

Spectral theory of the thermal Hamiltonian: 1D case

Giuseppe De Nittis and Vicente Lenz

Abstract. In 1964 J. M. Luttinger introduced a model for the quantum thermal transport. In this paper we study the spectral theory of the Hamiltonian operator associated with Luttinger’s model, with a special focus at the one-dimensional case. It is shown that the (so called) thermal Hamiltonian has a one-parameter family of self-adjoint extensions and the spectrum, the time-propagator group and the Green function are explicitly computed. Moreover, the scattering by convolution-type potentials is analyzed. Finally, also the associated classical problem is completely solved, thus providing a comparison between classical and quantum behavior. This article aims to be a first contribution in the construction of a complete theory for the thermal Hamiltonian.

Mathematics Subject Classification (2020). Primary: 81Q10; Secondary: 81Q05, 81Q15, 33C10.

Keywords. Thermal Hamiltonian, self-adjoint extensions, spectral theory, scattering theory.

Contents

1	Introduction	1416
2	The spectral theory of the operator Π	1426
3	The spectral theory of the thermal Hamiltonian	1438
4	The classical dynamics	1446
	Appendices	1455
A	Spectral theory of the momentum operator	1455
B	Technical tools	1459
	References	1466

1. Introduction

The aim of this introductory section is twofold: first of all, we will provide the physical background that motivates the study of the thermal Hamiltonian; secondly, we will present the mathematical problems and the main results achieved in this work.

1.1. Physical motivations. The motion of an electron inside the matter, and subjected to a static magnetic field B , is described by the (one-particle) Hamiltonian

$$H(A, V) := K(A) + V \quad (1.1)$$

where

$$K(A) := \frac{1}{2m} \left(p - \frac{e}{c} A \right)^2. \quad (1.2)$$

The parameters m and e describe the *mass* and the *charge* of the electron, respectively. The constant c is the (in vacuum) *speed of the light*. The static (effective) potential V takes care of the interaction of the electron with the atomic structure of the matter and causes only *elastic scattering*. The magnetic field enters in the *kinetic* term $K(A)$ through its *vector potential* A according to the equation $B = \nabla \times A$. In Quantum Mechanics the Hamiltonian $H(A, V)$ is interpreted as a differential operator acting on the Hilbert space $L^2(\mathbb{R}^d)$, where the differential part is provided by the momentum operator $p := -\hbar i \nabla$, \hbar being the *Planck constant*. The potentials $V = V(x)$ and $A = A(x)$ are functions of the position operator $x = (x_1, \dots, x_d)$, and act as multiplication operators.

The *transport phenomena* in the matter are analyzed by studying the response of the system to an external *perturbation* $F = F(x)$ [18, 19]. In the stationary regime, that is when all the transient effects due to the switching-on of the perturbation are suppressed, the system reacts by generating a (stationary) *drift current*. The latter can be computed (at least in the linear response regime, see e.g. [11]) starting from the full dynamics generated by the *perturbed* Hamiltonian

$$H(A, F, V) := H_0(A, F) + V. \quad (1.3)$$

In (1.3) the “free” Hamiltonian

$$H_0(A, F) := K(A) + F \quad (1.4)$$

describes the motion of an electron that moves in the empty space under the influence of the (external) fields generated by A and F . The potential V in (1.3) describes the interaction with the matter which generates elastic scattering of the

particle. Once the “free” dynamics generated by $H_0(A, F)$ is known, one can study the influence of the matter by means of the *scattering theory* [27, 31, 17] for the pair of operators $H_0(A, F)$ and $H(A, F, V)$.

The best studied case concerns the response of the system to the perturbation induced by a uniform electric field $E = (E_1, \dots, E_d)$. In this case the perturbation is described by the electrostatic potential

$$F_E(x) := -eE \cdot x = -e(E_1x_1 + \dots + E_dx_d)$$

and the associate perturbed Hamiltonian takes the form

$$H(A, F_E, V) := H_{\text{Stark}}(A) + V \tag{1.5}$$

where the “free” part is given by

$$H_{\text{Stark}}(A) := K(A) - eE \cdot x \tag{1.6}$$

according to (1.4). The operator $H_{\text{Stark}}(A)$ is known as (*magnetic*) *Stark Hamiltonian*. The non-magnetic case $H_{\text{Stark}}(A = 0) = \frac{p^2}{2m} - eE \cdot x$ has been extensively studied since the dawn of the Quantum Mechanics. Among the vast literature, we will refer to [4] for a concise and rigorous presentation of the spectral theory of $H_{\text{Stark}}(0)$ and the related scattering theory when the background potential V is taken in consideration. The spectral theory of $H_{\text{Stark}}(A)$ in presence of a uniform magnetic field is discussed in [12, 2], among others.

In order to study the *thermal transport* in the matter, Luttinger proposed a model which allows a “mechanical” derivation of the thermal coefficients [19]. Such a model has been then applied and generalized successfully by other authors like in [29, 30]. The essential point of Luttinger’s model is that the effect of the thermal gradient in the matter is replaced by a “fictitious” gravitational field, which can be easily described by a perturbation of the Hamiltonian in the spirit of (1.3) and (1.4). More precisely, one assumes that the particle is subject to a force which has the direction of the thermal gradient $\nabla\vartheta$ (where ϑ is the distribution of temperature) and which is proportional to the local content of energy divided by c^2 (in view of the mass-energy equivalence). The latter is given by the Hamiltonian (1.1) itself. Such a *thermal-gravitational field* is given by the potential

$$\begin{aligned} F_T &:= \frac{1}{2} \left[\left(\frac{\nabla\vartheta}{c^2} \cdot x \right) H(A, V) + H(A, V) \left(\frac{\nabla\vartheta}{c^2} \cdot x \right) \right] \\ &= \frac{\nabla\vartheta}{c^2} \cdot \frac{1}{2} \{ H(A, V), x \}, \end{aligned} \tag{1.7}$$

where the anti-commutator $\{\cdot, \cdot\}$ between $H(A, V)$ and x is needed to make H_T formally self-adjoint (i.e., symmetric). The total perturbed Hamiltonian $H(A, F_T, V)$, computed according to (1.3), can be written as

$$H(A, F_T, V) = H_T(A) + W(V) \quad (1.8)$$

where the “free” part, called (*magnetic*) *thermal Hamiltonian*,¹ is given by

$$H_T(A) := K(A) + \frac{\nabla\vartheta}{c^2} \cdot \frac{1}{2}\{K(A), x\} \quad (1.9)$$

and the effective *gravitational-matter potential* reads

$$W(V) := \left(1 + \frac{\nabla\vartheta}{c^2} \cdot x\right)V. \quad (1.10)$$

The thermal Hamiltonian $H_T(A)$ is the analog of the Stark Hamiltonian when the system is perturbed by the gravitational-thermal field instead of the electric field. For this reason, it seems natural to look for the extension of the results valid for the Stark Hamiltonian (e.g. [4, 12, 2]) to the case of the thermal Hamiltonian. This consists of two consecutive problems: (i) the analysis of the spectral theory of the “free” operator $H_T(A)$; (ii) the study of the scattering theory for the pair $H_T(A)$ and $H(A, F_T, V)$. Both of these problems seem not to have been studied yet in the literature, at least to the best of our knowledge. For this reason we devote this work at the analysis of the questions (i) and (ii) above, in the one-dimensional case. The multi-dimensional case will be treated in a future work.

1.2. Position of the spectral problem. In order to formulate the problems sketched above in a rigorous mathematical setting we will make some simplifications. The most relevant concerns the absence of the magnetic field. From here on, unless otherwise indicated, we will fix $A = 0$. It is worth mentioning that this is not a major restriction as long as one is interested only the one-dimensional regime. Indeed, in one spatial dimension the magnetic field is a pure gauge and can be removed with a unitary transformation.²

As usual in mathematics, we will normalize all the physical units: $2m = \hbar = c = e = 1$. Moreover, we will denote with $\lambda := |\nabla\vartheta| > 0$ the strength of the thermal gradient and with $\gamma := \lambda^{-1}\nabla\vartheta \in \mathbb{S}^{d-1}$ its direction. With these

¹ An equivalently appropriate name for $H_T(A)$ could be (*magnetic*) *Luttinger Hamiltonian*.

² This fact can be interpreted as a consequence of the *Stone-von Neumann theorem* (see e.g. [24]). Indeed, in one spatial dimension the pair $x, \pi_f := p + f(x)$ necessarily meets the canonical commutation rule and so it is unitarily equivalent to the canonical pair x, p .

simplifications the thermal Hamiltonian reads

$$H_T \equiv H_T(\lambda, \gamma) := p^2 + \frac{\lambda}{2}\{p^2, \gamma \cdot x\}. \quad (1.11)$$

The expression (1.11) is formal without the specification of the domain of definition of H_T . However, H_T is evidently well defined on the space of the compactly supported smooth function $\mathcal{C}_c^\infty(\mathbb{R}^d)$ or on the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. On these dense domains the operator (1.11) acts as

$$(H_T \psi)(x) := -(1 + \lambda \gamma \cdot x)(\Delta \psi)(x) - \lambda(\gamma \cdot \nabla \psi)(x) \quad (1.12)$$

where $\Delta := \sum_{j=1}^d \partial_{x_j}^2$ denotes the Laplacian and $\gamma \cdot \nabla := \sum_{j=1}^d \gamma_j \partial_{x_j}$. We can simplify the last expression with the help of two unitary transformations of the Hilbert space $L^2(\mathbb{R}^d)$. The first one is the rotation

$$(R_\gamma \psi)(x) := \psi(O_\gamma^{-1}x), \quad \psi \in L^2(\mathbb{R}^d) \quad (1.13)$$

where the orthogonal matrix O_γ meets the condition $O_\gamma \gamma = (1, 0, \dots, 0)$. A short computation shows that

$$(R_\gamma H_T R_\gamma^* \psi)(x) = -(1 + \lambda x_1)(\Delta \psi)(x) - \lambda \frac{\partial \psi}{\partial x_1}(x),$$

where x_1 denotes the first component of the position vector $x = (x_1, x_\perp) \in \mathbb{R}^d$ and $x_\perp := (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$ is its orthogonal complement. Evidently, the rotation O_γ has the role of aligning the thermal-gravitational field along the x_1 -axis.³ The second transformation is the translation

$$(S_\lambda \psi)(x_1, x_\perp) := \psi\left(x_1 - \frac{1}{\lambda}, x_\perp\right), \quad \psi \in L^2(\mathbb{R}^d) \quad (1.14)$$

and a direct calculation provides

$$(S_\lambda R_\gamma H_T R_\gamma^* S_\lambda^* \psi)(x) = \lambda \left(-x_1 (\Delta \psi)(x) - \frac{\partial \psi}{\partial x_1}(x) \right).$$

The operator on the brackets

$$(T \psi)(x) := -x_1 (\Delta \psi)(x) - \frac{\partial \psi}{\partial x_1}(x) \quad (1.15)$$

agrees with the formal anti-commutator

$$T \equiv \frac{1}{2}\{p^2, x_1\} := \frac{1}{2}(p^2 x_1 + x_1 p^2) \quad (1.16)$$

³ Clearly, in dimension $d = 1$ the thermal-gravitational field is trivially aligned with the only spatial axis and therefore R_γ reduces to the identity.

when evaluated on sufficiently regular functions like $\psi \in \mathcal{S}(\mathbb{R}^d)$. With a slight abuse of notation, we will often use the representation (1.16) for the operator T , instead of the more precise definition (1.15).

The unitary equivalence between H_T and T implies that the spectral theory of the thermal Hamiltonian H_T can be completely recovered from the spectral theory of the operator T . For this reason, one is led to the problem of determining if the operator T , initially defined by (1.15) on the dense domain $\mathcal{S}(\mathbb{R}^d)$, admits self-adjoint extensions and, in that case, to compute the related spectra.

For technical reasons, it results easier to face the equivalent problems in the Fourier space. Let $\mathcal{F}: L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ be the Fourier transform defined (just to fix the convention) by

$$(\mathcal{F}\psi)(k) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} dx e^{-ik \cdot x} \psi(x)$$

on the dense subspace $\psi \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Let $\Pi := \mathcal{F}T\mathcal{F}^*$ be the Fourier transformed version of the operator (1.15). A direct computation shows that for $\psi \in \mathcal{S}(\mathbb{R}^d)$

$$(\Pi\psi)(x) := i \left[x_1 \psi(x) + x^2 \frac{\partial \psi}{\partial x_1}(x) \right], \quad (1.17)$$

where $x^2 := \sum_{j=1}^d x_j^2$. The operator defined by (1.17) agrees with the formal expression

$$\Pi \equiv -\frac{1}{2} \{x^2, p_1\} := -\frac{1}{2} (x^2 p_1 + p_1 x^2) \quad (1.18)$$

on sufficiently regular functions.⁴

The representation (1.18) is quite intriguing if one compares the operator Π with the typical generator of C_0 -groups associated to \mathcal{C}^∞ -flows [1, Chapter 4]. At first glance, it would seem that the general theory of C_0 -groups applies to Π . However, a closer inspection to the \mathbb{R} -flow associated to Π shows that this is not the case in general (see Section 2.4 for more details). Therefore, the question of the self-adjointness of Π needs to be investigated with other tools.

The first fundamental question is whether the operator Π , initially defined by (1.17) on $\mathcal{S}(\mathbb{R}^d)$, admits self-adjoint extensions or not. This is fortunately true and easily demonstrable. Indeed, it is straightforward to check that Π , as defined by (1.17), is *symmetric* (hence *closable*) on $\mathcal{S}(\mathbb{R}^d)$, i.e.,

$$\langle \Pi\psi, \varphi \rangle = \langle \psi, \Pi\varphi \rangle, \quad \text{for all } \psi, \varphi \in \mathcal{S}(\mathbb{R}^d).$$

⁴ Formula (1.18) can be formally derived from (1.16) by using the well-known transformations of the canonical operators $\mathcal{F} p_j \mathcal{F}^* = x_j$ and $\mathcal{F} x_j \mathcal{F}^* = -p_j$ for all $j = 1, \dots, d$.

This observation allows us to identify Π with its closure (still denoted with the same symbol) defined on the domain

$$\mathcal{D}_0 := \overline{\mathcal{S}(\mathbb{R}^d)}^{\|\cdot\|_\Pi} \quad (1.19)$$

obtained by the closure of $\mathcal{S}(\mathbb{R}^d)$ with respect to the graph-norm

$$\|\psi\|_\Pi^2 := \|\psi\|^2 + \|\Pi\psi\|^2.$$

The existence of self-adjoint extensions of Π is justified by von Neumann's criterion [26, Theorem X.3]. Let Υ be the anti-unitary operator on $L^2(\mathbb{R}^d)$ defined by $(\Upsilon\psi)(x) = \overline{\psi(-x)}$. The domains $\mathcal{C}_c^\infty(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$ are left unchanged by Υ and a direct check shows that $\Upsilon\Pi = \Pi\Upsilon$ on these domains. This is sufficient to claim that:

Proposition 1.1. *The closed symmetric operator Π with domain \mathcal{D}_0 admits self-adjoint extensions.*

Proposition 1.1 allows a precise definition of the family of thermal Hamiltonians.

Definition 1.1 (thermal Hamiltonian). Let Π_θ be a given self-adjoint extension of the operator Π with domain $\mathcal{D}(\Pi_\theta) \supset \mathcal{D}_0$. Let $\mathcal{F}(\lambda, \gamma) := \mathcal{F}S_\lambda R_\gamma$ be the unitary operator given by the product of the Fourier transform \mathcal{F} , the translation S_λ defined by (1.14), and the rotation R_γ defined by (1.13). Then, the associated thermal Hamiltonian is the self-adjoint operator

$$H_{T,\theta}(\lambda, \gamma) := \lambda \mathcal{F}(\lambda, \gamma)^* \Pi_\theta \mathcal{F}(\lambda, \gamma), \quad \lambda > 0, \gamma \in \mathbb{S}^{d-1}$$

defined on the domain $\mathcal{D}(H_{T,\theta}) := \mathcal{F}(\lambda, \gamma)^*[\mathcal{D}(\Pi_\theta)]$.

Definition (1.1) reduces the question of the spectral theory of the thermal Hamiltonian to the analysis of the self-adjoint realizations of the operator Π . This is usually done by studying the deficiency subspaces

$$\mathcal{K}_\pm := \text{Ker}(i \mp \Pi^*).$$

The existence of the conjugation Υ for Π implies the equality of the *deficiency indices* $n_\pm := \dim(\mathcal{K}_\pm)$ (see [26, Theorem X.3]), which in turn ensures the existence of self-adjoint extensions. In order to build the spaces \mathcal{K}_\pm and to compute n_\pm , one needs to solve the equations $\Pi^*\psi = \pm i\psi$ which, in view

of (1.17), is equivalent of finding the *weak solutions* [25, Section V.4] to the differential equations

$$(x_1^2 + x_\perp^2) \frac{\partial \psi}{\partial x_1}(x_1, x_\perp) + (x_1 \mp 1) \psi(x_1, x_\perp) = 0 \quad \psi \in L^2(\mathbb{R}^d) \cap \mathcal{S}'(\mathbb{R}^d),$$

where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions.⁵ This problem will be solved for the one-dimensional case in Section 2.2.

1.3. Overview on the one-dimensional case. In Section 2.2, it is shown that the differential operator (1.17), in one spatial dimension ($d = 1$), admits a family of self-adjoint realizations parametrized by the angle $\theta \in \mathbb{S}^1$ (see Theorem 2.1). As a consequence, the domains $\mathcal{C}_c^\infty(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$ *can not* be cores for Π (in contrast to [1, Proposition 4.2.3]). However, it turns out that all these self-adjoint realizations Π_θ are equivalent in the sense that there are unitary operators L_θ such that $\Pi_\theta = L_\theta \Pi_0 L_\theta^*$. This fact immediately implies the independence of the spectrum by the particular self-adjoint realization. In particular, it results that the spectrum of every extension Π_θ is purely absolutely continuous and coincides with the real axis, i.e.,

$$\sigma(\Pi_\theta) = \sigma_{\text{a.c.}}(\Pi_\theta) = \mathbb{R}, \quad \text{for all } \theta \in \mathbb{S}^1. \quad (1.20)$$

We are now in position to state our first main result. Let us just recall that in dimension $d = 1$ the only relevant parameter in the definition of the thermal Hamiltonian is $\lambda > 0$ since no rotation R_γ is required (cf. note 3). Then, according to Definition 1.1, we can define the family of one-dimensional thermal Hamiltonians as

$$H_{T,\theta}(\lambda) := \lambda(\mathcal{F} S_\lambda)^* \Pi_\theta(\mathcal{F} S_\lambda), \quad \lambda > 0, \theta \in \mathbb{S}^1.$$

In view of the unitary equivalence of the various realizations Π_θ it follows that all the one-dimensional thermal Hamiltonians with a given coupling constant $\lambda > 0$ are unitarily equivalent. For this reason we can focus on the special realization with $\theta = 0$.

Theorem 1.1 (spectral theory in $d = 1$). *For every $\lambda > 0$, and up to a unitary equivalence, there is a unique one-dimensional thermal Hamiltonian on $L^2(\mathbb{R})$ defined by*

$$H_T \equiv H_T(\lambda) := \lambda(\mathcal{F} S_\lambda)^* \Pi_0(\mathcal{F} S_\lambda). \quad (1.21)$$

⁵ Similarly, one can consider weak solutions in $L^2(\mathbb{R}^d) \cap \mathcal{D}'(\mathbb{R}^d)$ where $\mathcal{D}'(\mathbb{R}^d) \supset \mathcal{S}'(\mathbb{R}^d)$ is the space of distributions.

The operator H_T is self-adjoint on its domain $\mathcal{D}(H_T) := (\mathcal{F} S_\lambda)^*[\mathcal{D}(\Pi_0)]$ and has purely absolutely continuous spectrum given by

$$\sigma(H_T) = \sigma_{\text{a.c.}}(H_T) = \mathbb{R}$$

independently of $\lambda > 0$.

The proof of Theorem (1.1) is a corollary Theorem 2.1 and of Definition 1.1. For the determination of the spectrum one uses the invariance of the spectrum under unitary equivalences and the *spectral mapping theorem*.

The operator H_T , defined by (1.21), will be called the *standard realization* of the one-dimensional thermal Hamiltonian (with coupling constant $\lambda > 0$). Theorem 1.1 expresses the fact that in dimension $d = 1$ there is a “unique” thermal Hamiltonian, at least in the sense that all relevant *physical quantities*, which by definition must be invariant under unitary equivalences, can be calculated from H_T .

Theorem 1.1 can be complemented with some more precise information. First of all, it is possible to have a precise description of the domain $\mathcal{D}(H_T)$ (cf. Section 3.1). Let

$$\mathcal{Q}(\mathbb{R}) := \left\{ \psi \in L^2(\mathbb{R}) \left| \int_{\mathbb{R}} dx x^2 |\psi(x)|^2 < +\infty \right. \right\} \quad (1.22)$$

be the natural domain of the position operator. Let

$$(B_\lambda \psi)(x) := \int_{\mathbb{R}} dy \mathcal{B}\left(x + \frac{1}{\lambda}, y\right) \psi(y) \quad (1.23)$$

be the unitary operator with integral kernel

$$\mathcal{B}(x, y) := i \frac{\text{sgn}(x) - \text{sgn}(y)}{2} J_0(2\sqrt{|xy|}), \quad (1.24)$$

where

$$\text{sgn}(x) := \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is the *sign function* and J_0 is the 0 Bessel function of the first kind,⁶ see [15]. Then, it holds true that

$$\mathcal{D}(H_T) = B_\lambda[\mathcal{Q}(\mathbb{R})].$$

⁶ The kernel (1.24) is reminiscent of the Hankel transform of order 0-th. This aspect is briefly discussed in Section B.3.

Moreover $\mathcal{D}(H_T)$ contains a dense core for H_T given by

$$\begin{aligned} \mathcal{D}_0(H_T) &:= \mathcal{S}(\mathbb{R}) + \mathbb{C}[\tilde{\kappa}_0] \\ &= \{\varphi \in L^2(\mathbb{R}) \mid \varphi = \psi + c\tilde{\kappa}_0, \psi \in \mathcal{S}(\mathbb{R}), c \in \mathbb{C}\} \end{aligned}$$

and on this core H_T acts according to (cf. Proposition 3.1)

$$(H_T(\psi + c\tilde{\kappa}_0))(x) := -(1 + \lambda x)\psi''(x) - \lambda\psi'(x) + c\tilde{\kappa}_1(x) \quad (1.25)$$

where the (normalized) functions $\tilde{\kappa}_0$ and $\tilde{\kappa}_1$ are explicitly given by

$$\begin{aligned} \tilde{\kappa}_0(x) &:= -\sqrt{\frac{8}{\pi}} \operatorname{sgn}\left(x + \frac{1}{\lambda}\right) \operatorname{kei}\left(2\sqrt{\left|x + \frac{1}{\lambda}\right|}\right), \\ \tilde{\kappa}_1(x) &:= \sqrt{\frac{8}{\pi}} \operatorname{ker}\left(2\sqrt{\left|x + \frac{1}{\lambda}\right|}\right), \end{aligned} \quad (1.26)$$

and $\operatorname{kei}(x)$ and $\operatorname{ker}(x)$ are the *irregular Kelvin functions* of 0-th order (see Section B.2 and references therein). It is worth noting that the function $\tilde{\kappa}_0$ introduces a jump discontinuity around the *critical point* $x_c = -\lambda^{-1}$. The Hamiltonian H_T , acting on $\tilde{\kappa}_0$, produces the wavefunction $\tilde{\kappa}_1$ which shows a logarithmic divergence around x_c . A similar singular behavior around the critical point x_c is detectable also in the classical dynamics (cf. Section 4).

The unitary propagator $U_T(t) := e^{-itH_T}$ acts as an integral operator

$$(U_T(t)\psi)(x) := \int_{\mathbb{R}} dy \mathcal{U}_{\lambda t}\left(x + \frac{1}{\lambda}, y + \frac{1}{\lambda}\right) \psi(y), \quad t \in \mathbb{R} \setminus \{0\} \quad (1.27)$$

with kernel given by (cf. Proposition 3.2)

$$\mathcal{U}_{\tau}(x, y) := \frac{\operatorname{sgn}(x) + \operatorname{sgn}(y)}{i2\tau} e^{i\frac{(x+y)}{\tau}} J_0\left(\frac{2}{\tau}\sqrt{|xy|}\right) \quad (1.28)$$

for all $\tau \in \mathbb{R} \setminus \{0\}$. Finally, the knowledge of the unitary propagator allows to compute the resolvent

$$R_{\zeta}(H_T) := (H_T - \zeta\mathbf{1})^{-1}, \quad \zeta \in \mathbb{C} \setminus \mathbb{R}$$

by means of the Laplace transformation (see Section 3.3). It turns out that also $R_{\zeta}(H_T)$ is an integral operator

$$(R_{\zeta}(H_T)\psi)(x) := \frac{1}{\lambda} \int_{\mathbb{R}} dy \mathcal{Z}_{\frac{\zeta}{\lambda}}\left(x + \frac{1}{\lambda}, y + \frac{1}{\lambda}\right) \psi(y), \quad (1.29)$$

with kernel $\mathcal{Z}_{\alpha}(x, y)$ given explicitly by the (long) formulas (3.7) and (3.8).

Theorem (1.1) provides also the first step for the one-dimensional scattering theory of the thermal Hamiltonian. Indeed, one infers from Theorem (1.1) that H_T does not admit bounded states and so generate a “free-like” dynamics. In this work only the scattering theory for a special type of *convolution* perturbations is discussed. The scattering theory for (physical) perturbations given by gravitational-matter potentials of type (1.10) presents several technical complications and will be treated in a separated work. By convolution perturbation we mean an integral operator W_g acting on $\psi \in L^2(\mathbb{R})$ as

$$(W_g \psi)(x) := \int_{\mathbb{R}} dy g(x-y)\psi(y) \quad (1.30)$$

where the kernel is chosen as $g \in L^1(\mathbb{R})$. Let us denote by

$$H_{T,g} := H_T + W_g \quad (1.31)$$

the perturbed operator. As usual the wave operators for the pair $(H_T, H_{T,g})$ are defined by

$$\Omega_g^\pm := \text{s-lim}_{t \rightarrow \pm} e^{itH_{T,g}} e^{-itH_T} \quad (1.32)$$

where s-lim denotes the limit taken with respect to the strong topology for bounded operators. In Section 3.4 the following result will be proven.

Theorem 1.2 (scattering theory for convolution perturbations in $d = 1$). *Let $g \in L^1(\mathbb{R})$ and W_g the associated convolution perturbation defined by (1.30). Then,*

- (i) *the perturbed operator $H_{T,g}$ defined by (1.31) is self-adjoint with domain $\mathcal{D}(H_T)$ and*

$$\sigma(H_{T,g}) = \sigma_{\text{a.c.}}(H_{T,g}) = \mathbb{R}.$$

Let \hat{g} be the Fourier transform of g and assume that there are constants $\varepsilon > 0$ and $C > 0$ such that $|\hat{g}(x)| \leq Cx^{\frac{3}{2}}$ for all $|x| < \varepsilon$. Then,

- (ii) *the wave operators Ω_g^\pm defined by (1.32) exist and are complete;*
 (iii) *the S-matrix $S_g := (\Omega_g^+)^* \Omega_g^-$ is a constant phase given by*

$$S_g = e^{-i \frac{\sqrt{2\pi}}{\lambda} \int_{\mathbb{R}} ds \frac{\hat{g}(s)}{s^2}}.$$

Structure of the paper. Section 2 is devoted to the study of the spectral theory of the auxiliary operator Π in the one-dimensional case. The spectral theory of the one-dimensional thermal Hamiltonian H_T is discussed in Section 3 along with a subsection on the scattering theory by a convolution-type potentials. The classical dynamics of the thermal Hamiltonian (in any dimension) is studied in Section 4. Finally Appendix A and Appendix B contain some review material and some technical computations needed to make the present work self contained.

Acknowledgements. Giuseppe De Nittis' research is supported by the grant *Fondecyt Regular* – 1190204. Giuseppe De Nittis and Vicente Lenz are indebted to Claudio Fernández, Marius Măntoiu, and Serge Richard for many stimulating discussions.

2. The spectral theory of the operator Π

We already know from the general discussion in Section 1.2 that the operator Π defined by (1.17) (or formally by (1.18)) is symmetric and in turn closable. Moreover, Proposition 1.1 ensures that Π admits self-adjoint extensions. While, on the one hand, these results are valid in every dimension, in this section we will classify all the self-adjoint extensions of Π in dimension $d = 1$ and we will describe the the spectral theory for this family of operators.

2.1. Equivalence with the momentum operator. In dimensional $d = 1$ the operator Π is initially defined by

$$(\Pi\psi)(x) = i \left[x\psi(x) + x^2 \frac{d\psi}{dx}(x) \right] = i x \frac{d}{dx} [x\psi(x)] \quad \psi \in \mathcal{S}(\mathbb{R}). \quad (2.1)$$

The last equality allows us to identify

$$\Pi \equiv -xpx$$

on sufficiently regular functions.

The operator (2.1) is symmetric, hence closable, and its closure (still denoted with Π) has domain \mathcal{D}_0 given by (1.19). In order to give a more precise characterization of \mathcal{D}_0 we will benefit from the transformation

$$(I\psi)(x) := \frac{1}{x} \psi\left(\frac{1}{x}\right), \quad \psi \in L^2(\mathbb{R}).$$

Lemma 2.1. *I is a unitary involution.*

Proof. A direct computation shows that

$$\begin{aligned} \|I\psi\|^2 &= \int_{\mathbb{R}} \frac{dx}{x^2} \left| \psi\left(\frac{1}{x}\right) \right|^2 = - \int_{-\infty}^{+\infty} d\left(\frac{1}{x}\right) \left| \psi\left(\frac{1}{x}\right) \right|^2 \\ &= - \int_{+\infty}^{-\infty} ds |\psi(s)|^2 = \|\psi\|^2. \end{aligned}$$

Then I , initially defined on every “good enough” dense domain, extends to an isometry on the whole $L^2(\mathbb{R})$. From its very definition, it follows that $I^2\psi = \psi$. This shows that I is an involution, and in particular it is invertible. As a consequence I is also unitary. \square

Instead of Π let us consider the transformed operator

$$\wp := I\Pi I \tag{2.2}$$

defined on the domain $\mathcal{D}(\wp) := I[\mathcal{D}_0]$. We use the standard notation $H^k(\Omega) := W^{k,2}(\Omega) \subset L^2(\Omega)$ for the k -th Sobolev space⁷ with respect to the open set $\Omega \subseteq \mathbb{R}$. Let

$$H_0^1(\mathbb{R}) := \{\phi \in H^1(\mathbb{R}) \mid \phi(0) = 0\}$$

be the space of the Sobolev functions on \mathbb{R} vanishing in $x = 0$. Let us point out that the latter requirement makes sense since Sobolev functions on \mathbb{R} are uniquely identifiable with continuous functions [8, Theorem 8.2]. In view of this remark we will tacitly identify Sobolev functions with their continuous representative so that the following inclusions $H_0^1(\mathbb{R}) \subset H^1(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$ hold.

Proposition 2.1. *The closed symmetric operator \wp defined by (2.2) coincides with the momentum operator on $H_0^1(\mathbb{R})$, namely*

$$(\wp\phi)(x) = -i\phi'(x), \quad \phi \in \mathcal{D}(\wp) = H_0^1(\mathbb{R})$$

where ϕ' is the weak derivative of ϕ .

Proof. The unitarity of I implies that the graph norms of \wp and Π are related by $\|\phi\|_{\wp} = \|I\phi\|_{\Pi}$ for all $\phi \in \mathcal{D}(\wp)$. This gives

$$\mathcal{D}(\wp) = I[\overline{\mathcal{S}(\mathbb{R})}^{\|\cdot\|_{\Pi}}] = \overline{I[\mathcal{S}(\mathbb{R})]}^{\|\cdot\|_{\wp}}.$$

⁷ For the theory of Sobolev spaces we refer the reader to [8, Chapters 8 and 9].

Let $\phi \in I[\mathcal{S}(\mathbb{R})]$. Since $I\phi \in \mathcal{S}(\mathbb{R})$, one infers from (2.1) that

$$(\Pi I\phi)(x) = ix \frac{d}{dx}[x(I\phi)(x)] = ix \frac{d}{dx}\left[\phi\left(\frac{1}{x}\right)\right] = -\frac{i}{x} \frac{d\phi}{dx}\left(\frac{1}{x}\right).$$

Therefore

$$(\wp\phi)(x) = (I(\Pi I\phi))(x) = -i \frac{d\phi}{dx}(x)$$

acts as the momentum operator on $I[\mathcal{S}(\mathbb{R})]$. This implies that the domain of the closed operator \wp is given by the closure of $I[\mathcal{S}(\mathbb{R})]$ with respect the Sobolev norm $\|\phi\|_{H^1}^2 := \|\phi\|^2 + \|\phi'\|^2$. Let $\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})$ be the set of smooth functions having compact support separated from the origin. It holds true that

$$\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\}) \subset I[\mathcal{S}(\mathbb{R})] \subset H_0^1(\mathbb{R}). \tag{2.3}$$

Indeed, let $\psi \in C_c^\infty(\mathbb{R} \setminus \{0\})$ supported in $[-b, -a] \cup [a, b]$ and $\phi := I\psi$. A direct inspection shows that ϕ is a smooth function supported in $[-a^{-1}, -b^{-1}] \cup [b^{-1}, a^{-1}]$. This allows to conclude that $I[\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})] \subseteq \mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})$. By exploiting the involutive character of I one gets $I[\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})] = \mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\}) \subset \mathcal{S}(\mathbb{R})$ and in turn $\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\}) \subset I[\mathcal{S}(\mathbb{R})]$. For the second inclusion let us take $\phi \in I(\mathcal{S}(\mathbb{R}))$ so that $\phi(x) = x^{-1}\psi(x^{-1})$ for some $\psi \in \mathcal{S}(\mathbb{R})$. Clearly, ϕ is smooth in $\mathbb{R} \setminus \{0\}$ and extends to a smooth function on \mathbb{R} such that $\phi^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. In particular $\phi \in H_0^1(\mathbb{R})$, implying the second inclusion $I[\mathcal{S}(\mathbb{R})] \subset H_0^1(\mathbb{R})$. To conclude the proof it is enough to show that the closure of the space $\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})$ with respect to the Sobolev norm $\|\cdot\|_{H^1}$ is (identifiable with) $H_0^1(\mathbb{R})$. Let $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_- := (-\infty, 0)$ and observe that

$$\begin{aligned} \overline{\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})}^{\|\cdot\|_{H^1}} &= \overline{\mathcal{C}_c^\infty(\mathbb{R}_-)}^{\|\cdot\|_{H^1}} \oplus \overline{\mathcal{C}_c^\infty(\mathbb{R}_+)}^{\|\cdot\|_{H^1}} \\ &= W_0^{1,2}(\mathbb{R}_-) \oplus W_0^{1,2}(\mathbb{R}_+) = H_0^1(\mathbb{R}), \end{aligned} \tag{2.4}$$

where the notation for $W_0^{1,2}(\Omega)$ was borrowed from [8, Section 8.3]. The last equality in (2.4) is a consequence of the fact that every element of $W_0^{1,2}(\mathbb{R}_\pm)$ can be uniquely identified with a continuous function that vanishes on the boundary $x = 0$ [8, Theorem 8.12]. The identification (2.4), along with the double inclusion (2.3), implies the desired result $\mathcal{D}(\wp) = \overline{I[\mathcal{S}(\mathbb{R})]}^{\|\cdot\|_{H^1}} = H_0^1(\mathbb{R})$. \square

The first consequence of Proposition 2.1 is a precise description of the domain of the closed operator Π , i.e.,

$$\mathcal{D}_0 = I[\mathcal{D}(\wp)] = \left\{ \psi \in L^2(\mathbb{R}) \mid \psi(x) = \frac{1}{x}\phi\left(\frac{1}{x}\right), \phi \in H_0^1(\mathbb{R}) \right\}. \tag{2.5}$$

Unlike the functions in $H_0^1(\mathbb{R})$, the elements of the domain \mathcal{D}_0 are generally not continuous and can show singularities in $x = 0$. An example is the function $\phi(x) := (1 + x^2)^{-\frac{1}{3}} e^{-\frac{1}{x^2}}$ which is evidently an element of $H_0^1(\mathbb{R})$. Its image $\psi(x) := (I\phi)(x) = (x^3 + x)^{-\frac{1}{3}} e^{-x^2}$ is divergent in $x = 0$. On the other hand elements of \mathcal{D}_0 have a decay at infinity which is at least of order 1.

Proposition 2.2. *Let $\psi \in \mathcal{D}_0$. Then it holds true that*

$$\lim_{|x| \rightarrow \infty} (x\psi(x)) = 0.$$

Proof. The claim follows from the characterization (2.5) which provides

$$\lim_{x \rightarrow \pm\infty} (x\psi(x)) = \lim_{t \rightarrow 0^\pm} \phi(t) = \phi(0) = 0.$$

In the last equality, the continuity of $\phi \in H_0^1(\mathbb{R})$ is used. □

2.2. Classification of self-adjoint extensions. We are now in position to study the self-adjoint realizations of Π . In view of the unitary transform I this is the same of studying the self-adjoint realization of the *singular* momentum operator \wp . The latter is a classical problem strongly related with the study of singular delta interactions for one-dimensional Dirac operators [16, 6, 9] (see also [3, Appendix J]).

Proposition 2.3. *The closed symmetric operator \wp has deficiency indices equal to 1. Therefore, the self-adjoint extensions of \wp are in one-to-one correspondence with the angles $\theta \in \mathbb{S}^1 \simeq [0, 2\pi)$. The self-adjoint extension \wp_θ has domain*

$$\mathcal{D}(\wp_\theta) := \left\{ \varphi \in L^2(\mathbb{R}) \mid \varphi = \phi + c\eta_\theta, \phi \in H_0^1(\mathbb{R}), c \in \mathbb{C} \right\},$$

where

$$\eta_\theta(x) := e^{-|x|} e^{i \operatorname{sgn}(x) \frac{\theta}{2}}$$

and acts has

$$\wp_\theta(\phi + c\eta_\theta) := -i\phi' + c\eta_{\theta+\pi}. \quad (2.6)$$

Finally, \wp_0 agrees with the standard momentum operator p with domain $H^1(\mathbb{R})$.

Proof. Since $\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\}) = \mathcal{C}_c^\infty(\mathbb{R}_-) \oplus \mathcal{C}_c^\infty(\mathbb{R}_+)$ is dense (with respect to the graph norm) in the domain of \wp , a standard argument shows that the adjoint operator \wp^* acts as the weak derivative on its domain $\mathcal{D}(\wp^*) := H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+)$ (see e.g. [25, Section VII.2]). The eigenvalue equations $\wp^* \phi_\pm = \pm i \phi_\pm$

for the deficiency subspaces correspond to the differential equations $\phi'_\pm = \mp\phi_\pm$ which admit in $\mathcal{D}(\wp^*)$ the unique (normalized) weak solutions

$$\phi_+(x) := \begin{cases} \sqrt{2}e^{-x} & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \quad \phi_-(x) := \begin{cases} 0 & \text{if } x > 0, \\ \sqrt{2}e^{+x} & \text{if } x < 0. \end{cases}$$

According to von Neumann's theory for self-adjoint extensions (cf. [26, Section X.1]) one has that the self-adjoint extensions of \wp are parametrized by the unitary maps from $\mathcal{K}_+ = \mathbb{C}[\phi_+] \simeq \mathbb{C}$ to $\mathcal{K}_- = \mathbb{C}[\phi_-] \simeq \mathbb{C}$. The later are identified by the angle $\theta \in \mathbb{S}^1 \simeq [0, 2\pi)$ according to $U_\theta\phi_+ := e^{-i\theta}\phi_-$. From the general theory [26, Theorem X.2] one has that the domain of the self-adjoint extension \wp_θ is made by functions of the type $\phi + c'(\phi_+ + e^{-i\theta}\phi_-) = \phi + c\eta_\theta$ with $\phi \in H_0^1(\mathbb{R})$ and $c, c' \in \mathbb{C}$ suitable complex coefficients. The action of \wp_θ on the elements of its domain is given by

$$\wp_\theta(\phi + c'(\phi_+ + e^{-i\theta}\phi_-)) = -i\phi' + ic'(\phi_+ - e^{-i\theta}\phi_-)$$

which translates into equation (2.6) in terms of the function η_θ . Evidently, the standard momentum operator p is a self-adjoint extension of \wp since $H_0^1(\mathbb{R}) \subset H^1(\mathbb{R})$. This extension corresponds to \wp_0 in view of the fact that $\eta_0 \in H^1(\mathbb{R})$. \square

Although the symmetric operator \wp admits several self-adjoint realizations, all these realizations are in a sense equivalent. To express this fact in a precise way we need to introduce the family of unitary operators L_θ defined by

$$(L_\theta\psi)(x) := e^{i\operatorname{sgn}(x)\frac{\theta}{2}}\psi(x), \quad \psi \in L^2(\mathbb{R}).$$

Proposition 2.4. *The unitary operators L_θ intertwine all the self-adjoint realizations of the operator \wp . More precisely one has that*

$$\wp_\theta = L_\theta p L_\theta^*, \quad \theta \in \mathbb{S}^1,$$

where $p = \wp_0$ is the standard momentum operator. As a consequence one has that

$$\sigma(\wp_\theta) = \sigma_{\text{a.c.}}(\wp_\theta) = \mathbb{R}, \quad \text{for all } \theta \in \mathbb{S}^1.$$

Proof. Let us show that $\mathcal{D}(\wp_\theta) = L_\theta[H^1(\mathbb{R})]$. Every $\psi \in H^1(\mathbb{R})$ can be decomposed as $(\psi - \psi(0)\eta_0) + \psi(0)\eta_0$. Evidently $\phi := \psi - \psi(0)\eta_0 \in H_0^1(\mathbb{R})$, $L_\theta\phi \in H_0^1(\mathbb{R})$ and $L_\theta\eta_0 = \eta_\theta$. Therefore, $L_\theta\psi \in \mathcal{D}(\wp_\theta)$ which implies $L_\theta[H^1(\mathbb{R})] \subseteq \mathcal{D}(\wp_\theta)$. On the other hand every $\varphi \in \mathcal{D}(\wp_\theta)$ can be decomposed as $\varphi = L_\theta(L_{-\theta}\phi + c\eta_0)$ whit $L_{-\theta}\phi \in H_0^1(\mathbb{R})$, and in turn $(L_{-\theta}\phi + c\eta_0) \in H^1(\mathbb{R})$

proving the inverse inclusion $\mathcal{D}(\wp_\theta) \subseteq L_\theta[H^1(\mathbb{R})]$. Now, let $\varphi \in \mathcal{D}(\wp_\theta)$. By exploiting the decomposition used above one has

$$(L_\theta pL_\theta^*)\varphi = L_\theta(pL_{-\theta}\phi + cp\eta_0) = -i\phi' + c\eta_{\theta+\pi},$$

where we used $(p\eta_0)(x) = \eta_0(x) e^{i\operatorname{sgn}(x)\frac{\pi}{2}}$ and $pL_{-\theta}\phi = L_{-\theta}p\phi$ in view of $\phi \in H_0^1(\mathbb{R})$. Hence, a comparison with (2.6) shows that $L_\theta pL_\theta^* = \wp_\theta$ on the domain $\mathcal{D}(\wp_\theta)$. \square

Remark 2.1. The unitary equivalence of the different realizations \wp_θ can be understood in terms of the celebrated *Stone–von Neumann theorem* (see e.g. [24]). Indeed, a direct computation shows that

$$(x\wp_\theta - \wp_\theta x)\varphi = i\varphi, \quad \varphi \in \mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\})$$

and $\mathcal{C}_c^\infty(\mathbb{R} \setminus \{0\}) = \mathcal{C}_c^\infty(\mathbb{R}_-) \oplus \mathcal{C}_c^\infty(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_-) \oplus L^2(\mathbb{R}_+) = L^2(\mathbb{R})$. Therefore, by continuous extension, one can unambiguously define the commutation relation $[x, \wp_\theta] = i\mathbf{1}$ which means that the pair (x, \wp_θ) satisfies the *canonical commutation relation*. As a result, the Stone–von Neumann theorem ensures that \wp_θ is unitarily equivalent to the standard momentum p .

Proposition 2.3 provides the key result for the complete description of the self-adjoint extensions of Π .

Theorem 2.1 (self-adjoint extensions: one-dimensional case). *The self-adjoint extensions of the closed symmetric operator Π initially defined by (2.1) are in one-to-one correspondence with the angles $\theta \in \mathbb{S}^1$. The self-adjoint extension Π_θ has domain*

$$\mathcal{D}(\Pi_\theta) := \{\varphi \in L^2(\mathbb{R}) \mid \varphi = \psi + c\zeta_\theta, \psi \in \mathcal{D}_0, c \in \mathbb{C}\},$$

where

$$\zeta_\theta(x) := \frac{1}{x} e^{-\frac{1}{|x|}} e^{i\operatorname{sgn}(x)\frac{\theta}{2}}$$

and acts as

$$\Pi_\theta(\psi + c\zeta_\theta) := \Pi\psi + c\zeta_{\theta+\pi}.$$

All the self-adjoint realizations are unitarily equivalent, i.e., $\Pi_\theta = L_\theta\Pi_0L_\theta^*$ for all $\theta \in \mathbb{S}^1$. Finally one has that

$$\sigma(\Pi_\theta) = \sigma_{\text{a.c.}}(\Pi_\theta) = \mathbb{R}, \quad \text{for all } \theta \in \mathbb{S}^1.$$

Proof. This is a direct consequence of the unitary equivalence established in Proposition 2.1 which allows to define the self-adjoint realizations of Π by $\Pi_\theta := I\wp_\theta I$. Therefore, the statement is nothing more than a rephrasing of Proposition 2.3 and Proposition 2.4. The formula $\Pi_\theta = L_\theta \Pi_0 L_\theta^*$ is justified by the commutation relation $L_\theta I = I L_\theta$. \square

In view of the unitary equivalence among all the self-adjoint realizations of Π we can focus the attention only in a “preferred” realization.

Definition 2.1 (standard realization). We will call $\Pi_0 = \Pi_{\theta=0}$ the *standard* self-adjoint realization of the operator initially defined by (2.1).

2.3. Boundary triplets. The problem of the determination of the self-adjoint realizations of \wp or Π can be investigated also with the theory of the *boundary triplets* [28, Chapter 14]. Let us start with the operator \wp and its adjoint \wp^* . According to [28, Definition 14.2], a boundary triplet for \wp^* is a triplet $(\mathbb{H}, \Gamma_0, \Gamma_1)$ made by an Hilbert space \mathbb{H} and linear maps Γ_0, Γ_1 from $\mathcal{D}(\wp^*)$ to \mathbb{H} that satisfy the *abstract Green’s identity*

$$\langle \wp^* \varphi, \psi \rangle - \langle \varphi, \wp^* \psi \rangle = \langle \Gamma_0 \varphi, \Gamma_1 \psi \rangle_{\mathbb{H}} - \langle \Gamma_1 \varphi, \Gamma_0 \psi \rangle_{\mathbb{H}}, \quad \text{for all } \varphi, \psi \in \mathcal{D}(\wp^*)$$

and the mapping $\mathcal{D}(\wp^*) \ni \varphi \mapsto (\Gamma_0 \varphi, \Gamma_1 \varphi) \in \mathbb{H} \times \mathbb{H}$ is surjective. Since the operator \wp^* acts as the weak derivative on its domain

$$\mathcal{D}(\wp^*) := H^1(\mathbb{R}_-) \oplus H^1(\mathbb{R}_+),$$

an integration by parts provides

$$\langle \wp^* \varphi, \psi \rangle - \langle \varphi, \wp^* \psi \rangle = i(\overline{\varphi(0^-)}\psi(0^-) - \overline{\varphi(0^+)}\psi(0^+)),$$

where $\varphi(0^\pm) := \lim_{x \rightarrow 0^\pm} \varphi(x)$ and similarly for $\psi(0^\pm)$. A comparison with the abstract Green’s identity shows that the triplet $(\mathbb{H}, \Gamma_0, \Gamma_1)$ can be fixed in the following way: $\mathbb{H} := \mathbb{C}$;

$$\Gamma_0 \varphi := \frac{\varphi(0^+) - \varphi(0^-)}{i\sqrt{2}}, \quad \Gamma_1 \varphi := \frac{\varphi(0^+) + \varphi(0^-)}{\sqrt{2}}.$$

The surjectivity condition is obviously satisfied. Observe that $\text{Ker}(\Gamma_0) \cap \text{Ker}(\Gamma_1) = H_0^1(\mathbb{R}) = \mathcal{D}(\wp)$. The self-adjoint extensions of \wp are in one-to-one correspondence with the self-adjoint operators on $\mathbb{H} = \mathbb{C}$ [28, Theorem 14.10]. More precisely, the self-adjoint extensions of \wp can be parametrized by a real number $\gamma \in \mathbb{R} \cup \{\infty\}$ which defines a restriction $\wp_\gamma := \wp^*|_{\mathcal{D}_\gamma}$ where the domain

$\mathcal{D}_\gamma \subset \mathcal{D}(\wp^*)$ is defined by

$$\begin{aligned} \mathcal{D}_\gamma &:= \{\varphi \in \mathcal{D}(\wp^*) \mid \gamma \Gamma_0 \varphi = \Gamma_1 \varphi\} \\ &= \{\varphi \in \mathcal{D}(\wp^*) \mid e^{-i \arctan(\frac{1}{\gamma})} \varphi(0^+) = e^{i \arctan(\frac{1}{\gamma})} \varphi(0^-)\}. \end{aligned} \quad (2.7)$$

A comparison with Proposition 2.3 shows that the self-adjoint extensions \wp_θ and \wp_γ are related by the equation $\theta(\gamma) = \arctan(\frac{1}{\gamma})$. In particular, the standard momentum is identified by $\gamma = \infty$ which corresponds to $\theta = 0$. Definition (2.7) provides the description of the domain of \wp_θ in terms of *boundary conditions*. The same can be done for the self-adjoint extensions Π_θ with the help of the unitary operator I . A direct computation shows that

$$\mathcal{D}(\Pi_\theta) := \{\varphi \in I[\mathcal{D}(\wp^*)] \mid e^{-i \frac{\theta}{2}} (x\varphi)(+\infty) = e^{+i \frac{\theta}{2}} (x\varphi)(-\infty)\},$$

where $(x\varphi)(\pm\infty) := \lim_{x \rightarrow \pm\infty} x\varphi(x)$.

2.4. Unitary propagator. Let

$$V_\theta(t) := e^{-it\Pi_\theta}, \quad t \in \mathbb{R} \quad (2.8)$$

be the *unitary propagator* defined by the self-adjoint operator Π_θ on $L^2(\mathbb{R})$. The description of $V_\theta(t)$ is provided in the following theorem.

Theorem 2.2. *Let $V_\theta(t)$ be the unitary group defined by (2.8). It holds true that*

$$(V_\theta(t)\psi)(t) = \frac{e^{i \frac{\theta}{2} (1 - \operatorname{sgn}(1-tx)) \operatorname{sgn}(x)}}{1-tx} \psi\left(\frac{x}{1-tx}\right), \quad \psi \in L^2(\mathbb{R}).$$

Proof. We can use the unitary equivalence $\Pi_\theta = IL_\theta p L_\theta^* I$ proved in Section 2.2. This implies that $V_\theta(t) = IL_\theta e^{-itp} L_\theta^* I$ along with the well-known fact $(e^{-itp} \psi)(x) = \psi(x-t)$. The proof of the claim follows by a direct computation. \square

For each $t \in \mathbb{R}$ let us consider the map $f_t: \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$f_t(x) := \begin{cases} \frac{x}{1-tx} & \text{if } x \in \mathbb{R} \setminus \{t^{-1}\}, \\ \infty & \text{if } x = t^{-1}, \\ -t^{-1} & \text{if } x = \infty, \end{cases} \quad (2.9)$$

with the convention that $\pm 0^{-1} \equiv \infty$. The family of these maps defines an \mathbb{R} -flow in the sense that the following relations hold:

$$\begin{cases} f_0 = \text{Id}, \\ f_{t_1} \circ f_{t_2} = f_{t_1+t_2}, & \text{for all } t, t_1, t_2 \in \mathbb{R}. \\ f_t^{-1} = f_{-t}, \end{cases} \quad (2.10)$$

The flow f_t allows to rewrite the action of $V_\theta(t)$ in the form

$$(V_\theta(t)\psi)(t) = e^{\frac{i}{2}(1-\text{sgn}(1-tx))(\text{sgn}(x)\theta+\pi)} \sqrt{(\partial_x f_t)(x)} \psi(f_t(x)). \tag{2.11}$$

When $\theta = \pi$ the exponential prefactor is 1 and equation (2.11) agrees with the definition of the C_0 -group associated to the flow f_t as defined in [1, Section 4.2]. It is interesting to notice that the flow f_t is not of class C^∞ and the generator of the flow

$$F(x) := \left. \frac{d f_t}{d t} \right|_{t=0} (x) = x^2$$

has an unbounded first derivative. Therefore the flow f_t doesn't meet the conditions of [1, Lemma 4.2.2 and Proposition 4.2.3]. The latter fact explains why [1, Proposition 4.2.3] doesn't apply to the operator $\Pi \equiv -\frac{1}{2}(pF(x) + F(x)p)$ which indeed is not essentially self-adjoint on $C_c^\infty(\mathbb{R})$.

2.5. Resolvent and Green function. The resolvent of the of the operator Π_θ can be derived from the resolvent of the standard momentum operator p by exploiting the various unitary equivalences described in Section 2.2. For every $\zeta \in \mathbb{C} \setminus \mathbb{R}$ the resolvent of Π_θ at ζ is defined as

$$R_\zeta(\Pi_\theta) := (\Pi_\theta - \zeta \mathbf{1})^{-1} = L_\theta I(p - \zeta \mathbf{1})^{-1} I L_\theta^*. \tag{2.12}$$

The next results shows that $R_\zeta(\Pi_\theta)$ is an integral operator.

Proposition 2.5. *Let $\zeta := \epsilon \pm i\delta \in \mathbb{C} \setminus \mathbb{R}$ with $\delta > 0$. The resolvent $R_\zeta(\Pi_\theta)$ acts as*

$$(R_\zeta(\Pi_\theta)\psi)(x) = \int_{\mathbb{R}} dy \mathcal{R}_\zeta^\theta(x, y)\psi(y), \quad \psi \in L^2(\mathbb{R})$$

with kernel given by

$$\mathcal{R}_{\epsilon \pm i\delta}^\theta(x, y) := \frac{e^{i(\text{sgn}(x)-\text{sgn}(y))\frac{\theta}{2}}}{\mp ixy} \Theta\left(\pm\left(\frac{1}{x} - \frac{1}{y}\right)\right) e^{i\epsilon\left(\frac{1}{x}-\frac{1}{y}\right)} e^{-\delta\left|\frac{1}{x}-\frac{1}{y}\right|}$$

where Θ is the **Heaviside function**.⁸

⁸ The Heaviside function is defined by

$$\Theta(x) := \begin{cases} 1 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Proof. The integral kernel \mathcal{R}_ζ^0 of the resolvent of Π_0 can be obtained from Green's function \mathcal{G}_ζ^0 of the standard momentum operator (see Appendix A.1). A direct computation provides

$$(\mathcal{R}_\zeta(\Pi_0)\psi)(x) = (I(p - \zeta\mathbf{1})^{-1}I\psi)(x) = \frac{1}{x} \int_{\mathbb{R}} dy \mathcal{G}_\zeta^0\left(\frac{1}{x}, y\right) \frac{1}{y} \psi\left(\frac{1}{y}\right).$$

The explicit expression of \mathcal{G}_ζ^0 given in (A.1) and a change of variable in the integral provide

$$\mathcal{R}_\zeta^0(x, y) := \frac{1}{xy} \mathcal{G}_\zeta^0\left(\frac{1}{x}, \frac{1}{y}\right).$$

Since L_θ is a multiplication operator, the relation between the kernels for $\theta = 0$ and $\theta \neq 0$ is simply given by

$$\mathcal{R}_\zeta^\theta(x, y) := e^{i(\operatorname{sgn}(x) - \operatorname{sgn}(y))\frac{\theta}{2}} \mathcal{R}_\zeta^0(x, y) = \frac{e^{i(\operatorname{sgn}(x) - \operatorname{sgn}(y))\frac{\theta}{2}}}{xy} \mathcal{G}_\zeta^0\left(\frac{1}{x}, \frac{1}{y}\right).$$

This concludes the proof. \square

It is worth noting that along the diagonal one has

$$\mathcal{R}_\zeta^\theta(x, x) = \operatorname{sgn}(\operatorname{Im}(\zeta)) \frac{i}{2x^2}$$

for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and $\theta \in \mathbb{S}^1$.

2.6. Spectral measure and density of states. Let μ_ψ^θ be the spectral measure of the operator Π_θ associated with the normalized state $\psi \in L^2(\mathbb{R})$. We know from Theorem 2.1 that Π_θ has a purely absolutely continuous spectrum which coincides with \mathbb{R} . This implies that the spectral measure μ_ψ^θ is purely absolutely continuous. More precisely one has that

$$\mu_\psi^\theta(d\epsilon) := f_\psi^\theta(\epsilon) d\epsilon$$

with $f_\psi^\theta \in L^1(\mathbb{R})$ a non-negative function. The next result provides a description of f_ψ^θ .

Proposition 2.6. *Let μ_ψ^θ be the spectral measure of the operator Π_θ associated with the (normalized) state $\psi \in L^2(\mathbb{R})$. Then, μ_ψ^θ is absolutely continuous with respect to the Lebesgue measure $d\epsilon$ in \mathbb{R} and*

$$\mu_\psi^\theta(d\epsilon) := |\hat{\phi}_\theta(\epsilon)|^2 d\epsilon \tag{2.13}$$

where $\hat{\phi}_\theta := \mathcal{F}(\phi_\theta)$ is the Fourier transform of the function

$$\phi_\theta(x) := (L_\theta^* I \psi)(x) = \frac{e^{-i \operatorname{sgn}(x) \frac{\theta}{2}}}{x} \psi\left(\frac{1}{x}\right).$$

Proof. From the unitary equivalence $\Pi_\theta = I L_\theta p L_\theta^* I$ one gets

$$F_\psi^\theta(\zeta) := \langle \psi, (\Pi_\theta - \zeta \mathbf{1})^{-1} \psi \rangle = \langle \psi, I L_\theta (p - \zeta \mathbf{1})^{-1} L_\theta^* I \psi \rangle = F_{\phi_\theta}^p(\zeta).$$

Following the arguments in Appendix A.2 on gets

$$\lim_{\delta \rightarrow 0^+} \frac{1}{\pi} \operatorname{Im}(F_\psi^\theta(\epsilon + i \delta)) = f_{\phi_\theta}^p(\epsilon) = |\hat{\phi}_\theta(\epsilon)|^2$$

where the last equality is justified by (A.3). This concludes the proof. □

In order to define the integrated density of states (IDOS) of Π_θ let us start by introducing the spectral projections P_ϵ^θ of Π_θ defined by

$$P_\epsilon^\theta := \begin{cases} \chi_{[0, \epsilon]}(\Pi_\theta) & \text{if } \epsilon > 0, \\ \chi_{[\epsilon, 0]}(\Pi_\theta) & \text{if } \epsilon < 0. \end{cases}$$

Let $(Q_\Lambda \psi)(x) = \chi_\Lambda(x) \psi(x)$ be the projection which restricts the functions $\psi \in L^2(\mathbb{R})$ on the interval $\Lambda = [a, b]$. Let us introduce the function $\mathcal{N}_\Lambda^\theta : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\mathcal{N}_\Lambda^\theta(\epsilon) := \frac{\operatorname{sgn}(\epsilon)}{|\Lambda|} \operatorname{Tr}(P_\epsilon^\theta Q_\Lambda). \tag{2.14}$$

Definition (2.14) is well posed in view of the following result:

Lemma 2.2. *Let $\Lambda := [a, b]$ with $ab > 0$. The operator $P_\epsilon^\theta Q_\Lambda$ is trace class and*

$$\mathcal{N}_\Lambda^\theta(\epsilon) = \frac{1}{ab} \frac{\epsilon}{2\pi}$$

independently of θ .

Proof. By combining the spectral theorem with the unitary equivalence between Π_θ and p one gets that $P_\epsilon^\theta = L_\theta I P_\epsilon I L_\theta^*$ where

$$P_\epsilon := \begin{cases} \chi_{[0, \epsilon]}(p) & \text{if } \epsilon > 0, \\ \chi_{[\epsilon, 0]}(p) & \text{if } \epsilon < 0. \end{cases} \tag{2.15}$$

This means that $P_\epsilon^\theta Q_\Lambda = L_\theta(I P_\epsilon I Q_\Lambda) L_\theta^* = L_\theta I(P_\epsilon I Q_\Lambda I) I L_\theta$. Thus, to prove that $P_\epsilon^\theta Q_\Lambda$ is trace-class it is sufficient to prove that $P_\epsilon(I Q_\Lambda I)$ is trace-class. Let $b > a > 0$ or $a < b < 0$. A direct computation shows that

$$(I Q_{[a,b]} I \psi)(x) = \chi_{[a,b]} \left(\frac{1}{x} \right) \psi(x) = \chi_{[b^{-1}, a^{-1}]}(x) \psi(x), \quad \psi \in L^2(\mathbb{R}),$$

namely $I Q_\Lambda I = Q_{\tilde{\Lambda}}$ with $\tilde{\Lambda} := [b^{-1}, a^{-1}]$. This implies that $P_\epsilon(I Q_\Lambda I) = P_\epsilon Q_{\tilde{\Lambda}}$ is trace-class in view of [27, Theorem XI 20]. Moreover, one has that

$$\mathcal{N}_\Lambda^\theta(\epsilon) = \frac{|\tilde{\Lambda}| \operatorname{sgn}(\epsilon)}{|\Lambda| |\tilde{\Lambda}|} \operatorname{Tr}(P_\epsilon^\theta Q_\Lambda) = \frac{a^{-1} - b^{-1}}{b - a} \mathcal{N}_{\tilde{\Lambda}}^\theta(\epsilon)$$

where $\mathcal{N}_{\tilde{\Lambda}}^\theta(\epsilon)$ is the local density of states for the operator p in the region $\tilde{\Lambda}$. The proof follows by using Lemma A.2. \square

The quantity $\mathcal{N}_\Lambda^\theta(\epsilon)$ measures the volumetric density of states up to the energy ϵ localized in the region Λ . States with negative energy are counted as “negative” states. Lemma 2.2 shows that this number is not homogeneous in space. One can ask how this number changes for fixed volume in function of the spatial localization. Let $\ell > 0$ and set $\Lambda_{x,\ell} := [x, x + \ell]$ when $x > 0$ or $\Lambda_{x,\ell} := [x - \ell, x]$ when $x < 0$. Then

$$\mathcal{N}_{\Lambda_{x,\ell}}^\theta(\epsilon) := \frac{1}{x^2 + |x|\ell} \frac{\epsilon}{2\pi}.$$

Since the density decreases as x^{-2} in function of the spatial localization and as ℓ^{-1} in function of the volume one immediately concludes that the majority of states are concentrated around $x = 0$ with a divergent density.

Ultimately, the spatial inhomogeneity of $\mathcal{N}_\Lambda^\theta$ is a consequence of the fact that Π_θ breaks the invariance under spatial translations. To define a density of states on the thermodynamic limit a precise prescription on how to carry out the spatial average is necessary. Let us define the *principal value integral density of states* (pv-IDOS) as

$$\operatorname{pv} - \mathcal{N}^\theta(\epsilon) := \lim_{L \rightarrow \infty} \frac{L}{2(L^2 - 1)} \operatorname{sgn}(\epsilon) \operatorname{Tr}(P_\epsilon Q'_L),$$

where $Q'_L := Q_{[-L,L]} - Q_{[-L^{-1}, L^{-1}]}$. From Lemma (2.2) one immediately gets that

$$\operatorname{pv} - \mathcal{N}^\theta(\epsilon) := \frac{\epsilon}{2\pi}.$$

3. The spectral theory of the thermal Hamiltonian

The thermal Hamiltonian H_T is defined by equation (1.21) as the conjugation of Π_0 through the unitary $\mathcal{F}S_\lambda$. For this reason the spectral theory of H_T (summarized by Theorem 1.1) is equivalent to the spectral theory of Π_0 studied in Section 2. The next section is mainly devoted to the translation of the results obtained for Π_0 to H_T by exploiting the precise form of the unitary $\mathcal{F}S_\lambda$.

3.1. Description of the domain. By construction the domain of H_T is given by

$$\mathcal{D}(H_T) := (\mathcal{F}S_\lambda)^*[\mathcal{D}(\Pi_0)] = (S_\lambda^*B\mathcal{F}^*)[H^1(\mathbb{R})]$$

with $B := \mathcal{F}^*I\mathcal{F}$. The last equality is justified by $\mathcal{D}(\Pi_0) = I[\mathcal{D}(\wp_0)]$ and $\mathcal{D}(\wp_0) = H^1(\mathbb{R})$. It is known that the Fourier transform of $H^1(\mathbb{R})$ is the domain of the position operator [26, Chapter IX] defined by (1.22). Therefore, the domain H_T is made by functions in $\mathcal{Q}(\mathbb{R})$ transformed by the operator S_λ^*B . The operators B and $B_\lambda := S_\lambda^*B$ have a description in terms of integral kernels.

Lemma 3.1. *On the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ the operator $B = \mathcal{F}^*I\mathcal{F}$ acts as an integral operator with kernel given by (1.24). As a consequence $B_\lambda := S_\lambda^*B$ acts according to (1.23).*

Proof. Let us start with the computation of the kernel of B acting on $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then $\mathcal{F}(\psi) \in L^2(\mathbb{R}) \cap \mathcal{C}_0(\mathbb{R})$, namely $\mathcal{F}(\psi)$ is a square-integrable continuous function that vanishes at infinity. For every $n \in \mathbb{N}$, let χ_{I_n} be the characteristic function of the interval $I_n := [-n, -n^{-1}] \cup [n^{-1}, n]$. Since $\mathcal{F}(\psi) - \mathcal{F}(\psi)\chi_{I_n} = \mathcal{F}(\psi)\chi_{I_n^c}$, where I_n^c is the complement of I_n , one can prove that $\mathcal{F}(\psi)\chi_{I_n} \rightarrow \mathcal{F}(\psi)$ in the L^2 -topology when $n \rightarrow +\infty$. Thus, the unitarity of the Fourier transform implies that $\psi_n \rightarrow \psi$ in the L^2 -topology where $\psi_n := \mathcal{F}^*(\mathcal{F}(\psi)\chi_{I_n}) = \psi * \mathcal{F}^*(\chi_{I_n})$ and $*$ denotes the convolution. Since B is a unitary operator one gets $B\psi_n \rightarrow B\psi$ with respect to the L^2 -topology. An explicit computation provides

$$\begin{aligned} (B\psi_n)(x) &= (\mathcal{F}^*I\mathcal{F}\psi_n)(x) \\ &= (\mathcal{F}^*I\mathcal{F}(\psi * \mathcal{F}^*\chi_{I_n}))(x) \\ &= (\mathcal{F}^*I(\mathcal{F}\psi)\chi_{I_n})(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} du e^{iux} \frac{1}{u} (\mathcal{F}\psi)\left(\frac{1}{u}\right) \chi_{I_n}\left(\frac{1}{u}\right) \end{aligned}$$

$$= \frac{1}{2\pi} \int_{I_n} du e^{iux} \frac{1}{u} \left(\int_{\mathbb{R}} dy e^{-i\frac{y}{u}} \psi(y) \right),$$

where in the last two equalities we used the fact that $I(\mathcal{F}\psi)\chi_{I_n}$ and ψ are L^1 -functions (this justifies the use of the integral representation of \mathcal{F} and \mathcal{F}^*) and the equality $\chi_{I_n}(u^{-1}) = \chi_{I_n}(u)$. Since the function $g_x(y, u) := \frac{1}{u} e^{ixu} e^{-i\frac{y}{u}} \psi(y)$ is absolutely integrable in $\mathbb{R} \times I_n$ one can invoke the Fubini–Tonelli theorem to change the order of integration. This provides

$$(B\psi_n)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} dy \psi(y) \left(\int_{I_n} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} \right). \quad (3.1)$$

Corollary B.1 says that

$$\lim_{n \rightarrow \infty} \int_{I_n} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} = 2\pi \mathcal{B}(x, y).$$

And

$$\left| \int_{I_n} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} \right| \leq 4\pi$$

for all $n > n_0$. In view of the bound above one can use Lebesgue's dominated convergence theorem in (3.1) providing the formula

$$\lim_{n \rightarrow +\infty} (B\psi_n)(x) = \int_{-\infty}^{+\infty} dy \mathcal{B}(x, y) \psi(y). \quad (3.2)$$

Equation (3.2) says that $B\psi_n$ converges pointwise to the integral in the right-hand side. Since $B\psi_n$ converges to $B\psi$ in the L^2 -topology it follows there exists a subsequence $B\psi_{n_k}$ which converges pointwise (almost everywhere) to $B\psi$ [8, Theorem 4.9 (a)]. Then the unicity of the limit assures that $B\psi$ coincides with the right-hand side of (3.2). The last part of the proof follows from the explicit computation

$$(B_\lambda \psi)(x) = (B\psi)\left(x + \frac{1}{\lambda}\right) = \int_{\mathbb{R}} dy \mathcal{B}\left(x + \frac{1}{\lambda}, y\right) \psi(y)$$

which provides equation (1.23). \square

Remark 3.1. Lemma 3.1 states that B_λ can be expressed as an integral operator only on the dense domain $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. For function in $\psi \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$ in principle, we do not have the right to write $B_\lambda \psi$ using the integral kernel. However, in the following, we will tacitly use the convention

$$(B_\lambda \psi)(x) \equiv \lim_{R \rightarrow \infty} \int_{-R}^{+R} dy \mathcal{B}\left(x + \frac{1}{\lambda}, y\right) \psi(y), \quad \text{if } \psi \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R}).$$

This identification must be understood as follows: (i) the product $\psi_R := \psi \chi_{[-R, +R]}$ is in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and so $B_\lambda \psi_R$ can be computed (pointwise) through the integral formula; (ii) $\psi_R \rightarrow \psi$, and in turn $B_\lambda \psi_R \rightarrow B_\lambda \psi$, in the L^2 -topology; (iii) then, the identification above makes sense almost everywhere on subsequences [8, Theorem 4.9 (a)].

Lemma 3.1 allows to describe the domain of H_T as follows:

$$\mathcal{D}(H_T) = \left\{ \psi \in L^2(\mathbb{R}) \mid \psi(x) = \int_{\mathbb{R}} dy \mathcal{B}_\lambda(x, y) \phi(y), \phi \in \mathcal{Q}(\mathbb{R}) \right\}.$$

An explicit computation (made of several changes of integration variable) shows that the generic element ψ in $\mathcal{D}(H_T)$ has the form

$$\psi(x) = \frac{1}{x + \frac{1}{\lambda}} \int_0^{+\infty} ds J_0(\sqrt{s}) \phi\left(\frac{s}{x + \frac{1}{\lambda}}\right), \quad \phi \in \mathcal{Q}(\mathbb{R}).$$

From (1.19) and Theorem 2.1 one infers that $\mathcal{S}(\mathbb{R}) \subset \mathcal{D}_0 \subset \mathcal{D}(\Pi_0)$ and $\mathcal{S}(\mathbb{R}) + \mathbb{C}[\zeta_0]$ is a core for Π_0 . Since $(\mathcal{F} S_\lambda)^*[\mathcal{S}(\mathbb{R})] = \mathcal{S}(\mathbb{R})$ in view of the invariance of the Schwartz space under the Fourier transform and the translations, it follows that

$$\mathcal{D}_0(H_T) := \mathcal{S}(\mathbb{R}) + \mathbb{C}[\tilde{\kappa}_0]$$

is a core for H_T , with $\tilde{\kappa}_0 := (B_\lambda \mathcal{F}^*) \eta_0$ (the function η_0 is described in Proposition 2.3). Moreover, the unitary transform $B_\lambda \mathcal{F}^*$ and Proposition 2.3 also justify (1.25) with $\tilde{\kappa}_1 := (B_\lambda \mathcal{F}^*) \eta_\pi$.

Proposition 3.1. *The functions $\tilde{\kappa}_0$ and $\tilde{\kappa}_1$ are given by the formulas (1.26).*

Proof. Let $\eta_0(x) = e^{-|x|}$ and $\eta_\pi(x) = i \operatorname{sgn}(x) e^{-|x|}$. The inverse Fourier transforms of these functions are given by

$$(\mathcal{F}^* \eta_0)(x) = \sqrt{\frac{2}{\pi}} \frac{1}{1 + x^2}, \quad (\mathcal{F}^* \eta_\pi)(x) = -\sqrt{\frac{2}{\pi}} \frac{x}{1 + x^2}.$$

Since $\mathcal{F}^*\eta_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the transformed function $B\mathcal{F}^*\eta_0$ can be computed via the integral kernel of B . Then Lemma B.2 provides

$$(B\mathcal{F}^*\eta_0)(x) = -i\sqrt{\frac{8}{\pi}} \operatorname{sgn}(x) \operatorname{kei}(2\sqrt{|x|}).$$

Since $\mathcal{F}^*\eta_1 \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$, the transformed function $B\mathcal{F}^*\eta_1$ as to be computed according to the prescription of Remark 3.1. In this case one has

$$(B\mathcal{F}^*\eta_\pi)(x) = -\sqrt{\frac{2}{\pi}} \lim_{R \rightarrow +\infty} \int_{-R}^{+R} dy \frac{\mathcal{B}(x, y)y}{1+y^2}.$$

However, as shown in the proof of Lemma B.2, the integrand is absolutely integrable for every values of x . This allows to forget the limit and one gets

$$(B\mathcal{F}^*\eta_\pi)(x) = i\sqrt{\frac{8}{\pi}} \operatorname{ker}(2\sqrt{|x|}).$$

Finally a translation by S_λ^* and a multiplication by $-i$ provide the formulas (1.26). \square

Remark 3.2 (other self-adjoint extensions). As for the operator Π discussed in Section 2, also the thermal Hamiltonian H_T admits a family of unitarily equivalent self-adjoint extension parametrized by $\theta \in \mathbb{S}^1$, and defined by

$$H_{T,\theta} := \lambda(\mathcal{F}S_\lambda)^* \Pi_\theta(\mathcal{F}S_\lambda).$$

Since $\Pi_\theta = L_\theta \Pi_0 L_\theta^*$ one obtains that $H_{T,\theta}$ is related to the standard thermal Hamiltonian H_T by the unitary equivalence

$$H_{T,\theta} := N_\theta H_T N_\theta^*$$

where $N_\theta := (\mathcal{F}S_\lambda)^* L_\theta(\mathcal{F}S_\lambda)$. An explicit computation provides that

$$N_\theta := \cos\left(\frac{\theta}{2}\right)\mathbf{1} - \sin\left(\frac{\theta}{2}\right)\mathfrak{H}$$

where \mathfrak{H} denotes the *Hilbert transform* defined by

$$(\mathfrak{H}\psi)(x) := \frac{1}{\pi} \int_{\mathbb{R}} dy \frac{\psi(y)}{x-y}$$

over sufficiently regular functions ψ , and with the integral taken as a Cauchy principal value.

3.2. Unitary propagator. Let

$$U_T(t) := e^{-itH_T}$$

the *unitary propagator* associated with the self-adjoint operator H_T . Using the various unitary equivalences that connect H_T with the momentum operator p one has that

$$U_T(t) = B_\lambda (\mathcal{F}^* e^{-i\lambda t p} \mathcal{F}) B_\lambda^* = S_\lambda^* (B e^{i\lambda t x} B) S_\lambda$$

where in the last equality we used $\mathcal{F}^* p \mathcal{F} = -x$. With the help of Lemma 3.1 we can compute the integral kernel of $U_T(t)$.

Proposition 3.2. *On the dense domain $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ the unitary propagator $U_T(t)$ with ($t \neq 0$) acts as an integral operator with kernel given by (1.27) and (1.28).*

Proof. Let us start by computing the kernel of $A_\tau := B e^{i\tau x} B$, with $\tau \in \mathbb{R} \setminus \{0\}$, on $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$,

$$(A_\tau \psi)(x) := \lim_{R \rightarrow +\infty} \int_{-R}^{+R} dy e^{i\tau y} \mathcal{B}(x, y) \left(\int_{\mathbb{R}} ds \mathcal{B}(y, s) \psi(s) \right).$$

The integral in the variable y is meant in the sense of a principal value in view of Remark 3.1. For every $x, \tau \in \mathbb{R}$ the function

$$g_{(x,\tau)}(s, y) := e^{i\tau y} \mathcal{B}(x, y) \mathcal{B}(y, s) \psi(s)$$

is absolutely integrable in $\mathbb{R} \times [-R, +R]$ since $|g_{(x,\tau)}| \leq |\mathcal{B}(x, y)| |\psi(s)|$. Then, we can invoke the Fubini–Tonelli theorem to change the order of integration

$$(A_\tau \psi)(x) = \lim_{R \rightarrow +\infty} \int_{\mathbb{R}} ds \mathcal{A}_\tau^R(x, s) \psi(s), \quad (3.3)$$

where

$$\mathcal{A}_\tau^R(x, s) := \int_{-R}^{+R} dy e^{i\tau y} \mathcal{B}(x, y) \mathcal{B}(y, s).$$

For $xs \neq 0$ the change of variables $u := -xy$ provides

$$\mathcal{A}_\tau^R(x, s) := \frac{\operatorname{sgn}(s) + \operatorname{sgn}(x)}{2x} \int_0^{+R|x|} du e^{-i\frac{\tau}{x}u} J_0(2\sqrt{u}) J_0\left(2\sqrt{\frac{|s|}{|x|}}\sqrt{u}\right).$$

By using formula [15, eq. 6.615] one gets

$$\lim_{R \rightarrow +\infty} \mathcal{A}_\tau^R(x, s) = -i \frac{\operatorname{sgn}(s) + \operatorname{sgn}(x)}{2\tau} e^{i \frac{x+s}{\tau}} I_0\left(-i \frac{2}{\tau} \sqrt{|xs|}\right).$$

Finally, the well-known relations $I_0(\pm i x) = J_0(\mp x) = J_0(x)$ valid for $x \geq 0$ provide

$$\lim_{R \rightarrow +\infty} \mathcal{A}_\tau^R(x, s) = \mathcal{U}_\tau(x, s), \quad (3.4)$$

where the kernel \mathcal{U}_τ is defined by (1.28). Equation (3.4) is valid also in the singular cases $xs = 0$. For instance, for $x = 0$ one gets after the usual change of coordinates

$$\lim_{R \rightarrow +\infty} \mathcal{A}_\tau^R(0, s) = \frac{1}{2s} \int_0^{+\infty} du e^{-i \frac{\tau}{s} u} J_0(2\sqrt{u}) = \mathcal{U}_\tau(0, s)$$

where the last equality is justified by [15, eq. 6.614(1)]. The case $s = 0$ is similar. In view of (3.4) we have the pointwise convergence

$$\lim_{R \rightarrow +\infty} \mathcal{A}_\tau^R(x, s) \psi(s) = \mathcal{U}_\tau(x, s) \psi(s).$$

and since $|\mathcal{U}_\tau(x, s)| \leq |\tau|^{-1}$ for all $(x, s) \in \mathbb{R}^2$ one has that the function $s \mapsto \mathcal{A}_\tau^R(x, s) \psi(s)$ is definitively dominated by the integrable function $s \mapsto |\tau|^{-1} \psi(s)$ (provided $\tau \neq 0$). This fact allows to use Lebesgue's dominated convergence theorem in (3.3), providing in this way

$$(A_\tau \psi)(x) = \int_{\mathbb{R}} ds \mathcal{U}_\tau(x, s) \psi(s). \quad (3.5)$$

Formula (1.27) is obtained by observing that $U_T(t) = S_\lambda^* A_{\lambda t} S_\lambda$. □

3.3. Resolvent and Green function. The resolvent of H_T can be computed as the Laplace transform of the unitary propagator $U_T(t)$ according to the well-known formula [17, eq. (1.28), p. 484]. For every $\zeta \in \mathbb{C} \setminus \mathbb{R}$ let

$$R_\zeta(H_T) := (H_T - \zeta \mathbf{1})^{-1}$$

be the resolvent of H_T . Then, it holds true that

$$R_\zeta(H_T) = i \int_0^{+\infty} dt e^{i \zeta t} U_T(t), \quad \operatorname{Im}(\zeta) > 0, \quad (3.6)$$

where the integral is interpreted as a strong Riemann integral $\lim_{\sigma \rightarrow +\infty} \int_0^\sigma$. The resolvent for $\text{Im}(\zeta) < 0$ can be obtained from the relation $R_{\bar{\zeta}}(H_T) = R_\zeta(H_T)^*$. The formula (3.6) is helpful to compute the integral kernel of $R_\zeta(H_T)$.

One can take advantage of the unitary equivalence $U_T(t) = S_\lambda^* A_{\lambda t} S_\lambda$ used in Proposition 3.2 to obtain $R_\zeta(H_T) = \lambda^{-1} S_\lambda^* Z_{\frac{\zeta}{\lambda}} S_\lambda$ with

$$Z_\alpha := i \int_0^{+\infty} d\tau e^{i\alpha\tau} A_\tau, \quad \text{Im}(\alpha) > 0.$$

Let $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. With the integral kernel of A_τ provided in (3.5) one can write

$$(Z_\alpha \psi)(x) := i \lim_{\sigma \rightarrow +\infty} \int_0^\sigma d\tau e^{i\alpha\tau} \left(\int_{\mathbb{R}} ds \mathcal{U}_\tau(x, s) \psi(s) \right).$$

Since the $J_0(\tau^{-1}) \sim \sqrt{\tau}$ if $\tau \rightarrow 0$ one can check that the function $h_x(\tau, s) := e^{i\alpha\tau} \mathcal{U}_\tau(x, s) \psi(s)$ meets the conditions of the Fubini–Tonelli theorem for the change of the order of integration. Moreover, one can take care of the limit in σ with the help of Lebesgue’s dominated convergence theorem. At the end of these manipulations one gets

$$(Z_\alpha \psi)(x) := \int_{\mathbb{R}} ds \mathcal{Z}_\alpha(x, s) \psi(s)$$

with kernel

$$\mathcal{Z}_\alpha(x, s) := i \int_0^{+\infty} d\tau e^{i\alpha\tau} \mathcal{U}_\tau(x, s) = (\text{sgn}(x) + \text{sgn}(y)) F_\alpha(x, y), \tag{3.7}$$

where

$$F_\alpha(x, y) := \frac{1}{2} \int_0^{+\infty} d\tau \frac{e^{i(\alpha\tau + \frac{(x+s)}{\tau})}}{\tau} J_0\left(\frac{2}{\tau} \sqrt{|xs|}\right).$$

Setting $\alpha := |\alpha| e^{i\phi}$, $0 < \phi < \pi$, the last integral can be integrated case by case using Macdonald’s and Nicholson’s formulas [13, Section 7.7.6] or [21, Section III, p. 98]. A different way of calculating the kernel (3.7) is sketched at the end of Appendix B.3. In both cases, after some tedious calculations, one gets

$$F_\alpha(x, y) := I_0(2\sqrt{|\alpha| \min\{|x|, |y|\}}) e^{i[\frac{\phi}{2} - \frac{\pi}{4}(\text{sgn}(x)+1)]} \times K_0(2\sqrt{|\alpha| \min\{|x|, |y|\}}) e^{i[\frac{\phi}{2} - \frac{\pi}{4}(\text{sgn}(x)+1)]}. \tag{3.8}$$

It is also possible to check directly that the kernel $\mathcal{Z}_\alpha(x, s)$ inverts in a distributional sense the operator $T - \alpha\mathbf{1}$.

3.4. Scattering by a convolution potential. Let $g \in L^1(\mathbb{R})$ and consider the associated convolution potential W_g defined by (1.30). Since W_g is a bounded operator of norm $\|W_g\| = \|g\|_1$ the perturbed operator $H_{T,g} := H_T + W_g$ is well defined as a self-adjoint operator on the domain $\mathcal{D}(H_T)$ as a consequence of the Kato–Rellich theorem [26, Theorem X.12]. The it makes sense to consider the scattering theory of the pair $(H_T, H_{T,g})$.

In view of the unitary equivalence $p = \frac{1}{\lambda} I \mathcal{F} S_\lambda H_T S_\lambda^* \mathcal{F}^* I$ between the momentum operator and H_T we can equivalently study the scattering theory of the pair (p, p_g) , where $p_g := p + M_g$ is the perturbation of the momentum given by the potential

$$M_g := \frac{1}{\lambda} I \mathcal{F} S_\lambda W_g S_\lambda^* \mathcal{F}^* I.$$

Lemma 3.2. *The potential M_g is the multiplication operator defined by*

$$(M_g \psi)(x) := \frac{\sqrt{2\pi}}{\lambda} \hat{g}\left(\frac{1}{x}\right) \psi(x), \quad \psi \in L^2(\mathbb{R}).$$

where \hat{g} denotes the Fourier transform of g .

Proof. By construction the convolution is invariant under translations. This means that $S_\lambda W_g S_\lambda^* = W_g$. Moreover the Fourier transform of a convolution gives a multiplication operator

$$(\mathcal{F} W_g \mathcal{F}^* \psi)(x) := \sqrt{2\pi} \hat{g}(x) \psi(x), \quad \psi \in L^2(\mathbb{R})$$

where \hat{g} denotes the Fourier transform of g . The proof is completed by observing recalling the definition of the involution I . \square

We are now in position to provide the proof of Theorem 1.2.

Proof of Theorem 1.2. Since $g \in L^1(\mathbb{R})$ then $\hat{g} \in C_0(\mathbb{R})$ (continuous functions vanishing at infinity) in view of the Riemann–Lebesgue lemma [26, Theorem IX.7]. This implies that the function $x \mapsto \hat{g}\left(\frac{1}{x}\right)$ belongs to $C(\mathbb{R}) \cap L^\infty(\mathbb{R})$. As a result the multiplicative potential M_g is bounded with norm $\|M_g\| = \frac{\sqrt{2\pi}}{\lambda} \|\hat{g}\|_\infty$ and the conditions of [17, Example 3.1, p. 530] are satisfied. Then, one obtain that p and p_g are unitarily equivalent. This also implies the unitary equivalence of H_T and $H_{T,g}$, and in turn item (i) of claim. In [17, Example 3.1, p. 530] it

is also proven the existence and the completeness for the waves operators associated to the pair (p, p_g) under the assumption that of the existence of the improper integrals

$$\begin{aligned} \lim_{x \rightarrow 0^+} \int_x^{+\infty} ds \hat{g}\left(\frac{1}{s}\right) &= \lim_{x \rightarrow +\infty} \int_0^x ds \frac{\hat{g}(s)}{s^2} \\ \lim_{x \rightarrow 0^-} \int_{-\infty}^x ds \hat{g}\left(\frac{1}{s}\right) &= \lim_{x \rightarrow -\infty} \int_x^0 ds \frac{\hat{g}(s)}{s^2}. \end{aligned}$$

This requires that $\hat{g} \rightarrow 0$ fast enough when $s \rightarrow 0^\pm$. This is guaranteed by the (not optimal) conditions required in the theorem statement. Invoking once again the unitary equivalence between p and H_T one obtain the existence and the completeness for the waves operators associated to the pair $(H_T, H_{T,g})$, proving in this way item (ii). Also for item (iii), in [17, Example 3.1, p. 530] is proven that the S -matrix for the pair (p, p_g) is a complex number given by

$$S_g := e^{-i \frac{\sqrt{2\pi}}{\lambda} \int_{\mathbb{R}} dx \hat{g}\left(\frac{1}{x}\right)} := e^{-i \frac{\sqrt{2\pi}}{\lambda} \int_{\mathbb{R}} ds \frac{\hat{g}(s)}{s^2}}.$$

Since a complex number is unchanged by unitary equivalences it follows that S_g is also the S -matrix for the pair $(H_T, H_{T,g})$. □

4. The classical dynamics

In this last section we will study the classical dynamics induced by a thermal gradient. The classic analogue of Luttinger’s model is provided by the Hamiltonian function

$$H_T(x, p) := (1 + \lambda \gamma \cdot x) \frac{p^2}{2m} = K(p) + \lambda T_\gamma(x, p), \tag{4.1}$$

with parameters $\lambda > 0$ and $\gamma \in \mathbb{S}^{d-1}$. The Hamiltonian H_T can be seen as the sum of the Hamiltonian of a *free* d -dimensional particle of mass m

$$K(p) := \frac{p^2}{2m} = \frac{1}{2m} \sum_{j=1}^N p_j^2$$

coupled through the *coupling constant* $\lambda > 0$ with the *thermal potential*

$$T_\gamma(x, p) := (\gamma \cdot x) K(p) = \left(\frac{p^2}{2m}\right) \sum_{j=1}^d \gamma_j x_j$$

along the direction $\gamma \in \mathbb{S}^{d-1}$. The coupling constant has the dimension of the inverse of a distance, namely $\lambda = \ell^{-1}$ with $\ell > 0$ the *typical length* of the thermal field. Therefore, the limit $\lambda \rightarrow 0$ describes the situation in which the typical length of the field is much larger than the typical length of the system (e.g. the size of the particle). The potential T_γ is an example of what is known as a *generalized potential*, namely a potential which depends not only on the position but also on the velocity.

4.1. Hamiltonian formalism and Newton equation. The Hamilton equations associated to (4.1) read

$$\begin{cases} \dot{x} = +\nabla_p H_T = \frac{(1 + \lambda\gamma \cdot x)}{m} \mathbf{p}, \\ \dot{p} = -\nabla_x H_T = -\lambda \frac{p^2}{2m} \gamma. \end{cases} \quad (4.2)$$

The first equation can be inverted out of the *critical plane*

$$\Xi_c := \{x \in \mathbb{R}^d \mid \gamma \cdot x + \ell = 0\} \quad (4.3)$$

and provides

$$p(x, \dot{x}) = \frac{m}{1 + \lambda\gamma \cdot x} \dot{x}. \quad (4.4)$$

One can restore the usual relation $p = m_T \dot{x}$ between momentum and velocity by introducing the *position-dependent mass* (PDM)

$$m_T(x) := \frac{m}{1 + \lambda\gamma \cdot x}.$$

It is interesting to notice that the Hamiltonian (4.1) can be rewritten as

$$H_T(x, p) = \frac{p^2}{2m_T(x)}, \quad (4.5)$$

namely as the Hamiltonian of a free particle with a PDM. The second equation of (4.2) can be rewritten as

$$\dot{p} = -\lambda \nabla_x T_\gamma. \quad (4.6)$$

A straightforward computation allows to derive Newton's laws from (4.2):

$$m\ddot{x} = \lambda(\gamma \cdot \dot{x})p - \lambda(1 + \lambda\gamma \cdot x) \frac{p^2}{2m} \gamma.$$

After introducing (4.4) in the last expression one obtains Newton's equation

$$m\ddot{x} = \lambda F_T(x, \dot{x})$$

where the *thermal force* (which has the dimensions of a force times a distance) is given by

$$F_T(x, \dot{x}) = m_T(x) \left[(\gamma \cdot \dot{x}) \dot{x} - \frac{\dot{x}^2}{2} \gamma \right]. \quad (4.7)$$

A way of interpreting this Newton's equation is to say that the motion of the PDM-particle is influenced by the effect of its own internally self-produced force field generated by the spatial dependence of the mass. The relation between the force F_T and the potential T_γ can be deduced by observing that

$$-\lambda \nabla_x T_\gamma = -\lambda \frac{m_T(x)^2}{m} \frac{\dot{x}^2}{2} \gamma \quad (4.8)$$

in view of the (4.6), (4.2), and (4.4), respectively. After some manipulation and the use of equation (4.4) one gets

$$F_T(x, p) = -\nabla_x T_\gamma(x, p) + R_T(x, p) \quad (4.9)$$

which shows that the thermal force is not simply given by $-\nabla_x T_\gamma$, as for ordinary conservative forces, but it includes an extra *reacting* term

$$R_T(x, p) := \frac{d}{dt} (\gamma \cdot x) p = m \frac{d}{dt} (\nabla_p T_\gamma(x, p)) \quad (4.10)$$

which is generally not aligned with the direction γ of the field.

4.2. Qualitative analysis. Let us start with the analysis of the qualitative behavior of the solution of the Hamiltonian system (4.1). To simplify the study let us fix convenient notations. The unit vector γ can be completed to an orthonormal basis by adding other $d - 1$ orthonormal vectors e_1, \dots, e_{d-1} . This allows to fix the generalized coordinates $x_0 := \gamma \cdot x$, $x_j := e_j \cdot x$, and the generalized momenta $p_0 := \gamma \cdot p$, $p_j := e_j \cdot p$ with $j = 1, \dots, d - 1$. In this coordinates the Hamiltonian (4.1) reads

$$H_T(x_0, p_1, \dots, p_d) = (1 + \lambda x_0) \frac{p^2}{2m} \quad (4.11)$$

and the Hamilton equations (4.2) become

$$\begin{cases} \dot{x}_j = (1 + \lambda x_0) \frac{p_j}{m}, \\ \dot{p}_j = -\delta_{0,j} \lambda \frac{p^2}{2m}, \end{cases} \quad j = 0, \dots, d - 1. \quad (4.12)$$

The integration of the equations for the “orthogonal” components of the momentum immediately leads to

$$p_j(t) = \wp_j = \text{const.}, \quad j = 1, \dots, d - 1.$$

This can be seen as a consequence of Noether's theorem applied to the invariance under translations of the Hamiltonian H_T along all the directions orthogonal to γ . Let us introduce the constant of motion

$$\wp_{\perp} := \left(\sum_{j=1}^{d-1} \wp_j^2 \right)^{\frac{1}{2}}$$

which quantifies the momentum in the orthogonal plane to the direction of the thermal field. The square of the momentum at any time takes the form

$$p^2(t) = p_0^2(t) + \wp_{\perp}^2. \quad (4.13)$$

The value of the parameter \wp_{\perp} strongly determines the behavior of the solutions of the system (4.12). To see this, one can observe that the Hamiltonian H_T is time-independent and therefore Noether's theorem provides a further constant of motion, i.e., the (total) energy

$$E_0 := (1 + \lambda \varrho_0) \frac{\wp_0^2 + \wp_{\perp}^2}{2m}$$

which is completely specified by the initial conditions

$$\varrho_0 := x_0(t = 0), \quad \wp_0 := p_0(t = 0).$$

The constraint

$$H_T(x(t), p(t)) = E_0, \quad \text{for all } t \in \mathbb{R} \quad (4.14)$$

can be used to obtain the equation

$$x_0(t) = \frac{1}{\lambda} \left(\frac{2mE_0}{p^2(t)} - 1 \right) = \frac{\wp_0^2 + \wp_{\perp}^2}{p_0^2(t) + \wp_{\perp}^2} \left(\frac{1}{\lambda} + \varrho_0 \right) - \frac{1}{\lambda}, \quad (4.15)$$

which provides the time evolution of x_0 once it is known the form of $p_0^2(t)$ and the initial conditions ϱ_0 and $\wp_0, \wp_1, \dots, \wp_{N-1}$. In addition to this, the constraint (4.14) also provides useful information for a qualitative study of the trajectory $x(t)$ of the particle. A comparison between (4.11) and (4.14) shows that the sign of E_0 only depends on the quantity $1 + \lambda \varrho_0$. More precisely, one has that

$$\pm E_0 \geq 0 \implies \pm \varrho_0 \geq \mp \ell.$$

Thus, the critical plane $\Xi_c \subset \mathbb{R}^N$ separates the space into two regions labelled by the sign of the energy E_0 . The full trajectory $x(t)$ of the particle is fully contained in only one of these two half-spaces according to the initial position ϱ_0 along

the direction γ at the initial time $t = 0$. Moreover, the trajectory can touch the critical plane only at the cost of a divergence in the value of the total momentum, $p^2 \rightarrow \infty$.

The existence of this critical impenetrable plane can be justified on the basis of Newton's law $m\ddot{x}_j = \lambda F_{T,j}$ where the force (4.7) is given for components by

$$F_{T,j} = \begin{cases} \frac{E_0}{2} - (1 + \lambda x_0) \frac{\wp_\perp^2}{m} & \text{if } j = 0, \\ (1 + \lambda x_0) \frac{p_0 \wp^j}{m} & \text{if } j = 1, \dots, d-1. \end{cases} \quad (4.16)$$

In the derivation of (4.16) from (4.7) we made use of (4.5) along with $m_T \dot{x} = p$ and the conservation laws (4.13) and (4.14). The component $F_{T,0}$ is proportional to E_0 very close to the critical plane ($1 + \lambda x_0 \sim 0$) and force the particle to stay inside the half-space where the particle was at the initial time. When $\wp_\perp^2 \neq 0$ the component $F_{T,0}$ changes sign sufficiently far from the critical plane and begins to attract the particle towards Ξ_c . This suggests that the motion of the particle must be bounded in the direction γ provided that the momentum has a non-vanishing component orthogonal to γ at the initial time. The components $F_{T,1}, \dots, F_{T,d-1}$ are due to the reaction term R_T (4.10). The conservation of the energy implies that $|p_0| \propto |1 + \lambda x_0|^{-\frac{1}{2}}$ for $x_0 \rightarrow -\ell$. Therefore the orthogonal components of F_T vanish when the particle approaches the critical plane.

4.3. Exceptional solutions. The Hamilton equations (4.12) (or, equivalently, equations (4.2)) admit the *exceptional* family of solutions $p(t) = 0$ and $x(t) = \varrho$ for all $t \in \mathbb{R}$ parametrized by all the possible initial positions $\varrho \in \mathbb{R}^d \setminus \Xi_c$ not belong to the critical plane. In this case the particle is at every moment at rest in a configuration of total zero energy $E_0 = 0$. This is not surprising even though the particle is immersed in the thermal field. In fact the force F_T produced by the field vanishes when $p = 0$. If at the initial time one has $\wp_j = 0$ for all $j = 0, \dots, d-1$ and $\varrho_0 \neq -\ell$, then $p^2 = 0$ for all $t \in \mathbb{R}$ (as a consequence of energy conservation) and therefore the particle is not subject to any force. This allows the particle to stay in equilibrium forever at the position ϱ .

Another family of exceptional solutions is again described by $x(t) = \varrho$ for all $t \in \mathbb{R}$ with the initial positions $\varrho \in \Xi_c$. Also in this case the particle remains at rest in a configuration of total zero energy $E_0 = 0$. However, since the particle lies in the critical plane the total momentum is not forced to be zero. While the component of the momentum orthogonal to γ is constant and quantified by \wp_\perp the component $p_0(t)$ evolves in time according to the Hamilton equation (4.12) (with solutions (4.24) if $\wp_\perp = 0$ or (4.17) when $\wp_\perp \neq 0$).

4.4. The general solution. Let us derive the general solution of the Hamiltonian system (4.12) under the generic assumption $\wp_{\perp} \neq 0$. In this case the differential equation for p_0 reads

$$\dot{p}_0 = -\lambda \frac{p_0^2 + \wp_{\perp}^2}{2m}$$

and is solved by

$$p_0(t) = \wp_{\perp} \tan\left(\phi - \lambda \frac{\wp_{\perp}}{2m} t\right), \quad (4.17)$$

where $\phi := \arctan\left(\frac{\wp_0}{\wp_{\perp}}\right)$ is determined by the initial conditions. Equation (4.17) shows that $p_0(t)$ diverges periodically at the critical times $t_c^{(n)} := t_c + nT$, $n \in \mathbb{Z}$, where

$$t_c := (2\phi - \pi) \frac{\ell m}{\wp_{\perp}}, \quad T := 2\pi \frac{\ell m}{\wp_{\perp}}$$

and $\ell = \lambda^{-1}$.

From (4.17) and (4.13) one immediately gets

$$p^2(t) = \frac{\wp_{\perp}^2}{\cos\left(\phi - \lambda \frac{\wp_{\perp}}{2m} t\right)^2}$$

and after some manipulations, equation (4.15) provides

$$x_0(t) = \varrho_0 + A_{\lambda} \left[\cos\left(\phi - \lambda \frac{\wp_{\perp}}{2m} t\right)^2 - \cos(\phi)^2 \right] \quad (4.18)$$

where we the amplitude A_{λ} is given by

$$A_{\lambda} := \ell \frac{2mE_0}{\wp_{\perp}^2} = \frac{\ell + \varrho_0}{\cos(\phi)^2}.$$

Equation (4.18) shows that the motion along the direction γ is bounded and more precisely is confined between the critical plane Ξ_c which is reached periodically at the critical times $t_c^{(n)}$ and the *extremal* plane

$$\Xi_e := \left\{ x \in \mathbb{R}^d \mid \gamma \cdot x = \varrho_0 + \left(\frac{\wp_0}{\wp_{\perp}}\right)^2 (\ell + \varrho_0) \right\} \quad (4.19)$$

which is reached periodically at the *extremal* times $t_e^{(n)} := t_e + nT$ where $t_e := 2\phi \frac{\ell m}{\wp_{\perp}}$.

By inserting the solution (4.18) in the differential equations for the other components of the position one gets

$$\dot{x}_j(t) = \lambda \frac{\wp^j}{m} A_{\lambda} \cos\left(\phi - \lambda \frac{\wp_{\perp}}{2m} t\right)^2, \quad j = 1, \dots, d-1.$$

For each j , the corresponding differential equation is integrated by

$$x_j(t) = \varrho_j + \lambda \frac{\wp_j}{2m} A_\lambda t - \frac{A_\lambda}{2} \frac{\wp_j}{\wp_\perp} \left[\sin \left(2\phi - \lambda \frac{\wp_\perp}{m} t \right) - \sin(2\phi) \right]. \quad (4.20)$$

Evidently the motion in the directions e_j is unbounded when $\wp_j \neq 0$ due to the linear term in t which describes a uniform motion with constant velocity $v_{j,\lambda} := \lambda A_\lambda \frac{\wp_j}{2m}$.

Let us introduce the unit vector $\nu := \wp_\perp^{-1} \sum_{j=1}^{d-1} \wp_j e_j$. By construction ν is orthogonal to γ and $\wp := \wp_0 \gamma + \wp_\perp \nu$ describes the initial momentum of the particle at $t = 0$. From (4.18) and (4.20) one gets that

$$x(t) = \varrho + A_\lambda (f_0(t) \gamma + f_\perp(t) \nu), \quad (4.21)$$

with $\varrho := \varrho \gamma + \sum_{j=1}^{d-1} \rho_j e_j$ the initial position and

$$\begin{aligned} f_0(t) &:= \cos \left(\phi - \lambda \frac{\wp_\perp}{2m} t \right)^2 - \cos(\phi)^2, \\ f_\perp(t) &:= \lambda \frac{\wp_\perp}{2m} t - \frac{1}{2} \left[\sin \left(2\phi - \lambda \frac{\wp_\perp}{m} t \right) - \sin(2\phi) \right]. \end{aligned}$$

Equation (4.21) shows that the motion of the particle is essentially two-dimensional. In fact the orbit $x(t)$ lies entirely in the affine plane spanned by μ and ν and passing through the initial position ρ .

Remark 4.1 (2D-case). In view of (4.21) the general motion of a particle in the thermal field is a two-dimensional motion provided that the initial momentum is not aligned with the direction of the field. Therefore, one can always identify the direction γ of the field and the direction ν of the orthogonal component of the initial momentum with the x -axis and the y -axis of \mathbb{R}^2 , respectively. This allows us to use the ‘‘cozy’’ notation $x(t)$ and $y(t)$ for the two projections of the trajectory along the direction γ and ν , respectively. Let $\wp = (\wp_x, \wp_y)$ be the components of the initial momentum projected along the two coordinate direction γ and ν . Let us consider here the special situation in which the total momentum is completely orthogonal to γ . This means that $\wp_0 = \wp_x = 0$ and $\wp_\perp = |\wp_y| = |\wp|$. This also implies that $\phi = \arctan(0) = 0$ and $A_\lambda = \ell + \varrho_x$ with $\varrho_x = \varrho_0$ is the x -component of the initial position $\varrho = (\varrho_x, \varrho_y)$. In this case the equations of motion for the position simplify to

$$x(t) = \varrho_x + (\ell + \varrho_x) \left[\cos \left(\lambda \frac{|\wp|}{2m} t \right)^2 - 1 \right], \quad (4.22)$$

$$y(t) = \varrho_y + (\ell + \varrho_x) \left[\lambda \frac{|\wp|}{2m} t + \frac{1}{2} \sin \left(\lambda \frac{|\wp|}{m} t \right) \right].$$

The time evolution of the momentum is described by the equations $p_x(t) = -|\wp| \tan\left(\lambda \frac{|\wp|}{2m} t\right)$ and $p_y(t) = \wp_y$.

4.5. The one-dimensional case. As discussed at the end of Section 4.4 (see Remark 4.1) the general motion of a particle in the thermal field is two-dimensional whenever $\wp_\perp \neq 0$. Therefore the condition $\wp_\perp = 0$, $\wp_0 \neq 0$ corresponds to considering the one-dimensional case. In fact, under these conditions, one immediately gets from (4.12) that $p_j(t) = \wp_j = 0$ for all $j = 1, \dots, d-1$. This in turn implies $\dot{x}_j = 0$ for $j = 1, \dots, d-1$ and so

$$x_j(t) := \varrho_j = \text{const.}, \quad j = 1, \dots, d-1.$$

This means that the only possible motion could take place exclusively in the direction γ , namely it is one-dimensional.

Without loss of generality let us assume that $\varrho_1 = \dots = \varrho_{d-1} = 0$ which means that $x_j(t) = 0 = p_j(t)$ for all $j = 1, \dots, d-1$. Given that, the only interesting degrees of freedom are x_0 and p_0 and we can simplify the notation identifying x_0 with x and p_0 with p . With this notation the (non-trivial) one-dimensional system of Hamilton equations reads

$$\begin{cases} \dot{x} = (1 + \lambda x) \frac{p}{m}, \\ \dot{p} = -\lambda \frac{p^2}{2m}. \end{cases} \quad (4.23)$$

The equation for the momentum immediately integrated by

$$p(t) = \ell \frac{\wp}{\frac{\wp}{2m} t + \ell} \quad (4.24)$$

with $\wp = p(0)$ the initial momentum. Notice that the value of the momentum diverges at the critical time $t_c := -\ell \frac{2m}{\wp}$.

The time evolution of the position can be derived directly from equation (4.15) which, after some algebraic manipulation, provides

$$x(t) = \frac{\ell + \varrho}{\ell^2} \left(\frac{\wp}{2m} t + \ell \right)^2 - \ell \quad (4.25)$$

with $\varrho = x(0)$ the initial position. The long time behavior of the trajectory is determined by the sign of the coefficient of t^2 in (4.25), namely by the sign of $\ell + \varrho$. It follows that

$$\lim_{|t| \rightarrow \infty} x(t) = \pm \infty \quad \text{if } \pm \varrho > \mp \ell.$$

The turning time in which the velocity changes sign is determined by $\dot{x}(t) = 0$ and a simple computation shows that this time coincides with the critical time t_c . Moreover, one has that $x(t_c) = -\ell$ independently of the initial value $\varrho_0 \neq -\ell$. In conclusion the critical plane Ξ_c separates the space into two regions and the trajectory $x(t)$ is fully contained in only one of these two half-spaces according to the initial position ϱ . Moreover, the trajectory can touch the critical plane only once at the critical time t_c . These results are in accordance with the qualitative analysis of Section 4.2.

4.6. The Lagrangian Formalism. By using the Legendre transformation

$$\mathcal{L}_T(x, \dot{x}) = \dot{x} \cdot p - H_T(x, p)$$

one can compute the Lagrangian of the system:

$$\mathcal{L}_T(x, \dot{x}) := \frac{1}{2} \frac{m}{1 + \lambda \gamma \cdot x} \dot{x}^2 = m_T(x) \frac{\dot{x}^2}{2}. \quad (4.26)$$

Expressions of the type (4.26) are well studied in the literature under the name of *quasi-free PDM Lagrangian* (see [20, 7, 22] and references therein). The canonical momentum

$$p(x, \dot{x}) := \nabla_{\dot{x}} \mathcal{L}_T(x, \dot{x}) = m_T(x) \dot{x}$$

is exactly that given by equation (4.4). To compute the Euler–Lagrange equations of motion we need also

$$\nabla_x \mathcal{L}_T(x, \dot{x}) = \nabla_x m_T(x) \frac{\dot{x}^2}{2} = -\lambda \frac{m_T(x)^2 \dot{x}^2}{m} \gamma.$$

A comparison with (4.8) shows that

$$\nabla_x \mathcal{L}_T = \dot{p} = -\nabla_x H_T$$

and this assures that the Euler–Lagrange equation

$$\frac{d}{dt}(\nabla_{\dot{x}} \mathcal{L}_T) = \nabla_x \mathcal{L}_T$$

is equivalent to the Hamilton system (4.2). An explicit computation provides

$$\begin{aligned} \frac{d}{dt}(\nabla_{\dot{x}} \mathcal{L}_T) &= m_T(x) \ddot{x} + \frac{d}{dt}(m_T(x)) \dot{x} \\ &= m_T(x) \ddot{x} - \lambda \frac{m_T(x)^2}{m} (\gamma \cdot \dot{x}) \dot{x} \end{aligned}$$

and putting all the pieces together one gets

$$m_T(x)\ddot{x} = \lambda \frac{m_T(x)^2}{m} (\gamma \cdot \dot{x}) \dot{x} - \lambda \frac{m_T(x)^2}{m} \frac{\dot{x}^2}{2} \gamma \quad (4.27)$$

which is equivalent to Newton's equation $m\ddot{x} = \lambda F_T$ with the force (4.7).

In the one-dimensional it is useful to use the change of Lagrangian coordinates $(x, \dot{x}) \mapsto (q, \dot{q})$ implemented by

$$x(q) := e^{\lambda q} - \frac{1}{\lambda}, \quad \dot{x}(q, \dot{q}) := \lambda e^{\lambda q} \dot{q}.$$

The inverse is given by

$$q(x) := \frac{1}{\lambda} \log \left(x + \frac{1}{\lambda} \right)$$

and shows that the change of coordinates between x and q is one-to-one only when $x \geq -\ell$. However, as seen in Section 4.2, this is exactly the range of values of interest for the problem. With this change of coordinates the Lagrangian becomes

$$\mathcal{L}'_T(q, \dot{q}) := m\lambda e^{\lambda q} \frac{\dot{q}^2}{2}. \quad (4.28)$$

and the associated Euler–Lagrange equation reads

$$\ddot{q} := -\lambda \frac{\dot{q}^2}{2}.$$

This equation immediately provides the time-behavior of the generalized velocity

$$\dot{q}(t) := \frac{\dot{q}_0}{1 + \frac{\dot{q}_0 \lambda}{2} t}$$

and a further integration gives

$$q(t) := q_0 + \frac{2}{\lambda} \log \left(1 + \frac{\dot{q}_0 \lambda}{2} t \right)$$

where q_0, \dot{q}_0 are the initial conditions. By coming back to the original variable one can recover the expression (4.25) for $x(t)$.

Appendices

A. Spectral theory of the momentum operator

Let $p = -i \frac{d}{dx}$ be the momentum operator with domain $H^1(\mathbb{R}) \subset L^2(\mathbb{R})$, and purely absolutely continuous spectrum $\sigma(p) = \sigma_{\text{a.c.}}(p) = \mathbb{R}$.

A.1. Green's function. With the help of the Fourier transform \mathcal{F} one gets [26, Theorem IX.29]

$$((p - \zeta \mathbf{1})^{-1} \psi)(x) = \int_{\mathbb{R}} dy \mathcal{G}_{\zeta}^0(x, y) \psi(y)$$

where Green's function of p is given by

$$\mathcal{G}_{\zeta}^0(x, y) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^* \left[\frac{1}{k - \zeta} \right] (x - y).$$

A straightforward computation involving contours integrals in the complex plane provides

$$\mathcal{G}_{\epsilon \pm i\delta}^0(x, y) = \pm i \Theta(\pm(x - y)) e^{i\epsilon(x-y)} e^{-\delta|x-y|} \quad (\text{A.1})$$

with $\epsilon \in \mathbb{R}$ and $\delta > 0$.

A.2. Spectral measure. Let μ_{ψ}^A be the spectral measure of the self-adjoint operator A associated with the state $\psi \in L^2(\mathbb{R})$. The function $F_{\psi}^A: \mathbb{C} \setminus \mathbb{R} \rightarrow \mathbb{C}$ defined by the scalar product

$$F_{\psi}^A(\zeta) := \langle \psi, (A - \zeta \mathbf{1})^{-1} \psi \rangle = \int_{\mathbb{R}} d\mu_{\psi}^A(\epsilon) \frac{1}{\epsilon - \zeta}$$

is called the *Borel–Stieltjes transformation* of the finite Borel measures μ_{ψ}^A . Since

$$\text{Im}(F_{\psi}^A(\zeta)) = \text{Im}(\zeta) \int_{\mathbb{R}} d\mu_{\psi}^A(\epsilon) \frac{1}{|\epsilon - \zeta|^2}$$

it follows that $F_{\psi}^A: \mathbb{C}^+ \rightarrow \mathbb{C}^+$ is a holomorphic map from the upper half plane \mathbb{C}^+ into itself. Such functions are called *Herglotz* or *Nevanlinna functions* (see [10, Section 1.4] or [5, Appendix]). A classical result by de la Vallée-Poussin assures that the limit $F_{\psi}^A(\epsilon) := \lim_{\delta \rightarrow 0^+} F_{\psi}^A(\epsilon + i\delta)$ exists and is finite for Lebesgue-almost every $\epsilon \in \mathbb{R}$. Moreover, the absolutely continuous part of the spectral measure μ_{ψ}^A can be recovered from the imaginary part of $F_{\psi}^A(\rho)$ according to the classical formula [10, Theorem 1.4.16.]

$$\mu_{\psi}^A|_{\text{a.c.}}(d\epsilon) = f_{\psi}^A(\epsilon) d\epsilon$$

with

$$f_{\psi}^A(\epsilon) := \lim_{\delta \rightarrow 0^+} \frac{1}{\pi} \text{Im}(F_{\psi}^A(\epsilon + i\delta)). \quad (\text{A.2})$$

In the case $A = p$ is the standard momentum operator one knows that the spectral measure is purely absolutely continuous, i.e., $\mu_\psi^p = \mu_\psi^p|_{\text{a.c.}}$. By the help of the Fourier transform \mathcal{F} one obtains that

$$F_\psi^p(\epsilon + i\delta) := \langle \psi, (p - (\epsilon + i\delta)\mathbf{1})^{-1}\psi \rangle = \int_{\mathbb{R}} dk \frac{|\hat{\psi}(k)|^2}{(k - \epsilon) - i\delta}$$

where $\hat{\psi} := \mathcal{F}(\psi)$ is the Fourier transform of ψ . The application of the formula (A.2) provides

$$f_\psi^p(\epsilon) = \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}} dk \frac{1}{\pi} \frac{\delta}{(k - \epsilon)^2 + \delta^2} |\hat{\psi}(k)|^2 = |\hat{\psi}(\epsilon)|^2$$

where in the last equality one used that $\frac{1}{\pi} \frac{\delta}{x^2 + \delta^2}$ converges in the distributional sense to $\delta(x)$ when $\delta \rightarrow 0^+$. In this way one recovers the well-known result

$$\mu_\psi^p(d\epsilon) = |\hat{\psi}(\epsilon)|^2 d\epsilon. \quad (\text{A.3})$$

A.3. Density of states. For $\epsilon \in \mathbb{R}$ let P_ϵ , be the spectral projection of p associated with the energy ϵ according to (2.15). Let $(Q_L \psi)(x) = \chi_{[-L, L]}(x)\psi(x)$ be the projection which restricts the functions $\psi \in L^2(\mathbb{R})$ on the set $[-L, L]$. The quantity

$$\mathcal{N}_L^p(\epsilon) := \frac{\text{sgn}(\epsilon)}{2L} \text{Tr}(P_\epsilon Q_L)$$

is well defined since $P_\epsilon Q_L$ is trace-class in view of [27, Theorem XI 20].

Lemma A.1. For every $\epsilon \in \mathbb{R}$ and $L > 0$ it holds true that

$$\mathcal{N}_L^p(\epsilon) = \frac{\epsilon}{2\pi}.$$

Proof. By introducing the local Fourier basis supported in $[-L, L]$

$$\psi_n^L(x) := \frac{\chi_{[-L, L]}(x)}{\sqrt{2L}} e^{i\pi \frac{n}{L}x}, \quad n \in \mathbb{Z}$$

one obtains that

$$\begin{aligned} \mathcal{N}_L^p(\epsilon) &= \frac{\text{sgn}(\epsilon)}{2L} \sum_{n \in \mathbb{Z}} \langle \psi_n^L, P_\epsilon \psi_n^L \rangle = \frac{1}{2L} \sum_{n \in \mathbb{Z}} \int_0^\epsilon \mu_{\psi_n^L}^p(d\epsilon') \\ &= \frac{1}{2L} \sum_{n \in \mathbb{Z}} \int_0^\epsilon d\epsilon' |\hat{\psi}_n^L(\epsilon')|^2 = \int_0^\epsilon d\epsilon' g_L(\epsilon') \end{aligned} \quad (\text{A.4})$$

where

$$\begin{aligned}
 g_L(\epsilon) &:= \frac{1}{2L} \sum_{n \in \mathbb{Z}} |\hat{\psi}_n^L(\epsilon)|^2 = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\frac{\sin(\epsilon L - \pi n)}{\epsilon L - \pi n} \right)^2 \\
 &= \frac{1}{2\pi} \left(\frac{\sin(\epsilon L)}{\pi} \right)^2 \sum_{n \in \mathbb{Z}} \left(\frac{\epsilon L}{\pi} - n \right)^{-2}.
 \end{aligned}
 \tag{A.5}$$

Observe that the exchange between the sum and the integral in the last equality of (A.4) is justified by the monotone convergence theorem and the computation (A.5). The formula $\sum_{n \in \mathbb{Z}} (a - n)^{-2} = \left(\frac{\pi}{\sin(a)}\right)^2$ [] provides $g_L(\epsilon) = \frac{1}{2\pi}$ independently of L . □

The *integrated density of states* (IDOS) $\mathcal{N}^p : \mathbb{R} \rightarrow \mathbb{R}$ is defined by the limit

$$\mathcal{N}^p(\epsilon) := \lim_{L \rightarrow +\infty} \mathcal{N}_L^p(\epsilon).$$

From Lemma A.1 one gets that

$$\mathcal{N}^p(\epsilon) = \frac{\epsilon}{2\pi} = \int_0^\epsilon d\epsilon' g(\epsilon')$$

where the last equality emphasizes the fact that \mathcal{N}^p can be obtained by integrating the constant *density of states* (DOS) $g(\epsilon) := \frac{1}{2\pi}$.

The definition of the IDOS can be generalized allowing sequences of increasing sets less symmetric than $[-L, L]$. This essentially boils down on the invariance of p under translations.

Lemma A.2. *For every $\epsilon \in \mathbb{R}$ and every interval $\Lambda := [a, b] \subset \mathbb{R}$ of finite volume $|\Lambda| = b - a$ it holds true that*

$$\mathcal{N}_\Lambda^p(\epsilon) := \frac{\text{sgn}(\epsilon)}{|\Lambda|} \text{Tr}(P_\epsilon Q_\Lambda) = \frac{\epsilon}{2\pi}$$

where Q_Λ is the projection on Λ .

Proof. Set $L := \frac{b-a}{2}$ and $d := -\frac{a+b}{2}$. Let U_d be the unitary operator defined by $(U_d \psi)(x) := \psi(x - d)$. A simple calculation provides $U_d Q_\Lambda U_d^* = Q_{[-L, L]} \equiv Q_L$. From the invariance of the trace under unitary equivalences and the fact that P_ϵ and U_d commute one gets

$$\mathcal{N}_\Delta^p(\epsilon) = \frac{\text{sgn}(\epsilon)}{|\Lambda|} \text{Tr}(U_d P_\epsilon Q_\Lambda U_d^*) = \frac{\text{sgn}(\epsilon)}{2L} \text{Tr}(P_\epsilon Q_L) = \mathcal{N}_L^p(\epsilon).$$

The claim follows from Lemma A.1. □

Remark A.1 (DOS of the Laplacian). The IDOS of the momentum p and of the Laplacian p^2 are easily related by observing that $\chi_{[0,\epsilon]}(x^2) = \chi_{[-\sqrt{\epsilon},\sqrt{\epsilon}]}(x)$. From this relation one deduces

$$\mathcal{N}^{p^2}(\epsilon) = \mathcal{N}^p(\sqrt{\epsilon}) - \mathcal{N}^p(-\sqrt{\epsilon}) = 2\mathcal{N}^p(\sqrt{\epsilon}) = \frac{\sqrt{\epsilon}}{\pi}, \quad \epsilon \geq 0.$$

The last equality allows to recover the well-known formula for the DOS of the Laplacian which is given by $g^{(2)}(\epsilon) := \frac{1}{2\pi} \frac{1}{\sqrt{\epsilon}}$.

B. Technical tools

B.1. Some principal value integrals. The central argument of this appendix is the study of the following *principal value* integral

$$\mathcal{P} \int_{\mathbb{R}} du f(u) := \lim_{\substack{R \rightarrow +\infty \\ r \rightarrow 0^+}} \int_{\mathbb{I}_{R,r}} du f(u)$$

where $\mathbb{I}_{R,r} := [-R, -r] \cup [+R, +r]$ for all $R > r > 0$.

Lemma B.1. *Let*

$$G_s^\pm(u) := \frac{e^{is(u \pm \frac{1}{u})}}{u}, \quad s \in \mathbb{R}.$$

Then the principal value of G_s^\pm is given by

$$\mathcal{P} \int_{\mathbb{R}} du G_s^\pm(u) = i(1 \pm 1)\pi \operatorname{sgn}(s) J_0(2|s|), \quad (\text{B.1})$$

where J_0 is the 0-th Bessel function of the first kind.

Proof. For the trivial case $s = 0$ one has that $G_0^\pm(u) = u^{-1}$ and

$$\int_{\mathbb{I}_{R,r}} \frac{du}{u} = 0, \quad \text{for all } R > r > 0$$

since the function u^{-1} is odd and the integration domain $\mathbb{I}_{R,r}$ is symmetric with respect the origin. It follows that the principal value of G_0^\pm is identically zero according to (B.1). For $s \neq 0$ we one has the symmetry

$$G_{-|s|}^\pm(u) = -G_{|s|}^\pm(-u)$$

which provides

$$\mathcal{P} \int_{\mathbb{R}} d u G_{-|s|}^{\pm}(u) = \mathcal{P} \int_{\mathbb{R}} d(-u) G_{|s|}^{\pm}(-u) = - \mathcal{P} \int_{\mathbb{R}} d u G_{|s|}^{\pm}(u). \tag{B.2}$$

The relation (B.2) guarantees that we can focus only on the case $s > 0$. In this case the computation of the principal value of G_s^{\pm} requires Cauchy’s residue theorem. The function G_s^{\pm} has a holomorphic extension to every bounded open subset of $\mathbb{C} \setminus \{0\}$ and has a singularity in 0. Let us start by computing the residue of G_s^{\pm} . From the formula of the generating function for Bessel functions [15, eq. 8.511 (1)] one obtains the Laurent series

$$G_s^-(u) = \sum_{n \in \mathbb{Z}} J_n(i 2s) u^{n-1} = \sum_{n \in \mathbb{Z}} i^n I_n(2s) u^{n-1}$$

where the J_n are the Bessel function of the first kind and the $I_n(z) := (-i)^n J_n(i z)$ are the modified Bessel functions of the first kind. The Laurent series for G_s^+ can be derived from the relation

$$G_s^+(u) = i G_{-i s}^-(i u)$$

and provides

$$G_s^+(u) = \sum_{n \in \mathbb{Z}} i^n J_n(2s) u^{n-1}.$$

By definition, the residue of G_s^{\pm} is the coefficient of its Laurent series for $n = -1$. This provides

$$\text{Res}_{u=0}(G_s^-) = I_0(2s), \quad \text{Res}_{u=0}(G_s^+) = J_0(2s).$$

From Cauchy’s residue theorem one gets

$$i 2\pi \text{Res}_{u=0}(G_s^{\pm}) = \oint_{\Gamma_{R,r}} d z G_s^{\pm}(z) = \left(\int_{\mathbb{I}_{R,r}} + \int_{C_R^+} + \int_{C_r^-} \right) d z G_s^{\pm}(z)$$

where $\Gamma_{R,r}$ is a positively (counterclockwise) oriented simple closed curve composed by the union of the domain $\mathbb{I}_{R,r}$ on the real line, the semicircle $C_r^- := \{r e^{i\theta} \mid \theta \in [-\pi, 0]\}$ in the lower half-plane and the semicircle $C_R^+ := \{R e^{i\theta} \mid \theta \in [0, \pi]\}$ in the upper half-plane. An explicit computation provides

$$\int_{C_R^+} d z G_s^{\pm}(z) = i \int_0^{+\pi} d \theta e^{i s(R \pm R^{-1}) \cos \theta} e^{-s(R \mp R^{-1}) \sin \theta},$$

and consequently one has the following estimate:

$$\left| \int_{\mathbb{C}_R^+} dz G_s^\pm(z) \right| \leq \int_0^{+\pi} d\theta e^{-s(R \mp R^{-1}) \sin \theta}.$$

Since $e^{-s(R \mp R^{-1}) \sin \theta} \rightarrow 0$ when $R \rightarrow +\infty$ for all $\theta \in (0, \pi)$, it follows from Lebesgue's dominated convergence theorem that

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{C}_R^+} dz G_s^\pm(z) = 0. \quad (\text{B.3})$$

A similar computation for the integral along \mathbb{C}_r^- provides

$$\int_{\mathbb{C}_r^-} dz G_s^\pm(z) = i \int_{-\pi}^0 d\theta e^{i s(r \pm r^{-1}) \cos \theta} e^{-s(r \mp r^{-1}) \sin \theta}.$$

After the change of coordinate $\theta \mapsto -\theta$ one gets

$$\left| \int_{\mathbb{C}_r^-} dz G_s^\pm(z) \right| \leq \int_0^{+\pi} d\theta e^{\mp s(r^{-1} \pm r) \sin \theta}.$$

The latest inequality along with Lebesgue's dominated convergence theorem provides

$$\lim_{r \rightarrow 0^+} \int_{\mathbb{C}_r^-} dz G_s^+(z) = 0, \quad (\text{B.4})$$

but we didn't get a similar result for $G_s^-(z)$. Putting together (B.3), (B.4), and the formula of the residue theorem one gets

$$\mathcal{P} \int_{\mathbb{R}} du G_s^+(u) = i 2\pi J_0(2s), \quad s > 0. \quad (\text{B.5})$$

Finally, from both estimates and the residue the following uniform bound for $r < 1 < R$ is obtained

$$\left| \int_{\mathbb{I}_{R,r}} dz G_s^+(z) \right| \leq 4\pi$$

For the case $s < 0$, the relation (B.2) immediately provides

$$\mathcal{P} \int_{\mathbb{R}} du G_s^+(u) = -i 2\pi J_0(2|s|), \quad s < 0. \quad (\text{B.6})$$

Equations (B.5) and (B.6) together provide the proof of the formula (B.1) for G_s^+ (which automatically includes also the case $s = 0$ discussed at the beginning). The case of G_s^- can be managed by the following application of Cauchy’s residue theorem:

$$0 = \oint_{\Sigma_{R,r}} dz G_s^-(z) = \left(\int_{\mathbb{I}_{R,r}} + \int_{C_R^+} - \int_{C_r^+} \right) dz G_s^-(z),$$

where $\Sigma_{R,r}$ is a positively (counterclockwise) oriented simple closed curve composed by the union of the domain $\mathbb{I}_{R,r}$ on the real line the semicircles $C_R^+ := \{R e^{i\theta} \mid \theta \in [0, \pi]\}$ and $C_r^+ := \{r e^{i\theta} \mid \theta \in [0, \pi]\}$ both in the upper half-plane. The zero on the right-hand side is justified by the fact that $\Sigma_{R,r}$ does not enclose the singularity of $G_s^-(z)$ and the negative sign on the last integral is due to the fact that the semicircle C_r^+ has the opposite orientation with respect to C_R^+ . Equation (B.3) takes care of the integral over C_R^+ . The integral over C_r^+ can be controlled by observing that

$$\left| \int_{C_r^+} dz G_s^\pm(z) \right| \leq \int_0^\pi d\theta e^{-s(r+r^{-1}) \sin \theta}.$$

and, in turn

$$\lim_{r \rightarrow 0^+} \int_{C_r^+} dz G_s^-(z) = 0, \quad s > 0, \tag{B.7}$$

as a consequence of Lebesgue’s dominated convergence theorem. Putting together (B.3) and (B.7) in Cauchy’s residue formula one gets

$$\mathcal{P} \int_{\mathbb{R}} du G_s^-(u) = 0, \quad s > 0. \tag{B.8}$$

Similarly to case (a), an analogous bound can also be obtained. Combining both results we have

$$\left| \int_{\mathbb{I}_{R,r}} dz G_s^\pm(z) \right| \leq 4\pi. \tag{B.9}$$

These same results also hold also for $s < 0$ in view of the relation B.2. □

Corollary B.1. *The formula*

$$\mathcal{P} \int_{\mathbb{R}} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} = i 2\pi \left(\frac{\text{sgn}(x) - \text{sgn}(y)}{2} \right) J_0(2\sqrt{|xy|})$$

holds true for all $(x, y) \in \mathbb{R}^2$. Moreover the uniform bound

$$\left| \int_{\mathbb{I}_{R,r}} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} \right| \leq 4\pi \quad (\text{B.10})$$

is valid for all $x, y \in \mathbb{R}$.

Proof. Let us start by considering the singular situations $xy = 0$. The case $x = 0 = y$ corresponds to

$$\mathcal{P} \int_{\mathbb{R}} \frac{du}{u} = 0,$$

as proved at the beginning of Lemma B.1. The case $y = 0$ is proportional to the (well-known) Fourier transform of the function u^{-1} and provides

$$\mathcal{P} \int_{\mathbb{R}} du \frac{e^{ixu}}{u} = -\sqrt{2\pi} \mathcal{F}\left(\frac{1}{u}\right) = i\pi \operatorname{sgn}(x).$$

The case $x = 0$ can be treated with the change of variables $u \mapsto -v^{-1}$ which provides

$$\mathcal{P} \int_{\mathbb{R}} du \frac{e^{-i\frac{y}{u}}}{u} = -\mathcal{P} \int_{\mathbb{R}} dv \frac{e^{iyv}}{v} = -i\pi \operatorname{sgn}(y).$$

The non singular situation $xy \neq 0$ can be separated in two different cases: (a) $xy > 0$, and (b) $xy < 0$.

Case (a). Let $a := \sqrt{xy}$. Then, after the change of variables $v := \frac{a}{|y|}u$, one has

$$\int_{\mathbb{I}_{R,r}} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} = \int_{\mathbb{I}_{R',r'}} dv \frac{e^{i\frac{x|y|}{a}v} e^{-i\operatorname{sgn}(y)\frac{a}{v}}}{v} = \int_{\mathbb{I}_{R',r'}} dv G_s^-(v),$$

where $R' := a|y|^{-1}R$, $r' := a|y|^{-1}r$ and $s = a \operatorname{sgn}(y)$. Then, Lemma B.1 provides

$$\mathcal{P} \int_{\mathbb{R}} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} = \mathcal{P} \int_{\mathbb{R}} dv G_s^-(v) = 0.$$

Case (b). Let $b := \sqrt{|xy|}$. Then, after the change of variables $v := \frac{b}{|y|}u$, one has

$$\int_{\mathbb{I}_{R,r}} du \frac{e^{ixu} e^{-i\frac{y}{u}}}{u} = \int_{\mathbb{I}_{R',r'}} dv \frac{e^{i\frac{x|y|}{b}v} e^{-i\operatorname{sgn}(y)\frac{b}{v}}}{v} = \int_{\mathbb{I}_{R',r'}} dv G_s^+(v),$$

where $R' := b|y|^{-1}R$, $r' := b|y|^{-1}r$ and $s = -b \operatorname{sgn}(y)$. Again Lemma B.1 provides

$$\mathcal{P} \int_{\mathbb{R}} du \frac{e^{ixu} e^{-i\frac{x}{u}}}{u} = \mathcal{P} \int_{\mathbb{R}} dv G_s^+(v) = -i 2\pi \operatorname{sgn}(y) J_0(2\sqrt{|xy|}).$$

The observation that $-2 \operatorname{sgn}(y) = \operatorname{sgn}(x) - \operatorname{sgn}(y)$ when $xy < 0$ completes this case. The uniform bound (B.10) is deduced directly from (B.9) and the particular case $x = y = 0$ □

B.2. Irregular Kelvin functions. A reference for the (*irregular*) Kelvin functions is [23, Chapter 55]. Here we are interested only on the irregular functions of 0-th order

$$\ker(x) := \ker_{\nu=0}(x), \quad \operatorname{kei}(x) := \operatorname{kei}_{\nu=0}(x).$$

We are interested in the behavior of these functions on the half line $\mathbb{R}_+ := [0, +\infty)$. Both $\ker(x)$ and $\operatorname{kei}(x)$ have an exponential decay of the type $\sim \sqrt{\frac{\pi}{2x}} e^{-\frac{x}{\sqrt{2}}}$ when $x \rightarrow +\infty$. The function $\operatorname{kei}(x)$ is regular in the origin where it takes the value $\operatorname{kei}(0) = -\frac{\pi}{4}$. The function $\ker(x)$ diverges at the origin as $\sim -\log(x)$. In particular one has that both the Kelvin functions are in $L^2(\mathbb{R}_+)$. The importance of the Kelvin functions for the present work is related to the next result.

Lemma B.2. *Let $\mathcal{B}(x, y)$ the kernel (1.24). Then, the following formulas hold true:*

$$\begin{aligned} \int_{\mathbb{R}} dy \frac{\mathcal{B}(x, y)}{1 + y^2} &= -i 2 \operatorname{sgn}(x) \operatorname{kei}(2\sqrt{|x|}), \\ \int_{\mathbb{R}} dy \frac{\mathcal{B}(x, y)y}{1 + y^2} &= -i 2 \ker(2\sqrt{|x|}). \end{aligned}$$

Proof. After the change of variable $s := xy$ one gets

$$J_1(x) := \int_{\mathbb{R}} dy \frac{\mathcal{B}(x, y)}{1 + y^2} = ix \int_{-\infty}^0 ds \frac{J_0(2\sqrt{|s|})}{x^2 + s^2}.$$

A second change of variable $s := -t^2$ provides

$$\begin{aligned} \mathcal{J}_1(x) &= i 2x \int_0^{+\infty} dt t \frac{J_0(2t)}{x^2 + t^4} \\ &= i 2 \operatorname{sgn}(x) \int_0^{+\infty} d\left(\frac{t}{\sqrt{|x|}}\right) \left(\frac{t}{\sqrt{|x|}}\right) \frac{J_0(2\sqrt{|x|}\frac{t}{\sqrt{|x|}})}{\left(\frac{t}{\sqrt{|x|}}\right)^4 + 1} \\ &= -i 2 \operatorname{sgn}(x) \operatorname{kei}(2\sqrt{|x|}), \end{aligned}$$

where the last equality is justified by [23, eq. 55:3:6].

The second formula can be proved with similar changes of variable and one gets

$$\begin{aligned} \mathcal{J}_2(x) &:= \int_{\mathbb{R}} dy \frac{\mathcal{B}(x, y)y}{1 + y^2} = i \int_{-\infty}^0 ds \frac{J_0(2\sqrt{|s|})s}{x^2 + s^2} \\ &= -i 2 \int_0^{+\infty} dt t^3 \frac{J_0(2t)}{x^2 + t^4} \\ &= -i 2 \int_0^{+\infty} d\left(\frac{t}{\sqrt{|x|}}\right) \left(\frac{t}{\sqrt{|x|}}\right)^3 \frac{J_0(2\sqrt{|x|}\frac{t}{\sqrt{|x|}})}{\left(\frac{t}{\sqrt{|x|}}\right)^4 + 1} \\ &= -i 2 \operatorname{ker}(2\sqrt{|x|}), \end{aligned}$$

where the last equality comes from [23, eq. 55:3:5]. □

B.3. Bessel equation and Hankel transform. According to (1.15), the eigenvalue equation associated to the one-dimensional version of the operator T is

$$x \frac{d^2 \psi}{dx^2}(x) + \frac{d \psi}{dx}(x) = -k \psi(x), \quad k \in \mathbb{R}. \quad (\text{B.11})$$

The change of coordinates $x(u, k) := \frac{u^2}{4k}$ produces

$$\frac{d^2 \phi}{du^2}(u) + \frac{1}{u} \frac{d \phi}{du}(u) + \phi(u) = 0 \quad (\text{B.12})$$

where $\phi(u) := \psi(x(u, k))$. The (B.12) are the Bessel equations of order 0 and the solutions are the function $J_0(u)$ and $Y_0(u)$ in the standard case and $K_0(u)$ and $I_0(u)$ in the modified case. The only solution which has no singularity is the J_0 . With this information, a physical (a.k.a. non singular) solution of (B.11) in the case $k > 0$ is

$$\psi_{k>0}(x) := \chi_{[0, +\infty)}(x) J_0(2\sqrt{|kx|}),$$

while in the opposite case $k < 0$ is

$$\psi_{k<0}(x) := \chi_{(-\infty,0]}(x) J_0(2\sqrt{|kx|}),$$

where χ_J is the characteristic function of the interval J . In the case $k = 0$ the general solution of (B.11) is $c_1 \log(|x|) + c_2$, then the physical solution can be chosen as the constant solution

$$\psi_{k=0}(x) := 1.$$

These solutions are not in $L^2(\mathbb{R})$ but they meet the (generalized) normalization condition

$$\int_{\mathbb{R}} dx \psi_k(x) \psi_{k'}(x) = \delta(k - k')$$

in view of [15, 6.512 (8)]. Let $f \in L^1(\mathbb{R})$ and define the generalized eigenfunction expansion

$$\begin{aligned} \psi_f(x) &:= \int_{\mathbb{R}} dk \psi_k(x) f(k) \\ &= \chi_{(-\infty,0]}(x) (\mathcal{H}_- f)(x) + \chi_{[0,+\infty)}(x) (\mathcal{H}_+ f)(x) \end{aligned}$$

where

$$(\mathcal{H}_{\pm} f)(x) := \int_0^{+\infty} dk J_0(2\sqrt{|kx|}) f(\pm k)$$

are (a variant of) the Hankel transform of f , see [14, p. 3]. For $f \in L^2(\mathbb{R})$ it is possible to prove that $\psi_f \in L^2(\mathbb{R})$. In this way the Hankel transform can be used to generalize the Fourier theory for the operator (1.16).

As a final remark, it is worth observing that the kernel (3.7) of the resolvent $(T - \alpha \mathbf{1})^{-1}$ can be obtained by the expansion on the basis ψ_k according to

$$\mathcal{Z}_{\alpha}(x, y) = \int_{\mathbb{R}} dk \frac{\psi_k(x) \psi_k(y)}{k - \alpha}, \quad \alpha \in \mathbb{C} \setminus \mathbb{R}.$$

The last expression can be integrated by means of the formulas [15, eq. 6.541 (1)].

References

- [1] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu, *C₀-groups, commutator methods and spectral theory of N-body Hamiltonians*. Progress in Mathematics, 135. Birkhäuser Verlag, Basel, 1996. [MR 1388037 Zbl 0962.47500](#)

- [2] R. Assel, M. Dimassi, and C. Fernandez, Some remarks on the spectrum of the magnetic Stark Hamiltonians. *Mathematics and Statistics* **2**, no. 2 (2014), 101–104.
- [3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable models in quantum mechanics*. Texts and Monographs in Physics. Springer-Verlag, New York, 1988. [MR 0926273](#) [Zbl 0679.46057](#)
- [4] J. E. Avron, and I. W. Herbst, Spectral and scattering theory of Schrödinger operators related to the Stark effect. *Comm. Math. Phys.* **52** (1977), no. 3, 239–254. [MR 0468862](#) [Zbl 0351.47007](#)
- [5] M. Aizenman and S. Warzel, *Random operators*. Disorder effects on quantum spectra and dynamics. Graduate Studies in Mathematics, 168. American Mathematical Society, Providence, R.I., 2015. [MR 3364516](#) [Zbl 1333.82001](#)
- [6] S. Benvegnù and L. Dąbrowski, Relativistic point interaction. *Lett. Math. Phys.* **30** (1994), no. 2, 159–167. [MR 1264996](#) [Zbl 0788.47054](#)
- [7] B. Bagchi, S. Das, S. Ghosh, and S. Poria, Nonlinear dynamics of a position-dependent mass-driven Duffing-type oscillator. *J. Phys. A* **46** (2013), no. 3, 032001, 6pp. [MR 3007509](#) [Zbl 1273.81102](#)
- [8] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. [MR 2759829](#) [Zbl 1220.46002](#)
- [9] R. Carlone, M. Malamud, and A. Posilicano, On the spectral theory of Gesztesy–Šeba realizations of 1-D Dirac operators with point interactions on a discrete set. *J. Differential Equations* **254** (2013), no. 9, 3835–3902. [MR 3029138](#) [Zbl 1277.34120](#)
- [10] M. Demuth and M. Krishna, Determining spectra in quantum theory. *Bull. Amer. Math. Soc. (N.S.)* **45** (2008), no. 3, 469–477. [MR 3077139](#) [Zbl 1082.81003](#)
- [11] G. De Nittis and M. Lein, *Linear response theory*. An analytic-algebraic approach. SpringerBriefs in Mathematical Physics, 21. Springer, Cham, 2017. [MR 3675696](#) [Zbl 1365.82002](#)
- [12] M. Dimassi and V. Petkov, Spectral problems for operators with crossed magnetic and electric fields. *J. Phys. A* **43** (2010), no. 47, 474015, 14pp. [MR 2738110](#) [Zbl 1205.35052](#)
- [13] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions*. Vol. II. Based, in part, on notes left by H. Bateman. McGraw-Hill Book Company, New York, etc., 1953. [MR 0058756](#) [Zbl 0052.29502](#)
- [14] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of integral transforms*. Vol. II. Based, in part, on notes left by H. Bateman. McGraw-Hill Book Company, New York, etc., 1954. [MR 0065685](#) [Zbl 0058.34103](#)
- [15] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*. Translated from the Russian. Translation edited and with a preface by A. Jeffrey and D. Zwillinger. Elsevier/Academic Press, Amsterdam, 2007. [MR 2360010](#) [Zbl 1208.65001](#)
- [16] F. Gesztesy and P. Šeba, New analytically solvable models of relativistic point interactions. *Lett. Math. Phys.* **13** (1987), no. 4, 345–358. [MR 0895297](#) [Zbl 0646.60107](#)

- [17] T. Kato, *Perturbation theory for linear operators*. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995. [MR 1335452](#) [Zbl 0836.47009](#)
- [18] J. M. Luttinger, Transport theory. In R. C. Clark and G. H. Derrick (eds.), *Mathematical methods in solid state and superfluid theory*. Oliver & Boyd, Edinburgh, 1998, 157–193.
- [19] J. M. Luttinger, Theory of thermal transport coefficients. *Phys. Rev.* **135** (1964), no. 6A, A1505-A1514.
- [20] S. H. Mazharimousavi and O. Mustafa, Classical and quantum quasi-free position-dependent mass: Pöschl–Teller and ordering ambiguity. *Phys. Scr.* **87** (2013), no. 5, 055008.
- [21] M. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*. Third enlarged edition Die Grundlehren der mathematischen Wissenschaften, 52. Springer-Verlag New York, New York, 1966. [MR 0232968](#) [Zbl 0143.08502](#)
- [22] O. Mustafa, Comment on “Nonlinear dynamics of a position-dependent mass-driven Duffing-type oscillator”. *J. Phys. A* **46** (2013), no. 36, 368001, 5 pp. [MR 3100609](#) [Zbl 1273.81104](#)
- [23] K. Oldham, J. Myland, and J. Spanier, *An atlas of functions*. Second edition. Springer, New York, 2009. [MR 2466333](#)
- [24] J. Rosenberg, A selective history of the Stone–von Neumann theorem. In R. S. Doran and R. V. Kadison (eds.), *Operator algebras, quantization, and noncommutative geometry*. A centennial celebration honoring John von Neumann and Marshall H. Stone. Proceedings of the AMS Special Session held in Baltimore, MD, January 15–16, 2003. Contemporary Mathematics, 365. American Mathematical Society, Providence, R.I., 2004, 331–353. [MR 2106827](#) [Zbl 1073.01020](#)
- [25] M. Reed and B. Simon, *Methods of modern mathematical physics*. I. Functional analysis. Academic Press, New York and London, 1972. [MR 0493419](#) [Zbl 0242.46001](#)
- [26] M. Reed and B. Simon, *Methods of modern mathematical physics*. II. Fourier analysis, self-adjointness. Academic Press, New York and London, 1975. [MR 0493420](#) [Zbl 0308.47002](#)
- [27] M. Reed and B. Simon, *Methods of modern mathematical physics*. III. Scattering theory. Academic Press, New York and London, 1979. [MR 0529429](#) [Zbl 0405.47007](#)
- [28] K. Schmüdgen, *Unbounded self-adjoint operators on Hilbert space*. Graduate Texts in Mathematics, 265. Springer, Dordrecht, 2012. [MR 2953553](#) [Zbl 1257.47001](#)
- [29] L. Smrčka and P. Štěředa, Transport coefficients in strong magnetic fields. *J. Phys. C: Solid State Phys.* **10** (1977), 2153–2161.
- [30] O. Vafek, A. Melikyan, and Z. Tešanović, Quasiparticle Hall transport of d -wave superconductors in the vortex state. *Phys. Rev. B* (2001) **64**, no. 22, 224508, 17 pp.

- [31] D. R. Yafaev, *Mathematical scattering theory*. General theory. Translated from the Russian by J. R. Schulenberger. Translations of Mathematical Monographs, 105. American Mathematical Society, Providence, R.I., 1992. [MR 1180965](#)
[Zbl 0761.47001](#)

Received March 2, 2020

Giuseppe De Nittis, Facultad de Matemáticas E Instituto de Física,
Pontificia Universidad Católica de Chile, Vicuña Mackenna 4860, Macul, Santiago, Chile
e-mail: gidenittis@mat.uc.cl

Vicente Lenz, DIAM, TUDelft, Building 36, Mekelweg 4, 2628 CD Delft, Netherlands
e-mail: v.lenz@tudelft.nl