text{ and } \lambda = 0, \end{cases}$$

is in $\mathcal{A}_H$.

Lemma 2.4 will be a consequence of Corollary 3.9, see Remark 3.10.

2.5. The Egorov Theorem on H-type groups. For $s \in \mathbb{R}$, we define the flow Ψ^s on $G \times (\mathfrak{z}^* \setminus \{0\})$ via

$$\begin{aligned} \Psi^s : G \times (\mathfrak{z}^* \setminus \{0\}) &\longrightarrow G \times (\mathfrak{z}^* \setminus \{0\}), \\ (x, \lambda) &\longmapsto (x \operatorname{Exp}(s Z^{(\lambda)}), \lambda). \end{aligned} \tag{2.24}$$

In particular, this map may be composed with symbols with support in $G \times (\mathfrak{z}^* \setminus \{0\})$ such as the symbols in $\mathcal{A}_H$. This and Lemma 2.4 allows us to define the following action on $\mathcal{A}_H^{(d)}$:

$$[t]\Phi^s: \mathcal{A}_H^{(d)} \longrightarrow \mathcal{A}_H^{(d)},$$
$$\sigma = \sum_n \Pi_n \sigma \Pi_n \longmapsto \Phi^s(\sigma) = \sum_n (\Pi_n \sigma \Pi_n) \circ \Psi^{(n+\frac{d}{2})s}, \qquad s \in \mathbb{R}. \quad (2.25)$$

Theorem 2.5. *Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(G)$ and $\psi^\varepsilon(t) = e^{i\varepsilon^{2-\tau} t \mathbb{L}_G} \psi_0^\varepsilon$ be the solution to* (2.18). *Let $\sigma = \sigma^{(d)} + \sigma^{(a)} \in \mathcal{A}_H$ be decomposed into H-diagonal and anti-H-diagonal as in Lemma 2.4. Let $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$.*

(i) *For the anti-H-diagonal part, we have*

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon(\sigma^{(a)})\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} \, dt = O(\varepsilon^{\min(\tau,1)}).$$

(ii) *For the H-diagonal part, we have the following alternative:*

(1) *if $\tau \in (0,2)$,*

$$\int_{\mathbb{R}} \theta'(t)(\mathrm{Op}_\varepsilon(\sigma^{(d)})\psi^\varepsilon(t), \psi^\varepsilon(t)) \, dt = O(\varepsilon^{\min(1,2-\tau)}),$$

(2) *if $\tau = 2$, for all $s \in \mathbb{R}$ (transport)*

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon(\sigma^{(d)})\psi^\varepsilon(t), \psi^\varepsilon(t)) \, dt$$
$$= \int_{\mathbb{R}} \theta(t+s)(\mathrm{Op}_\varepsilon(\Phi^{-s}(\sigma^{(d)}))\psi^\varepsilon(t), \psi^\varepsilon(t)) \, dt + O(\varepsilon),$$

(3) *if $\tau > 2$, for all $s \in \mathbb{R}$ (invariance)*

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon(\sigma^{(d)})\psi^\varepsilon(t), \psi^\varepsilon(t)) \, dt$$
$$= \int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon(\Phi^{-s}(\sigma^{(d)}))\psi^\varepsilon(t), \psi^\varepsilon(t)) \, dt + O(\varepsilon^{\min(1,\tau-2)}).$$

In parts (2) *and* (3), *we use the action Φ^s defined in* (2.25).

It may appear unusual to have an Egorov Theorem holding in the space of distributions in the time variable. However, it is already the case in the Euclidean case when one works with the propagator of a Schrödinger operator with matrix-valued potential $-\frac{\varepsilon^2}{2}\Delta\,\mathrm{Id} + V(x)$ with V matrix-valued (see [25, 19, 20]). The proof of this Theorem is postponed until Section 4.

2.6. Time averaged semi-classical measures and the quantum limits. We now want to pass to the limit $\varepsilon \to 0$ in expressions of the form (1.5) and to identify the limiting objects, together with their properties. For this purpose, we follow [18, Section 5] with slightly different notation and introduce vocabulary for operator valued measures. We recall that if Z is a locally compact Hausdorff set, we denote by $\mathcal{M}(Z)$ the set of finite Radon measures on Z and by $\mathcal{M}^+(Z)$ the subset of its positive elements.

Definition 2.6. Let Z be a complete separable metric space, and let $\xi \mapsto \mathcal{H}_\xi$ a measurable field of complex Hilbert spaces of Z.

- The set $\widetilde{\mathcal{M}}_{\mathrm{ov}}(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ is the set of pairs (γ, Γ) where γ is a positive Radon measure on Z and $\Gamma = \{\Gamma(\xi) \in \mathcal{L}(\mathcal{H}_\xi) : \xi \in Z\}$ is a measurable field of trace-class operators such that

$$\|\Gamma d\gamma\|_{\mathcal{M}} := \int_Z \mathrm{Tr}_{\mathcal{H}_\xi} |\Gamma(\xi)| d\gamma(\xi) < \infty. \tag{2.26}$$

- Two pairs (γ, Γ) and (γ', Γ') in $\widetilde{\mathcal{M}}_{\mathrm{ov}}(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ are equivalent when there exists a measurable function $f : Z \to \mathbb{C} \setminus \{0\}$ such that

$$d\gamma'(\xi) = f(\xi) d\gamma(\xi) \quad \text{and} \quad \Gamma'(\xi) = \frac{1}{f(\xi)} \Gamma(\xi)$$

for γ-almost every $\xi \in Z$. The equivalence class of (γ, Γ) is denoted by $\Gamma d\gamma$, and the resulting quotient set is denoted by $\mathcal{M}_{\mathrm{ov}}(Z, (\mathcal{H}_\xi)_{\xi \in Z})$. The norm $\|\cdot\|_{\mathcal{M}}$ extends as a norm on $\mathcal{M}_{\mathrm{ov}}(Z, (\mathcal{H}_\xi)_{\xi \in Z})$.

- A pair (γ, Γ) in $\widetilde{\mathcal{M}}_{\mathrm{ov}}(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ is positive when $\Gamma(\xi) \geq 0$ for γ-almost all $\xi \in Z$. In this case, we may write $(\gamma, \Gamma) \in \widetilde{\mathcal{M}}_{\mathrm{ov}}^+(Z, (\mathcal{H}_\xi)_{\xi \in Z})$, and $\Gamma d\gamma \geq 0$ for $\Gamma d\gamma \in \mathcal{M}_{\mathrm{ov}}^+(Z, (\mathcal{H}_\xi)_{\xi \in Z})$.

By convention and if not otherwise specified, a representative of the class $\Gamma d\gamma$ is chosen such that $\mathrm{Tr}_{\mathcal{H}_\xi} \Gamma = 1$. In particular, if $\mathcal{H}_\xi$ is 1-dimensional, $\Gamma = 1$ and $\Gamma d\gamma$ reduces to the measure $d\gamma$. One checks readily that $\mathcal{M}_{\mathrm{ov}}(Z, (\mathcal{H}_\xi)_{\xi \in Z})$ equipped with the norm $\|\cdot\|_{\mathcal{M}}$ is a Banach space.

When the field of Hilbert spaces is clear from the setting, we may write

$$\mathcal{M}_{\mathrm{ov}}(Z) = \mathcal{M}(Z, (\mathcal{H}_\xi)_{\xi \in Z}), \quad \text{and} \quad \mathcal{M}_{\mathrm{ov}}^+(Z) = \mathcal{M}^+(Z, (\mathcal{H}_\xi)_{\xi \in Z}),$$

for short. For instance, if $\xi \mapsto \mathcal{H}_\xi$ is given by $\mathcal{H}_\xi = \mathbb{C}$ for all ξ, then $\mathcal{M}(Z)$ coincides with the space of finite Radon measures on Z. Another example is when

Z is of the form $Z = Z_1 \times \widehat{G}$ where Z_1 is a complete separable metric space, and $\mathcal{H}_{(z_1,\lambda)} = \mathcal{H}_\lambda$, where the Hilbert space $\mathcal{H}_\lambda$ is associated with the representation of $\lambda \in \widehat{G}$ (that is, using the description in (2.10), $\mathcal{H}_\lambda$ is equivalent to $L^2(\mathfrak{p}_\lambda)$ if the representation corresponds to $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and $\mathcal{H}_{(0,\omega)} = \mathbb{C}$ if $\lambda = 0$ and the representation corresponds to $(0, \omega)$ with $\omega \in \mathfrak{v}^*$).

We will consider measurable bounded maps of the time variable, valued in the space of measures that are positive as scalar-valued or operator-valued measures:

Definition 2.7. If X denotes the Banach space $\mathcal{M}(Z)$ or more generally $\mathcal{M}_{\mathrm{ov}}(Z)$ as in Definition 2.6, then $L^\infty(\mathbb{R}, X^+)$ denotes the space of maps of $t \in \mathbb{R}$ and valued in X in $L^\infty(\mathbb{R}, X)$ with positive values for almost every $t \in \mathbb{R}$.

2.6.1. Time-averaged semi-classical measures. With a bounded family $(u^\varepsilon)_{\varepsilon>0}$ in $L^\infty(\mathbb{R}, L^2(G))$, we associate the quantities

$$\ell_\varepsilon(\theta, \sigma) = \int_\mathbb{R} \theta(t)(\mathrm{Op}_\varepsilon(\sigma)u^\varepsilon(t), u^\varepsilon(t))_{L^2(G)}\, dt, \quad \sigma \in \mathcal{A}_0, \theta \in L^1(\mathbb{R}), \quad (2.27)$$

the limits of which are characterized by a map in $L^\infty(\mathbb{R}, \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G}))$ as explained in the following statement (whose proof is given in Section 3.2):

Theorem 2.8. *Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(G))$. There exist a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in $(0, +\infty)$ with $\varepsilon_k \xrightarrow[k \to +\infty]{} 0$ and a map $t \mapsto \Gamma_t d\gamma_t$ in $L^\infty(\mathbb{R}, \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G}))$ such that we have for all $\theta \in L^1(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$,*

$$\int_\mathbb{R} \theta(t)(\mathrm{Op}_{\varepsilon_k}(\sigma)u^{\varepsilon_k}(t), u^{\varepsilon_k}(t))_{L^2(G)}\, dt$$
$$\xrightarrow[k \to +\infty]{} \int_{\mathbb{R} \times G \times \widehat{G}} \theta(t)\, \mathrm{Tr}(\sigma(x, \lambda)\Gamma_t(x, \lambda))d\gamma_t(x, \lambda)\, dt.$$

Given the sequence $(\varepsilon_k)_{k \in \mathbb{N}}$, the map $t \mapsto \Gamma_t d\gamma_t$ is unique up to equivalence. Besides, we have for almost all $t \in \mathbb{R}$

$$\|\Gamma_t d\gamma_t\|_{\mathcal{M}} = \int_{G \times \widehat{G}} \mathrm{Tr}(\Gamma_t(x, \lambda))\, d\gamma_t(x, \lambda) \le \limsup_{\varepsilon \to 0} \|u^\varepsilon\|_{L^\infty(\mathbb{R}, L^2(G))}.$$

We call the map $t \mapsto \Gamma_t d\gamma_t$ satisfying Theorem 2.8 (for some subsequence ε_k) a *time-averaged semi-classical measure* of the family $(u^\varepsilon(t))$. Note that we have not assumed any estimate of the form (1.2) on the family u^ε in order to define its

time averaged semi-classical measure; such additional property will however be useful to determine the limits of the time-averaged densities associated with u^ε as marginals of time-averaged semi-classical measures, as we shall see in Section 5.

Remark 2.9. (1) Note that this result can be generalised to any graded Lie group: for any bounded family $(u^\varepsilon)_{\varepsilon>0}$ in $L^\infty(\mathbb{R}, L^2(G))$, there exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ in $(0,+\infty)$ with $\varepsilon_k \xrightarrow[k\to+\infty]{} 0$ and a map $t \mapsto \Gamma_t d\gamma_t$ in $L^\infty(\mathbb{R}, \mathcal{M}_{ov}^+(G \times \widehat{G}))$ such that we have

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_{\varepsilon_k}(\sigma)u^{\varepsilon_k}(t), u^{\varepsilon_k}(t))_{L^2(G)}\, dt$$

$$\xrightarrow[k\to+\infty]{} \int_{\mathbb{R}\times G\times\widehat{G}} \theta(t)\, \mathrm{Tr}(\sigma(x,\pi)\Gamma_t(x,\pi))\, d\gamma_t(x,\pi)dt,$$

for every $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$.

(2) In the case of this article where G is H-type, the special structure of $\widehat{G}$ implies that $\Gamma_t d\gamma_t$ consists of two pieces, one localized above $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and another one which is scalar above $\mathfrak{v}^*$, see (2.10).

2.6.2. Semi-classical measures and Schrödinger equation. Our main theorem regarding semi-classical measures of solutions to the Schrödinger equation is the following:

Theorem 2.10. *Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(G)$ and*

$$\psi^\varepsilon(t) = e^{i\varepsilon^{2-\tau}t\mathbb{L}_G}\, \psi_0^\varepsilon$$

be the solution to (2.18). Then any semi-classical measure $t \mapsto \Gamma_t d\gamma_t \in L^\infty(\mathbb{R}, \mathcal{M}_{ov}^+(G \times \widehat{G}))$ as in Theorem 2.8 for the family $u^\varepsilon(t) = \psi^\varepsilon(t)$, satisfies the following additional properties.

(i) *For almost every $(t,x,\lambda) \in \mathbb{R}\times G\times\widehat{G}$, the operator $\Gamma_t(x,\lambda)$ commutes with $\widehat{\mathbb{L}}_G(\lambda) = H(\lambda)$:*

$$\Gamma_t(x,\lambda) = \sum_{n\in\mathbb{N}} \Gamma_{n,t}(x,\lambda) \quad \text{with } \Gamma_{n,t}(x,\lambda) := \Pi_n^{(\lambda)}\Gamma_t(x,\lambda)\Pi_n^{(\lambda)}, \quad (2.28)$$

where Π_n is the homogeneous symbol given by the spectral projection of $H(\lambda)$ for the eigenvalue $|\lambda|(2n+d)$ (see Section 2.3).

(ii) *For each $n \in \mathbb{N}$, the map $(t,x,\lambda) \mapsto \Gamma_{n,t}(x,\lambda)d\gamma_t(x,\lambda)$ defines a distribution on $\mathbb{R}\times G\times(\mathfrak{z}^* \setminus \{0\})$ valued in the finite dimensional space $\mathcal{L}(\mathcal{V}_n)$ which satisfies the following alternatives*:

(1) *if $\tau \in (0, 2)$,*

$$\partial_t (\Gamma_{n,t}(x, \lambda) d\gamma_t(x, \lambda)) = 0,$$

(2) *if $\tau = 2$,*

$$\left(\partial_t - \left(n + \frac{d}{2}\right) \mathcal{Z}^{(\lambda)}\right)(\Gamma_{n,t}(x, \lambda) d\gamma_t(x, \lambda)) = 0 \tag{2.29}$$

where $\mathcal{Z}^{(\lambda)} \in \mathfrak{z}$ is the vector corresponding to $|\lambda|^{-1}\lambda$ (see Section 2.1.1),

(3) *if $\tau > 2$, then the distribution $\Gamma_{n,t} d\gamma_t$ is invariant under the flow on $G \times (\mathfrak{z}^* \setminus \{0\})$*

$$\Psi^s : (x, \lambda) \longmapsto (x \operatorname{Exp}(s\mathcal{Z}^{(\lambda)}), \lambda),$$

(see (2.24)), and thus is equal to 0.

(iii) *Above $\lambda = 0$, the map $t \mapsto \varsigma_t$ in $L^\infty(\mathbb{R}, \mathcal{M}^+(G \times \mathfrak{v}^*))$ defined via*

$$d\varsigma_t(x, \omega) = \Gamma_t(x, (0, \omega)) d\gamma_t(x, (0, \omega)) \mathbf{1}_{\lambda=0}$$

satisfies the following alternatives:

(1) *if $\tau \in (0, 1)$, the map $t \mapsto \varsigma_t$ is constant from $\mathbb{R}$ to $\mathcal{M}^+(G \times \mathfrak{v}^*)$,*

(2) *if $\tau = 1$, then the map $t \mapsto \varsigma_t$ is weakly continuous from $\mathbb{R}$ to $\mathcal{M}^+(G \times \mathfrak{v}^*)$, and for all $t \in \mathbb{R}$,*

$$\partial_t \varsigma_t(x, \omega) - \omega \cdot V \varsigma_t(x, \omega) = 0 \tag{2.30}$$

where $\omega \cdot V = \sum_{j=1}^{d} \omega_j V_j \in \mathfrak{g}$,

(3) *if $\tau > 1$, the measures ς_t are invariant under the flow on $G \times \mathfrak{v}^*$*

$$\Xi^s : (x, \omega) \longmapsto (x \operatorname{Exp}(s\omega \cdot V), \omega), \tag{2.31}$$

and thus supported on $G \times \{\omega = 0\}$.

The existence of the semi-classical measure $\Gamma_t d\gamma_t$ follows from Theorem 2.8, while its additional properties come from the fact that $\psi^\varepsilon(t)$ solves the Schrödinger equation. Point (i) of Theorem 2.8 is a consequence of (i) of Theorem 2.5. It will then appear that we will need to only use symbols which commute with $H(\lambda)$.

Before closing this section, we discuss why the invariance of a semi-classical measure by vector fields imply that it is 0 (as in (3) of (ii)) or that its support has special properties (as in (3) of (iii)).

Proof of part (3) *in* (ii) *and part* (3) *in* (iii). The invariance properties have consequences that have been already studied in the Euclidean case in [9, Lemma 3.6]. We adapt them to the setting of (3) of (ii) in the following way. First let $N_{\widehat{G}} : \widehat{G} \to [0, +\infty)$ be defined via

$$N_{\widehat{G}}(\lambda) := \begin{cases} \sqrt{|\lambda|} & \text{if } \lambda \neq 0, \\ |\omega| & \text{if } \lambda = (0, \omega), \omega \in \mathfrak{v}^*, \end{cases}$$

by [18, Section 2.3], it is continuous. We also define the usual quasi-norm on G via

$$|\operatorname{Exp}(V + Z)| = (|X|^4 + |Z|^2)^{1/4}, \quad \text{with } V \in \mathfrak{v} \text{ and } Z \in \mathfrak{z}.$$

We can now define the continuous function $N : G \times \widehat{G} \to [0, +\infty)$ with

$$N(x, \lambda) = (|x|^4 + N_{\widehat{G}}(\lambda)^4)^{1/4}.$$

Let $x = \operatorname{Exp}(V + Z) \in G$ with $V \in \mathfrak{v}$ and $z \in \mathfrak{z}$. In view of

$$x \operatorname{Exp}(s \mathcal{Z}^{(\lambda)}) = \operatorname{Exp}(V + Z + s \mathcal{Z}^{(\lambda)})$$

we deduce

$$|x \operatorname{Exp}(s \mathcal{Z}^{(\lambda)})|^4 = |V|^4 + |Z + s \mathcal{Z}^{(\lambda)}|^2 + N_{\widehat{G}}(\lambda)^4.$$

As a consequence, if K is any compact subset of $G \times (\mathfrak{z}^* \setminus \{0\})$, then there exist constants $\alpha_1, \beta_1, \alpha_2, \beta_2, s_0 > 0$ such that

$$\alpha_1 |s|^{1/2} - \beta_1 < |x \operatorname{Exp}(s \mathcal{Z}^{(\lambda)})| < \alpha_2 |s|^{1/2} + \beta_2, \quad \text{for all } s \geq s_0 \text{ and } (x, \lambda) \in K,$$

which is enough for the proof of Lemma 3.6 in [9]. The measure $\operatorname{Tr}(\Gamma_t) d\gamma_t$ which is invariant under the flow Ψ^s is 0 above K.

A similar argument can be performed for (3) of (iii) since the only invariant set by the action of Ξ^s is the set $G \times \{0\}$ and if K is a compact subset of $G \times \mathfrak{v}^*$ such that $K \cap (G \times \{0\}) = \emptyset$, then there exists $\alpha_1, \beta_1, \alpha_2, \beta_2, s_0 > 0$ such that for $s \geq_0$,

$$\alpha_2 |s| - \beta_2 < |x \operatorname{Exp}(s \omega \cdot V)| < \alpha_2 |s| + \beta_2.$$

Therefore, the measure $\varsigma_t(x, \omega)$ is supported on $G \times \{\omega = 0\}$. $\square$

3. The C^*-algebra $\mathcal{A}$ associated with semi-classical symbols

In this section we introduce the C^*-algebra formalism which can be associated with semi-classical symbols. The properties of the algebra introduced in Section 3.1 are at the roots of our analysis and allow us to prove Theorem 2.8 in

Section [3.2]. The proofs of Theorems [2.5] and [2.10] will use several other ingredients. First, the proof of Egorov theorem [2.5] requires the analysis of the symbolic properties of the eigenprojectors $\Pi_n^{(\lambda)}$ performed in Section [3.4]. Then, in order to pass to the limits in the relations of the Egorov theorem [2.5], we will need to approximate general symbols in $\mathcal{A}_0$ by symbols belonging to the class $\mathcal{A}_H$, which is done in Section [3.5]. Finally, it will use more general symbols that commute with H, the space of which is studied in Section [3.6].

3.1. The C^*-algebra $\mathcal{A}$ and its states.

We introduce the algebra $\mathcal{A}$ which is the closure of $\mathcal{A}_0$ for the norm $\|\cdot\|_{\mathcal{A}}$ given by

$$\|\sigma\|_{\mathcal{A}} := \sup_{(x,\lambda) \in G \times \widehat{G}} \|\sigma(x,\lambda)\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))}. \tag{3.1}$$

Clearly, $\mathcal{A}$ is a sub-C^*-algebra of the tensor product of the commutative C^*-algebra $C_0(G)$ of continuous functions on G vanishing at infinity together with $L^\infty(\widehat{G})$. It turns out that one can identify its spectrum in the following way:

Proposition 3.1. *The set $\mathcal{A}$ is a separable C^*-algebra of type 1. It is not unital but admits an approximation of identity. Besides, if $\pi_0 \in \widehat{G}$ and $x_0 \in G$, then the mapping*

$$\mathcal{A}_0 \longrightarrow \mathcal{L}(\mathcal{H}_{\lambda_0}),$$
$$\sigma \longmapsto \sigma(x_0, \pi_0),$$

extends to a continuous mapping $\rho_{x_0,\pi_0} \colon \mathcal{A} \to \mathcal{L}(\mathcal{H}_{\pi_0})$ which is an irreducible non-zero representation of $\mathcal{A}$. Furthermore, the mapping

$$G \times \widehat{G} \longrightarrow \widehat{\mathcal{A}},$$
$$(x_0, \pi_0) \longmapsto \rho_{x_0,\pi_0},$$

is a homeomorphism which allows for the identification of $\widehat{\mathcal{A}}$ with $G \times \widehat{G}$.

The proof follows the lines of [18, Section 5]. It utilises the fact that, by definition, the C^* algebra $C^*(G)$ of the group G is the closure of $\mathcal{FS}(G)$ for $\sup_{\lambda \in \widehat{G}} \|\cdot\|_{\mathcal{L}(\mathcal{H}_\lambda)}$ and that the spectrum of $C^*(G)$ is $\widehat{G}$. This implies readily that $\mathcal{A}$ may be identified with the C^*-algebra of continuous functions which vanish at infinity on G and are valued on $C^*(G)$ and that its spectrum is as described in Proposition [3.1]. Furthermore, the algebraic span of the symbols of the form $\tau(x,\lambda) = a(x)b(\lambda)$ with $a(x)$ in $C_c^\infty(G)$ and the Fourier multiplier $b(\lambda)$ in $S^{-\infty}$ is dense in $\mathcal{A}$. Notice that their boundedness is easier to obtain since $\mathrm{Op}_\varepsilon(\tau)$ simply

is the composition of the operator of multiplication by $a(x)$ and of the Fourier multiplier $b(\varepsilon^2 \lambda)$, and one has

$$\| \mathrm{Op}_\varepsilon(\tau) \|_{\mathcal{L}(L^2(G))} \leq \sup_{x \in G, \lambda \in \widehat{G}} \| \tau \|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))}. \tag{3.2}$$

We can also describe the states of the C^*-algebra $\mathcal{A}$.

Proposition 3.2. *If ℓ is a state of the C^*-algebra $\mathcal{A}$, then there exists a pair (γ, Γ) unique up to its equivalence class in $\mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G})$ which satisfies*

$$\| \Gamma d\gamma \|_{\mathcal{M}} = \int_{G \times \widehat{G}} \mathrm{Tr}(\Gamma(x, \lambda)) \, d\gamma(x, \lambda) = 1, \tag{3.3}$$

and

$$\ell(\sigma) = \int_{G \times \widehat{G}} \mathrm{Tr}(\sigma(x, \lambda) \Gamma(x, \lambda)) \, d\gamma(x, \lambda), \quad \text{for all } \sigma \in \mathcal{A}. \tag{3.4}$$

Conversely, if $\Gamma \, d\gamma \in \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G})$ satisfies (3.3), then the linear form ℓ defined via (3.4) is a state of $\mathcal{A}$.

Proof. This proposition is a corollary of Proposition 4.1 in [17]. Its proof follows the lines of [18, Section 5] and is performed in details in the appendix of [17]. $\quad \square$

Having in mind that a state is a positive linear form of norm 1, the description of the states of the C^*-algebra $\mathcal{A}$ yields the following corollary:

Corollary 3.3. *The topological dual $\mathcal{A}^*$ of $\mathcal{A}$ may be identified as a Banach space with $\mathcal{M}_{\mathrm{ov}}(G \times \widehat{G})$ in the following way:*

- *if $\ell : \mathcal{A} \to \mathbb{C}$ is a continuous linear form, then there exists a unique element $\Gamma d\gamma \in \mathcal{M}_{\mathrm{ov}}(G \times \widehat{G})$ satisfying (3.4);*
- *conversely, (3.4) defines a continuous linear form ℓ on $\mathcal{A}$.*

Moreover, we have the following properties:

(1) $\| \ell \|_{\mathcal{A}^*} = \| \Gamma d\gamma \|_{\mathcal{M}}$,

(2) $\Gamma d\gamma \geq 0$ *if and only if* $\ell(\sigma^* \sigma) \geq 0$ *for every element σ in the C^*-algebra $\mathcal{A}$,*

(3) ℓ *is a state if and only if* $\Gamma d\gamma \geq 0$ *and (3.3) holds.*

3.2. Time dependent states and the proof of Theorem 2.8. With this precise description of the topological dual $\mathcal{A}^*$, one can now consider functionals defined on $\mathcal{C}_c^\infty(\mathbb{R}) \times \mathcal{A}_0$.

Proposition 3.4. *Let $\ell: \mathcal{C}_c^\infty(\mathbb{R}) \times \mathcal{A}_0 \to \mathbb{C}$ be a non-zero bilinear map satisfying*

$$\ell(\theta, \sigma) \leq \|\sigma\|_{\mathcal{A}} \|\theta\|_{L^1(\mathbb{R})}, \quad \text{for all } \sigma \in \mathcal{A}_0 \text{ and } \theta \in \mathcal{C}_c^\infty(\mathbb{R}), \qquad (3.5)$$

and

$$\ell(|\theta|^2, \sigma^*\sigma) \geq 0, \quad \text{for all } \sigma \in \mathcal{A}_0 \text{ and } \theta \in \mathcal{C}_c^\infty(\mathbb{R}). \qquad (3.6)$$

Then ℓ extends uniquely to a continuous bilinear map $\ell: L^1(\mathbb{R}) \times \mathcal{A} \to \mathbb{C}$ for which we keep the same notation. Furthermore, there exists a unique map $t \mapsto \Gamma_t d\gamma_t$ in $L^\infty(\mathbb{R}, \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G}))$ satisfying $\|\Gamma_t d\gamma_t\|_{\mathcal{M}} \leq 1$ for almost all $t \in \mathbb{R}$, and

$$\ell(\theta, \sigma) = \int_{\mathbb{R}} \theta(t) \int_{G \times \widehat{G}} \mathrm{Tr}(\sigma(x, \lambda) \Gamma_t(x, \lambda)) \, d\gamma_t(x, \lambda) \, dt$$

for all $\sigma \in \mathcal{A}$ and $\theta \in L^1(\mathbb{R})$.

Proof. The estimate in (3.5) implies that the bilinear map ℓ extends uniquely into a continuous bilinear map on $L^1(\mathbb{R}) \times \mathcal{A}$ for which we keep the same notation. Furthermore, for each $\sigma \in \mathcal{A}$, we identify the continuous linear map $\theta \mapsto \ell(\theta, \sigma)$ on the Banach space $L^1(\mathbb{R})$ with the function $\ell_\sigma \in L^\infty(\mathbb{R})$ given via

$$\ell(\theta, \sigma) = \int_{\mathbb{R}} \theta(t) \ell_\sigma(t) \, dt, \quad \text{for all } \theta \in L^1(\mathbb{R}).$$

Note that $\|\ell_\sigma\|_{L^\infty(\mathbb{R})} \leq \|\sigma\|_{\mathcal{A}}$ and that the map $\sigma \mapsto \ell_\sigma$ is a linear mapping on $\mathcal{A}$ to $L^\infty(\mathbb{R})$. Hence, we can view the map $L: t \mapsto (\sigma \mapsto \ell_\sigma(t))$ as a measurable bounded map from $\mathbb{R}$ to the Banach space $\mathcal{A}^*$, $\|L\|_{L^\infty(\mathbb{R}, \mathcal{A}^*)} \leq 1$. Moreover, the assumption in (3.6) implies that

$$\int_{\mathbb{R}} |\theta(t)|^2 \ell_{\sigma^*\sigma}(t) \, dt = \ell(|\theta|^2, \sigma^*\sigma) \geq 0, \quad \text{for all } \theta \in \mathcal{C}_c^\infty(\mathbb{R}),$$

hence $\ell_{\sigma^*\sigma}(t) \geq 0$ for almost every $t \in \mathbb{R}$. In other words, $\sigma \mapsto \ell_\sigma(t)$ is a positive linear form of the C^*-algebra $\mathcal{A}$ for almost every $t \in \mathbb{R}$. Hence, the map $L \in L^\infty(\mathbb{R}, \mathcal{A}^*)$ is valued in the set of positive linear forms on $\mathcal{A}$ with $\|L\|_{L^\infty(\mathbb{R}, \mathcal{A}^*)} \leq 1$. Corollary 3.3 with the identification of $\mathcal{A}^*$ with $\mathcal{M}_{\mathrm{ov}}(G \times \widehat{G})$ allows us to conclude. $\qquad \square$

With these concepts in mind, one can now sketch the proof of Theorem 2.8 as an adaptation of the similar arguments in [18, 17].

Sketch of proof of Theorem 2.8. If $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{R},L^2(G))} = 0$, then the map given by $\Gamma_t d\gamma_t = 0$ for all $t \in \mathbb{R}$ answers our problem. Hence, by dividing u^ε by $\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{R},L^2(G))}$ if necessary, we can assume that

$$\limsup_{\varepsilon\to 0}\|u^\varepsilon\|_{L^\infty(\mathbb{R},L^2(G))} = 1.$$

We then consider the quantities $\ell_\varepsilon(\theta,\sigma)$ defined in (2.27) and we observe the three following facts.

(1) For any $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$, the family $\ell_\varepsilon(\theta,\sigma)$ is bounded and there exists a subsequence $(\varepsilon_k(\sigma))_{k\in\mathbb{N}}$ such that $\ell_{\varepsilon_k(\sigma)}(\theta,\sigma)$ has a limit $\ell(\theta,\sigma)$.

(2) Using the separability of $\mathcal{C}_c^\infty(\mathbb{R})\times\mathcal{A}_0$ and a diagonal extraction, one can find a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that for all $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\sigma \in \mathcal{A}_0$, the sequence $(\ell_{\varepsilon_k}(\theta,\sigma))_{k\in\mathbb{N}}$ has a limit $\ell(\theta,\sigma)$ and the sequence $(\|u^{\varepsilon_k}\|_{L^\infty(\mathbb{R},L^2(G))})_{k\in\mathbb{N}}$ converges to 1.

(3) The map $(\theta,\sigma) \mapsto \ell(\theta,\sigma)$ constructed at point (2) satisfies (3.6). It also satisfies (3.5) for symbols of the form $\tau(x,\lambda) = a(x)b(\lambda)$, see (3.2), therefore for all $\sigma \in \mathcal{A}$ (since the algebraic span of such τ is dense in $\mathcal{A}$, see Section 3).

We conclude with Proposition 3.4. $\square$

3.3. Some comments on time-dependent states.

The result in Proposition 3.4 calls for some comments which will not be used in the following paper but are of interest in themselves. Indeed, with a straightforward adaptation of the arguments given for the proof of Proposition 3.1, we obtain an analogue description for the closure of $C^\infty(J)\otimes\mathcal{A}_0$ if J is a compact interval, and of $C_c^\infty(\mathbb{R})\otimes\mathcal{A}_0$ if $J = \mathbb{R}$. Note first that in both cases, the closure is for the norm

$$\|\tau\|_{C_0(J,\mathcal{A})} := \inf\left\{\sum_j \|\theta_j\|_{L^\infty(J)}\|\sigma_j\|_{\mathcal{A}} : \tau = \sum_j \theta_j\sigma_j\right\},$$

and we identify them respectively with the C^*-algebra $C(J,\mathcal{A})$ of continuous functions on J valued in $\mathcal{A}$ when J is a compact interval and with the Banach space $C_0(\mathbb{R},\mathcal{A})$ of continuous functions on $\mathbb{R}$ valued in $\mathcal{A}$ and vanishing at infinity when $J = \mathbb{R}$. In order to unify the presentation, we may write $C_0(J,\mathcal{A})$ for $C(J,\mathcal{A})$ when J is a compact interval of $\mathbb{R}$.

Then, an analysis similar to the one of the proof of Proposition 3.1 gives that the C^*-algebra $C_0(J,\mathcal{A})$ for J compact interval and for $J = \mathbb{R}$ is a separable

C^*-algebra of type 1. It is not unital but admits an approximation of identity. Its spectrum may be identified with $J \times G \times \widehat{G}$. Its states may be identified with the elements $\Gamma \, d\gamma$ in $\mathcal{M}_{\mathrm{ov}}^+(J \times G \times \widehat{G})$ satisfying

$$\int\limits_{J \times G \times \widehat{G}} \mathrm{Tr}(\Gamma(t, x, \lambda)) \, d\gamma(t, x, \lambda) = 1,$$

via

$$\ell(\sigma) = \int\limits_{J \times G \times \widehat{G}} \mathrm{Tr}(\sigma(t, x, \lambda)\Gamma(t, x, \lambda)) \, d\gamma(t, x, \lambda) \quad \text{for all } \sigma \in C_0(J, \mathcal{A}).$$

Finally, a map $\ell \colon \mathcal{C}_c^\infty(\mathbb{R}) \times \mathcal{A}_0 \to \mathbb{C}$ as in Proposition 3.4. extends uniquely into a continuous linear map on $L^1(\mathbb{R}) \times \mathcal{A}$, and also into a state of $C(J, \mathcal{A})$ up to the normalisation $|J|$ for any compact interval. Using the characterisation above and the uniqueness, we can define a pair $(\gamma, \Gamma) \in \widetilde{\mathcal{M}}_{\mathrm{ov}}^+(\mathbb{R} \times G \times \widehat{G})$, unique up to equivalence, such that

$$\ell(\theta, \sigma) = \int\limits_{\mathbb{R} \times G \times \widehat{G}} \theta(t) \, \mathrm{Tr}(\sigma(x, \lambda)\Gamma(t, x, \lambda)) \, d\gamma(t, x, \lambda)$$

for all $\sigma \in \mathcal{A}_0$ and $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$; here γ is a measure on $\mathbb{R} \times G \times \widehat{G}$. This is a weaker result than the one obtained in Proposition 3.4, which states that the measure above is absolutely continuous with respect to dt, and this explains why we proceed in this manner.

Note that we can proceed as in Proposition 3.2 and Corollary 3.3 to obtain a description of the dual of $C_0(J, \mathcal{A})$ that we will use later on:

Corollary 3.5. *The topological dual $C_0(J, \mathcal{A})^*$ of $C_0(J, \mathcal{A})$ may be identified as a Banach space with $\mathcal{M}_{\mathrm{ov}}(J \times G \times \widehat{G})$ in the following way*:

- *If $\ell \colon C_0(J, \mathcal{A}) \to \mathbb{C}$ is a continuous linear form, then there exists a unique element $\Gamma d\gamma \in \mathcal{M}_{\mathrm{ov}}(J \times G \times \widehat{G})$ satisfying* (3.4).
- *Conversely,* (3.4) *defines a continuous linear form ℓ on $C_0(J, \mathcal{A})$.*

Moreover, we have the following properties:

(1) $\|\ell\|_{C_0(J, \mathcal{A})^*} = \|\Gamma d\gamma\|_{\mathcal{M}}.$

(2) $\Gamma d\gamma \geq 0$ *if and only if $\ell(\sigma) \geq 0$ for all any positive element σ in the C^* algebra $C_0(J, \mathcal{A})$.*

3.4. Symbolic properties of the eigenprojectors. In this section, we analyse the fields of the spectral projectors $\Pi_n^{(\lambda)}$ of $H(\lambda)$. We use the notion of homogeneous symbols introduced in [18] (see Definition 4.1 therein), which express in terms of difference operators, the definition of which we shortly recall here (from [21]). Assuming that a basis $X_1, \ldots, X_{2d+p}$ of $\mathfrak{g}$ has been fixed, the difference operators Δ^α are related with the coordinate functions $\phi_1, \ldots, \phi_{2d+p}$ associated with the basis $X_1, \ldots, X_{2d+p}$, i.e. $x = \mathrm{Exp}(\phi_1(x)X_1 + \cdots + \phi_{2d+p}(x)X_{2d+p})$. Denote by $\Delta_{\phi_1}, \ldots, \Delta_{\phi_{2d+p}}$ the corresponding difference operators,

$$(\Delta_{\phi_j}\sigma)(x,\lambda) = \mathcal{F}(\phi_j \kappa_x) \quad \text{if } \sigma(x,\lambda) = \mathcal{F}\kappa_x(\lambda),$$

then Δ^α is the product of $|\alpha|$ of such difference operators. Note that two difference operators Δ_{ϕ_1} and Δ_{ϕ_2} of that form will commute since $\Delta_{\phi_1}\Delta_{\phi_2} = \Delta_{\phi_1\phi_2}$.

Following [21] (Section 5.2 for any graded nilpotent Lie group and Section 6.5 for the Heisenberg group), the class S^m of *symbols of order m* in G consists of fields of operators $\sigma(x,\lambda)$ such that for each $\alpha, \beta \in \mathbb{N}^n$ and $\gamma \in \mathbb{R}$ we have

$$\sup_{x\in G, \lambda\in\widehat{G}} \|(\mathrm{Id}+H(\lambda))^{\frac{[\alpha]-m+\gamma}{2}} X_x^\beta \Delta^\alpha \sigma(x,\lambda)(\mathrm{Id}+H(\lambda))^{-\frac{\gamma}{2}}\|_{\mathcal{L}(\mathcal{H}_\lambda)} < \infty. \quad (3.7)$$

Using the dilation induced on $\widehat{G}$ by the one of G, one defines for $m \in \mathbb{R}$, m-homogeneous fields of operators $\sigma(x,\lambda)$ by asking that $\sigma(x, \varepsilon \cdot \lambda) = \varepsilon^m \sigma(x,\lambda)$ for all $x \in G$, almost all $\lambda \in \widehat{G}$ and $d\varepsilon$-almost all $\varepsilon > 0$ (in the preceding formula, $\varepsilon \cdot \lambda = \varepsilon^2 \lambda$ for $\lambda \in \mathfrak{z}^*$ and $\varepsilon \cdot (0, \omega) = (0, \varepsilon\omega)$ for $\omega \in \mathfrak{v}^*$). In parallel to what is done in the Euclidean setting, one then defines *regular m-homogeneous symbols* as the set $\dot{S}^m$ of m-homogeneous fields of operators $\sigma(x,\lambda)$ which satisfy for any $\alpha, \beta \in \mathbb{N}^n, \gamma \in \mathbb{R}$:

$$\sup_{\substack{\lambda\in\widehat{G} \\ x\in G}} \|H(\lambda)^{\frac{[\alpha]-m+\gamma}{2}} X_x^\beta \Delta^\alpha \sigma(x,\lambda) H(\lambda)^{-\frac{\gamma}{2}}\|_{\mathcal{L}(\mathcal{H}_\lambda)} < \infty. \quad (3.8)$$

In both the inhomogeneous and homogeneous case, it was proved that an equivalent characterisation is (3.7) and (3.8) respectively, for all α, β but only $\gamma = 0$.

Proposition 3.6. *Let $n \in \mathbb{N}$. The spectral projectors $\Pi_n^{(\lambda)}$ associated with $H(\lambda)$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, form a field Π_n of operators which is a homogeneous symbol in $\dot{S}^0$.*

Consequently, for any function $\psi \in C^\infty(\mathbb{R})$ with $\psi \equiv 0$ on $(-\infty, 1/2)$ and $\psi \equiv 1$ on $(1, +\infty)$, $\psi(H)\Pi_n$ is in S^0. Moreover, for every $\lambda \in \mathfrak{z}^* \setminus \{0\}$, $\psi(uH(\lambda))\Pi_n$ converges to Π_n in the strong operator topology (SOT) of $L^2(\mathfrak{p}_\lambda)$ as $u \to 0$. Furthermore, $\psi(uH)\Pi_n$ converges to Π_n in SOT for $L^\infty(\widehat{G})$ as $u \to 0$.

Remark 3.7. A remark on the notations: we shall use Π_n when denoting the field of operators acting on $L^2(\mathfrak{p}_\lambda)$ and write $\Pi_n^{(\lambda)}$ when some $\lambda \in \mathfrak{z}^*$ is fixed.

The proof of Proposition 3.6 relies on the spectral expression:

$$\Pi_n^{(\lambda)} = \frac{1}{2i\pi} \oint_{\mathcal{C}_n} (|\lambda|^{-1}H(\lambda) - z)^{-1}dz, \tag{3.9}$$

where $\mathcal{C}_n$ is any circle of the complex plane with centre $2n + d$ and radius $\rho \in (0, 2)$, and the following lemma:

Lemma 3.8. *The field of operators* $|\lambda|I_{L^2(\mathfrak{p}_\lambda)}$, $\lambda \in \mathfrak{z}^* \setminus \{0\}$, *yields a homogeneous regular symbol in* $\dot{S}^2$.

Proof of Lemma 3.8. The first thing to notice is that $|\lambda|H(\lambda)^{-1}$ is a bounded operator (as a self-adjoint operator with bounded eigenvalues). Then, we look at $\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)}$. The kernel corresponding to the symbol $|\lambda|I_{L^2(\mathfrak{p}_\lambda)}$ is the distribution $\delta_{v=0} \otimes \mathcal{F}_{\mathfrak{z}}^{-1}|\lambda|$, so the corresponding convolution operator on G acts only on the central component. Furthermore $\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)} = 0$ for $q = v_j$. For $q = z_k$, $\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)}$ is up to a constant the group Fourier transform of the distribution $\delta_{v=0} \otimes \mathcal{F}_{\mathfrak{z}}^{-1}\partial_{\lambda_k}|\lambda|$, so

$$\begin{aligned}
\sup_{\lambda \in \mathfrak{z}^*\setminus\{0\}} \|\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)}\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} &= \|f \longmapsto f * (\delta_{v=0} \otimes \mathcal{F}_{\mathfrak{z}}^{-1}\partial_{\lambda_k}|\lambda|)\|_{\mathcal{L}(L^2(G))} \\
&\leq \|g \longmapsto g * (\mathcal{F}_{\mathfrak{z}}^{-1}\partial_{\lambda_k}|\lambda|)\|_{\mathcal{L}(L^2(\mathfrak{z}))} \\
&= \sup_{\lambda \in \mathfrak{z}^*\setminus\{0\}} \partial_{\lambda_k}|\lambda| < \infty.
\end{aligned}$$

For $q = z_j z_k$, $\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)}$ is up to a constant the group Fourier transform of the distribution $\delta_{v=0} \otimes \mathcal{F}_{\mathfrak{z}}^{-1}\partial_{\lambda_j}\partial_{\lambda_k}|\lambda|$. We see that for $q = z_k$, $[q] = 2$ and

$$H(\lambda)^{-1+\frac{[q]}{2}}\Delta_q|\lambda| = \Delta_q|\lambda|.$$

Therefore

$$\sup_{\lambda \in \mathfrak{z}^*} \|H(\lambda)^{-1+\frac{[q]}{2}}\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)}\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} < \infty.$$

Recursively, we obtain for any monomial q, we have the estimates required by (3.8):

$$\sup_{\lambda \in \mathfrak{z}^*} \|H(\lambda)^{-1+\frac{[q]}{2}}\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)}\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} < \infty,$$

where $[q]$ denotes the degree of homogeneity of q. This shows Lemma 3.8. $\square$

Proof of Proposition 3.6. In order to show that the field of operators consisting of the $\Pi_n^{(\lambda)}$ is in $\dot{S}^0$, it suffices to show that $(H(\lambda) - |\lambda|z)^{-1}$ is in $\dot{S}^2$ with uniform semi-norms estimates of the form (3.8) with respect to $z \in \mathcal{C}_n$ because of (3.9), Lemma 3.8 and

$$(|\lambda|^{-1}H(\lambda) - z)^{-1} = |\lambda|(H(\lambda) - |\lambda|z)^{-1}.$$

The rest of the statement will then follow by [18] (see Section 4.2 therein).

The semi-norms estimates of the form (3.8) with $\alpha = \beta = 0$ and $\gamma = 0$ are satisfied for all $z \in \mathcal{C}_n$ with

$$\sup_{\lambda \in \mathfrak{z}^* \setminus \{0\}} \|H(\lambda)(H(\lambda) - |\lambda|z)^{-1}\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))} \leq \sup_{n \in \mathbb{N}} \frac{2n + d}{2n + d - \rho} < \infty.$$

For $q = v_j$ or z_k, since $\Delta_q I_{L^2(\mathfrak{p}_\lambda)} = 0$, the Leibniz formula implies

$$\Delta_q(H(\lambda) - |\lambda|z)^{-1} = -(H(\lambda) - |\lambda|z)^{-1}\Delta_q(H(\lambda) - |\lambda|z)(H(\lambda) - |\lambda|z)^{-1}.$$

Note that

$$\Delta_q(H(\lambda) - |\lambda|z) = \Delta_q H(\lambda) - z\Delta_q|\lambda|I_{L^2(\mathfrak{p}_\lambda)},$$

and that both terms in the right-hand side are in $\dot{S}^{2-[q]}$ (for the first one see [18], Example 4.5, for the second by Lemma 3.8), so $\Delta_q(H(\lambda) - |\lambda|z) \in \dot{S}^{2-[q]}$. This and the estimates above yield that

$$\|H(\lambda)^{1+\frac{[q]}{2}}\Delta_q(H(\lambda) - |\lambda|z)^{-1}\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))}$$
$$\leq \|H(\lambda)(H(\lambda) - |\lambda|z)^{-1}\|^2_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))}$$
$$\cdot \|H(\lambda)^{\frac{[q]}{2}}\Delta_q(H(\lambda) - |\lambda|z)H(\lambda)^{-1}\|_{\mathcal{L}(L^2(\mathfrak{p}_\lambda))},$$

so its supremum over $\lambda \neq 0$ is finite. Proceeding recursively shows

$$(H(\lambda) - |\lambda|z)^{-1} \in \dot{S}^{-2}$$

and thus

$$(|\lambda|^{-1}H(\lambda) - z)^{-1} \in \dot{S}^0,$$

with uniform semi-norms estimates with respect to $z \in \mathcal{C}_n$. This yields the first part of the statement. The rest follows from the characterization of homogeneous symbols (see [18], Proposition 4.6 (2)) and properties of the sub-Laplacian ([18], Lemma 2.10 (3)). □

3.5. Approximation of H-diagonals and anti-H-diagonals symbols. We now use the symbolic properties of Section 3.4 for the projections $\Pi_n^{(\lambda)}$ to decompose symbols as described in Lemma 2.4.

We first notice that if $\sigma = \{\sigma(\lambda), \lambda \in \hat{G}\} \in S^{-\infty}$ is a smoothing symbol independent of x, we can then define the symbol $\sigma^{(n,n')} := \Pi_n \sigma \Pi_{n'}$ for each n, n'. For $n = n'$, $\sigma^{(n,n')}$ commutes with H; for general pairs of integers (n, n'), it satisfies

$$H(\lambda)\sigma^{(n,n')}(\lambda) = |\lambda|(2n+d)\sigma^{(n,n')}(\lambda)$$

and

$$\sigma^{(n,n')}H(\lambda) = |\lambda|(2n'+d)\sigma^{(n,n')}(\lambda).$$

However, $\sigma^{(n,n')}$ is usually not smoothing but may be weakly approximated by the smoothing symbols $\psi(uH)\Pi_n\sigma\Pi_{n'}\psi(uH)$ with $u \to 0$ with ψ as in Proposition 3.6.

Proposition 3.6 allows us to modify the previous construction to symbols which also depend on $x \in G$ in order to obtain the following weak approximation of the Π_n-projections of smoothing symbols:

Corollary 3.9. *We fix a smooth function $\psi: \mathbb{R} \to [0, 1]$ satisfying $\psi \equiv 0$ on $(-\infty, 1/2)$ and $\psi \equiv 1$ on $(1, +\infty)$. Let $\sigma \in S^{-\infty}$.*

(1) *For each $n, n' \in \mathbb{N}$ and $u \in (0, 1]$, the symbol*

$$\sigma^{(n,n',u)} := \psi(uH)\Pi_n\sigma\Pi_{n'}\psi(uH) \tag{3.10}$$

   is smoothing, i.e. $\sigma^{(n,n',u)} \in S^{-\infty}$. Moreover, for every $x \in G$ and $\lambda \in \mathfrak{z}^* \setminus \{0\}$, $\sigma^{(n,n',u)}(x, \lambda)$ converges to $\sigma^{(n,n')}(x, \lambda) := \Pi_n\sigma(\lambda)\Pi_{n'}$ in SOT of $L^2(\mathfrak{p}_\lambda)$ as $u \to 0$; and $\sigma^{(n,n',u)}(x, \cdot)$ converges to $\sigma^{(n,n')} = \Pi_n\sigma(x, \cdot)\Pi_{n'}$ in SOT for $L^\infty(\hat{G})$ as $u \to 0$, for every $x \in G$.

(2) *If $\sigma \in \mathcal{A}_0$, then $\sigma^{(n,n',u)} \in \mathcal{A}_0$ for every $n, n' \in \mathbb{N}$ and $u \in (0, 1)$.*

(3) *If σ is supported in $K \times (\mathfrak{z}^* \setminus \{0\})$ with K a compact of G and vanish identically near $\lambda = 0$, then $\sigma^{(n,n',u)} \in \mathcal{A}_H$ for every $n, n' \in \mathbb{N}$ and $u \in (0, 1)$.*

(4) *If $\sigma \in \mathcal{A}_H$, then we can write σ as the finite sum $\sigma = \sum_{n,n'} \sigma^{(n,n')}$ and for u small enough we have $\sigma^{(n,n')} = \sigma^{(n,n',u)}$.*

(5) *If $\sigma \in \mathcal{A}$, then $\sigma^{(n,n',u)} \in \mathcal{A}$ for every $n, n' \in \mathbb{N}$ and $u \in (0, 1)$. If in addition σ is a positive element in the C^*-algebra $\mathcal{A}$, then so is $\sigma^{(n,n',u)}$ for $n = n'$.*

Proof. Part (1) follows from Proposition 3.6 and [18]. Parts (2), (3), and (4) follows from part (1). Let $\sigma \in \mathcal{A}$. Since $\sigma \mapsto \sigma^{(n,n',u)}$ is linear and

$$\|\sigma^{(n,n',u)}(x,\cdot)\|_{L^\infty(\widehat{G})} \leq \|\sigma(x,\cdot)\|_{L^\infty(\widehat{G})},$$

$\sigma^{(n,n',u)} \in \mathcal{A}$. Furthermore, for any $\sigma \in \mathcal{A}$, $(\sigma^*\sigma)^{(n,n,u)}$ is positive in $C_0(G) \otimes L^\infty(\widehat{G})$, therefore also in $\mathcal{A}$. This shows part (5). $\square$

Remark 3.10. Note that part (4) of Corollary 3.9 proves Lemma 2.4.

The same ideas also gives the approximations of symbols in $\mathcal{A}_0$ by symbols in $\mathcal{A}_H$:

Corollary 3.11. *Let $\sigma \in \mathcal{A}_0$. Then there exists a sequence $(\sigma_n)_{n\in\mathbb{N}}$ of symbols in $\mathcal{A}_H$ converging to σ in the following sense*:

- *for each $\lambda \in \mathfrak{z}^* \setminus \{0\}$, we have the convergence $\sigma_n(x,\lambda) \longrightarrow \sigma(x,\lambda)$ in SOT of $L^2(\mathfrak{p}_\lambda)$ as $n \to +\infty$ uniformly in $x \in G$, and*
- *we have the convergence $\sigma_n(x,\cdot) \to \sigma(x,\cdot)$ in SOT of $L^\infty(\widehat{G})$ as $n \to +\infty$ uniformly in $x \in G$.*

Corollary 3.11 follows from Proposition 3.6 and Proposition 4.6 of [18] together with a smooth cut-off in $\lambda \in \mathfrak{z}^*$. The latter property is granted by the following lemma:

Lemma 3.12. *Let $\sigma \in \mathcal{A}_0$.*
(1) *For any $g \in \mathcal{S}(\mathfrak{z}^*)$, the symbol given by $g(\lambda)\sigma(x,\lambda)$ is in $\mathcal{A}_0$.*
(2) *Let $(g_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathfrak{z}^* \setminus \{0\})$ be a sequence of functions which are bounded by 1 and converges to the constant function 1 pointwise. Then $(g_n\sigma(x,\cdot))_{n\in\mathbb{N}}$ converges to $\sigma(x,\cdot)$ in SOT of $L^\infty(\widehat{G})$ uniformly in $x \in G$.*

Proof. Let κ_σ be the kernels associated with the symbols which may be identified with a map in $C_c^\infty(G;\mathcal{S}(G))$. Part (1) follows from the kernel κ_g associated with the symbol $g(\lambda)$ being the central distribution $\delta_{v=0} \otimes \mathcal{F}_{\mathfrak{z}}^{-1}g$ and the kernel associated with $g(\lambda)\sigma$ being $\kappa_g * \kappa_\sigma$. Part (2) follows from the Lebesgue dominated convergence theorem and the Plancherel formula. $\square$

Remark 3.13. If $\sigma = \{\sigma(x,\pi), (x,\pi) \in G \times \widehat{G}\} \in S^{-\infty}$, then constructing as above $\psi(uH)\Pi_n\sigma\Pi_n\psi(uH)$ yields a smoothing symbol commuting with $H(\lambda)$. The closure of their span form a much larger class than the class of spectral multipliers of $H(\lambda)$. Indeed, spectral multipliers are constant on the vector sets $\mathcal{V}_n$ which is not the case of $\psi(uH)\Pi_n\sigma\Pi_n\psi(uH)$. The reader can refer to [21, Chapter 4] for considerations on symbols that are functions of $H(\lambda)$.

3.6. The sub-C^*-algebra $\mathcal{B}$ of H-commuting symbols. In order to prove Theorem 2.10, we shall use symbols in $\mathcal{A}$ which commute with $H(\lambda)$. The set $\mathcal{B}$ of such symbols satisfies the following properties:

Lemma 3.14. *Let $\mathcal{B}$ be the sub-C^*-algebra of $\mathcal{A}$ consisting of symbols $\sigma \in \mathcal{A}$ which commute with $H(\lambda)$, i.e. $\sigma(x, \lambda)H(\lambda) = H(\lambda)\sigma(x, \lambda)$ for almost every $(x, \lambda) \in G \times \widehat{G}$.*

(1) *The space $\mathcal{B}$ contains the symbols of the form $a(x)\sigma(\lambda)$ with $a \in C_c^\infty(G)$ and σ smoothing and commuting with $H(\lambda)$, and the algebraic span of these symbols are dense in $\mathcal{B}$.*

(2) *The states of $\mathcal{B}$ are in one-to-one correspondence as in Proposition 3.2 with the measures $\Gamma d\gamma \in \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G})$ such that $\Gamma = \sum_{n \in \mathbb{N}} \Pi_n \Gamma \Pi_n$.*

Proof of Lemma 3.14. Part (1) is readily checked. Let us prove part (2). Let ℓ be a state of $\mathcal{B}$. For any $N \in \mathbb{N}$ and $u \in (0, 1)$, part (5) of Corollary 3.9 implies that

$$\ell_{N,u}(\sigma) := \sum_{n=0}^{N} \ell(\sigma^{(n,n,u)}),$$

defines a continuous linear functional $\ell_{N,u}$ on $\mathcal{A}$ which is a state or 0. We denote by $\Gamma_{N,u} d\gamma_{N,u} \in \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G})$ the measure corresponding to $\ell_{N,u}$ by Proposition 3.2. We observe that for any $N_1 \leq N_2$ and $2u_1 < u_2 < u_3/2$

$$\ell_{u_1,N_1}(\sigma) = \ell_{u_2,N_2}\Big(\sum_{n=0}^{N_1} \sigma^{(n,n,u_3)} \Big).$$

The uniqueness in Proposition 3.2 implies that there exists $\Gamma d\gamma \in \mathcal{M}_{\mathrm{ov}}^+(G \times \widehat{G})$ of mass 1, commuting with $H(\lambda)$ and such that

$$\Gamma_{N,u} d\gamma_{N,u} = \sum_{n=0}^{N} \psi(uH) \Pi_n \Gamma \Pi_{n'} \psi(uH) d\gamma.$$

This implies part (2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It readily follows from Lemma 3.14 that the pure states of the C^*-algebra $\mathcal{B}$ are given either by $|v\rangle\langle v|\delta_{x_0} \otimes \delta_{\pi^{\lambda_0}}$ for some $v \in \mathcal{V}_n$ when $\lambda_0 \in \mathfrak{z}^* \setminus \{0\}$ and $x_0 \in G$, or by $\delta_{x_0} \otimes \delta_{\pi^{(0,\omega_0)}}$ for some $\omega_0 \in \mathfrak{v}^*$ and $x_0 \in G$. One can thus describe easily the dual of $\mathcal{B}$ by identification with the subset

$$S := \{(x, \lambda_0, n) \in G \times \widehat{G} \times \mathbb{N} : n = 0 \text{ if } \lambda_0 \notin \mathfrak{z}^* \setminus \{0\}\}$$

of $G \times \widehat{G} \times \mathbb{N}$.

4. Proof of Theorems 2.5 and 2.10

The core of our results are Theorems 2.5 and 2.10 from which Theorem 1.1 will derive in Section 5.3, so we focus on these two first statements. Their proofs rely on a careful analysis of the commutator $[\mathrm{Op}_\varepsilon(\sigma), \varepsilon^2 \mathbb{L}_G]$ based on Equation (2.23). Considering anti-H-commuting symbols in Section 4.1 allows us to prove first point (i) of both Theorems 2.5 and 2.10. We successively prove points (ii) of the two theorems in Sections 4.2 and (iii) of Theorem 2.10 in Section 4.3 respectively.

4.1. Spectral decomposition of the time-averaged semi-classical measure.
Here, we prove part (i) of Theorems 2.10 and 2.5, together with Equation (2.28). We take $\sigma \in \mathcal{A}_0$. The Schrödinger equation (2.3) yields

$$i\varepsilon^\tau \frac{d}{dt}(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} = \left(\left[\mathrm{Op}_\varepsilon(\sigma), -\frac{\varepsilon^2}{2}\mathbb{L}_G \right]\psi^\varepsilon(t), \psi^\varepsilon(t) \right)_{L^2(G)}.$$

By use of (2.23), we obtain

$$\frac{1}{2}(\mathrm{Op}_\varepsilon([\sigma, H(\lambda)]\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)}$$
$$= \frac{1}{2}([\mathrm{Op}_\varepsilon(\sigma), -\varepsilon^2\mathbb{L}_G]\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} + O(\varepsilon) + O(\varepsilon)$$
$$= \frac{i}{2}\varepsilon^\tau \frac{d}{dt}(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} + O(\varepsilon).$$

Therefore,

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon([\sigma, H(\lambda)])\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} \, dt = O(\varepsilon^\tau) + O(\varepsilon). \qquad (4.1)$$

4.1.1. Proof of Theorem 2.10 (i). Firstly, taking the limit as $\varepsilon \to 0$ in (4.1) for a subsequence defining a semi-classical measure $\Gamma_t \, d\gamma_t$, it gives for all $\sigma \in \mathcal{A}_0$ and for all $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} \theta(t) \int_{G \times \widehat{G}} \mathrm{Tr}([\sigma(x, \lambda), H(\lambda)]\Gamma_t(x, \lambda)) \, d\gamma_t(x, \lambda) \, dt = 0.$$

We apply this to any symbol $\sigma^{(n,n',u)} \in \mathcal{A}_0$ given by (3.10), see Corollary 3.9. By Corollary 3.5, this implies that the element in $\mathcal{M}_{\mathrm{ov}}(\mathbb{R} \times G \times \widehat{G})$ given by

$$[H(\lambda), \Pi_{n'}\psi(uH(\lambda))\Gamma_t(x, \lambda)\psi(uH(\lambda))\Pi_n] \, d\gamma_t \, dt$$

is zero. For any pair of integers with $n \neq n'$, taking $u \to 0$ implies that $\Pi_{n'} \Gamma_t \Pi_n = 0$ for almost every $(x, \lambda) \in G \times \widehat{G}$, this shows point (i) of Theorem 2.10. As Γ_t is a positive compact operator, it admits the spectral decomposition (2.28).

4.1.2. Proof of Theorem 2.5 (i). Secondly, we can apply (4.1) for the symbols

$$\tilde{\sigma}^{(g,n,n',u)}(x, \lambda) := g(\lambda)(2|\lambda|(n - n'))^{-1}\sigma^{(n,n',u)}$$

where $\sigma^{(n,n',u)} \in \mathcal{A}_0$ is given by (3.10) and $g \in \mathcal{S}(\mathfrak{z}^*)$ supported away from 0 (by Lemma 3.12, this symbol is indeed in $\mathcal{A}_0$). We obtain that for $n \neq n'$

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon(g\sigma^{(n,n',u)})\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} dt = O(\varepsilon^{\min(1,\tau)}).$$

This implies point (i) of Theorem 2.5 for $\sigma \in \mathcal{A}_H$ by taking $g \in C_c^\infty(\mathfrak{z}^* \setminus \{0\})$ identically 1 on the λ-support of σ, and u small enough so that $\sigma^{(n,n')} = \sigma^{(n,n',u)}$ where $\sigma = \sum_{n,n'} \sigma^{(n,n')}$ with the notation of Corollary 3.9 (2).

4.1.3. A more precise computation after integration in time. In order to prove the rest of Theorems 2.5 and 2.10, we write down more precisely the equalities we have obtained above using (2.23)

$$
\begin{aligned}
i\varepsilon^\tau \frac{d}{dt}&(\mathrm{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} \\
&= \frac{1}{2}(\mathrm{Op}_\varepsilon([\sigma, H(\lambda)])\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} \\
&\quad - \varepsilon(\mathrm{Op}_\varepsilon(V \cdot \pi^\lambda(V)\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)} \\
&\quad - \frac{\varepsilon^2}{2}(\mathrm{Op}_\varepsilon(\mathbb{L}_G\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t))_{L^2(G)}.
\end{aligned}
\tag{4.2}
$$

It is actually convenient to use the notation $\ell_\varepsilon(\theta, \sigma)$ introduced in (2.27). With these notations, equation (4.2) implies

$$-i\varepsilon^\tau \ell_\varepsilon(\theta', \sigma) = \frac{1}{2}\ell_\varepsilon(\theta, [\sigma, H(\lambda)]) - \varepsilon\ell_\varepsilon(\theta, V \cdot \pi^\lambda(V)\sigma) - \frac{\varepsilon^2}{2}\ell_\varepsilon(\theta, \mathbb{L}_G\sigma). \tag{4.3}$$

By Theorem 2.8, we know that up to extraction of a subsequence ε_k, $\ell_\varepsilon(\theta, \sigma)$ has a limit that we denote by $\ell_\infty(\theta, \sigma)$.

4.2. Proof of (ii) Theorems 2.10 and 2.5. In this paragraph, we prove parts (ii) in Theorems 2.10 and 2.5. We consider symbols supported away from $\lambda = 0$ and commuting with $H(\lambda)$. We are going to use some properties that are summarised in the following technical lemma, the proof of which is in Appendix B.

Lemma 4.1. *If $\sigma_0 \in \mathcal{A}_0$ commutes with H and if $g \in \mathcal{S}(\mathfrak{z}^*)$ is supported away from 0, then the symbol $\sigma \in \mathcal{A}_0$ given via $\sigma(x, \lambda) = g(\lambda)\sigma_0(x, \lambda)$ satisfies the following properties*:

(1) *the symbol σ_1 given via*

$$\sigma_1(x, \lambda) = -\frac{1}{2i|\lambda|} \sum_{j=1}^{d} (P_j \pi^\lambda(Q_j) - Q_j \pi^\lambda(P_j))\sigma(x, \lambda)$$

   is in $\mathcal{A}_0$;

(2) *for any $(x, \lambda) \in G \times \widehat{G}$,*

$$[\sigma_1(x, \lambda), H(\lambda)] = V \cdot \pi^\lambda(V)\sigma(x, \lambda);$$

(3) *for any $n \in \mathbb{N}$, $(x, \lambda) \in G \times \widehat{G}$,*

$$\Pi_n(V \cdot \pi^\lambda(V)\sigma_1(x, \lambda))\Pi_n = \frac{1}{2}\left(\left(n + \frac{d}{2}\right)i\mathcal{Z}^{(\lambda)} - \frac{1}{2}\mathbb{L}_G\right)\Pi_n\sigma(x, \lambda)\Pi_n.$$

4.2.1. The case $\tau \in (0, 2)$ – Proof of part (ii) (1) of Theorems 2.10 and 2.5.
We continue with the setting of Lemma 4.1, i.e. a symbol $\sigma(x, \lambda)$ commuting with $H(\lambda)$ and of the form $\sigma(x, \lambda) = g(\lambda)\sigma_0(x, \lambda)$ with $g \in \mathcal{S}(\mathfrak{z}^*)$ supported away from 0, and $\sigma_0 \in \mathcal{A}_0$. Then, parts (1) and (2) of Lemma 4.1 together with equation (4.3) applied to the symbol σ_1 yield the development of the term:

$$\begin{aligned}
\ell_\varepsilon(\theta, V \cdot \pi^\lambda(V)\sigma) &= \ell_\varepsilon(\theta, [\sigma_1, H(\lambda)]) \\
&= -2i\varepsilon^\tau \ell_\varepsilon(\theta', \sigma_1) + 2\varepsilon\ell_\varepsilon(\theta, V \cdot \pi^\lambda(V)\sigma_1) + \varepsilon^2 \ell_\varepsilon(\theta, \mathbb{L}_G\sigma_1).
\end{aligned}$$

Plugging this into (4.3) shows that

$$\begin{aligned}
\ell_\varepsilon(\theta', \sigma) &= i\varepsilon^{-\tau}\left(-\varepsilon\ell_\varepsilon(\theta, V \cdot \pi^\lambda(V)\sigma) - \frac{\varepsilon^2}{2}\ell_\varepsilon(\theta, \mathbb{L}_G\sigma)\right) \\
&= -2\varepsilon\ell_\varepsilon(\theta', \sigma_1) - 2i\varepsilon^{2-\tau}\ell_\varepsilon(\theta, V \cdot \pi^\lambda(V)\sigma_1) \\
&\quad -\frac{i}{2}\varepsilon^{2-\tau}\ell_\varepsilon(\theta, \mathbb{L}_G\sigma) - i\varepsilon^{3-\tau}\ell_\varepsilon(\theta, \mathbb{L}_G\sigma_1) \\
&= O(\varepsilon) + O(\varepsilon^{2-\tau}).
\end{aligned} \tag{4.4}$$

Proceeding as in Section 4.1, we obtain

$$\ell_\varepsilon(\theta', \Pi_n^\lambda \sigma \Pi_n^\lambda) = O(\varepsilon) + O(\varepsilon^{2-\tau})$$

and passing to the limit, we obtain $\partial_t(\Gamma_t d\gamma_t) = 0$. This proves parts (ii) (1) of Theorems 2.5 and 2.10, and concludes the analysis in the case $\tau \in (0, 2)$.

4.2.2. A more precise computation when $\tau \geq 2$. Before focusing on the cases $\tau \geq 2$, let us make more explicit the computation in Section 4.2.1 (which is valid for any τ) by setting for any σ as in Lemma 4.1:

$$j_\varepsilon(\theta, \sigma) := 2\ell_\varepsilon(\theta, V \cdot \pi^\lambda(V)\sigma_1) + \frac{1}{2}\ell_\varepsilon(\theta, \mathbb{L}_G\sigma). \tag{4.5}$$

Rewriting (4.4), we obtain

$$j_\varepsilon(\theta, \sigma) = i\varepsilon^{\tau-2}\ell_\varepsilon(\theta', \sigma) + 2i\varepsilon^{\tau-1}\ell_\varepsilon(\theta', \sigma_1) - \varepsilon\ell_\varepsilon(\theta, \mathbb{L}_G\sigma_1). \tag{4.6}$$

Theorem 2.5 (i) and Lemma 4.1 (3) together with (4.5) give for any $\sigma \in \mathcal{A}_H$,

$$j_\varepsilon(\theta, \sigma) = i\sum_{n\in\mathbb{N}}\ell_\varepsilon\left(\theta, \left(n+\frac{d}{2}\right)\mathcal{Z}^{(\lambda)}\Pi_n\sigma\Pi_n\right) + O(\varepsilon^{\min(1,\tau)}), \tag{4.7}$$

as the terms in $\mathbb{L}_G$ cancel each other.

We set for any σ as in Lemma 4.1:

$$j_\infty(\theta, \sigma) := \lim_{\varepsilon_k\to 0} j_{\varepsilon_k}(\theta, \sigma).$$

Theorem 2.8 and (4.6) imply

$$j_\infty(\theta, \sigma) = 0 \quad \text{when} \quad \tau > 2, \tag{4.8}$$

while passing to the limit by Theorem 2.8, we have for any $\tau > 0$:

$$j_\infty(\theta, \sigma)$$
$$= \int_\mathbb{R} \theta(t) \int_{G\times\hat{G}} i\sum_{n=0}^\infty \left(n+\frac{d}{2}\right) \text{Tr}((\mathcal{Z}^{(\lambda)}\Pi_n\sigma(x,\lambda)\Pi_n)\Gamma_{n,t}(x,\lambda))d\gamma_t(x,\lambda)\,dt. \tag{4.9}$$

4.2.3. Proof of parts (2) and (3) of Theorem 2.5 (ii). We now consider $\sigma \in \mathcal{A}_H^{(d)}$. We may assume that $\sigma = \Pi_n\sigma\Pi_n$ for some fixed $n \in \mathbb{N}$.

In the case $\tau = 2$, the computations in Section 4.2.2 give (using first equation (4.6) and then (4.7))

$$\frac{d}{ds}\ell^\varepsilon(\theta(\cdot+s), \sigma\circ\Psi^{-(n+\frac{d}{2})s})$$
$$= \ell^\varepsilon(\theta'(\cdot+s), \sigma\circ\Psi^{-(n+\frac{d}{2})s})$$
$$+ \ell^\varepsilon\left(\theta(\cdot+s), -\left(n+\frac{d}{2}\right)\mathcal{Z}^{(\lambda)}\sigma\circ\Psi^{-(n+\frac{d}{2})s}\right)$$

$$\begin{aligned} &= -ij_\varepsilon(\theta(\cdot + s), \sigma \circ \Psi^{-(n+\frac{d}{2})s}) \\ &\quad + \ell_\varepsilon\left(\theta(\cdot + s), -\left(n + \frac{d}{2}\right)Z^{(\lambda)}\sigma \circ \Psi^{-(n+\frac{d}{2})s}\right) + O(\varepsilon) \\ &= O(\varepsilon). \end{aligned}$$

An integration over s shows part (2).

Part (3) for $\tau > 2$ is proved in a similar manner: using again (4.6) and (4.7)

$$\begin{aligned} \frac{d}{ds}\ell_\varepsilon(\theta, \sigma \circ \Psi^{-(n+\frac{d}{2})s}) &= \ell_\varepsilon\left(\theta, -\left(n + \frac{d}{2}\right)Z^{(\lambda)}\sigma \circ \Psi^{-(n+\frac{d}{2})s}\right) \\ &= ij_\varepsilon(\theta, \sigma \circ \Psi^{-(n+\frac{d}{2})s}) + O(\varepsilon) \\ &= O(\varepsilon^{\min(1,\tau-2)}). \end{aligned} \qquad (4.10)$$

This concludes the proof of Theorem 2.5.

4.2.4. Proof of part (2) of Theorem 2.10 (ii). Here $\tau = 2$. For any $\sigma \in \mathcal{A}_H$ satisfying $\sigma = \Pi_n\sigma\Pi_n$ for some fixed $n \in \mathbb{N}$, the considerations in Section 4.2.2 (see equations (4.4), (4.5), and (4.9)) give

$$\ell_\infty(\theta', \sigma) = \ell_\infty\left(\theta, \left(n + \frac{d}{2}\right)Z^{(\lambda)}\sigma\right) \quad \text{if } \tau = 2.$$

The weak density of $\mathcal{A}_H$ in $\mathcal{A}_0$ (see Corollary 3.9) implies Equation (2.29), as stated in part (2) of Theorem 2.10 (ii).

4.2.5. Proof of part (3) Theorem 2.10 (ii). Here $\tau > 2$. For any $\sigma \in \mathcal{A}_H$ satisfying $\sigma = \Pi_n\sigma\Pi_n$ for some fixed $n \in \mathbb{N}$, passing to the limit in (4.10) give (in view of (4.8))

$$\ell_\infty(\theta, Z^{(\lambda)}\sigma) = 0.$$

The weak density of $\mathcal{A}_H$ in $\mathcal{A}_0$ (see Corollary 3.9) implies that $\Gamma_{n,t}(x,\lambda)d\gamma_t(x,\lambda)$ is invariant under $Z^{(\lambda)}$ away from $\lambda = 0$. As this is true for every $n \in \mathbb{N}$, the measure $\mathbf{1}_{\lambda\neq 0}\operatorname{Tr}(\Gamma_t)d\gamma_t = \mathbf{1}_{\lambda\neq 0}\sum_{n\in\mathbb{N}}\operatorname{Tr}(\Gamma_{n,t})d\gamma_t$ is invariant under the flow Ψ^s defined in (2.24). This concludes the proof of Theorem 2.10 (ii) in view of the discussion at the end of Section 2.6.2.

4.3. Proof of Theorem 2.10 (iii). We recall that the measure $\Gamma_t d\gamma_t$ is scalar above $\{\lambda = 0\}$, and we can define the measure

$$d\varsigma_t(x,\omega) := \Gamma_t(x,(0,\omega))d\gamma_t(x,(0,\omega))\mathbf{1}_{\lambda=0}.$$

In order to compute this measure, we observe the following fact.

Lemma 4.2. *The map* $\sigma \mapsto \sigma|_{G \times \{\lambda = 0\}}$ *is a* C^* *algebra morphism from* $\mathcal{A}$ *to* $C_0(G \times \mathfrak{v}^*)$. *This map as well as its restriction to the sub-C^*-algebra* $\mathcal{B}$ *of H-commuting symbols* (*see Section* 3.6) *are surjective.*

We observe that the map $\Theta \colon \sigma \mapsto \sigma|_{G \times \{\lambda = 0\}}$ maps $\mathcal{A}_0$ to $\mathcal{C}_c^\infty(G; \mathcal{S}(\mathfrak{v}^*))$ in the following way

$$\Theta(\sigma)(x, \omega) := \sigma(x, (0, \omega)) = \int_G \kappa_x(v, z) e^{-i\omega \cdot v} \, dv \, dz,$$

where we write $\sigma \in \mathcal{A}_0$ as $\sigma(x, \lambda) = \hat{\kappa}_x(\lambda)$ with the map $x \mapsto \kappa_x$ in $\mathcal{C}_c^\infty(G; \mathcal{S}(G))$. In fact, $\Theta(\mathcal{A}_0) = C_c^\infty(\mathcal{S}(\mathfrak{v}^*))$ and we can easily construct a right inverse via

$$(x \longmapsto \phi_x(\omega)) \longmapsto \mathcal{F}_G(\mathcal{F}_{\mathfrak{v}^*}^{-1} \phi_x \mathcal{F}_{\mathfrak{z}^*}^{-1} \chi(z)),$$

where $\chi \in \mathcal{C}_c^\infty(\mathfrak{z}^*)$ with $\chi(0) = 1$. From this, we easily check that Θ extends to a C^*-algebra morphism from $\mathcal{A}$ onto $\mathcal{C}_0(G \times \mathfrak{v}^*)$. However, we now give below another argument for the surjectivity of Θ which has the advantage that it also holds for its restriction to $\mathcal{B}$.

Proof of Lemma 4.2. Using the notation just above, $\Theta \colon \sigma \mapsto \sigma|_{G \times \{\lambda = 0\}}$ maps $\mathcal{A}_0$ to $\mathcal{C}_c^\infty(G; \mathcal{S}(\mathfrak{v}^*))$ and extends to a C^*-algebra morphism from $\mathcal{A}$ to $C_0(G \times \mathfrak{v}^*)$. The set $\Theta(\mathcal{A})$ is a sub-$\mathcal{C}^*$-algebra of $\mathcal{C}_0(G \times \mathfrak{v}^*)$, their spectrum are included accordingly with equality if and only if they are equal. Any state of $\mathcal{C}_0(G \times \mathfrak{v}^*)$ is given by a measure in $\mathcal{M}^+(G \times \mathfrak{v}^*)$ which may be viewed as an operator valued measure in $\mathcal{M}_{\text{ov}}^+(G \times \widehat{G})$ vanishing on $G \times \{\lambda \neq 0\}$. This shows that the spectrum of the commutative algebra $\Theta(\mathcal{A})$ is $G \times \mathfrak{v}^*$, so $\Theta(\mathcal{A}) = C_0(G \times \mathfrak{v}^*)$. The same argument holds for $\mathcal{B}$. □

For any $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ and any $\sigma \in \mathcal{A}_0 \cap \mathcal{B}$, we decompose the integrals in $\ell_\infty(\theta', \sigma)$ and $\ell_\infty(\theta, V \cdot \pi^\lambda(V)\sigma)$ over $G \times \widehat{G}$ as the sum of two integrals, one over $G \times \{\lambda \neq 0\}$ and one over $G \times \{\lambda = 0\} \sim G \times \mathfrak{v}^*$. The two integrals over $G \times \{\lambda \neq 0\}$ are zero by part (ii) and by Lemma 4.1 respectively. Then, using this when passing to the limit in (4.3), we have the following alternatives.

(1) If $\tau \in (0, 1)$, the relation $\ell_\infty(\theta', \sigma) = 0$ gives

$$\int_{\mathbb{R}} \theta'(t) \int_{G \times \mathfrak{v}^*} \sigma(x, (0, \omega)) d\varsigma_t(x, \omega) \, dt = 0, \quad \text{for all } \sigma \in \mathcal{A}_0 \cap \mathcal{B},$$

whence $\partial_t \varsigma_t = 0$ in the sense of distributions by Lemma 4.2.

(2) If $\tau = 1$, using (2.17), we obtain $\ell_\infty(\theta', \sigma) = \ell_\infty(\theta, V \cdot \pi^\lambda(V)\sigma)$, whence

$$\int_\mathbb{R} \theta'(t) \int_{G \times \mathfrak{v}^*} \sigma(x, (0, \omega)) d\varsigma_t(x, \omega) \, dt$$

$$= \int_\mathbb{R} \theta(t) \int_{G \times \mathfrak{v}^*} \omega \cdot V\sigma(x, (0, \omega)) d\varsigma_t(x, \omega) \, dt,$$

from which we deduce in the sense of distributions by Lemma 4.2

$$\partial_t \varsigma_t = \omega \cdot V \varsigma_t.$$

(3) If $\tau > 1$, using again (2.17) and Lemma 4.2 yields $\ell_\infty(\theta, V \cdot \pi^\lambda(V)\sigma) = 0$ and this implies that the measure $\varsigma_t(x, \omega)$ is invariant under the flow Ξ^s, $s \in \mathbb{R}$, defined by (2.31) And this invariance translates in terms of localisation of the support of ς_t in view of the discussion of the end of Section 2.6.2, whence part (iii) (3) of Theorem 2.10 .

In the situation when $\tau \in (0, 1]$, part (iii) (1) and (2) of Theorem 2.10 comes from the resolution of the transport equations satisfied by ς_t by using the continuity of $t \mapsto \varsigma_t$ that is proved in the next section. This concludes the proof of Theorem 2.10.

4.4. An improvement in the case $\tau \in (0, 1]$. Here, we show the following improvement for the case $\tau \in (0, 1]$:

Proposition 4.3. *Assume* $\tau \in (0, 1]$ *and consider, as in Theorem* 2.10*, a semi-classical measure* $t \mapsto \Gamma_t d\gamma_t \in L^\infty(\mathbb{R}, \mathcal{M}^+_{ov}(G \times \widehat{G}))$ *corresponding to the family of solutions to* (2.18) *for an initial data* $(\psi_0^\varepsilon)_{\varepsilon>0}$ *which is a bounded family in* $L^2(G)$. *Then, for any* $\sigma \in \mathcal{B}$, *the map* $t \mapsto \int_{G \times \widehat{G}} \mathrm{Tr}(\sigma \Gamma_t) \, d\gamma_t$ *is locally Lipschitz on* $\mathbb{R}$.

The proof of Proposition 4.3 relies on (4.2) and Lemma 3.14.

Remark 4.4. (1) The weak continuity of the map $t \mapsto \Gamma_t d\gamma_t$ granted in Proposition 4.3 allows us to solve the transport equations of (ii) point (1), and so for (i) and (ii) point (2).

(2) The proof of Proposition 4.3 implies that, under the assumptions of Proposition 4.3, if ε_k is the sub-sequence realising a semi-classical measure $\Gamma_t d\gamma_t$, then we have for all $t \in \mathbb{R}$

$$(\mathrm{Op}_\varepsilon(\sigma)\psi^{\varepsilon_k}(t), \psi^{\varepsilon_k}(t)) \xrightarrow[k \to +\infty]{} \int_{G \times \widehat{G}} \mathrm{Tr}(\sigma(x, \lambda)\Gamma_t(x, \lambda)) \, d\gamma_t(x, \lambda), \quad (4.11)$$

for all $\sigma \in \mathcal{B}$, meaning that one can pass to the limit t-by-t and not only when averaged in time (at the difference to the case $\tau > 2$ for example).

Proof of Proposition 4.3. For any $\sigma \in \mathcal{B}$, (4.2) gives

$$\frac{d}{dt}\ell_{\varepsilon,t}(\sigma) = O_\sigma(\varepsilon^{1-\tau}), \quad \text{where } \ell_{\varepsilon,t}(\sigma) := (\text{Op}_\varepsilon(\sigma)\psi^\varepsilon(t), \psi^\varepsilon(t)), \qquad (4.12)$$

in the sense that the map $t \mapsto \frac{d}{dt}\ell_{\varepsilon,t}(\sigma)$ is uniformly bounded with respect to t in a bounded interval of $\mathbb{R}$ and for $\varepsilon \in (0,1)$, which implies that the map $t \mapsto \ell_{\varepsilon,t}(\sigma)$ is equicontinuous. Consider a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ in $(0,1)$ converging to 0 as $j \to \infty$ and realizing the semi-classical measure in part (i) and part (ii) (1) of Theorem 2.10. By the Arzelà–Ascoli theorem and (4.12), we can extract a subsequence $(\varepsilon_{j_k})_{k \in \mathbb{N}}$ such that, as $k \to \infty$, $\varepsilon_{j_k} \to 0$ and $(\ell_{\varepsilon_{j_k},.}(\sigma))_{k \in \mathbb{N}}$ converges to a continuous function $t \mapsto \ell_t(\sigma)$ locally uniformly on $\mathbb{R}$ for all $\sigma \in \mathcal{B}$ (this requires to consider a dense subset of $\mathcal{B}$ and a diagonal extraction procedure). We proceed as in the proof of Theorem 2.8 (see also [18, 17]) using the C^*-algebra $\mathcal{B}$ with its properties in Lemma 3.14 instead of $\mathcal{A}$: either $L := \limsup_{k \to 0} \|\psi_0^{\varepsilon_{j_k}}\|_{L^2(G)} = 0$ and $\ell_t = 0$ or $L^{-1}\ell_t$ is a state of $\mathcal{B}$ for each $t \in \mathbb{R}$. Let $\Gamma_t d\gamma_t \in \mathcal{M}_{\text{ov}}(G \times \widehat{G})$ with $\Gamma_t = \sum_{n \in \mathbb{N}} \Pi_n \Gamma_t \Pi_n$ corresponding to ℓ_t. By construction, $\Gamma_t d\gamma_t$ coincides with the semi-classical measure in part (i) and part (ii) (1) of Theorem 2.10, whence the result.   □

5. Uniform ε-oscillation and marginals of semi-classical measures

We prove here Theorem 1.1 in Section 5.3, using the notion of ε-oscillation explained in Section 5.2, which allows to relate the weak limits of energy density and marginals of semi-classical measures, that we first study in Section 5.1.

5.1. Marginals of semi-classical measures. In this section, we describe the consequence of Theorem 2.10 for the marginals of semi-classical measures for the solutions to the Schrödinger equation. Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(G))$, $\Gamma_t d\gamma_t$ and ε_k the semi-classical measure and the subsequence associated with $(e^{i\varepsilon^{2-\tau}\mathbb{L}_G}\psi_0^\varepsilon)$ by Theorem 2.10 whose notations we use. We also consider the corresponding measurable map $t \mapsto \varrho_t$ in $L^\infty(\mathbb{R}; \mathcal{M}^+(G))$ given by the marginal on G of the measure $\text{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)$, that is, formally

$$\varrho_t := \int_{\widehat{G}} \text{Tr}(\Gamma_t)d\gamma_t(x,d\lambda),$$

or more precisely, for all $\phi \in \mathcal{C}_c^\infty(G)$ and $\theta \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R} \times G} \theta(t)\phi(x)d\varrho_t(x)\, dt = \int_{\mathbb{R}} \theta(t) \int_{G \times \widehat{G}} \phi(x)\operatorname{Tr}(\Gamma_t(x,\lambda))d\gamma_t(x,\lambda)\, dt.$$

We define two measurable maps in $L^\infty(\mathbb{R}; \mathcal{M}^+(G))$ via

$$\varrho_t^{\mathfrak{z}^*}(x) := \int_{\mathfrak{z}^* \setminus \{0\}} \mathbf{1}_{\lambda \neq 0} \operatorname{Tr}\Gamma_t(x,\lambda)d\gamma_t(x,\lambda) = \sum_{n \in \mathbb{N}} \int_{\mathfrak{z}^* \setminus \{0\}} \mathbf{1}_{\lambda \neq 0} \operatorname{Tr}\Gamma_{n,t}(x,\lambda)\, d\gamma_t(x,\lambda),$$

$$\varrho_t^{\mathfrak{v}^*}(x) := \int_{\mathfrak{v}^*} \mathbf{1}_{\lambda = 0} \operatorname{Tr}\Gamma_t(x,(0,\omega))d\gamma_t(x,(0,\omega)) = \int_{\mathfrak{v}^*} \varsigma_t(x,d\omega),$$

and, because of the decomposition of $\widehat{G}$ recalled in Section 2.2.2, $\varrho_t = \varrho_t^{\mathfrak{z}^*} + \varrho_t^{\mathfrak{v}^*}$. Besides, Theorem 2.10 directly implies the next proposition.

Proposition 5.1. *In the setting just above, we have the following properties.*

(1) *If $\tau \in (0,1]$, then the maps $t \mapsto \rho_t$, $t \mapsto \rho_t^{\mathfrak{z}^*}$ and $t \mapsto \rho_t^{\mathfrak{v}^*}$ are weakly continuous from $\mathbb{R}$ to $\mathcal{M}(G)$.*

(2) *The measures $\varrho_t^{\mathfrak{z}^*}$ satisfy the following properties*:

    (a) *if $\tau \in (0,2)$, then $\partial_t \varrho_t^{\mathfrak{z}^*} = 0$ in the sense of distribution on $\mathbb{R} \times G$;*

    (b) *if $\tau = 2$, then*

$$\varrho_t^{\mathfrak{z}^*}(x) = \sum_{n \in \mathbb{N}} \int_{\mathfrak{z}^* \setminus \{0\}} \gamma_{n,t}(x,d\lambda)$$

    where $\gamma_{n,t}(x,\lambda) := \mathbf{1}_{\lambda \neq 0} \operatorname{Tr}\Gamma_{n,t}(x,\lambda)d\gamma_t(x,\lambda)$, satisfies the transport equation (1.4) in the sense of distributions on $\mathbb{R} \times G \times (\mathfrak{z}^* \setminus \{0\})$;

    (c) *if $\tau > 2$, then $\varrho_t^{\mathfrak{z}^*} = 0$.*

(3) *The measures $\varrho_t^{\mathfrak{v}^*}$ satisfy the following properties*:

    (a) *if $\tau \in (0,1)$, then, for all $t \in \mathbb{R}$, $\varrho_t^{\mathfrak{v}^*} = \varrho_0^{\mathfrak{v}^*}$;*

    (b) *if $\tau = 1$, then, for all $t \in \mathbb{R}$,*

$$\varrho_t^{\mathfrak{v}^*}(x) = \int_{\mathfrak{v}^*} \varsigma_0(x\operatorname{Exp}(t\omega \cdot V), d\omega).$$

    (c) *if $\tau > 1$, then $\varrho_t^{\mathfrak{v}^*}(x) = \varsigma_t(x, \{\omega = 0\})$.*

This statement is the core of the proof of Theorem 1.1 by use of the concept of ε-oscillation that we now discuss.

5.2. ε-oscillating families. We stick here to a general framework and consider a family $(u^\varepsilon(t))_{\varepsilon>0}$ bounded in $L^\infty(\mathbb{R}, L^2(G))$. Our aim is to link here the weak limits of the measure $|u^\varepsilon(t,x)|^2 dx dt$ and the semi-classical measures of the family $(u^\varepsilon(t))_{\varepsilon>0}$. In analogy with [23] and Section 4.4 of [18] for H-type groups, we introduce the notion of uniform strict ε-oscillations for time-dependent families of $L^\infty(\mathbb{R}, L^2(G))$.

Definition 5.2. Let $(u^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(G))$. We shall say that $(u^\varepsilon)_{\varepsilon>0}$ is uniformly ε-*oscillating* when, for all $T > 0$,

$$\limsup_{\varepsilon\to 0} \sup_{t\in[-T,T]} \|\mathbf{1}_{-\varepsilon^2 \mathbb{L}_G > R} u^\varepsilon(t)\|_{L^2(G)} \xrightarrow[R\to+\infty]{} 0.$$

If, moreover,

$$\limsup_{\varepsilon\to 0} \sup_{t\in[-T,T]} \|\mathbf{1}_{-\varepsilon^2 \mathbb{L}_G < \delta} u^\varepsilon(t)\|_{L^2(G)} \xrightarrow[\delta\to 0]{} 0,$$

then the family $(u^\varepsilon)_{\varepsilon>0}$ is said *uniformly strictly ε-oscillating*.

The interest of the notion of ε-oscillation relies on the fact that it gives an indication of the size of the oscillations that have to be taken into account. It legitimates the use of semi-classical pseudodifferential operators and semi-classical measures in order to describe the time-averaged densities (1.1) of these families. Indeed, we have the following proposition.

Proposition 5.3. (1) *Let* $(u^\varepsilon)_{\varepsilon>0} \in L^\infty(\mathbb{R}, L^2(G))$ *be a uniformly ε-oscillating family admitting a time-averaged semi-classical measure* $t \mapsto \Gamma_t d\gamma_t$ *for the sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$. *Then for all* $\phi \in \mathcal{C}_c^\infty(G)$ *and* $\theta \in L^1(\mathbb{R})$,

$$\lim_{k\to+\infty} \int_{\mathbb{R}\times G} \theta(t)\phi(x)|u^{\varepsilon_k}(t,x)|^2 \, dx \, dt$$

$$= \int_{\mathbb{R}} \theta(t) \int_{G\times\widehat{G}} \phi(x) \operatorname{Tr}(\Gamma_t(x,\lambda)) \, d\gamma_t(x,\lambda) \, dt,$$

(2) *If moreover* $(u^\varepsilon)_{\varepsilon>0}$ *is uniformly strictly ε-oscillating, then the semi-classical measure does not charge the trivial representation* $1_{\widehat{G}}$ *in the sense that*

$$\gamma_t(G \times \{1_{\widehat{G}}\}) = 0 \quad \text{for almost every } t \in \mathbb{R}.$$

Proof. Let $\phi \in \mathcal{C}_c^\infty(G)$ and let $\theta \in L^1(\mathbb{R})$. We can write for any $R > 0$

$$\int_{\mathbb{R}\times G} \theta(t)\phi(x)|u^\varepsilon(t,x)|^2 \, dt \, dx = I_{1,\varepsilon,R} + I_{2,\varepsilon,R},$$

where

$$I_{j,\varepsilon,R} := \int_{\mathbb{R}} \theta(t)(\mathrm{Op}_\varepsilon(\sigma_{j,R})u^\varepsilon(t), u^\varepsilon(t))_{L^2(G)} \, dt, \quad j = 1, 2,$$

with

$$\sigma_{1,R}(x,\lambda) := \phi(x)\chi\left(\frac{\varepsilon^2}{R}H(\lambda)\right) \quad \text{and} \quad \sigma_{2,R}(x,\lambda) := \phi(x)(1 - \chi)\left(\frac{\varepsilon^2}{R}H(\lambda)\right),$$

having fixed a function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi = 0$ on $]-\infty, 1]$ and $\chi = 1$ on $[2, +\infty[$. Note that the symbol $\sigma_{2,R}$ is in $\mathcal{A}_0$ while, using the notation of [18], the symbol $\sigma_{1,R}$ is in S^0. For $I_{1,\varepsilon,R}$, let us first assume that θ is compactly supported in $[-T, T]$ for some $T > 0$. We have

$$|I_{1,\varepsilon,R}| \leq \|\theta\|_{L^1(\mathbb{R})}\|\phi\|_{L^\infty(G)}\|u^\varepsilon\|_{L^\infty(\mathbb{R},L^2(G))} \sup_{t\in[-T,T]} \|\chi(-\varepsilon^2 R^{-1}\mathbb{L}_G)u^\varepsilon\|_{L^2(G)},$$

so

$$\lim_{R\to+\infty} \limsup_{k\to\infty} I_{1,\varepsilon_k,R} = 0.$$

since $(u^\varepsilon)_{\varepsilon>0}$ is ε-oscillating and $0 \leq \chi \leq \mathbf{1}_{x>1}$. By density of $C_c^\infty(\mathbb{R})$ in $L^1(\mathbb{R})$, this is also true for any $\theta \in L^1(\mathbb{R})$.

For $I_{2,\varepsilon,R}$, by Theorem 2.8, we have

$$\lim_{k\to\infty} I_{2,\varepsilon_k,R} = \int_{\mathbb{R}} \theta(t) \int_{G\times\widehat{G}} \mathrm{Tr}(\sigma_{2,R}(x,\lambda)\Gamma_t(x,\lambda)) \, d\gamma_t(x,\lambda) \, dt.$$

Since we have

$$|\mathrm{Tr}(\sigma_{2,R}(x,\lambda)\Gamma_t(x,\lambda))| \leq \|\phi\|_{L^\infty(G)} \mathrm{Tr}(\Gamma_t(x,\lambda)),$$

and

$$\lim_{R\to+\infty} \mathrm{Tr}(\sigma_{2,R}(x,\lambda)\Gamma_t(x,\lambda)) = \phi(x) \mathrm{Tr}(\Gamma_t(x,\lambda)),$$

the Lebesgue dominated convergence theorem implies

$$\lim_{R\to+\infty} \lim_{k\to\infty} I_{2,\varepsilon_k,R} = \int_{\mathbb{R}} \theta(t) \int_{G\times\widehat{G}} \phi(x) \mathrm{Tr}(\Gamma_t(x,\lambda)) \, d\gamma_t(x,\lambda) \, dt.$$

This yields part (1).

For part (2), we see that if moreover $(u^\varepsilon)_{\varepsilon>0}$ is uniformly strictly ε-oscillating family, then for any $\theta \in C_c(\mathbb{R})$ and $\phi \in C_c^\infty(G)$, the expression

$$\int_{\mathbb{R}} \theta(t)(\phi(x)(1 - \chi)\left(-\frac{\varepsilon^2}{\delta}\mathbb{L}_G\right)u^\varepsilon(t), u^\varepsilon(t))_{L^2(G)} \, dt$$

is bounded by

$$\|\theta\|_{L^1(\mathbb{R})} \sup_{t \in \operatorname{supp} \theta} \left\|(1-\chi)\left(-\frac{\varepsilon^2}{\delta}\mathbb{L}_G\right)u^\varepsilon(t)\right\|_{L^2(G)} \sup_{t \in \mathbb{R}} \|u^\varepsilon(t)\|_{L^2(G)}\|\phi\|_{L^\infty(G)},$$

which tends to 0 when $\varepsilon = \varepsilon_k$ with $k \to +\infty$ and then $\delta \to 0$. However, by Theorem 2.8, the limit of the same expression as $\varepsilon = \varepsilon_k$ with $k \to +\infty$ is

$$\int_{\mathbb{R}} \theta(t) \int_{G \times \widehat{G}} \operatorname{Tr}(\phi(x)(1-\chi)\left(-\frac{1}{\delta}\mathcal{F}\mathbb{L}_G(\lambda)\right)\Gamma_t(x,\lambda))\, d\gamma_t(x,\lambda)\, dt,$$

which, by Lebesgue's dominated convergence theorem, converges as $\delta \to 0$ to

$$\int_{\mathbb{R}} \theta(t) \int_{G \times \widehat{G}} \phi(x) 1_{\pi = 1_{\widehat{G}}}\, d\gamma_t(x,\lambda)\, dt,$$

since $(1-\chi)(\frac{1}{\delta}\mathcal{F}\mathbb{L}_G)(\pi)$ tends to 0 in SOT for any non-trivial representation $\pi \in \widehat{G}$ while at $\pi = 1_{\widehat{G}}$ it is equal to 1. Consequently this last expression is zero, and this concludes the proof of Proposition 5.3. $\qquad\square$

The fact that a family is uniformly ε-oscillating can be derived from Sobolev bounds.

Proposition 5.4. (1) *If there exists $s > 0$ and $C > 0$ such that*

$$\sup_{t \in [-T,T]} \|(-\varepsilon^2 \mathbb{L}_G)^{\frac{s}{2}} u^\varepsilon(t)\|_{L^2(G)} \le C, \quad \text{for all } \varepsilon > 0,$$

then $(u^\varepsilon)_\varepsilon$ is ε-oscillating.

(2) *If there exists $s > 0$ and $C > 0$ such that*

$$\sup_{t \in [-T,T]} \|(-\varepsilon^2 \mathbb{L}_G)^{\frac{s}{2}} u^\varepsilon(t)\|_{L^2(G)} + \sup_{t \in [-T,T]} \|(-\varepsilon^2 \mathbb{L}_G)^{-\frac{s}{2}} u^\varepsilon(t)\|_{L^2(G)} \le C,$$

for all $\varepsilon > 0$, then $(u^\varepsilon)_{\varepsilon > 0}$ is strictly ε-oscillating.

Proof. We use Plancherel formula (2.12) and the facts that for $s > 0$ and χ as in the proof of Proposition 5.3,

$$\chi\left(-\frac{\varepsilon^2}{R}\mathbb{L}_G\right) \le \frac{(-\varepsilon^2 \mathbb{L}_G)^{\frac{s}{2}}}{R^s}\chi\left(-\frac{\varepsilon^2}{R}\mathbb{L}_G\right) \le \frac{(-\varepsilon^2 \mathbb{L}_G)^{\frac{s}{2}}}{R^s}$$

and

$$(1-\chi)\left(-\frac{\varepsilon^2}{\delta}\mathbb{L}_G\right) \le \delta^s(-\varepsilon^2 \mathbb{L}_G)^{-\frac{s}{2}}(1-\chi)\left(-\frac{\varepsilon^2}{\delta}\mathbb{L}_G\right) \le \delta^s(-\varepsilon^2 \mathbb{L}_G)^{-\frac{s}{2}}. \qquad\square$$

For families of solutions to the Schrödinger equation (2.18), the uniform ε-oscillating property is inherited from the initial data. Indeed, using that operators of the form $\chi(-\frac{\varepsilon^2}{R}\mathbb{L}_G)$ commute with the sublaplacian, the following result follows by energy estimates as a consequence of Proposition 5.4:

Proposition 5.5. *Let* $(\psi_0^\varepsilon)_{\varepsilon>0}$ *be a bounded family in* $L^\infty(\mathbb{R}, L^2(G))$. *If it satisfies*

$$there\ exist\ s, C > 0,\ such\ that\ \varepsilon > 0,$$

$$\|(-\varepsilon^2\mathbb{L}_G)^{\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(G)} \le C,$$

then the family of solutions $(\psi^\varepsilon)_{\varepsilon>0}$ *to the equation* (2.18) *for the initial data* ψ_0^ε *is uniformly* ε*-oscillating. Moreover, if* $(\psi_0^\varepsilon)_{\varepsilon>0}$ *satisfies* (1.2)*, then* $(\psi^\varepsilon)_{\varepsilon>0}$ *is uniformly strictly* ε*-oscillating.*

From Propositions 5.3 (2) and 5.5, it follows that:

Corollary 5.6. *We continue with the setting and the notation in Proposition* 5.1*. Assume in addition that* $(\psi_0^\varepsilon)_{\varepsilon>0}$ *satisfies* (1.2)*. If* $\tau > 1$ *then* $\varrho_t^{\mathfrak{v}^*} = 0$.

5.3. Proof of Theorem 1.1. Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^\infty(\mathbb{R}, L^2(G))$ satisfying (1.2). We set $\psi^\varepsilon(t) = e^{i\frac{t}{2\varepsilon^\aleph}\mathbb{L}_G}\psi_0^\varepsilon$. Then $\psi^\varepsilon(t)$ satisfies the semi-classical Schrödinger equation (2.18) with $\tau = \aleph + 2$, and $(\psi^\varepsilon)_{\varepsilon>0}$ is uniformly strictly ε-oscillating by Proposition 5.5.

We consider a weak limit of $|e^{i\frac{t}{2\varepsilon^\aleph}\mathbb{L}_G}\psi_0^\varepsilon(x)|^2\,dx\,dt$ for a converging subsequence (ε_j). Up to another extraction of a subsequence, it admits a semi-classical measure $\Gamma_t\,d\gamma_t$ as in Theorem 2.10. By Proposition 5.3, the marginals ρ_t defined in Proposition 5.1 coincide with the weak limit of $|e^{i\frac{t}{2\varepsilon^\aleph}\mathbb{L}_G}\psi_0^\varepsilon(x)|^2\,dx\,dt$. The result now readily follows from Proposition 5.1 and Corollary 5.6.

Remark 5.7. The case $\aleph \in (-2, -1]$ in Theorem 1.1 holds under the weaker hypothesis

$$there\ exist\ s, C > 0\ such\ that\ \|(-\varepsilon^2\mathbb{L}_G)^{\frac{s}{2}}\psi_0^\varepsilon\|_{L^2(G)} \le C.$$

Appendices

A. Dispersion in the Euclidean case

We describe here the analogue of Theorem 1.1 in the Euclidean setting for the Laplace operator $\Delta = \sum_{1\le j\le d}\partial_{x_j}^2$. The assumption (1.2) then writes in the

same manner replacing the sub-Laplacian by the Laplace operator Δ. We point out that the result below is only an elementary version of results that hold in more general setting and for more general Hamiltonian, including integrable systems (see [2, 9]). We use the semi-classical measures as introduced in the 90's in [23, 24, 25, 29].

Lemma A.1. *Let $(\psi_0^\varepsilon)_{\varepsilon>0}$ be a bounded family in $L^2(\mathbb{R}^d)$ satisfying (1.2) for the Laplace operator Δ. Then any limit point of the measure $|\mathrm{e}^{-i\frac{t}{2\varepsilon^\aleph}\Delta}\psi_0^\varepsilon|^2 dx dt$ is of the form $\varrho_t(x)dt$ where ϱ_t is a measure on $\mathbb{R}^d$. Besides,*

(1) *if $\aleph \in (-2,-1)$, then $\partial_t\varrho_t = 0$;*

(2) *if $\aleph = -1$ then there exists a positive Radon measure μ_0 on $\mathbb{R}^d \times \mathbb{R}^d$ such that*
$$\varrho_t(x) = \int_{\mathbb{R}^d} \mu_0(x - t\xi, d\xi);$$

(3) *if $\aleph > -1$ then $\varrho_t = 0$.*

Proof. A simple proof of this fact can be given by use of semi-classical measures. Denoting by $\mathrm{Op}_\varepsilon(a)$ the semi-classical pseudodifferential operator of symbol a, a semi-classical measure μ_t of the family $\psi^\varepsilon(t,x) := \mathrm{e}^{-i\frac{t}{2\varepsilon^\aleph}\Delta}\psi_0^\varepsilon$ is such that for a subsequence ε_k, for all $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$,

$$\int_{\mathbb{R}} \theta(t)(\mathrm{Op}_{\varepsilon_k}(a)\psi^{\varepsilon_k}(t), \psi^{\varepsilon_k}(t))\, dt \xrightarrow[k\to+\infty]{} \int_{\mathbb{R}\times\mathbb{R}^{2d}} \theta(t)a(x,\xi)\mu_t(dx,d\xi)\, dt. \quad \text{(A.1)}$$

The fact that $(\psi_0^\varepsilon)_{\varepsilon>0}$ satisfies (1.2) implies that it is a strictly ε-oscillating family and that it is also the case for the family $(\psi^\varepsilon)_{\varepsilon>0}$. One then has $\mu_t(\{\xi = 0\}) = 0$ and for the subsequence ε_k of (A.1), $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}\times\mathbb{R}^d} \theta(t)\phi(x)|\psi^{\varepsilon_k}(t,x)|^2\, dx\, dt \xrightarrow[k\to+\infty]{} \int_{\mathbb{R}\times\mathbb{R}^{2d}} \theta(t)\phi(x)\mu_t(dx,d\xi)\, dt.$$

The knowledge of the semi-classical measures determines all the limit points of the energy density.

Let us now take $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, we observe that

$$\frac{d}{dt}(\mathrm{Op}_\varepsilon(a)\psi^\varepsilon(t), \psi^\varepsilon(t)) = \frac{1}{i\,\varepsilon^\kappa}\left(\left[\mathrm{Op}_\varepsilon(a), -\frac{\varepsilon^2}{2}\Delta\right]\psi^\varepsilon(t), \psi^\varepsilon(t)\right).$$

Since

$$\left[\mathrm{Op}_\varepsilon(a), -\frac{\varepsilon^2}{2}\Delta\right] = i\varepsilon\,\mathrm{Op}_\varepsilon(\xi\cdot\nabla a) + \varepsilon^2\,\mathrm{Op}_\varepsilon(\Delta a),$$

we obtain immediately the following description:

(1) for $\kappa \in (0, 1)$,

$$(\mathrm{Op}_\varepsilon(a)\psi^\varepsilon(t), \psi^\varepsilon(t)) = (\mathrm{Op}_\varepsilon(a)\psi_0^\varepsilon, \psi_0^\varepsilon) + O(\varepsilon^{1-\kappa}),$$

whence $\mu_t(x, \xi) = \mu_0(x, \xi)$ for all times $t \in \mathbb{R}$;

(2) for $\kappa = 1$, the map $t \mapsto \mu_t$ is weakly continuous form $\mathbb{R}$ to $\mathcal{M}^+(\mathbb{R}^{2n})$ and can be realised by the same subsequence ε_k for all $t \in [0, T]$, $T > 0$ with

$$\partial_t \mu_t(x, \xi) = \xi \cdot \nabla_x \mu_t(x, \xi) \quad \text{in the sense of distributions};$$

(3) for $\kappa > 1$, we observe that

$$\frac{d}{ds}(\mathrm{Op}_\varepsilon(a(x + s\xi, \xi))\psi^\varepsilon(t), \psi^\varepsilon(t))\Big|_{s=0}$$
$$= (\mathrm{Op}_\varepsilon(\xi \cdot a)\psi^\varepsilon(t), \psi^\varepsilon(t))$$
$$= \varepsilon^{\kappa-1}\frac{d}{dt}(\mathrm{Op}_\varepsilon(a)\psi^\varepsilon(t), \psi^\varepsilon(t)) + O(\varepsilon).$$

In the last case, we deduce that for $\theta \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\int \theta(t)\frac{d}{ds}(\mathrm{Op}_\varepsilon(a(x + s\xi, \xi))\psi^\varepsilon(t), \psi^\varepsilon(t))\Big|_{s=0} dt$$
$$= -i\varepsilon^{\kappa-1}\int \theta'(t)(\mathrm{Op}_\varepsilon(a)\psi^\varepsilon(t), \psi^\varepsilon(t)) dt = O(\varepsilon^{\kappa-1}).$$

Therefore, the measure μ_t is invariant under the flow $(x, \xi) \mapsto (x + s\xi, \xi)$ and, since μ_t is of finite mass, μ_t is supported on $\{\xi = 0\}$, whence $\mu_t = 0$. $\qquad\square$

B. Proof of Lemma 4.1 and Hermite functions

The proof of Lemma 4.1 uses the bracket structure of $\mathfrak{g}$ via the two following lemmata:

Lemma B.1. *For any $\lambda \in \mathfrak{z}^* \setminus \{0\}$ and $j = 1, \ldots, d$, we have*

$$[H(\lambda), \pi^\lambda(P_j)] = -2i|\lambda|\pi^\lambda(Q_j) \quad and \quad [H(\lambda), \pi^\lambda(Q_j)] = 2i|\lambda|\pi^\lambda(P_j).$$

Proof of Lemma B.1. By (2.19), we have

$$[\mathbb{L}_G, P_{j_0}] = \sum_{j=1}^d ([P_j^2, P_{j_0}] + [Q_j^2, P_{j_0}]).$$

Working modulo $\ker \lambda$, by (2.7), P_{j_0} commutes with Q_j if $j \neq j_0$ and with any P_j, so we have

$$
\begin{aligned}
[\mathbb{L}_G, P_{j_0}] &= [Q_{j_0}^2, P_{j_0}] + \ker \lambda \\
&= Q_{j_0}[Q_{j_0}, P_{j_0}] + [Q_{j_0}, P_{j_0}]Q_{j_0} + \ker \lambda \\
&= -2Q_{j_0} Z^{(\lambda)} + \ker \lambda
\end{aligned}
$$

having used (2.6). We now apply π^λ, we use (2.16) and (2.15) (which implies that for $Z \in \ker \lambda$, $\pi^\lambda(Z) = 0$). This yields the first equality of the statement. The second equality is proved in a similar way. $\qquad\square$

Lemma B.2. *Considering for $\lambda \in \mathfrak{z}^* \setminus \{0\}$,*

$$
T := \Big(\sum_{j_1=1}^{2d} V_{j_1} \pi^\lambda(V_{j_1}) \Big) \circ \Big(\sum_{j_2=1}^{d} (P_{j_2} \pi^\lambda(Q_{j_2}) - Q_{j_2}\pi^\lambda(P_{j_2})) \Big),
$$

we have

$$
\Pi_n^{(\lambda)} T \Pi_n^{(\lambda)} = \frac{|\lambda|}{2}\Big(\Big(n + \frac{d}{2} \Big) Z^{(\lambda)} + \frac{i}{2}\mathbb{L}_G \Big) \Pi_n^{(\lambda)}.
$$

The proof of Lemma B.2 will use the properties of a special family of $H(\lambda)$-eigenfunctions we now recall. The family of Hermite functions $(h_n)_{n \in \mathbb{N}}$ given by

$$
h_n(\xi) = \frac{(-1)^n}{\sqrt{2^n n! \sqrt{\pi}}} e^{\frac{\xi^2}{2}} \frac{d}{d\xi}(e^{\xi^2}), \quad n \in \mathbb{N},
$$

is an orthonormal basis of $L^2(\mathbb{R})$ which satisfies

$$
-h_n''(\xi) + \xi^2 h_n(\xi) = (2n + 1)h_n(\xi).
$$

Hence, for each multi-index $\alpha \in \mathbb{N}^d$, the function h_α defined by

$$
h_\alpha(\xi) := \prod_{j=1}^{d} h_{\alpha_j}(\xi_j), \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d,
$$

is an eigenfunction of the operator $H(\lambda)$ (see Section 2.3):

$$
H(\lambda)h_\alpha = |\lambda|(2|\alpha| + d)h_\alpha.
$$

The eigenvalues $|\lambda|(2|\alpha| + d)$, $\alpha \in \mathbb{N}^d$, describe the entire spectrum of $H(\lambda)$ since the functions $h_\alpha, \alpha \in \mathbb{N}^d$ form an orthonormal basis of $L^2(\mathbb{R}^d)$.

For each $\lambda \in \mathfrak{z}^* \setminus \{0\}$, the symplectic structure on $\mathfrak{v}$ given by $B(\lambda)$ naturally suggests to consider a complex structure by setting

$$R_j := \frac{1}{2}(P_j - iQ_j), \quad \text{and} \quad \bar{R}_j := \frac{1}{2}(P_j + iQ_j).$$

By (2.16), the operators

$$\pi^\lambda(R_j) = \frac{\sqrt{|\lambda|}}{2}(\partial_{\xi_j} + \xi_j) \quad \text{and} \quad \pi^\lambda(\bar{R}_j) = \frac{\sqrt{|\lambda|}}{2}(\partial_{\xi_j} - \xi_j)$$

are the creation–annihilation operators associated with the harmonic oscillator $H(\lambda)$. The well-known recursive relations of the Hermite functions

$$h'_n(\xi) = \sqrt{\frac{n}{2}}h_{n-1}(\xi) - \sqrt{\frac{n+1}{2}}h_{n+1}(\xi),$$

$$\xi h_n(\xi) = \sqrt{\frac{n}{2}}h_{n-1}(\xi) + \sqrt{\frac{n+1}{2}}h_{n+1}(\xi),$$

gives for each $\lambda \in \mathfrak{z}^* \setminus \{0\}$, $n \in \mathbb{N}$ and $j = 1, \dots, d$,

$$\pi^\lambda(R_j)h_\alpha = \frac{\sqrt{|\lambda|}}{2}\sqrt{2\alpha_j}h_{\alpha-\mathbf{1}_j}$$

$$\pi^\lambda(\bar{R}_j)h_\alpha = -\frac{\sqrt{|\lambda|}}{2}\sqrt{2(\alpha_j+1)}h_{\alpha+\mathbf{1}_j}.$$

Consequently, we have $\pi^\lambda(\bar{R}_j)(\mathcal{V}_n) = \mathcal{V}_{n+1}$ and $\pi^\lambda(R_j)(\mathcal{V}_n) = \mathcal{V}_{n-1}$ with the convention that $\mathcal{V}_{-1} = \{0\}$. Moreover

$$\pi^\lambda(R_{j_1})\pi^\lambda(\bar{R}_{j_2})h_\alpha = -\frac{|\lambda|}{2}(\alpha_{j_2}+1)^{1/2}(\alpha_{j_1})^{1/2}h_{\alpha+\mathbf{1}_{j_2}-\mathbf{1}_{j_1}}$$

$$\pi^\lambda(\bar{R}_{j_1})\pi^\lambda(R_{j_2})h_\alpha = -\frac{|\lambda|}{2}(\alpha_{j_1}+1)^{1/2}(\alpha_{j_2})^{1/2}h_{\alpha+\mathbf{1}_{j_1}-\mathbf{1}_{j_2}}.$$

$$(B.1)$$

Proof of Lemma B.2. We observe that

$$\sum_{j=1}^{2d} V_j\pi^\lambda(V_j) = \sum_{j=1}^{d}(P_j\pi^\lambda(P_j) + Q_j\pi^\lambda(Q_j))$$

$$= 2\sum_{j=1}^{d}(R_j\pi^\lambda(\bar{R}_j) + \bar{R}_j\pi^\lambda(R_j)),$$

$$(B.2)$$

and

$$\sum_{j=1}^{d}(P_j\pi^\lambda(Q_j) - Q_j\pi^\lambda(P_j)) = -2i\sum_{j=1}^{d}(R_j\pi^\lambda(\bar{R}_j) - \bar{R}_j\pi^\lambda(R_j))$$

whence

$$T = -4i \sum_{j_1, j_2} (R_{j_1} \pi^\lambda(\bar{R}_{j_1}) + \bar{R}_{j_1} \cdot \pi^\lambda(R_{j_1}))(R_{j_2} \pi^\lambda(\bar{R}_{j_2}) - \bar{R}_{j_2} \pi^\lambda(R_{j_2})).$$

The properties (B.1) of the R_j's given above yield

$$\Pi_n T \Pi_n = -4i \sum_{j=1}^{d} \Pi_n (-R_j \bar{R}_j \pi^\lambda(\bar{R}_j)\pi^\lambda(R_j) + \bar{R}_j R_j \pi^\lambda(R_j)\pi^\lambda(\bar{R}_j))\Pi_n$$

$$= -4i \sum_{j=1}^{d} \sum_{|\alpha|=n} \Pi_n \Big(\frac{|\lambda|}{2}\alpha_j R_j \bar{R}_j - \frac{|\lambda|}{2}(\alpha_j + 1)\bar{R}_j R_j \Big)\Pi_\alpha$$

since $\Pi_n = \sum_{|\alpha|=n} \Pi_\alpha$ where $\Pi_\alpha = |h_\alpha\rangle\langle h_\alpha|$. Using (2.6) and

$$R_j \bar{R}_j = \frac{1}{4}(P_j^2 + Q_j^2) + \frac{i}{4}[P_j, Q_j]$$

and

$$\bar{R}_j R_j = \frac{1}{4}(P_j^2 + Q_j^2) - \frac{i}{4}[P_j, Q_j],$$

we obtain

$$\sum_{j=1}^{d} (\alpha_j R_j \bar{R}_j - (\alpha_j + 1)\bar{R}_j R_j) = \frac{i}{4} z^{(\lambda)}(2|\alpha| + d) - \frac{1}{4}\mathbb{L}_G,$$

and the result follows.   □

We can now show Lemma 4.1.

Proof of Lemma 4.1. By Lemma 3.12 (1), $\sigma \in \mathcal{A}_0$. To prove part (1), we first observe that, since the endomorphism $B(\lambda)$ defined via (2.1) is represented by J in the $(P_1, \ldots, P_d, Q_1, \ldots, Q_d)$-basis, we have in vector notation

$$\sum_{j=1}^{d}(P_j \pi^\lambda(Q_j) - Q_j \pi^\lambda(P_j)) = \begin{pmatrix} P \\ Q \end{pmatrix}^t J \begin{pmatrix} \pi(P) \\ \pi(Q) \end{pmatrix}$$

$$= |\lambda|^{-1} V^t B(\lambda)\pi(V)$$

$$= |\lambda|^{-1} \sum_{j,k} B(\lambda)_{j,k} V_j \pi(V_k).$$

Hence we can write

$$\sigma_1(x,\lambda) = \frac{-1}{2i\,|\lambda|^2} \sum_{j,k=1}^{2d} B(\lambda)_{j,k} \pi^\lambda(V_k) V_j \sigma(x,\pi)$$

$$= \sum_{j,k=1}^{2d} \pi^\lambda(V_k) V_j\, g_{j,k}(\lambda)\sigma_0(x,\pi),$$

As g is smooth and supported away from 0 and $B(\lambda)$ depends linearly on $\lambda \in \mathfrak{z}^* \setminus \{0\}$, each function $g_{j,k} := B_{j,k} g$ is smooth on $\mathfrak{z}^* \setminus \{0\}$; it is also Schwartz as g is Schwartz. By Lemma 3.12 (1), each symbol $g_{j,k}(\lambda)\sigma_0$ is in $\mathcal{A}_0$ so $\sigma_1 \in \mathcal{A}_0$. This shows part (1).

Part (2) follows from the observation that, as σ commutes with $H(\lambda)$,

$$[\sigma_1(x,\lambda), H(\lambda)] = \frac{-1}{2i\,|\lambda|}\Big(\sum_{j=1}^{d} P_j[\pi^\lambda(Q_j), H(\lambda)] - Q_j[\pi^\lambda(P_j), H(\lambda)]\Big)\sigma(x,\lambda).$$

We then use Lemma B.1 and obtain

$$P_j[\pi^\lambda(Q_j), H(\lambda)] - Q_j[\pi^\lambda(P_j), H(\lambda)] = -2i\,|\lambda|(P_j\pi^\lambda(P_j) + Q_j\pi^\lambda(Q_j)),$$

which allows us to conclude with (B.2).

Part (3) follows from the commutativity of σ with H and Lemma B.2. $\square$

References

[1] N. Anantharaman, F. Faure, and C. Fermanian Kammerer, *Chaos en mécanique quantique*. Lectures from the Mathematical Days (X-UPS 2014) held at l'École Polytechnique, Palaiseau, April 28–29, 2014. Edited by P. Harinck, A. Plagne, and C. Sabbah. Éditions de l'École Polytechnique, Palaiseau, 2014. MR 3379817 Zbl 1310.37002

[2] N. Anantharaman, C. Fermanian Kammerer and F. Macià, Semiclassical completely integrable systems: long-time dynamics and observability via two-microlocal Wigner measures. *Amer. J. Math.* **137** (2015), no. 3, 577–638. MR 3357117 Zbl 1319.35207

[3] N. Anantharaman and F. Macià, The dynamics of the Schrödinger flow from the point of view of semiclassical measures. In A. H. Barnett, C. S. Gordon, P. A. Perry, and A. Uribe (eds.), *Spectral geometry*. Proceedings of the International Conference held at Dartmouth College, Hanover, N.H., July 19–23, 2010. Proceedings of Symposia in Pure Mathematics, 84. American Mathematical Society, Providence, R.I., 2012, 93–116. MR 2985311 Zbl 1317.53080

[4] H. Bahouri, C. Fermanian Kammerer, and I. Gallagher, Phase space analysis on the Heisenberg group. *Astérisque* **342** (2012), vi+127 pp. MR 2952066 Zbl 1246.35003

[5] H. Bahouri, C. Fermanian Kammerer, and I. Gallagher, Dispersive estimates for the Schrödinger operator on step 2 stratified Lie Groups. *Anal. PDE* **9** (2016), no. 3, 545–574. MR 3518529 Zbl 06600505

[6] H. Bahouri, P. Gérard, and C.-J. Xu, Espaces de Besov et estimations de Strichartz généralisées sur le groupe de Heisenberg, *J. Anal. Math.* **82** (2000), 93–118. MR 1799659 Zbl 0965.22010

[7] R. Beals and P. Greiner, Calculus on Heisenberg manifolds. Annals of Mathematics Studies, 119. Princeton University Press, Princeton, N.J., 1988. MR 0953082 Zbl 0654.58033

[8] N. Burq and C. Sun, Time optimal observability for Grushin Schrödinger equation. Preprint, 2019. arXiv:1910.03691

[9] V. Chabu, C. Fermanian Kammerer and F. Macià, Semiclassical analysis of dispersion phenomena. In J. Delgado and M. Ruzhansky (eds.), *Analysis and partial differential equations*: *perspectives from developing countries.* Imperial College London, UK, 2016. Including selected papers from the Conference "Analysis and Partial Differential Equations", September 26–30, 2016, and the Conference "Noncommutative Analysis and Partial Differential Equations", April 11–15, 2016. Springer Proceedings in Mathematics & Statistics, 275. Springer, Cham, 2019, 84–108. MR 3923334 Zbl 1414.35189

[10] M. Christ, D. Geller, P. Glowacki, and L. Polin, Pseudodifferential operators on groups with dilations. *Duke Math. J.* **68** (1992), no. 1, 31–65. MR 1185817 Zbl 0764.35120

[11] Y. Colin de Verdière, Ergodicité et fonctions propres du laplacien. *Comm. Math. Phys.* **102** (1985), no. 3, 497–502. MR 0818831 Zbl 0592.58050

[12] Y. Colin de Verdière, L. Hillairet, and E. Trélat, Spectral asymptotics for sub-Riemannian Laplacians. I: quantum ergodicity and quantum limits in the 3D contact case. *Duke Math. J.* **167** (2018), no. 1, 109–174. MR 3743700 Zbl 1388.35137

[13] M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci, H-type groups and Iwasawa decompositions. *Adv. Math.* **87** (1991), no. 1, 1–41. MR 1102963 Zbl 0761.22010

[14] M. Del Hiero, Dispersive and Strichartz estimates on H-type groups. *Studia Math.* **169** (2005), no. 1, 1–20. MR 2139639 Zbl 1076.22010

[15] J. Dixmier, C^*-*algebras.* Translated from the French by F. Jellett. North-Holland Mathematical Library, 15. North-Holland Publishing Co., Amsterdam etc., 1977. MR 0458185 Zbl 0372.46058

[16] S. Dyatlov, L. Jin, and S. Nonnenmacher, Control of eigenfunctions on surfaces of variable curvature. Preprint, 2019. arXiv:1906.08923

[17] C. Fermanian Kammerer and V. Fischer, Semi-classical analysis on H-type groups. *Sci. China Math.* **62** (2019), no. 6, 1057–1086. MR 3951881 Zbl 1430.35284

[18] C. Fermanian Kammerer and V. Fischer, Defect measures on graded lie groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **21** (2020), 207–291. MR 4288603 Zbl 07373217

[19] C. Fermanian Kammerer and C. Lasser, Propagation through generic level crossings: a surface hopping semigroup. *SIAM J. Math. Anal.* **40** (2008), no. 1, 103–133. MR 2403314 Zbl 1168.35043

[20] C. Fermanian Kammerer and C. Lasser, An Egorov theorem for avoided crossings of eigenvalue surfaces. *Comm. Math. Phys.* **353** (2017), no. 3, 1011–1057. MR 3652482 Zbl 1375.35428

[21] V. Fischer and M. Ruzhansky, *Quantization on nilpotent Lie groups.* Progress in Mathematics, 314. Birkhäuser/Springer, Cham, 2016. MR 3469687 Zbl 1347.22001

[22] D. Geller, *Analytic pseudodifferential operators for the Heisenberg group and local solvability.* Mathematical Notes, 37. Princeton University Press, Princeton, N.J., 1990. MR 1030277 Zbl 0695.47051

[23] P. Gérard, Mesures semi-classiques et ondes de Bloch. In *Séminaire sur les Équations aux Dérivées Partielles. 1990–1991.* École Polytechnique, Centre de Mathématiques, Palaiseau, 1991, Exp. No. XVI, 19 pp. MR 1131589 Zbl 0739.35096

[24] P. Gérard and E. Leichtnam, Ergodic properties of eigenfunctions for the Dirichlet problem. *Duke Math. J.* **71** (1993), no. 2, 559–607. MR 1233448 Zbl 0788.35103

[25] P. Gérard, P. A. Markowich, N. J. Mauser, and F. Poupaud, Homogenization limits and Wigner transforms. *Comm. Pure Appl. Math.* **50** (1997), no. 4, 323–379. MR 1438151 Zbl 0881.35099

[26] B. Helffer, A. Martinez, and D. Robert, Ergodicité et limite semi-classique. *Comm. Math. Phys.* **109** (1987), no. 2, 313–326. MR 0880418 Zbl 0624.58039

[27] P. E. T. Jorgensen, Representations of differential operators on a Lie group. *J. Functional Analysis* **20** (1975), no. 2, 105–135.. MR 0383469 Zbl 0311.43003

[28] A. Kaplan, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms. *Trans. Amer. Math. Soc.* **258** (1980), no. 1, 147–153. MR 0554324 Zbl 0393.35015

[29] P.-L. Lions et T. Paul, Sur les mesures de Wigner. *Rev. Mat. Iberoamericana* **9** (1993), no. 3, 553–618. MR 1251718 Zbl 0801.35117

[30] F. Macià, High-frequency propagation for the Schrödinger equation on the torus. *J. Funct. Anal.* **258** (2010), no. 3, 933–955. MR 2558183 Zbl 1180.35438

[31] F. Macià, Semiclassical measures and the Schrödinger flow on Riemannian manifolds. *Nonlinearity* **22** (2009), no. 5, 1003–1020. MR 2501034 Zbl 1166.81020

[32] F. Macià, The Schrödinger flow in a compact manifold: high-frequency dynamics and dispersion. In M. Ruzhansky and J. Wirth (eds.), *Modern aspects of the theory of partial differential equations.* Operator Theory: Advances and Applications, 216. Advances in Partial Differential Equations (Basel). Birkhäuser/Springer Basel AG, Basel, 2011, 275–289. MR 2858875

[33] F. Macià and G. Rivière, Concentration and non-concentration for the Schrödinger evolution on Zoll manifolds. *Comm. Math. Phys.* **345** (2016), no. 3, 1019–1054. MR 3519588 Zbl 1348.58017

[34] D. Manchon, Weyl symbolic calculus on any Lie group. *Acta Appl. Math.* **30** (1993), no. 2, 159–186. MR 1204731 Zbl 0779.22005

[35] D. Manchon, Calcul symbolique sur les groupes de Lie nilpotents et applications. *J. Funct. Anal.* **102** (1991), no. 1, 206–251. MR 1138843 Zbl 0745.22006

[36] A. Melin, Parametrix constructions for some classes of right-invariant differential operators on the Heisenberg group. *Comm. Partial Differential Equations* **6** (1981), no. 12, 1363–1405. MR 0640161 Zbl 0486.35007

[37] A. Melin, Parametrix constructions for right invariant differential operators on nilpotent groups. *Ann. Global Anal. Geom.* **1** (1983), no. 1, 79–130. MR 0739894 Zbl 0524.58044

[38] N. Savale, Spectrum and abnormals in sub-Riemannian geometry: the 4D quasi-contact case. Preprint, 2019. arXiv:1909.00409

[39] A. I. Shnirelman, Ergodic properties of eigenfunctions. *Uspehi Mat. Nauk* **29** (1974), no. 6(180), 181–182. MR 0402834 Zbl 0788.35103

[40] R. Strichartz, Sub-Riemannian geometry. *J. Differential Geom.* **24** (1986), no. 2, 221–263. MR 0862049 Zbl 0609.53021

[41] M. E. Taylor, Noncommutative microlocal analysis. I. *Mem. Amer. Math. Soc.* **52** (1984), no. 313, ɪᴠ+182 pp. MR 0764508 Zbl 0554.35025 Revised version, 1984: https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/NCMLMS.pdf

[42] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.* **55** (1987), no. 4, 919–941. MR 0916129 Zbl 0643.58029

[43] S. Zelditch, Index and dynamics of quantized contact transformations. *Ann. Inst. Fourier* (*Grenoble*) **47** (1997), no. 1, 305–363. MR 1437187 Zbl 0865.47018

[44] S. Zelditch and M. Zworski, Ergodicity of eigenfunctions for ergodic billiards. *Comm. Math. Phys.* **175** (1996), no. 3, 673–682. MR 1372814 Zbl 0840.58048

Clotilde Fermanian-Kammerer,
Univ Paris-Est Créteil, CNRS, Lama, F-94010, Créteil, France,
Univ Gustave Eiffel, LAMA, F-77447 Marne-la-Vallée, France

e-mail: clotilde.fermanian@u-pec.fr

Véronique Fischer, Department of Mathematical Sciences, University of Bath,
Bath, BA2 7AY, UK

e-mail: v.c.m.fischer@bath.ac.uk