

A Szegő type theorem and distribution of symplectic eigenvalues

Rajendra Bhatia,¹ Tanvi Jain,² and Ritabrata Sengupta³

Abstract. We study the properties of stationary G-chains in terms of their generating functions. In particular, we prove an analogue of the Szegő limit theorem for symplectic eigenvalues, derive an expression for the entropy rate of stationary quantum Gaussian processes, and study the distribution of symplectic eigenvalues of truncated block Toeplitz matrices. We also introduce a concept of symplectic numerical range, analogous to that of numerical range, and study some of its basic properties, mainly in the context of block Toeplitz operators.

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1. Introduction

A quantum state ρ in a bosonic Fock space $\Gamma(\mathbb{C}^k)$ is a positive semidefinite operator with trace one. Let $q_1, p_1; \dots; q_k, p_k$ be k pairs of position-momentum observables of a quantum system with k degrees of freedom satisfying the canonical commutation relations. We introduce the observables $(X_1, X_2, \dots, X_{2k-1}, X_{2k}) = (q_1, p_1, \dots, q_k, p_k)$. Then if ρ has finite second moments, we write the covariance matrix of ρ as $A = [\text{Cov}_\rho(X_i, X_j)]_{i,j=1}^{2k}$, where

$$\text{Cov}_\rho(X_i, X_j) = \text{Tr} \frac{1}{2}(X_i X_j + X_j X_i)\rho - (\text{Tr} X_i \rho)(\text{Tr} X_j \rho).$$

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The complete Heisenberg uncertainty principle for all the position and momentum observables assumes the form of the following matrix inequality:

$$A + \frac{i}{2}J_{2k} \geq 0, \quad (1.1)$$

where $J_{2k} = \underbrace{J_2 \oplus \cdots \oplus J_2}_{k \text{ times}}$ with $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Following the terminology in [19], we call a real $2k \times 2k$ positive definite matrix A satisfying inequality (1.1) a *G-matrix*. A standard result in quantum theory states that a k -mode, mean zero, Gaussian quantum state is uniquely represented by its covariance matrix, which is a G-matrix. Conversely any $2k \times 2k$ G-matrix is the covariance matrix of a unique (up to permutation) k -mode mean zero quantum Gaussian state in the Fock space $\Gamma(\mathbb{C}^k)$, see [17, 11]. Finite mode quantum Gaussian states and quantum Gaussian processes have been extensively studied in quantum optics, quantum probability, and quantum information – both in theory as well as in experiments. A comprehensive survey of Gaussian states and their properties can be found in the two books of Holevo [10, 11]. For their applications to quantum information theory the reader is referred to the survey article by Weedbrook et al. [29], Holevo’s book [11], and the new book of Serafini [23].

In the present paper, our concern is with a stationary quantum Gaussian process. This is a chain of finite mode (k mode) quantum Gaussian states exhibiting stationarity. Let $\{\rho_n\}$ be a chain of quantum Gaussian states with covariance matrices $\{T_n\}$. The stationarity property means that each T_n is a positive definite block Toeplitz matrix such that T_n is the leading principal sub-matrix of T_{n+1} . This sequence $\{T_n\}$ gives rise to an infinite block Toeplitz matrix Σ . We call this chain $\{\rho_n\}$ of quantum Gaussian states a *stationary* quantum Gaussian process and the infinite matrix Σ a *G-chain* [19]. Thus a G-chain Σ is an infinite block Toeplitz matrix. The classical version of such objects has been well studied in probability theory. (See for instance [12].) A study of the quantum version has been initiated in [19, 20]. In order to study G-chains, we need to study properties of infinite block Toeplitz matrices with blocks of size $2k \times 2k$. Every leading $n \times n$ principal block sub-matrix gives a covariance matrix of an nk -mode quantum Gaussian state. Toeplitz matrices play an important part in the study of stationary processes in classical probability theory as well. See, e.g., Grenander and Szegő [8].

Among real positive definite matrices, G-matrices are characterised by a simple property of their symplectic eigenvalues. *Williamson’s theorem* [32] tells us that for every $2k \times 2k$ real positive definite matrix A , there exists a symplectic

matrix M such that

$$MAM^T = d_1(A)I_2 \oplus \cdots \oplus d_k(A)I_2,$$

where $d_1(A) \leq \cdots \leq d_k(A)$ are positive numbers uniquely determined by A . These are uniquely determined by A . We call these numbers the *symplectic eigenvalues* of A . We can see that a matrix A is a G-matrix if and only if all its symplectic eigenvalues $d_j(A) \geq \frac{1}{2}$. There has recently been considerable interest in the study of various properties of symplectic eigenvalues (see for instance [1, 4, 6, 9]), due to their close connection with quantum optics and thermodynamics [11, 15].

Given a k -mode quantum Gaussian state with covariance matrix A , the *von Neumann entropy* of the state is given by

$$S(A) = \sum_{j=1}^k \frac{2d_j(A) + 1}{2} H\left(\frac{2d_j(A) - 1}{2d_j(A) + 1}\right), \quad (1.2)$$

where H is the Shannon entropy function given by

$$H(t) = -t \log t - (1 - t) \log(1 - t), \quad 0 \leq t \leq 1,$$

and

$$H(0) = H(1) = 0.$$

See [5, 18], or [23], pp. 61–62. Let T_n be the covariance matrix of a k -mode stationary quantum Gaussian process, truncated at level n . The *entropy rate* of the process is defined as

$$\lim_{n \rightarrow \infty} \frac{S(T_n)}{n}.$$

An important problem in information theory has been the study of the entropy rate of any given stationary process. This can be very complicated [3, 7]. The entropy rate for a certain type of stationary quantum Gaussian process was calculated in [19]. We compute the entropy rate for a more general class, namely, the class of bounded partially symmetric stationary quantum Gaussian processes. Let $\Sigma = [A_{i-j}]$ be a G-chain corresponding to a stationary quantum Gaussian process. We call this process *bounded* if Σ is a bounded operator on l_{2k}^2 (the space of square summable sequences of elements of \mathbb{C}^{2k}). In this case Σ is a Toeplitz operator generated by a matrix symbol \tilde{A} in $L_{2k \times 2k}^\infty$. The process is *partially symmetric* if $A_{-n} = A_n$ for all $n \in \mathbb{N}$. We show that a stationary quantum Gaussian process is partially symmetric and bounded if and only if its corresponding G-chain is generated by an \tilde{A} in $L_{2k \times 2k}^\infty$ such that $\tilde{A}(\theta)$ is a G-matrix for almost all θ . The computation of the entropy rate requires a study of

the distribution of symplectic eigenvalues of block Toeplitz matrices. To achieve this we prove a symplectic analogue of a fundamental theorem for the distribution of eigenvalues of Toeplitz matrices, well-known as the Szegő limit theorem [8, 14]. The classical Szegő theorem can be stated as follows. *Suppose $\varphi: (-\pi, \pi) \rightarrow \mathbb{R}$ is an essentially bounded function, and (T_n) is the sequence of Hermitian Toeplitz matrices generated by φ . Then for every function f , continuous on the interval $[\text{ess inf } \varphi, \text{ess sup } \varphi]$, one has*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\lambda_j(T_n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi(x)) \, dx, \quad (1.3)$$

where $\lambda_j(T_n)$, $j = 1, 2, \dots, n$, are the eigenvalues of T_n . Many different versions and proofs of this theorem are available in the literature [2, 22, 24, 25, 26, 27, 28, 30, 31]. We prove an analogue of this theorem for symplectic eigenvalues, and apply this to compute the entropy rate and to study the distribution of symplectic eigenvalues of block Toeplitz matrices. In particular we prove that the union of the set of all symplectic eigenvalues of truncated $n \times n$ block Toeplitz matrices $T_n(\tilde{A})$ is dense in the set of all symplectic eigenvalues of $\tilde{A}(\theta)$ where $\tilde{A}(\theta)$ varies over the essential range of \tilde{A} .

In classical operator theory, the numerical range is an important and useful concept. We introduce an analogous notion of the symplectic numerical range and study its basic properties. We show that the closure of the symplectic numerical range of an operator is convex and contains the symplectic spectrum. We give a relationship between the symplectic numerical ranges of truncated block Toeplitz matrices and their symbol. This, in turn, helps us to have a better understanding of the distribution of symplectic eigenvalues of the truncated block Toeplitz matrices.

The paper is organised as follows. We give some basic notations and results in Section 2, introduce the notion of symplectic numerical range in Section 3, and study some of its basic properties, especially in the context of block Toeplitz operators. In Section 4 we prove a symplectic analogue of Szegő limit theorem, and give its applications.

2. Preliminaries

We begin with some basic facts about Toeplitz operators. For proofs and other details, the reader may refer to the book of Böttcher and Silbermann [2].

Let $L_{k \times k}^\infty$ denote the set of all functions $\tilde{A} = [\tilde{a}_{ij}]$ from $[-\pi, \pi]$ to the set of all $k \times k$ complex matrices, with $\tilde{A}(-\pi) = \tilde{A}(\pi)$ and \tilde{a}_{ij} essentially bounded for

all $i, j = 1, \dots, k$. For an \tilde{A} in $L_{k \times k}^\infty$, we define

$$\|\tilde{A}\| = \operatorname{ess\,sup}_{\theta \in [-\pi, \pi]} \|\tilde{A}(\theta)\|, \quad (2.1)$$

where $\|\tilde{A}(\theta)\|$ denotes the operator norm of $\tilde{A}(\theta)$. It is easy to see that $\|\cdot\|$ is a norm on $L_{k \times k}^\infty$. The space $L_{k \times k}^\infty$ is a C^* -algebra with the usual operations. Let L_k^2 be the set of all functions $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_k)$ from $[-\pi, \pi]$ to \mathbb{C}^k with $\tilde{x}(-\pi) = \tilde{x}(\pi)$ and $\tilde{x} \in L^2$ for all $i = 1, \dots, k$. The space L_k^2 is a Hilbert space with the inner product

$$\langle \tilde{x}, \tilde{y} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \tilde{x}(\theta), \tilde{y}(\theta) \rangle \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{i=1}^k \overline{\tilde{x}_i(\theta)} \tilde{y}_i(\theta) \, d\theta. \quad (2.2)$$

With each \tilde{A} in $L_{k \times k}^\infty$, we can associate the multiplication map $M_{\tilde{A}}$ on L_k^2 defined as

$$M_{\tilde{A}}(\tilde{x})(\theta) = \tilde{A}(\theta)\tilde{x}(\theta), \quad (2.3)$$

where \tilde{x} is here understood as a column vector. It can be verified that $M_{\tilde{A}}$ is a bounded linear operator on L_k^2 , and $\|M_{\tilde{A}}\| \leq \|\tilde{A}\|$. The space $\{M_{\tilde{A}}: \tilde{A} \in L_{k \times k}^\infty\}$ is a C^* -algebra, and the map $\tilde{A} \mapsto M_{\tilde{A}}$ is a surjective isomorphism. This implies that

$$\|M_{\tilde{A}}\| = \|\tilde{A}\|$$

for all \tilde{A} in $L_{k \times k}^\infty$.

Next let l_k^2 be the set of all sequences of vectors $\hat{x} = (x_0, x_1, x_2, \dots)$, $x_i \in \mathbb{C}^k$, such that $\sum_{i=0}^{\infty} \|x_i\|^2 < \infty$. Here the norm $\|x_i\|$ is the Euclidean norm of $x_i = (x_i^{(1)}, \dots, x_i^{(k)})$. The space l_k^2 is a Hilbert space with the inner product given by

$$\langle \hat{x}, \hat{y} \rangle = \sum_{i=0}^{\infty} \langle x_i, y_i \rangle. \quad (2.4)$$

Clearly this inner product induces the l^2 norm on l_k^2 . We denote this norm by $\|\cdot\|_2$. In a similar way, $l_k^2(\mathbb{Z})$ is the Hilbert space of all square summable doubly infinite sequences \hat{x} of vectors with the l^2 norm.

Throughout this paper, we denote the elements of $L_{k \times k}^\infty(L_k^2)$ by $\tilde{A}, \tilde{B}, \dots$ ($\tilde{u}, \tilde{x}, \dots$), the elements of l_k^2 by \hat{u}, \hat{x}, \dots , and the usual matrices (vectors) by A, B, \dots (u, x, \dots), unless we mention otherwise.

Let $\tilde{A} \in L_{k \times k}^\infty$. For each $n \in \mathbb{Z}$, let A_n be the n th Fourier coefficient of \tilde{A} given by

$$A_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{A}(\theta) e^{-in\theta} \, d\theta.$$

Suppose that $L(\tilde{A})$ is the doubly infinite $k \times k$ block Toeplitz matrix $[A_{i-j}]_{i,j=-\infty}^{\infty}$. Since $l_k^2(\mathbb{Z})$ and L_k^2 are isomorphic Hilbert spaces, we can identify $L(\tilde{A})$ with the linear operator $M_{\tilde{A}}$ defined in (2.3). Let $T(\tilde{A})$ be the infinite block Toeplitz matrix $[A_{i-j}]_{i,j=0}^{\infty}$. This is a principal submatrix of $L(\tilde{A})$. If for $n \in \mathbb{N}$, \bar{P}_n is the projection operator on $l_k^2(\mathbb{Z})$ defined as

$$\begin{aligned} \bar{P}_n(\dots, x_{-n}, x_{-(n-1)}, \dots, x_0, \dots, x_n, \dots) \\ = (\dots, 0, 0, x_{-(n-1)}, \dots, x_0, \dots, x_n, \dots), \end{aligned}$$

then $\bar{P}_n L(\tilde{A}) \bar{P}_n$ converges strongly to $L(\tilde{A})$, and for every n , $\bar{P}_n L(\tilde{A}) \bar{P}_n = T(\tilde{A})$. For \tilde{A} in $L_{k \times k}^{\infty}$, we say $T(\tilde{A})$ is the infinite block Toeplitz matrix generated by \tilde{A} and \tilde{A} is the *symbol* of the block Toeplitz operator $T(\tilde{A})$.

Proposition 2.1. *Let $\Sigma = [A_{i-j}]_{i,j=0}^{\infty}$ be an infinite block Toeplitz matrix. Then Σ is a bounded linear operator on l_k^2 if and only if $\Sigma = T(\tilde{A})$ for some \tilde{A} in $L_{k \times k}^{\infty}$. In this case $\|\Sigma\| = \|\tilde{A}\|$. For every $\hat{x} \in l_k^2$,*

$$\langle \hat{x}, T(\tilde{A})\hat{x} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \tilde{x}(\theta), \tilde{A}(\theta)\tilde{x}(\theta) \rangle d\theta, \quad (2.5)$$

where \tilde{x} is the element of L_k^2 defined as $\tilde{x}(\theta) = \sum_{n=0}^{\infty} x_n e^{in\theta}$.

A Hermitian operator T on a Hilbert space \mathcal{H} is said to be *positive semidefinite* if $\langle x, Tx \rangle \geq 0$ for all x in \mathcal{H} . If equality here holds only for the null vector, then T is said to be *positive definite*. If the space \mathcal{H} is finite-dimensional, a positive semidefinite operator is positive definite if and only if it is invertible. This is not the case when \mathcal{H} is infinite-dimensional (consider, e.g., the operator $T = \text{diag}(1, 1/2, 1/3, \dots)$ on the space l_2). So, we will use the term *positive invertible* for an operator that is positive definite and invertible. Let $T_n(\tilde{A})$ be the truncated $n \times n$ block Toeplitz matrix $[A_{i-j}]_{i,j=0}^{n-1}$. The operator $T(\tilde{A})$ is positive semidefinite if and only if all $T_n(\tilde{A})$ are positive semidefinite.

The *essential range* of \tilde{A} is given by the set of all $k \times k$ matrices B such that for every $\epsilon > 0$, $m(\{t: \|\tilde{A}(t) - B\| < \epsilon\}) > 0$. Here $m(\cdot)$ denotes the Lebesgue measure. We denote the essential range of \tilde{A} by $\mathcal{R}(\tilde{A})$. Clearly the essential range of \tilde{A} is closed in the space of $k \times k$ matrices and is contained in the closure of the range of \tilde{A} . So, if $\tilde{A} \in L_{k \times k}^{\infty}$, then $\mathcal{R}(\tilde{A})$ is compact. Also if $X \subseteq [-\pi, \pi]$ is any set such that $\tilde{A}(X) \cap \mathcal{R}(\tilde{A}) = \emptyset$, then $m(X) = 0$.

Proposition 2.2. *Let $\tilde{A} \in L_{k \times k}^\infty$. Then $T(\tilde{A})$ is a positive semidefinite operator on l_k^2 if and only if all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are positive semidefinite. Consequently the matrices $T_n(\tilde{A})$ are positive semidefinite for all n if and only if all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are positive semidefinite.*

If $T(\tilde{A})$ is positive invertible, then all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are positive definite.

The following proposition gives an equivalent condition for $T_n(\tilde{A})$ to be positive definite for each n . See [16].

Proposition 2.3. *For every $\tilde{A} \in L_{k \times k}^\infty$, $T_n(\tilde{A})$ is positive definite for every n if and only if all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are positive semidefinite, and $\tilde{A}(\theta)$ are positive definite for all θ in some subset of $[-\pi, \pi]$ that has positive measure*

We call a Toeplitz operator $\Sigma = [A_{i-j}]$ *partially symmetric* if each A_n is a real matrix and $A_{-n} = A_n$ for all $n \in \mathbb{N}$. An element \tilde{A} of $L_{k \times k}^\infty$ is *even* if $\tilde{A}(-\theta) = \tilde{A}(\theta)$ for almost all $\theta \in [-\pi, \pi]$.

Proposition 2.4. *For any \tilde{A} in $L_{k \times k}^\infty$, the following statements are equivalent:*

- (i) $T(\tilde{A})$ is partially symmetric;
- (ii) \tilde{A} is even and every matrix $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ is real;
- (iii) the infinite matrix $T(\tilde{A})$ is real and every matrix $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ is real.

Here we point out that symplectic eigenvalues of $T_n(\tilde{A})$ and $\tilde{A}(\theta)$ are defined only when $T(\tilde{A})$ is a partially symmetric operator on l_{2k}^2 , and $T_n(\tilde{A})$ and $\tilde{A}(\theta)$ are positive definite.

A stationary G-chain $\Sigma = [A_{i-j}]$ is *bounded* if it is bounded as a linear operator on l_{2k}^2 , and is *partially symmetric* if it is a partially symmetric linear operator. The following theorem gives a characterisation of a partially symmetric bounded stationary G-chain in terms of its symbol.

Theorem 2.5. *Let Σ be an infinite real matrix. Then Σ is a partially symmetric bounded stationary G-chain if and only if it is generated by an \tilde{A} in $L_{2k \times 2k}^\infty$ such that $\tilde{A}(\theta)$ is a G-matrix for all $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$.*

Proof. By Proposition 2.1 Σ is a bounded linear operator on l_{2k}^2 if and only if $\Sigma = T(\tilde{A})$ for some \tilde{A} in $L_{2k \times 2k}^\infty$. By Propositions 2.2 and 2.4 we know that $\tilde{A}(\theta)$ are real positive semidefinite matrices for all $\tilde{A}(\theta) \in \mathcal{R}(\tilde{A})$ if and only if $T(\tilde{A})$ is partially symmetric and positive semidefinite. Let $\Sigma_0 = T(\tilde{A}) + \frac{1}{2}J_\infty$, where

J_∞ is the infinite block diagonal matrix $\bigoplus_{\mathbb{N}} J_2$. Clearly Σ_0 is the infinite $2k \times 2k$ block Toeplitz matrix corresponding to the sequence $\langle B_n \rangle_{n \in \mathbb{Z}}$, where

$$B_n = \begin{cases} A_n & n \neq 0, \\ A_0 + \frac{l}{2} J_{2k} & n = 0. \end{cases}$$

One can see that Σ_0 is generated by the function $\tilde{B} = \tilde{A} + \frac{l}{2} J_{2k}$. Now $\tilde{A}(\theta)$ is a G-matrix if and only if $\tilde{B}(\theta)$ is positive semidefinite. Similarly, $T(\tilde{A})$ is a G-chain if and only if $T_n(\tilde{B})$ is positive semidefinite for every $n \in \mathbb{N}$. Hence we obtain the theorem by using Proposition 2.2. \square

3. Symplectic numerical range

Let \mathcal{H} be a real separable Hilbert space. We denote the direct sum $\mathcal{H} \oplus \mathcal{H}$ by $\tilde{\mathcal{H}}$. It is easy to see that the space $\tilde{\mathcal{H}}$ is isomorphic to $\bigoplus_{\mathbb{N}} \mathcal{K}$ where \mathcal{K} is a two dimensional real Hilbert space and the operator $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ on $\tilde{\mathcal{H}}$ is orthogonally equivalent to $\bigoplus_{\mathbb{N}} J_2$. Henceforth we will identify $\tilde{\mathcal{H}}$ with $\bigoplus_{\mathbb{N}} \mathcal{K}$ and the operator J with $\bigoplus_{\mathbb{N}} J_2$.

Definition 3.1. Let A be a positive definite operator on $\tilde{\mathcal{H}}$. We define the symplectic numerical range of A to be the set

$$W_s(A) = \left\{ \frac{1}{2}(\langle u, Au \rangle + \langle v, Av \rangle) : \langle u, Jv \rangle = 1, u, v \in \tilde{\mathcal{H}} \right\}.$$

This is a subset of $(0, \infty)$. It is unbounded as the set of vectors (u, v) with $\langle u, Jv \rangle = 1$ is unbounded. An infinite dimensional version of Williamson's theorem was proved in [21]: for any positive invertible operator A on $\tilde{\mathcal{H}}$ there exists a positive invertible operator P on \mathcal{H} and a symplectic transformation $L: \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ such that

$$A = L \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} L^T.$$

The *symplectic spectrum* of A is the spectrum of the positive invertible operator P . If A is a $2n \times 2n$ real positive definite matrix, then its symplectic spectrum is the set of its symplectic eigenvalues $\{d_1(A), \dots, d_n(A)\} \subseteq (0, \infty)$. We denote by $\sigma_s(A)$ the symplectic spectrum of A .

Proposition 3.2. *Let \mathcal{H} be a real separable Hilbert space and A a bounded positive invertible operator on $\tilde{\mathcal{H}}$. Then*

- (i) $W_s(A) = W_s(MAM^T)$ for every symplectic transformation M ;
- (ii) $\sigma_s(A) \subseteq \overline{W_s(A)} = [\inf \sigma_s(A), \infty)$;
- (iii) if \mathcal{H} is finite-dimensional, then $W_s(A)$ is the closed set $[d_1(A), \infty)$, where $d_1(A)$ is the minimum symplectic eigenvalue of A .

Proof. Part (i) follows from the fact that $\langle Mu, JMv \rangle = \langle u, Jv \rangle$ for every symplectic transformation M .

Let $u, v \in \tilde{\mathcal{H}}$ be such that $\langle u, Jv \rangle = 1$. Let

$$\alpha = \frac{1}{2}(\langle u, Au \rangle + \langle v, Av \rangle), \quad \alpha_1 = \frac{1}{2}\langle u, Au \rangle, \quad \alpha_2 = \frac{1}{2}\langle v, Av \rangle.$$

For any $t > 0$, $\langle tu, Jv/t \rangle = 1$. Let

$$\alpha(t) = \frac{\langle tu, A(tu) \rangle + \langle v/t, A(v/t) \rangle}{2} = t^2\alpha_1 + \frac{1}{t^2}\alpha_2.$$

Clearly $\alpha(t)$ is continuous in t and $\alpha(1) = \alpha$. Since $\lim_{t \rightarrow \infty} \alpha(t) = \infty$, by the intermediate value theorem, $[\alpha, \infty) \subseteq W_s(A)$. Thus $\overline{W_s(A)} = [\inf W_s(A), \infty)$. Now, let P be the positive invertible operator on \mathcal{H} such that

$$A = L\hat{P}L^T,$$

where L is a symplectic transformation on $\tilde{\mathcal{H}}$ and $\hat{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$. We know that $\sigma_s(A) = \sigma(P) = \sigma(\hat{P})$. Thus, we only need to show that $\inf W_s(A) = \inf \sigma(\hat{P})$. Let u, v be two distinct unit vectors in $\tilde{\mathcal{H}}$. Without loss of generality, we can assume that $\langle u, Jv \rangle > 0$. Clearly $\langle u, Jv \rangle \leq 1$. Let $u_0 = u/\sqrt{\langle u, Jv \rangle}$ and $v_0 = v/\sqrt{\langle u, Jv \rangle}$. Then $\langle u_0, Jv_0 \rangle = 1$, and

$$\frac{\langle u_0, Au_0 \rangle + \langle v_0, Av_0 \rangle}{2} = \frac{\langle u, Au \rangle + \langle v, Av \rangle}{2\langle u, Jv \rangle} \geq \frac{\langle u, Au \rangle + \langle v, Av \rangle}{2}. \quad (3.1)$$

Since the left-hand side of (3.1) belongs to $W_s(\hat{P})$ and the right-hand side to $W(\hat{P})$, it follows that

$$\inf W_s(\hat{P}) \geq \inf W(\hat{P}). \quad (3.2)$$

Now let x be any unit vector in \mathcal{H} , and let $u = \frac{1}{\sqrt{2}}(x \oplus x)$ and $v = \frac{1}{\sqrt{2}}(-x \oplus x)$. Then $\langle u, Jv \rangle = 1$. We see that

$$\langle x, Px \rangle = \frac{\langle u, \hat{P}u \rangle + \langle v, \hat{P}v \rangle}{2}.$$

This implies that

$$\inf W(P) \geq \inf W_s(\hat{P}). \quad (3.3)$$

Combining (3.2) and (3.3), and using the fact that $\inf W(\hat{P}) = \inf W(P) = \inf \sigma(P)$, we obtain (ii).

When \mathcal{H} is finite-dimensional, we have

$$d_1(A) = \min_{\substack{u, v \in \mathcal{H} \\ \langle u, Jv \rangle = 1}} \frac{\langle u, Au \rangle + \langle v, Av \rangle}{2}.$$

(See Theorem 5 of [1].) This gives part (iii). \square

Let $\tilde{A} \in L_{2k \times 2k}^\infty$ be such that all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are real positive definite. Then the symplectic numerical range of \tilde{A} is the set

$$W_s(\tilde{A}) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\langle \tilde{u}(\theta), \tilde{A}(\theta)\tilde{u}(\theta) \rangle + \langle \tilde{v}(\theta), \tilde{A}(\theta)\tilde{v}(\theta) \rangle}{2} d\theta : \right. \\ \left. \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \tilde{u}(\theta), J\tilde{v}(\theta) \rangle d\theta = 1, \tilde{u}, \tilde{v} \in L_{2k}^2 \right\}.$$

We next give a relationship between $W_s(\tilde{A})$ and $W_s(\tilde{A}(\theta))$ for $\tilde{A}(\theta) \in \mathcal{R}(\tilde{A})$.

Theorem 3.3. *Let \tilde{A} be an element of $L_{2k \times 2k}^\infty$ such that all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are real positive definite. The set $W_s(\tilde{A})$ is the same as the closed convex hull of $\bigcup_{\tilde{A}(\theta) \in \mathcal{R}(\tilde{A})} W_s(\tilde{A}(\theta))$.*

Proof. Let $B = \tilde{A}(\theta) \in \mathcal{R}(\tilde{A})$ and $\mu \in W_s(B)$. Then there exists a pair (u, v) in $\mathbb{R}^{2k} \times \mathbb{R}^{2k}$ such that $\langle u, J_{2k}v \rangle = 1$ and $\mu = \frac{1}{2}(\langle u, Bu \rangle + \langle v, Bv \rangle)$. For $n \in \mathbb{N}$, let S_n be the set

$$S_n = \left\{ t : \|\tilde{A}(t) - B\| < \frac{1}{n(\|u\|^2 + \|v\|^2)} \right\},$$

and let $m_n = m(S_n)$, the measure of S_n . Since $B \in \mathcal{R}(\tilde{A})$, $m_n > 0$ for every n . Define the vector functions \tilde{u}_n and \tilde{v}_n on $[-\pi, \pi]$ as

$$\tilde{u}_n(t) = \begin{cases} \sqrt{\frac{2\pi}{m_n}} u & t \in S_n, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{v}_n(t) = \begin{cases} \sqrt{\frac{2\pi}{m_n}} v & t \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly \tilde{u}_n and \tilde{v}_n are in L_{2k}^2 , and $\langle \tilde{u}_n, J_{2k}\tilde{v}_n \rangle = 1$. Let

$$\mu_n = \frac{1}{2}(\langle \tilde{u}_n, \tilde{A}\tilde{u}_n \rangle + \langle \tilde{v}_n, \tilde{A}\tilde{v}_n \rangle).$$

Then $\mu_n \in W_s(\tilde{A})$, and we have

$$\begin{aligned}
 |\mu_n - \mu| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\langle \tilde{u}_n(t), \tilde{A}(t)\tilde{u}_n(t) \rangle + \langle \tilde{v}_n(t), \tilde{A}(t)\tilde{v}_n(t) \rangle) dt \right. \\
 &\quad \left. - \frac{1}{2} (\langle u, Bu \rangle + \langle v, Bv \rangle) \right| \\
 &= \left| \frac{1}{m_n} \int_{S_n} \frac{1}{2} (u, \tilde{A}(t)u) + \langle v, \tilde{A}(t)v \rangle dt \right. \\
 &\quad \left. - \frac{1}{m_n} \int_{S_n} \frac{1}{2} (\langle u, Bu \rangle + \langle v, Bv \rangle) dt \right| \\
 &= \left| \frac{1}{m_n} \int_{S_n} \frac{1}{2} (\langle u, (\tilde{A}(t) - B)u \rangle + \langle v, (\tilde{A}(t) - B)v \rangle) dt \right| \\
 &\leq \frac{1}{2m_n} (\|u\|^2 + \|v\|^2) \int_{S_n} \|\tilde{A}(t) - B\| dt \\
 &\leq \frac{1}{2n}.
 \end{aligned}$$

This proves $\mu_n \rightarrow \mu$. Hence $W_s(B) \subseteq \overline{W_s(\tilde{A})}$. Since $W_s(\tilde{A})$ is convex, the closed convex hull of $\bigcup_{B \in \mathcal{R}(\tilde{A})} W_s(B)$ is contained in $W_s(\tilde{A})$.

To prove the reverse inclusion, we use the fact that every element of $W_s(\tilde{A})$ is a limit of finite sums of the form

$$\sum_j \frac{\alpha_j}{4\pi} (\langle u_j, \tilde{A}(\theta_j)u_j \rangle + \langle v_j, \tilde{A}(\theta_j)v_j \rangle),$$

where $\tilde{A}(\theta_j) \in \mathcal{R}(\tilde{A})$, and $\alpha_j \geq 0$ are such that $\sum_j \alpha_j \langle u_j, Jv_j \rangle = 1$. Let $\beta_j = \langle u_j, Jv_j \rangle$. Without loss of generality we may assume that $\beta_j \geq 0$ for all j . Replacing u_j by $\sqrt{\beta_j}u_j$ and v_j by $\sqrt{\beta_j}v_j$ we can take $\langle u_j, J_{2k}v_j \rangle = 1$ for every j , and $\sum \alpha_j = 1$. This shows that every element of $W_s(\tilde{A})$ is a limit of convex combinations of elements of $\bigcup_{\tilde{A}(\theta) \in \mathcal{R}(\tilde{A})} W_s(\tilde{A}(\theta))$. \square

Let $\hat{u}, \hat{v} \in l_{2k}^2$, $\hat{u} = (u_1, u_2, \dots)$ and $\hat{v} = (v_1, v_2, \dots)$. Define $\tilde{u}(\theta) = \sum u_n e^{in\theta}$ and $\tilde{v}(\theta) = \sum v_n e^{in\theta}$. Clearly

$$\langle \hat{u}, J\hat{v} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle \tilde{u}(\theta), J_{2k}\tilde{v}(\theta) \rangle d\theta.$$

Using (2.5) we see that

$$W_s(T(\tilde{A})) \subseteq W_s(\tilde{A})$$

for every partially symmetric, bounded, positive invertible operator $T(\tilde{A})$ on l_{2k}^2 . Since $T_n(\tilde{A})$ is a principal submatrix of $T(\tilde{A})$, we have

$$W_s(T_n(\tilde{A})) \subseteq W_s(T_{n+1}(\tilde{A})) \subseteq W_s(T(\tilde{A})) \subseteq W_s(\tilde{A}). \quad (3.4)$$

Let $\tilde{A} \in L_{2k \times 2k}^\infty$ be such that all matrices $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ are real positive definite. Let

$$m_{\tilde{A}} = \operatorname{ess\,inf}_{\theta \in [-\pi, \pi]} d_1(\tilde{A}(\theta)). \quad (3.5)$$

Using Theorem 3.3 we see that

$$m_{\tilde{A}} = \inf W_s(\tilde{A}). \quad (3.6)$$

Theorem 3.4. *Let $T(\tilde{A})$ be a partially symmetric, bounded, positive invertible operator on l_{2k}^2 . Let $n \in \mathbb{N}$, and let d be a symplectic eigenvalue of $T_n(\tilde{A})$. Then $d \geq m_{\tilde{A}}$. If $d = m_{\tilde{A}}$, then $d_1(\tilde{A}(\theta))$ is the constant $m_{\tilde{A}}$ for almost all θ .*

Proof. Let d be a symplectic eigenvalue of $T_n(\tilde{A})$. Then there exist vectors $\hat{u} = (u_1, \dots, u_n)$, $\hat{v} = (v_1, \dots, v_n)$, $u_j, v_j \in \mathbb{R}^{2k}$ with $\langle \hat{u}, J \hat{v} \rangle = 1$ and

$$d = \frac{1}{2}(\langle \hat{u}, T_n(\tilde{A})\hat{u} \rangle + \langle \hat{v}, T_n(\tilde{A})\hat{v} \rangle).$$

Let \tilde{u} and \tilde{v} be the elements of L_{2k}^2 given by $\tilde{u}(\theta) = \sum u_j e^{ij\theta}$, $\tilde{v}(\theta) = \sum v_j e^{ij\theta}$. Then

$$\begin{aligned} d - m_{\tilde{A}} &= \frac{1}{2}(\langle \hat{u}, T_n(\tilde{A})\hat{u} \rangle + \langle \hat{v}, T_n(\tilde{A})\hat{v} \rangle) - m_{\tilde{A}} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2}(\langle \tilde{u}(\theta), \tilde{A}(\theta)\tilde{u}(\theta) \rangle + \langle \tilde{v}(\theta), \tilde{A}(\theta)\tilde{v}(\theta) \rangle) \right. \\ &\quad \left. - m_{\tilde{A}} \langle \tilde{u}(\theta), J_{2k}\tilde{v}(\theta) \rangle \right] d\theta. \end{aligned}$$

Using (3.6), we know that the above integrand is nonnegative almost everywhere. Hence, we have $d \geq m_{\tilde{A}}$. Also $d = m_{\tilde{A}}$ if and only if

$$\frac{1}{2}(\langle \tilde{u}(\theta), \tilde{A}(\theta)\tilde{u}(\theta) \rangle + \langle \tilde{v}(\theta), \tilde{A}(\theta)\tilde{v}(\theta) \rangle) = m_{\tilde{A}} \langle \tilde{u}(\theta), J_{2k}\tilde{v}(\theta) \rangle$$

for almost all θ . So the last statement of the theorem by using Proposition 3.2(iii) and (3.5). \square

4. A Szegő type theorem for symplectic eigenvalues and applications

We first recall some basic facts about symplectic eigenvalues. See [1, 4], and [13] for details. A positive number d is a symplectic eigenvalue of a $2k \times 2k$ positive definite matrix A if and only if $\pm d$ are the eigenvalues of the (non-Hermitian) matrix $\iota J_{2k} A$. Thus each symplectic eigenvalue d_i of A lies in the interval $[0, \|A\|]$. Let $T(\tilde{A})$ be a bounded, partially symmetric, positive invertible operator on l_{2k}^2 generated by \tilde{A} . Let $d_1^{(n)} \leq \dots \leq d_{nk}^{(n)}$ denote the symplectic eigenvalues of $T_n(\tilde{A})$ arranged in increasing order. Since $\tilde{A} \in L_{2k \times 2k}^\infty$, each $d_i^{(n)} \leq \|T_n(\tilde{A})\| \leq \|\tilde{A}\|$.

Theorem 4.1. *Let Σ be a partially symmetric, bounded positive invertible operator on l_{2k}^2 generated by \tilde{A} . Let $d_1^{(n)} \leq \dots \leq d_{nk}^{(n)}$ denote the symplectic eigenvalues of $T_n(\tilde{A})$. Then for every function f continuous on $[0, \|\tilde{A}\|]$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} f(d_j^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^k f(d_j(\tilde{A}(\theta))) d\theta. \quad (4.1)$$

Proof. By Theorem 6.24 of [2], we know that if $\tilde{B} \in L_{2k \times 2k}^\infty$, and $\lambda_1^{(n)}, \dots, \lambda_{2nk}^{(n)}$ are the eigenvalues of the $n \times n$ truncated block Toeplitz matrix $T_n(\tilde{B})$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{2nk} (\lambda_j^{(n)})^m = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^{2k} (\lambda_j(\tilde{B}(\theta)))^m d\theta, \quad (4.2)$$

for every nonnegative integer m . Suppose $\tilde{B} = \iota J_{2k} \tilde{A}$. Then $T_n(\tilde{B}) = \iota J_{2nk} T_n(\tilde{A})$, and the eigenvalues $\lambda_1^{(n)}, \dots, \lambda_{2nk}^{(n)}$ of $T_n(\tilde{B})$ are $\pm d_1^{(n)}, \dots, \pm d_{nk}^{(n)}$. Also the eigenvalues $\lambda_j(\tilde{B}(\theta))$ of $\tilde{B}(\theta)$ are $\pm d_j(\tilde{A}(\theta))$. Hence for every non-negative integer m ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} (d_j^{(n)})^{2m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^k (d_j(\tilde{A}(\theta)))^{2m} d\theta. \quad (4.3)$$

By linearity, we can extend (4.3) to polynomials in $(d_j^{(n)})^2$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} p(d_j^{(n)2}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^k p(d_j^2(\tilde{A}(\theta))) d\theta \quad (4.4)$$

for every polynomial p . Each $d_j^{(n)} \in [0, \|\tilde{A}\|]$, and hence $(d_j^{(n)})^2 \in [0, \|\tilde{A}\|^2]$. Let $q(x)$ be any polynomial and let $s(x) = q(\sqrt{x})$. Clearly s is continuous on $[0, \|\tilde{A}\|^2]$. For a given $\epsilon > 0$, we can find a polynomial p such that

$$\sup_{x \in [0, \|\tilde{A}\|^2]} |s(x) - p(x)| < \epsilon.$$

Since $d_j(\tilde{A}(\theta)) \leq \|\tilde{A}(\theta)\|$ and $\|\tilde{A}(\theta)\| \leq \|\tilde{A}\|$ for almost all θ ,

$$\begin{aligned} & \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} s(d_j^2(\tilde{A}(\theta))) \, d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} p(d_j^2(\tilde{A}(\theta))) \, d\theta \right| \\ & \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|s(d_j^2(\tilde{A}(\theta))) - p(d_j^2(\tilde{A}(\theta)))\| \, d\theta \leq \epsilon, \end{aligned} \quad (4.5)$$

holds for all $j = 1, \dots, k$. Similarly,

$$|s(d_j^{(n)2}) - p(d_j^{(n)2})| < \epsilon \quad (4.6)$$

for all $j = 1, \dots, nk$. Combining the relations (4.4), (4.5), and (4.6), we see that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} s(d_j^{(n)2}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} s(d_j^2(\tilde{A}(\theta))) \, d\theta.$$

Since $s(x^2) = q(x)$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} q(d_j^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(d_j(\tilde{A}(\theta))) \, d\theta. \quad (4.7)$$

Now let f be any continuous function on $[0, \|\tilde{A}\|]$. Then by using the Weierstrass approximation theorem and arguing as above, we can show that (4.1) holds for f . \square

Remark 4.2. We know that d is a symplectic eigenvalue of a positive definite matrix A if and only if $\pm d$ are eigenvalues of the non-Hermitian matrix iJA . In [26] Tilli has proved a very general version of Szegő's limit theorem for non-Hermitian block Toeplitz matrices. It is possible to derive Theorem 4.1 from Tilli's general results. The proofs of the general version are, naturally, more intricate. We have given a short self-contained presentation for the special case we need.

The entropy rate of a stationary quantum Gaussian process with the associated G-chain $T(\tilde{A})$ is given by the formula

$$\mathfrak{S}(T(\tilde{A})) = \lim_{n \rightarrow \infty} \frac{S(T_n(\tilde{A}))}{n},$$

where $S(T_n(\tilde{A}))$ denotes the entropy of the quantum Gaussian state with the corresponding G-matrix $T_n(\tilde{A})$. As a consequence of Theorem 4.1, we obtain a

closed expression for the entropy rate of a partially symmetric, bounded stationary quantum Gaussian process in terms of the entropies of the Gaussian states with G-matrices $\tilde{A}(\theta)$.

Corollary 4.3. *Let $T(\tilde{A})$ be a partially symmetric, bounded stationary G-chain generated by \tilde{A} . The entropy rate $\mathfrak{S}(T(\tilde{A}))$ of the corresponding stationary Gaussian process is*

$$\mathfrak{S}(T(\tilde{A})) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\tilde{A}(\theta)) \, d\theta. \quad (4.8)$$

Proof. Define the function $f: [0, \|\tilde{A}\|] \rightarrow \mathbb{R}$ as

$$f(x) = \begin{cases} \left(x + \frac{1}{2}\right) \log \left(x + \frac{1}{2}\right) - \left(x - \frac{1}{2}\right) \log \left(x - \frac{1}{2}\right) & \text{if } \frac{1}{2} < x \leq \|\tilde{A}\|, \\ 0 & \text{if } 0 \leq x \leq \frac{1}{2}. \end{cases}$$

Using (1.2) we can see that the entropy of any G-matrix B can be written as

$$S(B) = \sum_{d \in \sigma_s(B)} f(d). \quad (4.9)$$

Hence the entropy rate $\mathfrak{S}(T(\tilde{A}))$ of $T(\tilde{A})$ is given by

$$\mathfrak{S}(T(\tilde{A})) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} f(d_j(T_n(\tilde{A}))). \quad (4.10)$$

Since f is continuous and $\tilde{A} \in L_{2k \times 2k}^\infty$, we can apply Theorem 4.1 to get

$$\mathfrak{S}(T(\tilde{A})) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{k} \sum_{j=1}^k f(d_j(\tilde{A}(\theta))) \, d\theta.$$

Using the formula (4.9) for the sum inside this integral, we obtain (4.8). \square

The entropy rate of a special kind of stationary quantum Gaussian process has been computed in the paper [19]. There the authors considered a block Toeplitz matrix $T_n(\tilde{A})$ given by

$$T_n(\tilde{A}) = \begin{cases} A & n = 0, \\ p_{|n|} B & \text{otherwise;} \end{cases}$$

where A and B are $2k \times 2k$ real symmetric matrices such that $A + tB$ is a G-matrix for each $t \in [-2, 2]$, and $\{p_1, p_2, \dots\}$ is a probability distribution over $\{1, 2, \dots\}$. In this case $\tilde{A}(\theta)$ takes the form $(A + \sum_{j \in \mathbb{Z} \setminus \{0\}} p_{|j|} B e^{t_j \theta})$, $\theta \in [-\pi, \pi]$. Our Corollary 4.3 gives a much more general result.

In the rest of the paper, we use Theorem 4.1 to study the distribution of symplectic eigenvalues of truncated block Toeplitz matrices. Henceforth $T(\tilde{A})$ is a partially symmetric, bounded, positive invertible operator on l_{2k}^2 generated by \tilde{A} . Recall the definition of $m_{\tilde{A}}$ given in (3.5).

Theorem 4.4. *For each $m \in \mathbb{N}$, $\lim_{n \rightarrow \infty} d_m^{(n)} = m_{\tilde{A}}$. Consequently*

$$\overline{\bigcup_{n \in \mathbb{N}} W_s(T_n(\tilde{A}))} = \overline{W_s(\tilde{A})}. \quad (4.11)$$

Proof. For every $n \in \mathbb{N}$, $T_n(\tilde{A})$ is a principal submatrix of $T_{n+1}(\tilde{A})$. Hence from the relation (42) of [1] we see that

$$0 \leq d_m^{(n+1)} \leq d_m^{(n)} \quad \text{for all } n \geq m.$$

Hence $\lim_{n \rightarrow \infty} d_m^{(n)}$ exists. Suppose this equals r . By definition $r \geq m_{\tilde{A}}$. Suppose $r > m_{\tilde{A}}$. Define f on $[0, \|\tilde{A}\|]$ as

$$f(x) = \begin{cases} 0 & x > r \text{ or } x < m_{\tilde{A}} - 1, \\ x - m_{\tilde{A}} + 1 & m_{\tilde{A}} - 1 \leq x \leq m_{\tilde{A}}, \\ \frac{x - r}{m_{\tilde{A}} - r} & m_{\tilde{A}} \leq x \leq r. \end{cases}$$

Clearly f is continuous on $[0, \|\tilde{A}\|]$, and the formula (4.1) holds for f . But $f(d_j^{(n)}) = 0$ for all $j = m, \dots, nk$ and for all $n \geq m$. So the left hand side of (4.1) is zero. But since $m_{\tilde{A}} = \text{ess inf } d_1(\tilde{A}(\theta))$ and f is positive for $[m_{\tilde{A}}, r]$, we have $f(d_j(\tilde{A}(\theta))) > 0$ for θ in a set of positive measure. Hence the right hand side of (4.1) is strictly positive. This is a contradiction.

By Proposition 3.2 (iii), we know that $W_s(T_n(\tilde{A})) = [d_1^{(n)}, \infty)$. Hence

$$\overline{\bigcup_{n \in \mathbb{N}} W_s(T_n(\tilde{A}))} = [\lim_{n \rightarrow \infty} d_1^{(n)}, \infty) = [m_{\tilde{A}}, \infty).$$

Since $m_{\tilde{A}} = \inf W_s(\tilde{A})$ and $W_s(\tilde{A})$ is convex, this proves (4.11) \square

Lemma 4.5. *Let K be a compact subset of \mathbb{R} , and let $c_n(K)$ be the cardinality of the set $\{j: d_j^{(n)} \in K\}$. Then*

$$\lim_{n \rightarrow \infty} \frac{c_n(K)}{n} = \frac{1}{2\pi} \sum_{j=1}^k m\{\theta \in [-\pi, \pi]: d_j(\tilde{A}(\theta)) \in K\}. \quad (4.12)$$

Proof. Let g be the distance function defined as

$$g(x) = |x - K| := \inf\{|x - t|: t \in K\}.$$

For any $\epsilon > 0$ define the function $f_\epsilon: [0, \|\tilde{A}\|] \rightarrow \mathbb{R}$ as $f_\epsilon(x) = e^{-\frac{g(x)}{\epsilon}}$. Clearly f_ϵ is continuous and $f_\epsilon(x) = 1$ if and only if $x \in K$. We can see that as $\epsilon \rightarrow 0$, f_ϵ converges to the characteristic function χ_K in the L^1 norm. Hence

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{j=1}^k f_\epsilon(d_j(\tilde{A}(\theta))) \, d\theta &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_j \chi_K(d_j(\tilde{A}(\theta))) \, d\theta \\ &= \frac{1}{2\pi} \sum_j m(\{\theta: d_j(\tilde{A}(\theta)) \in K\}). \end{aligned}$$

Also

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^{nk} f_\epsilon(d_j^{(n)}) = \lim_{n \rightarrow \infty} \sum_j \chi_K(d_j^{(n)}) = \lim_{n \rightarrow \infty} \frac{c_n(K)}{n}.$$

Applying Theorem 4.1 with $f = f_\epsilon$ and taking $\epsilon \rightarrow 0$ we get (4.12). \square

We know that the map d_j that takes a positive definite matrix B to its j th minimum symplectic eigenvalue $d_j(B)$ is continuous [1]. Since $\theta \mapsto \tilde{A}(\theta)$ is measurable on $[-\pi, \pi]$, the composite map $\theta \mapsto d_j(\theta) = d_j(\tilde{A}(\theta))$ is also measurable.

Let \mathcal{R}_j denote the essential range of the map $d_j(\theta)$ and let $\mathcal{R} = \bigcup_{j=1}^k \mathcal{R}_j$. Since $\tilde{A} \in L^\infty_{2k \times 2k}$, the set \mathcal{R} is compact.

Lemma 4.6. *For every $\tilde{A}(\theta)$ in $\mathcal{R}(\tilde{A})$ and $1 \leq j \leq k$, $d_j(A(\theta))$ is in \mathcal{R}_j .*

Proof. Let $B = \tilde{A}(\theta)$ be any element of $\mathcal{R}(\tilde{A})$. We show that $d_j(B) \in \mathcal{R}_j$. Let $\epsilon > 0$. Since the map d_j is continuous on positive definite matrices, we can find a $\delta > 0$ such that

$$\|\tilde{A}(t) - B\| < \delta \implies |d_j(\tilde{A}(t)) - d_j(B)| < \epsilon. \quad (4.13)$$

By the definition of the essential range of \tilde{A} , the set $S = \{t: \|\tilde{A}(t) - B\| < \delta\}$ has positive measure. Let T be the set

$$T = \{t: |d_j(\tilde{A}(t)) - d_j(B)| < \epsilon\}.$$

By (4.13) we see that $S \subseteq T$. Hence T also has positive measure. This shows that $d_j(B) \in \mathcal{R}_j$. \square

For any subset X of \mathbb{R} let $\mathcal{B}(X, \delta)$ be its δ -neighbourhood:

$$\mathcal{B}(X, \delta) = \{x \in \mathbb{R}: |x - s| < \delta \text{ for some } s \in X\}.$$

Let \mathcal{D}_n be the set of symplectic eigenvalues of $T_n(\tilde{A})$. Let $\mathcal{D} = \bigcup_n \mathcal{D}_n$

Theorem 4.7. *The set \mathcal{D} is dense in \mathcal{R} . Further for each $\delta > 0$ let X_δ be the set*

$$X_\delta = [m_{\tilde{A}}, \|\tilde{A}\|] \setminus \mathcal{B}(\mathcal{R}, \delta). \quad (4.14)$$

Then

$$\lim_{n \rightarrow \infty} \frac{c_n(X_\delta)}{n} = 0. \quad (4.15)$$

Proof. Since \mathcal{R} is compact, we can apply Lemma 4.5 to get

$$\lim_{n \rightarrow \infty} \frac{c_n(\mathcal{R})}{n} = \frac{1}{2\pi} \sum_{j=1}^k m(\{\theta: d_j(\tilde{A}(\theta)) \in \mathcal{R}\}). \quad (4.16)$$

Suppose \mathcal{D} is not a dense subset of \mathcal{R} . Then there exist $j \in \{1, 2, \dots, k\}$ $x \in \mathcal{R}_j$ and $\epsilon > 0$ such that $\mathcal{B}(x, \epsilon) \cap \mathcal{D} = \emptyset$. Since $x \in \mathcal{R}_j$ the set

$$S = \{t: |x - d_j(\tilde{A}(t))| < \epsilon\}$$

has a positive measure. Let $Y = \mathcal{R} \setminus \mathcal{B}(x, \epsilon)$. The set Y is compact, hence by (4.12) we have

$$\lim_{n \rightarrow \infty} \frac{c_n(Y)}{n} = \frac{1}{2\pi} \sum_{j=1}^k m\{t: d_j(\tilde{A}(t)) \in Y\}. \quad (4.17)$$

Since S has positive measure,

$$m\{t: d_j(\tilde{A}(t)) \in \mathcal{R}\} > m\{t: d_j(\tilde{A}(t)) \in Y\}.$$

This shows that the right hand side of (4.16) is strictly greater than the right hand side of (4.17). But since $\mathcal{D} \cap \mathcal{B}(x, \epsilon) = \emptyset$, $c_n(Y) = c_n(\mathcal{R})$ for all n . So, the left

hand sides of (4.16) and (4.17) are equal. This is a contradiction. Hence \mathcal{D} must be dense in \mathcal{R} .

The set X_δ is compact, and hence (4.12) holds when $K = X_\delta$. Since $X_\delta \cap \mathcal{R}_j = \emptyset$ for every j ,

$$m\{\theta: d_j(\tilde{A}(\theta)) \in X_\delta\} = 0.$$

This proves (4.15). □

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Rajendra Bhatia, Department of Mathematics, Ashoka University,
Rajiv Gandhi Education City, P.O. Rai, Sonapat, Haryana 131 029, India
e-mail: rajendra.bhatia@ashoka.edu.in

Tanvi Jain, Theoretical Statistics and Mathematics Unit, Indian Statistical Institute,
Delhi Centre, 7 S J S Sansanwal Marg, New Delhi 110 016, India
e-mail: tanvi@isid.ac.in

Ritabrata Sengupta, Department of Mathematical Science,
Indian Institute of Science, Education, & Research (IISER), Berhampur,
Transit campus – Govt. ITI, NH 59, Berhampur 760 010, Ganjam, Odisha, India
e-mail: rb@iiserbpr.ac.in