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Quantum ergodicity for pseudo-Laplacians

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Abstract. We prove quantum ergodicity for the eigenfunctions of the pseudo-Laplacian on surfaces with hyperbolic cusps and ergodic geodesic flows.

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Quantum ergodicity states that for quantum systems with ergodic classical flow, almost all high-frequency eigenfunctions are equidistributed in phase-space. Quantum unique ergodicity corresponds to equidistribution of all high-frequency eigenfunctions. The main examples are given by compact Riemannian manifolds (X, g) with ergodic geodesic flows, where one considers eigenfunctions of the Laplacian Δ_g associated to the metric, and negatively curved metrics are the typical models for ergodic geodesic flows.

The first results in this direction are due to Shnirelman [12], and later by Zelditch [13] and Colin de Verdière [6], who proved quantum ergodicity for closed manifolds with ergodic geodesic flows. In the case of manifolds with boundary, similar results were shown by Gérard and Leichtnam [9] and Zelditch and Zworski [16].

In this work, we consider instead non-compact manifolds, and the first examples one has in mind are surfaces with finite volume. In general, non-compactness often produces an essential spectrum for the Laplacian, and this is indeed the case for the simplest model of finite volume surfaces, namely hyperbolic surfaces realised as quotients $\Gamma\backslash\mathbb{H}^2$ of the hyperbolic plane by Fuchsian subgroups with a finite index. In this setting, there is however a way to get rid of this essential spectrum by a simple modification of the Laplacian, that is called *pseudo-Laplacian*, introduced by Colin de Verdière [3, 4]. This operator was very useful for obtaining a meromorphic extension of the Eisenstein series and the resolvent of the Laplacian, with important generalisation to the higher rank case by Müller [11]. In this case of hyperbolic surfaces with cusps, Zelditch's approach [14] gave additional results about the equidistribution of Eisenstein series and cusp forms (eigenfunctions corresponding to the embedding eigenvalues). That result has been recently generalised by Bonthonneau and Zelditch [2] to variable curvature and all dimensions. The strongest quantum ergodicity result was proved in the case of arithmetic surfaces by Lindenstrauss (see [10] and the references given there for earlier contributions).

The problem of quantum ergodicity for the eigenfunctions of the pseudo-Laplacian was first proposed by S. Zelditch [15]. This pseudo-Laplacian, that we will denote Δ_c, is defined as an unbounded operator on L^2 with domain D_c, and it has discrete spectrum. Here, Op_h will denote a semiclassical quantization for symbols that are compactly supported in space, see Section 1. Our main result is the following quantum ergodicity statement for this operator:

Theorem 0.1. *Let X be a Riemannian surface with a finite number of constant curvature hyperbolic cusps such that the geodesic flow on S^*X is ergodic. Let $u_j \in \mathcal{D}_c$ be an orthonormal family of eigenfunctions of $\Delta_c - \frac{1}{4}$ with eigenvalues $\lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \leq \cdots$, covering all the eigenvalues of $\Delta_c - \frac{1}{4}$ except a finite number of non-positive ones. Let $a \in S^0(T^*X)$ be compactly supported in space.*

Then, as $\lambda \to \infty$, we have

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle \mathrm{Op}_{h_j}(a) u_j, u_j \rangle - \int_{S^*X} a \right|^2 \longrightarrow 0,$$

where $N(\lambda) = |\{j : \lambda_j \leq \lambda\}|$ and $h_j = \lambda_j^{-1}$.

For a precise review of the geometry of the considered Riemannian manifolds, we refer to Section 1.1, while the definition of the pseudo-Laplacian will be given in Section 1.2. There are very natural examples of such manifolds given by negatively curved surfaces with finite volume and hyperbolic cusps.

Let us make several remarks about the Theorem. First, by a standard argument (see for instance [17, Section 15.5]) Theorem 0.1 implies that

$$\langle \mathrm{Op}(a) u_j, u_j \rangle \longrightarrow \int_{S^*X} a$$

for a sequence of density one, when a has compact support. Moreover, since we are only interested in quantizing symbols with a compact support in the space variable, we can use a standard quantization procedure, see for instance [17, Section 14.2]. That means however that the estimates are not uniform far in the cusp.

In the same geometric setting, we also mention that there are other works by Dyatlov [8] and Bonthonneau [1] on the microlocal limits of non-L^2 eigenfunctions of the Laplacian but with complex eigenvalues, where one instead get a sort of "quantum unique ergodicity."

For simplicity, the proof will be presented in the case where there is one cusp, the argument being the same with several cusps. The method of proof follows the scheme from [16] and has two steps.

1) A pointwise "ellipticity bound" that states that the eigenfunctions are microlocalized on the cosphere bundle. This implies that in the limit $\lambda \to \infty$,

$$M(a,\lambda)^2 := \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \left| \langle \mathrm{Op}_{h_j}(a) u_j, u_j \rangle - \int_{S^*X} a \right|^2$$

is controlled by $\|a_{|S^*X}\|_{L^2}^2$.

2) Taking a symbol with average zero, we propagate it by the geodesic flow to get a new symbol that is small on the cosphere bundle (by the L^2 ergodic theorem). We prove, using a "flow invariance" result, that this does not modify $M(a,\lambda)$.

We stress that working with a pseudo-Laplacian entails new difficulties that are not apparent in the compact setting. For the first step, since we are working with a pseudo-Laplacian, the pointwise ellipticity bound (and the subsequent microlocalization) works only outside the singular circle, and we need to prove that the needed correction is small enough. This requires a precise control of the eigenfunctions of the pseudo-Laplacian.

For the second step, it is important to notice that the eigenfunctions we are interested in are not eigenfunctions of the propagator we are using for the proof. We are able to prove that $M(a,\lambda)$ does not change much when replacing a by $a \circ \Phi^t$ if Φ^t is the geodesic flow, but we have to assume that the symbol a is supported quite far away from the singular circle. Since the admissible support has full measure, the L^2 control of $\limsup M(a,\lambda)$ we still get at step 1 leads to the same result.

Finally, in a compact manifold, $M(1,\lambda)$ is defined and easily shown to vanish, which is the last step of the proof of the main theorem of [16]. Since we only use symbols with a compact support in space, we cannot use this argument. Our proof has a third step which consists in finding symbols a with average close to 1 such that $\limsup M(a,\lambda)$ is arbitrarily close to zero. For that purpose, we shall prove that the modes of the eigenfunctions of the pseudo-Laplacian are microlocalized in the cusp.

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1. Preliminaries

1.1. Notations. We let X be a Riemannian surface with one hyperbolic cusp, i.e. a cusp with constant curvature. This means that X can be split into two parts, $X = X_0 \cup X_1$, where X_1 is a compact Riemannian surface with boundary, and $X_0 = (c_0, \infty)_r \times (\mathbb{R}/\mathbb{Z})_\theta$ with metric $dr^2 + e^{-2r} d\theta^2$. Using the notation $\xi dr + \eta d\theta$ for cotangent vectors in X_0, the Hamiltonian induced by the metric in the cusp is given by

$$p(r, \theta; \xi, \eta) = \xi^2 + e^{2r}\eta^2.$$

In X_0, my function $u \in L^2_{\mathrm{loc}}(X_0)$ can be expanded into Fourier series in the θ variable:

$$u(r, \theta) = \sum_{n \in \mathbb{Z}} u_n(r) e^{2i\pi n\theta},$$

where the u_n are in $L^2_{\mathrm{loc}}((c_0, \infty); e^{-r} dr)$. The metric induces a natural measure μ, called *Liouville measure,* on the unit cotangent bundle S^*X and for simplicity we shall normalize it so that it is a probability measure. The projection $T^*X \to X$ on the base will be denoted by π. Finally, $C > 0$ will denote a generic constant that is independent of the parameters we consider (except when indicated), and that will change from line to line.

As a Riemannian surface, X has a positive Laplacian, which we denote as Δ.

1.2. Definition of the pseudo-Laplacian

Definition 1.1. Let $c > c_0$. Let us denote $L^2_{0,c}$ (resp. $H^1_{0,c}$) the space of all $u \in L^2(X)$ (resp. $u \in H^1(X)$) such that $u_0(r) = 0$ for every $r \geq c$. The pseudo-Laplacian Δ_c is the unbounded non-negative self-adjoint operator on $L^2_{0,c}$ defined using the Friedrichs method by the quadratic form

$$q(u) = \int_X |\nabla^g u|_g^2 \, d\mathrm{v}_g \quad \text{for all } u \in H^1_{0,c}.$$

The Riemannian measure $d\mathrm{v}_g$ and the gradient are with respect to g.

We note that the spaces $L^2_{0,c}$ and $H^1_{0,c}$ are closed vector subspaces of $L^2(X)$, and $H^1(X)$. The circle $r = c$ in X_0 will be referred to as *the singular circle*.

The following results are proved in [5, Theorem 2].

Proposition 1.2. *The operator Δ_c is an unbounded, non-negative, self-adjoint operator with compact resolvent and discrete spectrum.*

We will denote $(u_j)_j$ an orthonormal family of eigenfunctions with positive eigenvalues of $\Delta_c - \frac{5}{18}$, that is, $\Delta_c u_j = \big(\lambda_j^2 + \frac{1}{4}\big)u_j$, where $(\lambda_j)_j$ is a positive, non-decreasing sequence going to $+\infty$ such that $\lambda_0 \geq \frac{1}{6}$. Note that the orthogonal of $\operatorname{Span}\{u_j,\ j \geq 0\}$ in $L^2_{0,c}$ is a finite-dimensional space that possesses an orthonormal basis of eigenfunctions of Δ_c. We will denote, for each $j \geq 0$, $h_j = \lambda_j^{-1}$.

Note that we extend Δ_c as an unbounded self-adjoint operator from L^2 to L^2 with compact resolvent by declaring that $\Delta_c v = 0$ whenever $v \in L^2(X)$ has support in $\{r \geq c\}$ and v does only depend on r.

1.3. Review of semiclassical analysis. We shall use the following semiclassical quantization procedure, which is similar to [17, Chapter 14.2]: we fix a locally finite cover by countably many relatively compact open sets U_i of X, $i \geq 0$, with diffeomorphisms $\varphi_i\colon U_i \to V_i$, where the $V_i \subset \mathbb{R}^2$ are open sets, and take a partition of unity $(\chi_i^2)_i$ associated with it. A compactly supported symbol $a \in S^m_{\mathrm{comp}}(X)$ is a smooth function $a \in C^\infty(T^*X)$ whose support projects to X into a compact set and satisfying uniform bounds

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}\langle \xi\rangle^{m-|\beta|}$$

for all multi-indices α, β. Then for any symbol $a \in S^m_{\mathrm{comp}}(X)$ with compact space support, and $h > 0$, we define

$$\mathrm{Op}_h(a) := \sum_i \chi_i(\varphi_i)^*((\varphi_i)_* a)^w(x, hD)(\varphi_i)_*\chi_i.$$

where

- $(\varphi_i)_* a$ is the symbol defined on $T^*V_i = V_i \times \mathbb{R}^2$ by composing a with the enhanced symplectomorphism $\varphi_i^{-1}\colon T^*V_i \to T^*U_i$;
- for $f\colon U_i \to \mathbb{C}$ and $x \in V_i$,

$$(\varphi_i)_* f(x) = \frac{f \circ \varphi_i^{-1}(x)}{|\det d_{\varphi_i^{-1}(x)}\varphi_i|^{1/2}};$$

- for $f: V_i \to \mathbb{C}$ and $x \in U_i$,

$$(\varphi_i)^* f(x) = |\det d_x \varphi_i|^{1/2} (f \circ \varphi_i(x)).$$

When $U_i \cap \pi(\mathrm{supp}(a)) = \emptyset$ (which always happen but for a finite number of i), i does not contribute to the sum, because $((\varphi_i)_* a) = 0$. In any case, $(\varphi_i)_* a \in S^m_{\mathrm{loc}}(V_i)$.

The specific choice of the partition of unity is not important, because the difference between two different such quantizations is then an $\mathcal{O}(h)_{L^2 \to L^2}$ for any $S^0_{\mathrm{comp}}(X)$ symbol. We shall thus make the following choices:

- $V_0 = V_1 = (c - 2\varepsilon, c + 2\varepsilon) \times (0, 1)$;
- $U_0 = (c - 2\varepsilon, c + 2\varepsilon) \times (0, 1) \subset (c_0, \infty) \times \mathbb{R}/\mathbb{Z}$ and φ_0 is the identity;
- $U_1 = (c - 2\varepsilon, c + 2\varepsilon) \times \left(-\frac{1}{2}, +\frac{1}{2}\right) \subset (c_0, \infty) \times \mathbb{R}/\mathbb{Z}$ and φ_1 is a shift in the second coordinate only;
- $\chi_0 \circ \varphi_0^{-1} = \chi_1 \circ \varphi_1^{-1}$;
- for every $i \geq 2$, $\chi_i = 0$ in $\{c - \varepsilon \leq r \leq c + \varepsilon\}$;
- every V_i is convex (so that we can apply verbatim the formula for Weyl quantization given that we have partitions on unity).

With this procedure most of the useful properties (about composition, Lie brackets, L^2 operators bounds for S^0 quantized symbols) hold: the proofs from [17, Chapters 14, 15] still apply when the symbols are compactly supported in X, however the constants depend on the size of the supports. We notice, however, that this quantization has an *exact* duality property (because our pull-backs and push-forwards are actually acting on half-densities and therefore are L^2-isometries) and, applying [17, Theorem 4.3], an exact identity $\mathrm{Op}_h(a)u = au$ if a is a S^0 symbol depending only on the space variables. Moreover, this procedure also behaves reasonably well with respect to the Laplacian: if ψ is a smooth compactly supported function on X, then $\mathrm{Op}_h(\psi(x)|\xi|^2)$ is $h^2\psi(x)\Delta$ up to higher-order terms (that is, an $\mathcal{O}_{L^2 \to H^{-1}}(h)$ operator, with the constant depending only on ψ).

Finally, this quantization has the following *exact* trace formula:

Proposition 1.3. *Let $a \in C_c^\infty(T^*X)$. Then $\mathrm{Op}_h(a)$ is trace class and*

$$\mathrm{Tr}_{L^2 \to L^2}(\mathrm{Op}_h(a)) = \frac{1}{(2\pi h)^2} \int\limits_{T^*X} a(x, \xi)\, dx d\xi.$$

Proof. Let $i \geq 0$ and let us consider $A_i = \chi_i (\varphi_i)^* ((\varphi_i)_* a)^w (x, hD)(\varphi_i)_* \chi_i$. Let $I = (\varphi_i)_*: L^2(U_i) \to L^2(V_i)$ be the isometry previously defined (with inverse $(\varphi_i)^*$), R be the natural restriction $L^2(X) \to L^2(U_i)$, $\chi_i' = \chi_i \circ \varphi_i^{-1} \in C_c^\infty(V_i)$, $a_i = ((\varphi_i)_*)a$, so that $A_i = R^* I^*(\chi_i' a_i^w(x, hD)\chi_i') IR$. Thus A_i is trace class if $\chi_i' a_i^w(x, hD)\chi_i'$ is trace class, and these operators have the same trace.

Now $\chi_i' a_i^w(x, hD)\chi_i' \colon L^2(V_i) \to L^2(V_i)$ is a kernel operator with smooth compactly supported kernel $(x, y) \mapsto \chi_i'(x)\chi_i'(y)\hat{a}_i\big(\frac{x+y}{2}, x - y\big)$, where $\hat{a}_i$ is the inverse Fourier transform of a_i with respect to its second variable. Its trace is the same as that of the operator $A_i' \colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ with the same (smooth, compactly supported) kernel (meaning that one operator is trace class iff the other is, and if so, their traces are equal). By [7, (9.3)], A_i' is trace class with trace $\int_{\mathbb{R}^2} \chi'(x)^2 \hat{a}_i(x, 0)dx = (2\pi h)^{-2} \int_{\mathbb{R}^4} \chi_i'(x)^2 a_i(x, \xi)dx d\xi = (2\pi h)^{-2} \int_{T^*U_i} \chi_i^2(x) a(x, \xi)dx d\xi$ because the variable change $a \mapsto a_i$ is induced by a symplectomorphism. To conclude, we note that $A_i = 0$ when U_i does not meet the support of a (which occurs every time but finitely many) and that $\sum_i \chi_i^2 = 1$. □

2. Estimates on the singular circle

2.1. Riemannian Laplacian of the eigenfunctions. In this section, we study the family $\big(\Delta - \lambda_j^2 - \frac{1}{4}\big)u_j$. We will denote by δ_c the Lebesgue measure with total mass 1 on the circle $r = c$ of the cusp of X.

The following lemma is an easy application of Stokes's theorem.

Lemma 2.1. *Let φ be a smooth function on X such that $\varphi(r, \theta) = \tilde{\varphi}(r)$ in the cusp on $r > c - \varepsilon$ for some $0 < \varepsilon < c - c_0$, where $\tilde{\varphi} \colon (c_0, \infty) \to \mathbb{C}$ is smooth. Assume that $\tilde{\varphi}(c) = 0$, then $\Delta(\mathbf{1}_{r \geq c}\varphi) = \mathbf{1}_{r \geq c}\Delta\varphi - e^{-c}\tilde{\varphi}'(c)\delta_c$.*

Now, we can write the Laplacian of u_j as a function of its zero mode.

Corollary 2.2. *For $j \geq 0$ and $h_j = \lambda_j^{-1}$, we have $(u_j)_0(r) = \alpha_j e^{r/2} \sin \frac{r-c}{h_j}$ for some $\alpha_j \in \mathbb{R}$ in the region $c_0 < r \leq c$, it vanishes when $r \geq c$, and*

$$\Big(\Delta - \lambda_j^2 - \frac{1}{4}\Big)u_j = Q_j \delta_c,$$

where $Q_j = +e^{-c/2}\frac{\alpha_j}{h_j}$.

Proof. On $\{c_0 < r < c\}$, (u_j) is an eigenfunction of the positive Riemannian Laplacian Δ with eigenvalue $\lambda_j^2 + \frac{1}{4}$. In our coordinates, $\Delta = -\partial_r^2 + \partial_r - e^{2r}\partial_\theta^2$. On the zero Fourier mode of u_j, ∂_θ acts as 0, thus $(-\partial_r^2 + \partial_r)(u_j)_0 = \big(\lambda_j + \frac{1}{4}\big)^2 (u_j)_0$.

Setting $(u_j)_0(r) = v_0(r)e^{r/2}$ yields $-\partial_r^2 v_0 = \lambda_j^2 v_0$. The boundary condition $(u_j)_0(c) = 0$ then gives the formula for $(u_j)_0$.

From the proof of Theorem 4 in [5],

$$u_j' := u_j + \mathbf{1}_{r\geq c}\alpha_j e^{r/2}\sin\frac{r-c}{h_j}$$

is a non-L^2 eigenfunction of the positive Laplacian with eigenvalue $\lambda_j^2 + \frac{1}{4}$. Therefore, using lemma 2.1,

$$\begin{aligned}(\Delta - \Delta_c)u_j &= \Big(\Delta - \lambda_j^2 - \frac{1}{4}\Big)u_j\\ &= \Big(\Delta - \lambda_j^2 - \frac{1}{4}\Big)(u_j - u_j')\\ &= -\alpha_j\Big(\Delta - \lambda_j^2 - \frac{1}{4}\Big)\Big(\mathbf{1}_{r\geq c}e^{r/2}\sin\frac{r-c}{h_j}\Big)\\ &= +e^{-c/2}\frac{\alpha_j}{h_j}\delta_c.\end{aligned}$$

This completes the proof. □

To estimate the Δu_j, we need an adequate description of the constants α_j from Corollary 2.2.

Proposition 2.3. *There exists a smooth compactly supported function $\tilde{\phi}$ on X, and a sequence $(I_j)_{j\geq 0}$ such that*

$$I_j\alpha_j = \langle u_j, \tilde{\phi}\rangle$$

(*it is the L^2 inner product*) *for every $j \geq 0$ and*

$$I_j = -h_j + \mathcal{O}(h_j^\infty).$$

Proof. Let ϕ be any smooth compactly supported function on $\mathbb{R}$ such that

- $\phi = 0$ on $\big(-\infty, \frac{c_0+c}{2}\big)$;
- $\phi(c) = 1$;
- for every $p \geq 1$, $\phi^{(2p)}(c) = 0$.

Let $\tilde{\phi}(r,\theta) = e^{r/2}\phi(r)$ (and $\tilde{\phi}$ is zero outside the cusp), such that $\tilde{\phi}$ is well-defined on X, smooth, compactly supported. Now, since $\tilde{\phi}$ has no non-zero θ-Fourier mode, using its support property and the nature of the hyperbolic metric, we know that $\langle u_j, \tilde{\phi}\rangle = \alpha_j I_j$, where

$$\begin{aligned} I_j &= \int_{(c_0+c)/2}^{c} \phi(r) \sin\frac{r-c}{h_j} dr \\ &= \int_{-\infty}^{c} \phi(r) \sin\frac{r-c}{h_j} dr \\ &= -h_j + h_j \int_{-\infty}^{c} \phi'(r) \cos\frac{r-c}{h_j}. \end{aligned}$$

Set $\phi_1(r) = \phi(r+c)$. There exists $\phi_2 \in C_c^\infty(\mathbb{R};\mathbb{R})$ such that $\phi_2(r) = \phi_1'(r)$ if $r \le 0$ and $\phi_2(r) = \phi_1'(-r)$ if $r \ge 0$ (recall that all derivatives of odd order of ϕ_1' vanish at 0). Then

$$\begin{aligned} 2(I_j + h_j) &= h_j \int_{-\infty}^{0} \phi_2(r) e^{ir/h_j} dr + h_j \int_{0}^{\infty} \phi_2(r) e^{ir/h_j} dr \\ &= h_j (\mathcal{F}\phi_2)\Big(\frac{1}{h_j}\Big) = \mathcal{O}(h_j^\infty) \end{aligned}$$

and we are done. □

Corollary 2.4. *We have*

$$\sum_j |Q_j|^2 h_j^4 < \infty.$$

Proof. Since $h_j \sim -I_j$ as j goes to infinity, and since $Q_j = F\alpha_j(h_j)^{-1}$ for some constant F, we find that $|Q_j|^2 h_j^4$ is positive and is equivalent to $|F|^2\alpha_j^2 I_j^2 = |F|^2 \langle u_j, \tilde{\phi}\rangle^2$ (with the above notations). Now, since the $(u_j)_j$ form an orthonormal family in L^2,

$$\sum_j \langle u_j, \tilde{\phi}\rangle^2 \le \|\tilde{\phi}\|_{L^2}^2 < \infty,$$

which proves the claim. □

2.2. Pseudo-differential operators acting on δ_c. Following up on the previous subsection, we have:

Proposition 2.5. *Let $a \in S^{-2}_{\mathrm{comp}}(X)$ with $\pi(\mathrm{supp}(a)) \subset \{c-\varepsilon < r < c+\varepsilon\}$. Then, for some $C > 0$ not depending on a, for every $1 > h > 0$,*

$$\| \operatorname{Op}_h(a)\delta_c\|^2_{L^2} \leq \frac{C}{h} \int\limits_{\mathbb{R}/\mathbb{Z}} \int\limits_{\mathbb{R}} |a|^2(c,\theta,\xi,0)\,d\xi\,d\theta + C\|a\|^2_{S^{-2}},$$

where $\|\cdot\|_{S^{-2}}$ is some S^{-2} seminorm (in every estimate of that kind in the following, the seminorm will have to be universal).

Lemma 2.6. *Let $a \in S^{-2}_{\mathrm{comp}}(\mathbb{R}^2)$, with $\pi(\mathrm{supp}(a)) \subset (c-\varepsilon, c+\varepsilon) \times (0,1)$. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be smooth and zero outside $(0,1)$. Let $\langle \nu, \varphi\rangle = \int_0^1 \chi(\theta)\varphi(c,\theta)d\theta$. Then, for some constant $C > 0$ depending only on seminorms of χ,*

$$\|a^w(x,hD)\nu\|^2_{L^2} \leq \frac{C}{h} \int\limits_{\substack{0<\theta<1\\ \xi\in\mathbb{R}}} |a(c,\theta,\xi,0)|^2|\chi(\theta)|^2\,d\theta\,d\xi + C\|a\|^2_{S^{-2}}$$

Proof. We may assume that a is compactly supported, if we find out that C does not depend on the support of a. A computation gives

$$\begin{aligned}
&(2h\pi)^4\|a^w(x,hD)\nu\|^2_{L^2}\\
&= \int\limits_0^1 \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^6} \chi(\phi)\chi(\phi')a\Big(\frac{r+c}{2}, \frac{\theta+\phi}{2}, \xi, \eta\Big)\bar{a}\Big(\frac{r+c}{2}, \frac{\theta+\phi'}{2}, \xi', \eta'\Big)\\
&\qquad \times \exp\Big(\frac{i}{h}\big((r-c)(\xi-\xi') + \theta(\eta-\eta')\\
&\qquad\qquad + (\phi'\eta' - \phi\eta)\big)\Big) dr\,d\phi\,d\phi'\,d\eta\,d\eta'\,d\xi'\,d\xi\,d\theta\\
&:= \int\limits_0^1 \int\limits_{\mathbb{R}} I(h,\xi,\theta)\,d\xi\,d\theta
\end{aligned}$$

Let $\varphi_{\xi,\theta}(r,\xi',\phi,\phi',\eta,\eta') = (r-c)(\xi-\xi') + \theta(\eta-\eta') + (\phi'\eta' - \phi\eta)$. The gradient of $\varphi_{\xi,\theta}$ vanishes only when $\xi' = \xi$, $\eta = \eta' = 0$, $\phi' = \phi = \theta$, $r = c$. At that point, the Hessian matrix of $\varphi_{\xi,\theta}$ is

$$\begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix},$$

so it has full rank and we see from the stationary phase method (say, [17, Theorem 3.16]), that for some constants F, C,

$$|I(h,\xi,\theta) - Fh^3|a(c,\theta,\xi,0)\chi(\theta)|^2| \leq Ch^4 \frac{\|a\|^2_{S^{-2}}}{1+|\xi|^2}.$$

The conclusion is easily drawn from this. □

Now let us prove proposition 2.5:

Proof. Let ψ be a smooth function on X such that $\psi = 1$ on $\{2|r-c| < \varepsilon\}$, and with support in $\{|r-c| < \varepsilon\}$. Then write $a = a(1-\psi) + a\psi$. The support $\pi(\mathrm{supp}(a(1-\psi)))$ is at distance at least $\frac{\varepsilon}{2}$ from $\{r = c\}$. Therefore, $\|\operatorname{Op}_h(a(1-\psi))\delta_c\|^2_{L^2} \leq C\|a\|^2_{S^{-2}}$, for some universal constant $C > 0$. Now, apply lemma 2.6 to the explicit quantization (as explained in Section 1.2) of $a\psi$ (where the only non-vanishing terms are for the charts 0 and 1). □

3. Ellipticity and variance bound

In this section, we complete what we have called in the introduction the first step of the proof. We use the results of the former section, as well as an ellipticity estimate similar to the one from [16], to prove that the microlocalization of the eigenfunctions on the energy surface still holds, albeit on average only.

Definition 3.1. We define, for any symbol $a \in S^0_{\mathrm{comp}}(X)$ and for any $h > 0, \lambda > 0$,

$$N(\lambda) := |\{j,\ \lambda_j \leq \lambda\}|, \tag{1}$$

$$Y(a,h) := h\sqrt{\sum_{h/2\leq h_j\leq 2h} \|\operatorname{Op}_{h_j}(a)u_j\|^2_{L^2}}, \tag{2}$$

$$M(a,\lambda) := \sqrt{\frac{1}{N(\lambda)}\sum_{j,\lambda_j\leq\lambda}\left|\langle \operatorname{Op}_{h_j}(a)u_j, u_j\rangle - \int_{S^*X} a\right|^2}. \tag{3}$$

Remark. The bound $M(a+b,t) \leq M(a,t) + M(b,t)$ holds, and similarly for Y.

Let us mention the following very important result:

Proposition 3.2 (Weyl law for Pseudo-Laplacians). *There is a constant $C > 0$ such that $N(\lambda) \sim C\lambda^2$ as $\lambda \to \infty$. As a consequence, there are $C_1 > 0, C_2 > 0$ such that for all $h > 0$ small*

$$C_1 h^{-2} \leq |\{j \in \mathbb{N};\ h/2 \leq h_j \leq 2h\}| \leq C_2 h^{-2}.$$

Proof. Actually, $N(\lambda)$ is, up to some additive constant, the number of eigenvalues of Δ_c that are not greater than $\lambda^2 + \frac{1}{4}$. The result is then proved in [5, Theorem 6]. $\square$

3.1. Ellipticity in the mean

Lemma 3.3. *Let $a \in S^0_{\mathrm{comp}}(X)$ be a symbol and assume that $a_{|S^*X} = 0$. Then*

$$\| \mathrm{Op}_{h_j}(a)u_j\|^2_{L^2} \leq C|Q_j|^2 h_j^3 \int\limits_{\mathbb{R}/\mathbb{Z}} \int\limits_{\mathbb{R}} \frac{|a(c,\theta,\xi,0)|^2}{(|\xi|^2-1)^2} d\xi d\theta + \mathcal{O}(h_j^2), \tag{4}$$

where the constant in the $\mathcal{O}(h_j^2)$ depends only on some S^0 seminorms of a and on $\pi(\mathrm{supp}(a))$, C is universal and Q_j is the constant of Corollary 2.2.

Proof. Write

$$a(x,\xi) = b(x,\xi)(\chi(x)(|\xi|^2-1)),$$

where a and b have same (compact) space support and b is S^{-2}, χ is a smooth compactly supported function on X that is 1 on the support of a and on the singular circle. Then $\mathrm{Op}_{h_j}(a) = \mathrm{Op}_{h_j}(b)\chi(x)(h_j^2\Delta - 1) + \mathcal{O}(h_j)$, the $\mathcal{O}$ referring to $L^2 \to L^2$ operator norm, and the constant depends only on S^0 seminorms of a and on $\pi(\mathrm{supp}(a))$. From corollary 2.2, $(h_j^2\Delta - 1)u_j = Q_j h_j^2 \delta_c + \frac{h_j^2}{4} u_j$. Thus $\mathrm{Op}_{h_j}(a)u_j = Q_j h_j^2 \mathrm{Op}_{h_j}(b)\delta_c + \mathcal{O}_{L^2}(h_j)$.

Let ψ be a smooth function on X that is 1 everywhere, except on the set $\{c-\varepsilon < r < c+\varepsilon\}$, and that is zero on $\{2|r-c| < \varepsilon\}$. Then

$$\mathrm{Op}_{h_j}(a)u_j = Q_j h_j^2 \mathrm{Op}_{h_j}(b\psi)\delta_c + Q_j h_j^2 \mathrm{Op}_{h_j}(b(1-\psi))\delta_c + \mathcal{O}_{L^2}(h_j),$$

with the $\mathcal{O}(h_j)$ depending only on S^0 seminorms of a and $\pi(\mathrm{supp}(a))$, yielding

$$\begin{aligned}\| \mathrm{Op}_{h_j}(a)u_j\|^2_{L^2} \leq{}& 2|Q_j|^2 h_j^4 \| \mathrm{Op}_{h_j}(b(1-\psi))\delta_c\|^2_{L^2} \\ &+ 4|Q_j|^2 h_j^4 \| \mathrm{Op}_{h_j}(b\psi)\delta_c\|^2_{L^2} + \mathcal{O}_{L^2}(h_j^2).\end{aligned}$$

The wave front set of δ_c is $\{r = c, \eta = 0\}$, which does not meet the phase space support of $b\psi$. Thus $\mathrm{Op}_{h_j}(b\psi)\delta_c$ is a smooth $\mathcal{O}_{L^2}(h_j)$ function. The constant depends only on S^{-2} seminorms of $b\psi$, $\pi(\mathrm{supp}(b\psi))$ and $\varepsilon/2$, a lower bound on the distance between $\mathrm{supp}(b\psi)$ and the wave front set of δ_c: so, in the end, it depends only on $\pi(\mathrm{supp}(a))$ and some S^0 seminorms of a. Finally, Proposition 2.5 gives us the upper bound

$$\| \mathrm{Op}_h(b(1-\psi))\delta_c\|^2_{L^2} \leq \frac{C}{h_j} \int\limits_{\mathbb{R}/\mathbb{Z}} \int\limits_{\mathbb{R}} |b(1-\psi)|^2(c,\theta,\xi,0) d\xi d\theta + C\|a\|^2_{S^{-2}},$$

and we conclude by saying that $(|Q_j|^2 h_j^4)_j$, being a ℓ^1 sequence because of Corollary 2.4, is bounded by a constant depending only on X and c. $\square$

Proposition 3.4 (ellipticity in the mean). *Let $a_j \in S^0_{\mathrm{comp}}(X)$ for every $j \geq 0$ and assume that $\bigcup_j (\pi(\mathrm{supp}(a_j))) \subset \mathcal{K}$ for some fixed compact set $\mathcal{K} \subset X$ and that the family is bounded in S^0. Assume that for each j, $(a_j)_{|S^*X} = 0$. Then*

$$h^2 \sum_{h/2 \leq h_j \leq 2h} \| \mathrm{Op}_{h_j}(a_j) u_j \|^2_{L^2} \leq C h \sup\{\|a_j\|^2_{S^0}\} + \mathcal{O}(h^2),$$

where $\|\cdot\|_{S^0}$ is some S^0 seminorm, C is universal, and the constant in the $\mathcal{O}(h_j^2)$ depends only on $\mathcal{K}$ and $\sup\{\|a_j\|^2_{S^0}\}$.

Proof. Let I be the supremum over $j \geq 0$ of the

$$C \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}} \frac{|a_j(c, \theta, \xi, 0)|^2}{(|\xi|^2 - 1)^2} d\xi d\theta,$$

where C is the constant in (4) and K is the constant in the $\mathcal{O}(h_j^2)$ of (4), which depends only on $\mathcal{K}$ and $\sup \|a_j\|^2_{S^0}$ for some S^0 seminorm. Then, using Weyl's law and corollary 2.4,

$$h^2 \sum_{h/2 \leq h_j \leq 2h} \| \mathrm{Op}_{h_j}(a_j) u_j \|^2_{L^2} \leq h \sum_{h/2 \leq h_j \leq 2h} [2I |Q_j|^2 h_j^4 + 2K h^3] \leq 2hC'I + K'h^2,$$

and we conclude by considering that for some suitable $\|\cdot\|_{S^0}$, $I \leq \sup\{\|a_j\|^2_{S^0}\}$. $\square$

3.2. Bound on the variance.

In this section, we shall prove some bounds on the variance $M(a, \lambda)$ defined in Definition 3.1.

Lemma 3.5. *Let $b \in C_c^\infty(T^*X)$. Then, there is some universal constant C such that for all $h > 0$ small*

$$h^2 \sum_j \| \mathrm{Op}_h(b) u_j \|^2_{L^2} \leq C \int_{T^*X} |b|^2 + \mathcal{O}(h).$$

Proof. We write

$$\begin{aligned} h^2 \sum_j \| \mathrm{Op}_h(b) u_j \|^2_{L^2} \leq h^2 \| \mathrm{Op}_h(b) \|^2_{\mathrm{HS}} &= h^2 \,\mathrm{Tr}(\mathrm{Op}_h(b)^* \mathrm{Op}_h(b)) \\ &= h^2 \,\mathrm{Tr}(\mathrm{Op}_h(|b|^2 + f_h)), \end{aligned}$$

where f_h are smooth functions such that $\operatorname{supp} f_h \subset \operatorname{supp} b$ is compact, and $\|f_h\|_{S^p} = \mathcal{O}(h)$ for each $p \in \mathbb{Z}$. Here HS means the Hilbert–Schmidt norm. We finally apply the trace formula of Proposition 1.3. $\square$

Proposition 3.6. *Let $a \in S^0_{\mathrm{comp}}(X)$. There is a universal constant C such that for all $h > 0$ small,*

$$Y(a,h)^2 \leq C \int_{S^*M} |a|^2 + \mathcal{O}(h),$$

where Y is defined in Definition 3.1 by (2).

Proof. Let us denote $f_\tau(x,\xi) = f(x,\tau\xi)$ for any $\tau > 0$ and any symbol f. Let $b \in \mathcal{C}^\infty_{\mathrm{c}}(T^*X)$ be such that $b(x,\xi) = \chi(|\xi|)a\big(x, \frac{\xi}{|\xi|}\big)$, where $\chi \in C^\infty_c((0,\infty))$ is non-negative and such that $\chi = 1$ on $[2/5, 5/2]$. From Proposition 3.4 (with the symbols $a - b$), we get $Y(a-b,h) = \mathcal{O}(h^{1/2})$, so

$$Y(a,h)^2 \leq 2Y(b,h)^2 + \mathcal{O}(h^{1/2})^2 = 2Y(b,h)^2 + \mathcal{O}(h).$$

So we may focus on $Y(b,h)$.

Now, since $\mathrm{Op}_{h\tau}(f) = \mathrm{Op}_h(f_\tau)$ (because of the quantization procedure), let us denote $\tau_j = h_j/h$. Then,

$$\begin{aligned} Y(b,h)^2 &= h^2 \sum_{h/2 \leq h_j \leq 2h} \|\mathrm{Op}_h(b_{\tau_j})u_j\|^2_{L^2} \\ &\leq 2h^2 \sum_{h/2 \leq h_j \leq 2h} \|\mathrm{Op}_{h_j}(b - b_{1/\tau_j})u_j\|^2_{L^2} + 2h^2 \sum_{h/2 \leq h_j \leq 2h} \|\mathrm{Op}_h(b)u_j\|^2_{L^2}. \end{aligned}$$

Since

$$\int_{T^*X} |b|^2 \leq C \int_{S^*X} |a|^2,$$

by the previous lemma, it is enough to prove that

$$h^2 \sum_{h/2 \leq h_j \leq 2h} \|\mathrm{Op}_{h_j}(b - b_{1/\tau_j})u_j\|^2_{L^2} = \mathcal{O}(h).$$

Now, since $1/2 \leq \tau_j \leq 2$, the $b - b_{1/\tau_j}$ are bounded in S^0, have a uniform support in space $\pi(\operatorname{supp}(b - b_{1/\tau_j})) \subset \pi(\operatorname{supp}(a))$, vanish on $\{4/5 < |\xi| < 5/4\} \supset S^*X$, the result follows from Proposition 3.4. $\square$

Proposition 3.7 (variance bound). *If $a \in S^0_{\mathrm{comp}}(X)$, then for some universal constant C, as $\lambda \to \infty$,*

$$M(a,\lambda)^2 \leq C \int_{S^*X} |a|^2 + \mathcal{O}(\lambda^{-1})$$

Proof. Let $h = 1/(2\lambda)$. Let
$$I := \int_{S^*X} |a|^2,$$
let $C, K > 0$ denote the constants from Proposition 3.6, C being as in the proposition, and K being the constant from the $\mathcal{O}(h)$ (for large enough h, $Y(a,h) = 0$ so K exists; however, K depends on a). We get

$$\begin{aligned} N(\lambda)M(a,\lambda)^2 &= \sum_{\lambda_j \le \lambda} \left| \langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle - \int_{S^*X} a \right|^2 \\ &\le \sum_{\lambda_j \le \lambda} 2\left| \int_{S^*X} a \right|^2 + 2|\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle|^2 \\ &\le \sum_{\lambda_j \le \lambda} 2I + 2\| \mathrm{Op}_{h_j}(a)u_j\|_{L^2}^2 \end{aligned}$$

by Cauchy–Schwarz (since S^*X has measure 1)
$$= 2N(\lambda)I + 2\sum_{h_j \ge 2h} \| \mathrm{Op}_{h_j}(a)u_j\|_{L^2}^2$$

Now, we split the $h_j \ge 2h$ into intervals of $I_k = 4^k h\left[\frac{1}{2}, 2\right]$ for $k \ge 1$. As the h_j decrease, there is a maximal k_0 such that I_{k_0} contains some h_j.

By definition of Y and by Proposition 3.6,

$$\sum_{h_j \in I_k} \| \mathrm{Op}_{h_j}(a)u_j\|_{L^2}^2 = \frac{Y(a, 4^k h)^2}{(4^k h)^2} \le 16^{-k} h^{-2} C I + 16^{-k} h^{-2} 4^k h K.$$

It follows

$$\begin{aligned} \sum_{h_j \ge 2h} \| \mathrm{Op}_{h_j}(a)u_j\|_{L^2}^2 &\le C I h^{-2} \sum_{1 \le k \le k_0} 16^{-k} + K h^{-1} \sum_{1 \le k \le k_0} 4^{-k} \\ &\le C I h^{-2} + K h^{-1} = 4 C I \lambda^2 + 2K\lambda, \end{aligned}$$

thus $N(\lambda)M(a,\lambda)^2 \le 2(N(\lambda) + 4C\lambda^2)I + 4K\lambda$. We conclude using again Proposition 3.2. □

4. Egorov theorem

This section deals with step 2: similarly to [16], we want to prove that propagating some symbol a through the geodesic flow does not change $M(a, \lambda)$ too much. The main difference here is the fact that the operator we study (the pseudo-Laplacian) is not the generator of the propagator we use. From a geometric point of view, we solve this by requiring that our symbols have a support far from the singular circle.

The main result of this section is proposition 4.4, which gives a precise statement about the idea above.

4.1. A good set for propagation.

We define

$$\Sigma_T = \{(z,\zeta) \in T^*X\colon \text{for all } |t| \le T, \Phi^t(z,\zeta) \notin T^*X_0 \cap \{\eta = 0\}\},$$

where Φ^t is the geodesic flow on T^*X and X_0 the region where the coordinate η is defined in the cusp.

Proposition 4.1. *Σ_T is an open set of full measure.*

Proof. The flow $\Phi\colon (t,(z,\zeta)) \in [-T,T]\times T^*X \mapsto \Phi^t(z,\zeta)$ is continuous, thus the inverse image of the closed set $S = \{\zeta \in T^*X,\ \zeta \in T^*X_0 \cap \{\eta = 0\}\}$ is closed. As $[-T,T]$ is compact, the projection $p\colon [-T,T]\times T^*X \to T^*X$ is closed, so $S_T = p(\Phi^{-1}(S))$ is a closed subset of T^*X. Therefore $\Sigma_T = T^*X\backslash S_T$ is an open subset of T^*X. It remains to prove the "full measure" part: one easily notes that

$$\Sigma_T = \bigcap_{|r|\le T, r\in\mathbb{Q}} \Phi^{-r}(T^*X\backslash(T^*X_0 \cap \{\eta = 0\})).$$

Now, $T^*X_0 \cap \{\eta = 0\}$ has null measure, thus its complement has full measure, and so has $\Phi^r(T^*X\backslash(T^*X_0 \cap \{\eta = 0\}))$ because Φ^r is a diffeomorphism. □

4.2. Flow invariance of the eigenfunctions

Lemma 4.2. *Let $T > 0$, let $a \in C_c^\infty(T^*X)$ with $\operatorname{supp}(a) \subset \Sigma_T$. Then there exists some constant $C > 0$, depending only on a and T, such that for every $j \ge 0$, and every $0 \le t \le T$,*

$$|\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle - \langle \mathrm{Op}_{h_j}(a\circ\Phi^t)u_j, u_j\rangle| \le Ch_j.$$

Proof. Let $P := (h_j^2\Delta - 1)$; then we have

$$Pu_j = h_j^2 Q_j \delta_c + \frac{h_j^2}{4}u_j,$$

and δ_c is H^{-1} (see the beginning of Section 2.1 for the definition) and $(h_j^2 Q_j)_j$ is in $\ell^2(\mathbb{N})$ (Proposition 2.4) hence bounded by D.

Let $s_j(t) := \langle \mathrm{Op}_{h_j}(a \circ \Phi^t)u_j, u_j \rangle$; every s_j is smooth, and

$$\begin{aligned}
\partial_t s_j(t) &= \langle \mathrm{Op}_{h_j}(\{p, a \circ \Phi^t\})u_j, u_j\rangle \\
&= -ih_j^{-1}\langle [P, \mathrm{Op}_{h_j}(a \circ \Phi^t)]u_j, u_j\rangle + ih_j\langle R(h_j,t)u_j, u_j\rangle \\
&= \frac{-i}{h_j}\langle P\, \mathrm{Op}_{h_j}(a \circ \Phi^t)u_j, u_j\rangle - \frac{-i}{h_j}\langle \mathrm{Op}_{h_j}(a \circ \Phi^t)Pu_j, u_j\rangle \\
&\quad + h_j\langle R(h_j,t)u_j, u_j\rangle \\
&= U - V + W.
\end{aligned}$$

In this computation, for every $0 \le t \le T$, $R(h,t) \in \Psi_h^{-\infty}(X)$ is a pseudodifferential operator satisfying

$$\sup\{\|R(h,t)\|_{L^2 \to L^2};\ 0 < h < h_0, 0 \le t \le T\} < \infty.$$

Therefore $|W| \le C_1 h_j$.

Next, we get

$$\begin{aligned}
|V| &= \left| \frac{1}{h_j}\langle \mathrm{Op}_{h_j}(a \circ \Phi^t)Pu_j, u_j\rangle \right| \\
&\le h_j|Q_j\langle \mathrm{Op}_{h_j}(a \circ \Phi^t)\delta_c, u_j\rangle| + \frac{h_j}{4}|\langle \mathrm{Op}_{h_j}(a)u_j, u_j\rangle|.
\end{aligned}$$

Let us notice that $\mathrm{WF}_h(\delta_c) \subset \{r = c, \eta = 0\}$, whereas $\mathrm{WF}_h(\mathrm{Op}_h(a \circ \Phi^t)) \subset \Phi^{-t}(\Sigma_T)$ is at positive distance from $\{\eta = 0\}$ (uniformly in $|t| \le T$). Therefore, $\|\mathrm{Op}_{h_j}(a \circ \Phi^t)\delta_c\|_{L^2} \le C_j'(t)h_j^2$, where $(C_j'(t))_{j, 0 \le t \le T}$ is bounded by $B_1 > 0$, so that $(h_j^2 Q_j C_j'(t))_{j \ge 1, 0 \le t \le T}$ is bounded by $B_2 = B_1 D > 0$. Thus for some constant $B > 0$ depending only on a and T, $|V| \le Bh_j$.

Finally, using our quantization procedure's characteristics,

$$\begin{aligned}
ih_j U &= \langle P\, \mathrm{Op}_{h_j}(a \circ \Phi^t)u_j, u_j\rangle \\
&= \langle u_j, \mathrm{Op}_{h_j}(a \circ \Phi^t)^* Pu_j\rangle \\
&= \langle u_j, \mathrm{Op}_{h_j}(\bar{a} \circ \Phi^t)Pu_j\rangle,
\end{aligned}$$

so the same argument as for V can be used to get $|ih_j U| \le B' h_j^2$ hence $|U| \le B' h_j$. So we have $|\partial_t s_j(t)| \le (C_1 + B + B')h_j = Ah_j$ uniformly in $0 \le t \le T$, where clearly C_1, B, B' depend only on a and T. This yields $|s_j(t) - s_j(0)| \le AT h_j$ when integrating. $\square$

A direct consequence is the following:

Corollary 4.3. *Let $T > 0$, let $a \in C_c^\infty(T^*X)$ with* $\mathrm{supp}(a) \subset \Sigma_T$. *There exists some constant $C > 0$ such that for every $j \geq 0$,*

$$|\langle \mathrm{Op}_{h_j}(a - \langle a\rangle_T)u_j, u_j\rangle| \leq C h_j$$

with $\langle a\rangle_T := \frac{1}{T}\int_0^T a \circ \Phi^t dt$.

An easy argument then yields (taking into account the fact that $a - \langle a\rangle_T$ has average zero):

Proposition 4.4. *Let $T > 0$, let $a \in C_c^\infty(T^*X)$ with* $\mathrm{supp}(a) \subset \Sigma_T$. *Then, as $\lambda \to \infty$, $M(a - \langle a\rangle_T, \lambda) \to 0$.*

5. Analysis far in the cusp

If we joined the main results of Sections 3 and 4, we would be able to prove the main theorem for symbols with average zero. This will be done in section 6.

But if we want to prove the main theorem for general symbols in $S^0_{\mathrm{comp}}(X)$, we have to find some symbols with non-zero average for which the result holds. A direct proof turns out to be difficult: so we will exhibit symbols s with average arbitrarily close to some non-zero constant and such that $\limsup_{\lambda\to\infty} M(s, \lambda)$ is arbitrarily close to 0.

Before, we need to introduce some cutoff functions: let us set some $R > e^{c_0}$, and χ_R be a smooth nondecreasing function such that

$$\chi_R(r) = 1 \text{ on } [R+1, \infty), \quad \chi_R(r) = 0 \text{ on } (-\infty, R]. \tag{5}$$

Let $\phi_R\colon X \to [0;1]$ be a smooth function that is zero outside the cusp and such that if $r > c_0, \theta \in \mathbb{R}/\mathbb{Z}, \phi_R(r,\theta) = \chi_R(r)$. Note that $1 - \phi_R^4 \in C_c^\infty(X)$.

We will show the following:

Proposition 5.1. *There exists a universal constant $C > 0$ such that for any $R > e^c$,*

$$\limsup_{\lambda\to\infty} M(1 - \phi_R^8, \lambda) \leq C e^{-R/4}.$$

Our first step is to understand where the mass of the u_j is localized.

Let us write, for every $j \geq 0, r > c_0, \theta \in \mathbb{R}/\mathbb{Z}$,

$$u_j(r,\theta) = e^{r/2} \sum_{k\in\mathbb{Z}} v_{j,k}(r) e^{2ik\pi\theta}.$$

This is similar to the expansion of u_j as a Fourier series in θ, but the coefficients were renormalized, so that $\int_{\theta\in\mathbb{R}/\mathbb{Z}} |u_j|^2 \, d\mathrm{v}_g = \sum_{k\in\mathbb{Z}} |v_{j,k}|^2(r)\, dr$. Note that by definition of the pseudo-Laplacian, for $r > c$, $v_{j,0}(r) = 0$.

Let χ_1 be a smooth nondecreasing function such that $\chi_1 = 0$ on $(-\infty, 0]$, and $\chi_1 = 1$ on $[1,\infty)$. We will now denote $\chi_{h,k}(r) = \chi_1(-\ln 2|k|h\pi - r + R/2)$. This function is going to be used to weaken the growth of the function $(2\pi kh)^2 e^{2r}$, which is not a symbol: indeed, $\chi_{h,k}(r) = 0$ as soon as $(2\pi hk)^2 e^{2r} \geq e^R$.

Let χ_2 be a smooth non-negative compactly supported function such that $\chi_2 = 1$ on $[-3,3]$, and $\chi_2 = 0$ outside $(-4,4)$, and $|\chi_2| \leq 1$.

We first have the following straightforward formula: for $u \in H^1(X)$, in the cusp,

$$|\nabla^g u|_g^2(r,\theta) = |(\partial_r u)(r,\theta)|^2 + e^{2r}|(\partial_\theta u)(r,\theta)|^2.$$

Corollary 5.2. *For $j \geq 0$, the following bounds hold true*:

$$\sum_{k\in\mathbb{Z}} (2\pi k)^2 \int_{c_0}^{\infty} e^{2r}|v_{j,k}|^2(r)\, dr \leq \lambda_j^2 + \frac{1}{4}, \tag{6}$$

$$\sum_{k\in\mathbb{Z}} \int_{c_0}^{\infty} |v_{j,k}|^2(r)\, dr \leq 1, \tag{7}$$

$$\sum_{k\in\mathbb{Z}} \int_{c_0}^{\infty} |v'_{j,k}|^2(r)\, dr \leq 2\lambda_j^2 + 1 \tag{8}$$

Proof. For the bound (6), we estimate

$$\begin{aligned}
\lambda_j^2 + \frac{1}{4} &= \langle u_j, \Delta_c u_j\rangle = \|\nabla u_j\|_{L^2}^2 \\
&\geq \int_{r>c_0} |\nabla^g u_j|_g^2 \, d\mathrm{v}_g \\
&\geq \int_{r>c_0} \left(\int_{\mathbb{R}/\mathbb{Z}} e^{2r}|\partial_\theta u_j(r,\theta)|^2 d\theta \right) e^{-r} dr \\
&\geq \sum_{k\in\mathbb{Z}} (2k\pi)^2 \int_{r>c_0} e^{2r}|v_{j,k}|^2(r)\, dr,
\end{aligned}$$

because of the normalization of $v_{j,k}$. The bound (7) is a direct consequence of the fact that $\|u_j\|_{L^2}^2 = 1$. As for (8), note that in the cusp,

$$(\partial_r u_j)(r,\theta) = e^{r/2} \sum_{k\in\mathbb{Z}} \Big(\frac{1}{2} v_{j,k} + v'_{j,k}\Big) e^{2ik\pi\theta},$$

so

$$\lambda_j^2 + \frac{1}{4} \geq \int_{c_0}^{\infty} \int_{\mathbb{R}/\mathbb{Z}} e^{-r} |\partial_r u_j(r,\theta)|^2 \, d\theta dr \geq \sum_{k\in\mathbb{Z}} \int_{c_0}^{\infty} \Big| \frac{v_{j,k}}{2} + v'_{j,k}\Big|^2 \, dr$$
$$\geq \sum_{k\in\mathbb{Z}} \int_{c_0}^{\infty} \Big(\frac{1}{2} |v'_{j,k}|^2 - \Big|\frac{v_{j,k}}{2}\Big|^2 \Big) \, dr \geq \frac{1}{2} \sum_{k\in\mathbb{Z}} \int_{c_0}^{\infty} |v'_{j,k}|^2(r) \, dr - \frac{1}{4}$$

and the proof is complete. □

Let now $h > 0$ be very small, and $\lambda = h^{-1}$. Let $\omega_j^2 = (h\lambda_j)^2$, for every j such that $h/2 \leq h_j \leq 2h$ (we say that j is *in the range*): then, ω_j is between $1/2$ and 2.

Lemma 5.3. *In the cusp region $r > c_0$,*

$$(-h^2\partial_r^2 - \omega_j^2 + (2kh\pi)^2 e^{2r}) v_{j,k} = 0.$$

We can therefore extend $v_{j,k}$ as a smooth function of the whole real line with the same properties.

This will not be used in the paper, but one can notice that with the substitution $w(r) = v_{j,k}(2k\pi e^r)$, the differential equation becomes $x^2 w''(x) + x w'(x) + (\lambda_j^2 - x^2) w(x) = 0$, thus making w a solution of the modified Bessel differential equation with pure imaginary parameter.

The following lemma means that microlocally, the mass of $v_{j,k}$ is concentrated near the curve $\xi^2 + (2kh\pi)^2 e^{2r} = \omega_j^2$:

Lemma 5.4. *There exists a constant C_R depending only on R (and not on h) such that for every j in the range, for every $1 \leq |k| \leq 3\lambda$,*

$$\| \mathrm{Op}_h((1-\chi_2)(\xi)\chi_R^2(r)\chi_{h,k}(r)^2)(\chi_R v_{j,k})\|_{L^2}^2$$
$$\leq C_R h^2 \|v_{j,k}\|_{L^2(c_0,\infty)}^2 + C_R h^4 \|v'_{j,k}\|_{L^2(c_0,\infty)}^2.$$

Proof. Let us denote here

$$f_{j,k}^{(1)} := \mathrm{Op}_h(\chi_R(r)\chi_{h,k}(r)(\xi^2 - \omega_j^2 + (2kh\pi)^2 e^{2r}))(\chi_R v_{j,k})$$
$$= \chi_R(r)\chi_{h,k}(r)(-2h^2\chi'_R(r) v'_{j,k} - h^2 \chi''_R(r) v_{j,k}),$$

and

$$f_{j,k} := \mathrm{Op}_h\left(\chi_R(r)\chi_{h,k}(r)\frac{1-\chi_2(\xi)}{\xi^2-\omega_j^2+(2kh\pi)^2e^{2r}}\right)f^{(1)}_{j,k}.$$

Using [17, Theorem 4.23], (inside the symbol, when $(2|k|h\pi)e^r > e^{R/2}$, the $\chi_{h,k}$ factor makes everything vanish; moreover, when $|\xi| < 3$, the $1-\chi_2$ factor makes everything vanish as well, so that the denominator of the symbol in the definition of $f_{j,k}$ is always bounded below by 5, so all relevant S^0 seminorms are bounded uniformly in j, k, h, as long as $h \le |k|h \le 3$ and j is in the range) we may write, for some constant $K_R > 0$ depending only on R:

$$\begin{aligned}\Big\|\frac{1}{h^2}f_{j,k}\Big\|^2_{L^2} &= \frac{1}{h^4}\Big\|\mathrm{Op}_h\left(\chi_R(r)\chi_{h,k}(r)\frac{1-\chi_2(\xi)}{\xi^2-\omega_j^2+(2hk\pi)^2e^{2r}}\right)f^{(1)}_{j,k}\Big\|^2_{L^2}\\ &\le \frac{1}{h^4}\Big\|\mathrm{Op}_h\left(\chi_R(r)\chi_{h,k}(r)\frac{1-\chi_2(\xi)}{\xi^2-\omega_j^2+(2hk\pi)^2e^{2r}}\right)\Big\|^2_{L^2\to L^2}\|f^{(1)}_{j,k}\|^2\\ &\le K_R(\|v_{j,k}\|^2_{L^2((c_0,\infty)}+\|v'_{j,k}\|^2_{L^2((c_0,\infty))}).\end{aligned}$$

Now, when h is small, $1 \le |k| \le 3\lambda$, j is in the range, the symbols

$$\begin{aligned}a_- &:= \chi_R(r)\chi_{h,k}(r)\frac{1-\chi_2(\xi)}{\xi^2-\omega_j^2+(2kh\pi)^2e^{2r}},\\ a_+ &:= \chi_R(r)\chi_{k,h}(r)(\xi^2-\omega_j^2+(2kh\pi)^2e^{2r}),\\ a_* &:= a_-a_+\end{aligned}$$

are bounded by a constant depending only on R in respectively the class of symbols $S(\langle\xi\rangle^{-2})$, $S^0(\langle\xi\rangle^2)$, $S(1)$ (using the notation of [17, Section 4.4.1]), thus by [17, Theorems 4.18, 4.23] we have $\|\mathrm{Op}_h(a_-)\mathrm{Op}_h(a_+)-\mathrm{Op}_h(a_*)\|_{L^2\to L^2} \le C_Rh$, for some constant $C_R > 0$ depending only on R. Therefore, we get that for some constant $C_R > 0$ depending only on R,

$$\begin{aligned}\|\mathrm{Op}_h(a_*)(\chi_Rv_{j,k})\|^2_{L^2} &\le 2C_Rh^2\|\chi_Rv_{j,k}\|^2_{L^2}+2\|f_{j,k}\|^2_{L^2}\\ &\le C_R(h^2\|v_{j,k}\|^2_{L^2(c_0,+\infty)}+h^4\|v'_{j,k}\|^2_{L^2(c_0,+\infty)}).\end{aligned}\qquad\square$$

Corollary 5.5. *There exists a constant C_R depending only on R such that, when j is in the range and h is small,*

$$\sum_{1\le|k|\le3\lambda}\|\chi_R(r)\,\mathrm{Op}_h((1-\chi_2)(\xi)\chi_R^2(r)\chi^2_{h,k}(r))(\chi_Rv_{j,k})\|^2_{L^2} \le C_Rh^2.$$

Proof. It is a consequence of the previous lemma and of Corollary 5.2, more precisely estimates (7) and (8). □

The following result is the main property of localization we were aiming at. It tells us that each $\phi_R^4 u_j$ is localized along his lowest modes in the cusp, and each of these modes is microlocalized in a compact zone that depends very little on j: that is, $v_{j,k}$ is microlocalized in the zone $|\xi| \leq 4$, $R \leq r \leq R/2 - \ln 2h|k|\pi$.

Let us first define the operator $A_{h,k} := \mathrm{Op}_h(\chi_2(\xi)\chi_R^2(r)\chi_{h,k}(r)^2)$.

Proposition 5.6. *Let $j \geq 0$ be in the range. For some universal constant $C > 0$, and some constant $C_R > 0$ depending only on $R > c$,*

$$\|\phi_R^4 u_j\|_{L^2}^2 \leq 4 \sum_{1\leq |k|\leq 3\lambda} \|\chi_R(r) A_{h,k}(\chi_R v_{j,k})\|_{L^2}^2 + Ce^{-R} + C_R h^2.$$

We split the proof in several steps.

Lemma 5.7. *Let $j \geq 0$ be in the range. Then*

$$\sum_{|k|>3\lambda} \|\chi_R^4 v_{j,k}\|_{L^2}^2 \leq e^{-2R}.$$

Proof. Using (6), we obtain the sequence of inequalities

$$\begin{aligned}
\sum_{|k|>3\lambda} \|\chi_R^4 v_{j,k}\|_{L^2}^2 &\leq \frac{1}{36} \sum_{|k|>3\lambda} e^{-2R} \frac{(2k\pi)^2}{\lambda^2} \int_R^\infty e^{2r} |v_{j,k}|^2 \, dr \\
&\leq \frac{1}{36 e^{2R} \lambda^2} \sum_{k\in\mathbb{Z}} (2\pi k)^2 \int_{c_0}^\infty e^{2r} |v_{j,k}|^2 \, dr \\
&\leq e^{-2R} \frac{\lambda_j^2 + \frac{1}{4}}{36\lambda^2} \leq e^{-2R}
\end{aligned}$$

which proves the claim. □

Lemma 5.8. *Let $j \geq 0$ be in the range. Then the following holds true*:

$$\sum_{1\leq |k|\leq 3\lambda} \|\chi_R^4 (1 - \chi_{h,k}^2) v_{j,k}\|_{L^2}^2 \leq 40 e^{-R} + 3h^2.$$

Proof. Using again (6), if b_k is the maximum of $R/2 - 1 - \ln 2\pi|k|h$ and c_0,

$$\begin{aligned}\sum_{1\le|k|\le3\lambda}\|\chi_R^4(1-\chi_{h,k}^2)v_{j,k}\|_{L^2}^2 &\le \sum_{1\le|k|\le3\lambda}\int_{b_k}^{\infty}|v_{j,k}|^2(r)\,dr\\ &\le \sum_{1\le|k|\le3\lambda}\int_{b_k}^{\infty}e^{-R}(2e\pi kh)^2e^{2r}|v_{j,k}|^2(r)\,dr\\ &\le e^2h^2e^{-R}\sum_{k\in\mathbb{Z}}(2\pi k)^2\int_{c_0}^{\infty}e^{2r}|v_{j,k}|^2(r)\,dr\\ &\le 40e^{-R}+\frac{1}{4}e^{2-R}h^2\le 40e^{-R}+3h^2\end{aligned}$$

which proves the claim. □

Now, we can prove Proposition 5.6:

Proof of Proposition 5.6. One easily sees that

$$\|\phi_R^4u_j\|_{L^2}^2=\sum_{|k|\ge1}\int_R^{\infty}\chi_R(r)^8|v_{j,k}|^2(r)\,dr\le\sum_{|k|\ge1}\|\chi_R^4v_{j,k}\|_{L^2}^2.$$

Now, we split the sum between the $|k|>3\lambda$, the sum of which is not greater than e^{-2R} (Lemma 5.7), and the $1\le|k|\le3\lambda$. Moreover, if $1\le|k|\le3\lambda$,

$$\|\chi_R^4v_{j,k}\|_{L^2}^2\le2\|\chi_R^4(1-\chi_{h,k}^2)v_{j,k}\|_{L^2}^2+2\|\chi_R^4\chi_{h,k}^2v_{j,k}\|_{L^2}^2.$$

From Lemma 5.8 we see that the first term contributes at most $6h^2+80e^{-R}$, so we have to bound the second term. Now,

$$\begin{aligned}2\|\chi_R^4\chi_{h,k}^2v_{j,k}\|_{L^2}^2&\le4\|\chi_RA_{h,k}(\chi_Rv_{j,k})\|_{L^2}^2\\ &\quad+4\|\chi_R\operatorname{Op}_h((1-\chi_2(\xi))\chi_{h,k}^2(r)\chi_R(r)^2)(\chi_Rv_{j,k})\|_{L^2}^2.\end{aligned}$$

We saw from Corollary 5.5 that the second term is at most C_Rh^2, where $C_R>0$ depends only on R, and this ends the proof. □

Now, we turn the pointwise localization estimate we have on the $\phi_R^4u_j$ into an average estimate on j. Thanks to Hilbert–Schmidt norm estimates (operators will always be considered as from the relevant L^2 spaces into themselves) we obtain significantly better results.

Let us define the operator $A'_{h,k} := A_{h,k}\chi_R$, then let

$$Aw(r,\theta) := \chi_R(r)e^{r/2}\sum_{1\le|k|\le 3\lambda}(A_{h,k}(\chi_R w_k))(r)e^{2ik\pi\theta}$$

for every $r > c_0, \theta \in \mathbb{R}/\mathbb{Z}$, where $w(r,\theta) = e^{r/2}\sum_{k\in\mathbb{Z}} w_k(r)e^{2ik\pi\theta}$.

Proposition 5.9. *There exist constants $C > 0$ universal, and $C_R > 0$ depending on R only such that*

$$h^2\sum_{h/2\le h_j\le 2h}\|\phi_R^4 u_j\|_{L^2}^2 \le C_R h^2 + Ce^{-R} + 4h^2\sum_{1\le|k|\le 3\lambda}\|A'_{k,h}\|_{\mathrm{HS}}^2.$$

Proof. From Proposition 5.6, we know that for any j in the range, $\|\phi_R^4 u_j\|^2 \le 200e^{-R} + C_R h^2 + 4\|Au_j\|_{L^2}^2$. So, using Weyl's law, and the fact the (u_j) are orthonormal,

$$h^2\sum_{h/2\le h_j\le 2h}\|\phi_R^4 u_j\|_{L^2}^2 \le Ce^{-R} + C_R h^2 + 4h^2\|A\|_{\mathrm{HS}}^2.$$

Now, let (f_p) be an orthonormal basis of $L^2(R,\infty)$, let (g_q) be an orthonormal basis of $\{f \in L^2(X), \mathbf{1}_{r>R}f = 0\}$. Then the family of all $e^{r/2}f_p(r)e^{2ik\pi\theta}$ and the g_q, is an orthonormal basis of $L^2(X)$. Therefore, when f is an element of this orthonormal basis, we realize that only when $f = e^{r/2}f_p(r)e^{2ik\pi\theta}$, $1 \le |k| \le 3\lambda$, $\|Af\|^2$ does not vanish. In that case, it will always be lower or equal than $\|(\chi_R \circ A_{h,k} \circ \chi_R)f_p\|_{L^2(\mathbb{R})}^2$. From this, it follows that

$$\|A\|_{\mathrm{HS}}^2 \le \sum_{1\le|k|\le 3\lambda}\|\chi_R \circ A_{k,h} \circ \chi_R\|_{\mathrm{HS}}^2 \le \sum_{1\le|k|\le 3\lambda}\|A'_{h,k}\|_{\mathrm{HS}}^2,$$

which completes the proof. □

Now, it is easy to give an upper bound on the Hilbert–Schmidt norm of the operators, and to turn it into a complete estimate:

Proposition 5.10. *The following bound holds true for* $1 \le |k| \le 3\lambda$:

$$\|A'_{h,k}\|_{\mathrm{HS}}^2 \le 4(h\pi)^{-1}(-\ln 2|k|h\pi - R/2)^+$$

Proof. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $\hat\psi$ be its Fourier transform with respect to its second variable. Let $T = \mathrm{Op}_h(\psi)\chi_R$. For any $f \in L^2(\mathbb{R})$,

$$\begin{aligned}2h\pi Tf(x) &= \int_{\mathbb{R}^2} e^{i\frac{\xi(x-y)}{h}}\psi\Big(\frac{x+y}{2},\xi\Big)\chi_R(y)f(y)\,dyd\xi\\ &= \int_{\mathbb{R}}\hat\psi\Big(\frac{x+y}{2},\frac{y-x}{h}\Big)\chi_R(y)f(y)\,dy,\end{aligned}$$

therefore

$$\begin{aligned}\|2h\pi T\|^2_{\mathrm{HS}} &= \int\limits_{\mathbb{R}^2} \Big|\hat{\psi}\Big(\frac{x+y}{2}, \frac{y-x}{h}\Big)\Big|^2 \chi_R^2(y)\, dxdy \\ &\leq \int\limits_{\mathbb{R}^2} \Big|\hat{\psi}\Big(\frac{x+y}{2}, \frac{y-x}{h}\Big)\Big|^2 dxdy \\ &\leq h \int\limits_{\mathbb{R}^2} |\hat{\psi}(x,y)|^2 dydx = 2h\pi \|\psi\|^2_{L^2}.\end{aligned}$$

Now, we obtain

$$\begin{aligned}&\|\chi_2(\xi)\chi_R^2(r)\chi_1^2(-\ln 2|k|h\pi - r + R/2)\|^2_{L^2} \\ &\quad\leq \int\limits_{\mathbb{R}^2} \mathbf{1}_{\{|\xi|\leq 4\}}\mathbf{1}_{\{r\geq R\}}\mathbf{1}_{\{r\leq R/2-\ln 2|k|h\pi\}}\, d\xi dr \leq 8(-\ln 2|k|h\pi - R/2)^+.\end{aligned}$$

This completes the proof. □

Corollary 5.11. *The following estimate holds true*:

$$4\pi h^2 \sum_{1\leq|k|\leq 3\lambda} \|A'_{k,h}\|^2_{\mathrm{HS}} \leq Ce^{-R/2} + C_R h,$$

where $C > 0$ is universal and C_R depends only on R.

Proof. Using Stirling's formula, assuming that $2\pi h e^{R/2} \leq 1$ (else the sum is zero and the bound holds), for some universal constant C,

$$\begin{aligned}&\frac{h\pi}{8}\sum_{1\leq|k|\leq 3\lambda} \|A'_{k,h}\|^2_{HS} \\ &\quad\leq \sum_{1\leq k\leq 3\lambda}\Big(-\ln 2kh\pi - \frac{R}{2}\Big)^+ \\ &\quad\leq \sum_{1\leq k\leq (2\pi)^{-1}\lambda e^{-R/2}} -\ln 2kh\pi - R/2 \\ &\quad\leq -\frac{\lambda e^{-R/2}}{2\pi}\ln 2h\pi - \frac{\lambda e^{-R/2}}{2\pi}\frac{R}{2} \quad - \sum_{1\leq k\leq (2\pi)^{-1}\lambda e^{-R/2}} \ln k \\ &\quad\leq \frac{\lambda e^{-R/2}}{2\pi}\ln\frac{\lambda e^{-R/2}}{2\pi} - \frac{\lambda e^{-R/2}}{2\pi}\ln\Big(\frac{\lambda e^{-R/2}}{2\pi}\Big) + \frac{2\lambda e^{-R/2}}{2\pi} + C \\ &\quad\leq \frac{e^{-R/2}}{h\pi} + \frac{Ch}{h}.\end{aligned}$$

□

Corollary 5.12. *For some universal constant $C > 0$ and some constant C_R depending only on R, one has*

$$Y'(\phi_R^4, h) := h^2 \sum_{h/2 \le h_j \le 2h} \|\phi_R^4 u_j\|_{L^2}^2 \le Ce^{-R/2} + C_R h.$$

Proof. It is a consequence of all that precedes. □

Proof of Proposition 5.1. We write

$$\begin{aligned} M(1-\phi_R^8, \lambda)^2 &= \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \left| \langle \mathrm{Op}_{h_j}(1-\phi_R^8) u_j, u_j \rangle - \int_{S^*X} (1-\phi_R^8) \right|^2 \\ &\le \frac{1}{N(\lambda)} \sum_{\lambda_j \le \lambda} \left| \int_X \phi_R^8 |u_j|^2 \, d\mathrm{v}_g - \int_{S^*X} \phi_R^8 \right|^2 \\ &\le \frac{2}{N(\lambda)} \sum_{\lambda_j \le \lambda} \left| \int_X \phi_R^8 |u_j|^2 \, d\mathrm{v}_g \right|^2 + 2\Big(\int_{S^*X} \phi_R^8 \Big)^2 \\ &= \frac{2}{N(\lambda)} \sum_{\lambda_j \le \lambda} \left| \int_X \phi_R^8 |u_j|^2 \, d\mathrm{v}_g \right|^2 + \frac{2}{\mathrm{Vol}(X)} \Big(\int_X \phi_R^8 d\mathrm{v}_g \Big)^2. \end{aligned}$$

The second term is lower than some Ce^{-2R} for some constant C independent of R, because $d\mathrm{v}_g(r,\theta) = e^{-r} dr d\theta$. The first term is not greater than

$$\frac{1}{N(\lambda)} \sum_{1 \le 4^k \le 4h_0\lambda} 2^{2-4k} \lambda^2 Y'\Big(\phi_R^4, \frac{2^{2k-1}}{\lambda}\Big)^2,$$

which by Corollary 5.12 is not greater than (using again Weyl's law)

$$\frac{C}{\lambda^2} \sum_{1 \le 4^k \le 4h_0\lambda} Ce^{-R/2} 2^{2-4k} \lambda^2 + C_R 2^{1-2k} \lambda \le Ce^{-R/2} + C_R \frac{1}{\lambda}.$$

This concludes the proof. □

6. Proof of the main theorem

Let $a \in S^0_{\mathrm{comp}}(X)$ and assume first that $\int_{S^*X} a \, d\mu = 0$. We let $T > 0$ and $\varepsilon > 0$. We may write $a = a_1 + a_2$, where $a_1 \in C_c^\infty(T^*X)$ and satisfies $\mathrm{supp}(a_1) \subset \Sigma_T$, and $\int_{S^*X} |a_2|^2 \le \varepsilon^2$. Then, as $\lambda \to \infty$,

$$\begin{aligned} M(a,\lambda) &\le M(a_1, \lambda) + C\varepsilon + O(\lambda^{-1}) \le M(\langle a_1 \rangle_T, \lambda) + C\varepsilon + o(1) \\ &\le M(\langle a \rangle_T, \lambda) + 2C\varepsilon + o(1) \le C(\|\langle a \rangle_T\|_{L^2(S^*X)} + 2\varepsilon) + o(1), \end{aligned}$$

where we applied Proposition 3.7 to a_2, then Proposition 4.4 to a_1, then Proposition 3.7 to $\langle a_2\rangle_T$ and to $\langle a\rangle_T$. We first take the lim sup as $\lambda \to \infty$, then let ϵ go to zero, and finally let T go to ∞, and the L^2-ergodic theorem proves the result.

In the general case, let $a \in S^0_{\mathrm{comp}}(X)$ and let $\alpha := \int_{S^*X} a$. Let us denote, for every $R > e^{c_0}$, $I_R := \int_{S^*X} \phi_R^8$. Then $I_R = \mathcal{O}(e^{-R})$, thus, if R is large enough, $a_R = a - \frac{\alpha}{1-I_R}(1-\phi_R^8)$ belongs to $S^0_{\mathrm{comp}}(X)$ with

$$\int_{S^*X} a_R = 0.$$

Thus we have $M(a_R,\lambda) \to 0$ as $\lambda \to \infty$. Now, by Proposition 5.1

$$\limsup_{\lambda\to\infty} M\Big(\frac{\alpha}{1-I_R}(1-\phi_R^8),\lambda\Big) \le \frac{C|\alpha|}{1-I_R}e^{-R/4}.$$

Thus,

$$\limsup_{\lambda\to\infty} M(a,\lambda) \le \frac{C|\alpha|}{1-I_R}e^{-R/4}.$$

Letting $R \to \infty$ yields $M(a,\lambda) \to 0$ and the proof of Theorem 0.1 is complete.

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