

Spectral invariants of Dirichlet-to-Neumann operators on surfaces

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Abstract. We obtain a complete asymptotic expansion for the eigenvalues of the Dirichlet-to-Neumann maps associated with Schrödinger operators on compact Riemannian surfaces with boundary. For the zero potential, we recover the well-known spectral asymptotics for the Steklov problem. For non-zero potentials, we obtain new geometric invariants determined by the spectrum of what we call the parametric Steklov problem. In particular, for constant potentials parametric Steklov problem, the total geodesic curvature on each connected component of the boundary is a spectral invariant. Under the constant curvature assumption, this allows us to obtain some interior information from the spectrum of these boundary operators.

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1. Introduction and main result

1.1. The Dirichlet-to-Neumann map. Let (Ω, g) be a compact Riemannian surface with smooth boundary Σ and $\tau \in C^\infty(\Omega; \mathbb{R})$. For $\lambda \in \mathbb{R}$, the Dirichlet-to-Neumann map on Ω

$$DN_\lambda := DN_\lambda(\Omega; \tau): C^\infty(\Sigma) \longrightarrow C^\infty(\Sigma)$$

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is defined as $\text{DN}_\lambda u = \partial_\nu \tilde{u}$, where ∂_ν is the outward pointing normal derivative on Σ and \tilde{u} is the solution to the problem

$$\begin{cases} (\Delta_g + \lambda\tau)\tilde{u} = 0 & \text{in } \Omega, \\ \tilde{u} = u & \text{on } \Sigma. \end{cases}$$

Given Ω and τ , the map DN_λ is well defined for all $\lambda \in \mathcal{V} \subset \mathbb{R}$, where $\mathbb{R} \setminus \mathcal{V}$ is a discrete set consisting in the Dirichlet eigenvalues of the linear operator pencil $\Delta_g + \lambda\tau$, i.e., the values of λ such that the problem

$$\begin{cases} (\Delta_g + \lambda\tau)u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma, \end{cases}$$

admits a non-trivial solution. For fixed $\lambda \in \mathcal{V}$, the Dirichlet-to-Neumann map is a self-adjoint elliptic pseudodifferential operator of order one with principal symbol $|\xi|_g$, see [21]. Its spectrum is discrete and accumulating only at infinity,

$$\sigma_0(\Omega; \tau; \lambda) \leq \sigma_1(\Omega; \tau; \lambda) \leq \dots \nearrow \infty.$$

These eigenvalues are solutions to the eigenvalue problem

$$\begin{cases} (\Delta_g + \lambda\tau)u = 0 & \text{in } \Omega, \\ \partial_\nu u = \sigma u & \text{on } \Sigma. \end{cases} \quad (1)$$

A survey of the general properties of the Steklov problem, i.e., the problem for $\lambda = 0$, is found in [9]. When $\lambda = 0$, since τ becomes irrelevant, we simply label the eigenvalues $\sigma_j(\Omega, 0)$.

1.2. Spectral asymptotics. Since DN_λ is a self-adjoint elliptic pseudodifferential operator of order one with principal symbol $|\xi|_g$, it follows from Weyl's law with sharp remainder (see [11]) that for any λ, τ the eigenvalues satisfy

$$\sigma_j(\Omega; \tau; \lambda) = \frac{\pi j}{\text{per}(\Sigma)} + R(j, \lambda, \tau),$$

where $\text{per}(\Sigma)$ denotes the length of Σ and the remainder $R(j, \lambda, \tau)$ is a bounded function of j for every fixed λ and τ . Our aim is to obtain a complete asymptotic expansion for $\sigma_j(\Omega; \tau; \lambda)$. In order to state our results, we generalise the standard notion of asymptotic expansions.

Definition 1.1. Let $\{a_j\}, \{b_j\}$ be two sequences of real numbers. We say that they are *asymptotically equivalent*, and write $a_j \sim b_j$, if for all $N \in \mathbb{N}$ there exist C_N and J_N such that for all $j \geq J_N$,

$$|a_j - b_j| \leq C_N j^{-N}.$$

More generally, suppose that for every $N \in \mathbb{N}$, $\{B_j^{(N)}\}$ is a sequence of real numbers. We write

$$a_j \sim B_j^{(\infty)}$$

if for every $N \in \mathbb{N}$, there exist C_N and J_N such that for all $j \geq J_N$,

$$|a_j - B_j^{(N-1)}| \leq C_N j^{-N}.$$

We observe that the last part of the definition corresponds to the usual notion of asymptotic expansion

$$a_j \sim \sum_{n=K}^{\infty} b_n j^{-n}$$

if we take $B_j^{(N)} = \sum_{n=K}^N b_n j^{-n}$ for every $N \in \mathbb{N}$, but our definition allows for a larger class of asymptotic behaviour.

When Ω is simply connected, Rozenblum [20] and Guillemin and Melrose (see [5]) proved independently that the eigenvalues of DN_0 were asymptotically double and satisfy the precise asymptotics

$$\sigma_{2j}(\Omega) \sim \sigma_{2j-1}(\Omega) \sim \frac{2\pi j}{\text{per}(\Sigma)} = \sigma_{2j}\left(\frac{\text{per}(\Sigma)}{2\pi}\mathbb{D}\right).$$

In other words, σ_{2j} and σ_{2j-1} have the same complete asymptotic expansion, and every term except the principal term in that expansion vanishes. That principal term is given by the Steklov eigenvalues for the disk with the same perimeter as Ω . Our first result is an extension of this result to the Dirichlet-to-Neumann operators associated to Schrödinger operators.

Theorem 1.2. *Let (Ω, g) be a simply connected compact Riemannian surface with smooth boundary Σ . For $\lambda \in \mathbb{R} \cap \mathcal{V}$, the eigenvalues of $\text{DN}_\lambda(\Omega; \tau)$ are asymptotically double and admit a complete asymptotic expansion given by*

$$\sigma_{2j} \sim \sigma_{2j-1} \sim \frac{j}{L} + \sum_{n=1}^{\infty} s_n(\lambda; \Omega) j^{-n}, \quad (2)$$

where $L = \frac{\text{per}(\Sigma)}{2\pi}$. The coefficients s_n are polynomials in λ of degree at most n with vanishing constant coefficients. They depend on both τ and the metric in an arbitrarily small neighbourhood of Σ . If $\tau \equiv 1$, then the first two terms are given by

$$s_1(\lambda; \Omega) = -\frac{\lambda L}{2}, \quad s_2(\lambda; \Omega) = \frac{\lambda L}{4\pi} \int_{\Sigma} k_g \, ds \quad (3)$$

where k_g is the geodesic curvature on Σ .

Just as with the Steklov problem, the righthand side of (2) is in fact an asymptotic expansion for an eigenvalue problem defined on the disk. When Ω is not simply connected, the situation is not so simple. Our main theorem shows that in such a case, the spectrum is asymptotically equivalent to the spectrum of a Dirichlet-to-Neumann operator associated to a Schrödinger operator on a disjoint union of disks. When $\lambda = 0$, Girouard, Parnowski, Polterovich, and Sher proved the same result in [8], whereas Arias-Marco, Dryden, Gordon, Hassannezhad, Ray, and Stanhope proved in [2] the equivalent statement for the eigenvalues of DN_0 on orbisurfaces.

To state the theorem, we introduce notation for a union of non-decreasing sequences, reordered to be non-decreasing. Let $\Xi = \{\xi^{(1)}, \dots, \xi^{(\ell)}\}$ be a finite set of non-decreasing sequences of real numbers accumulating at ∞ . We denote by $S(\Xi)$ the sequence $\xi^{(1)} \cup \dots \cup \xi^{(\ell)}$ rearranged in monotone non-decreasing order. Here, the union is understood as union of multisets, in other words repeated elements are kept with their multiplicity.

Theorem 1.3. *Let (Ω, g) be a compact Riemannian surface whose smooth boundary Σ has ℓ connected components $\Sigma_1, \dots, \Sigma_\ell$ with perimeters $\text{per}(\Sigma_m) = 2\pi L_m$, $1 \leq m \leq \ell$. Let $\lambda \in \mathbb{R} \cap \mathcal{V}$ and $\tau \in C^\infty(\Omega)$.*

(A) *For every $1 \leq m \leq \ell$, there are a metric g_m on the unit disk, a collar neighbourhood Υ_m of Σ_m , a collar neighbourhood $\tilde{\Upsilon}_m$ of \mathbb{S}^1 , and $\tau_m \in C^\infty(\mathbb{D})$ such that*

(a) *there is an isometry $\varphi_m: (\tilde{\Upsilon}_m, g_m) \rightarrow (\Upsilon_m, g)$;*

(b) *the restriction of τ_m to $\tilde{\Upsilon}_m$ is the pullback by φ_m of τ , in other words $\tau_m|_{\tilde{\Upsilon}_m} = \varphi_m^* \tau|_{\Upsilon_m}$;*

(c) *putting*

$$\Omega_{\#} = \bigsqcup_{\ell=1}^m (\mathbb{D}, g_m)$$

and $\tau_{\#} \in C^{\infty}(\Omega_{\#})$ to be equal to τ_m on each components of $\Omega_{\#}$,

$$\sigma_j(\Omega; \tau; \lambda) \sim \sigma_j(\Omega_{\#}; \tilde{\tau}; \lambda). \quad (4)$$

(B) Any other metric h on $\Omega_{\#}$ and function $\tau' \in C^{\infty}(\Omega_{\#})$ that satisfies (a) and (b) leads to the same asymptotic equivalence (4).

(C) For every $1 \leq m \leq \ell$ and $N \geq 1$, define the sequence $\xi^{(m,N)}$ as $\xi_0^{(m,N)} = 0$ and for $j \geq 1$,

$$\xi_{2j}^{(m,N)} = \xi_{2j-1}^{(m,N)} := \frac{j}{L_m} + \sum_{n=1}^N s_n^{(m)}(\lambda; \Omega) j^{-n}, \quad (5)$$

where the coefficients $s_n^{(m)}$ depend only on λ , τ and the metric in an arbitrarily small neighbourhood of Σ_m in the same way as in (2) (including the case when $\tau \equiv 1$). Let $\Xi^{(N)} = \{\xi^{(1,N)}, \dots, \xi^{(\ell,N)}\}$. For $\lambda \in \mathbb{R} \cap \mathcal{V}$, the eigenvalues of $\text{DN}_{\lambda}(\Omega; \tau)$ are asymptotically given by

$$\sigma_j \sim S(\Xi^{(\infty)})_j. \quad (6)$$

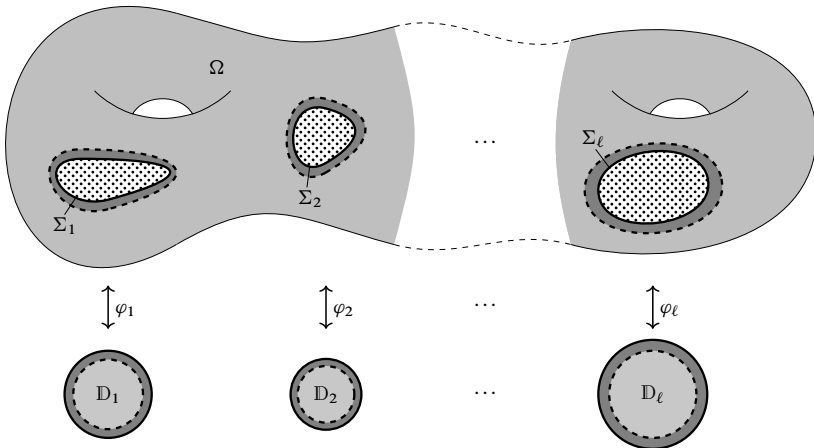


Figure 1. Construction of collar neighbourhoods in statement (A) of Theorem 1.3.

Remark 1.4. In the previous theorem, statements (B) and (C) are direct consequence of statement (A) and Theorem 1.2, along with the observation that the spectrum of DN_λ on a disjoint union of surfaces is equal to the union of the spectra, repeating multiplicity. As such, we will only need to prove statement (A). Nevertheless, statements (B) and (C) are useful to have explicitly for our applications to inverse problems.

Note that Theorem 1.3 obviously implies Theorem 1.2. However, the statement for simply connected surfaces is cleaner and obtained as an intermediate step in proving Theorem 1.3. Hence we state them separately.

1.3. Inverse spectral geometry. Inverse problems consist in recovering data of some PDE – the domain of definition Ω , the metric, the potential, etc. – from properties of the operator alone, and inverse spectral geometry consists in recovering that data from the spectrum only. One of the seminal questions in that field was asked for the Dirichlet Laplacian by Mark Kac in [15] and answered negatively by Gordon, Webb and Wolpert in [10]: “Can one hear the shape of a drum?” For this reason, we often say that any geometric data that one can recover from the spectrum of an operator can be “heard”.

It is long known and follows from Weyl’s law that the total boundary length can be heard from DN_λ . It also follows from the standard theory of the wave trace asymptotics as developed by Duistermaat and Guillemin [4] that the length spectrum – that is the length of the closed geodesics – of the boundary Σ can be heard as well. For DN_0 , it is shown in [8] that we can recover the number of connected components, as well as their lengths. It is also shown that from polynomial eigenvalue asymptotics alone in dimension two nothing more can be recovered. This can be seen as a consequence of Theorems 1.2 and 1.3 since the coefficients s_n and $s_n^{(m)}$ are all polynomials in λ that vanish when $\lambda = 0$.

For DN_0 , to extract more information different authors have turned to spectral quantities that have a more global nature. In [18], Polterovich and Sher obtain an asymptotic expansion as $t \rightarrow 0$ for the heat trace of DN_0 . From the coefficients, they obtain that the total mean curvature is a spectral invariant for $d \geq 3$. See also the work of Liu [17] for further improvements. In the case of $DN_\lambda(\Omega; \tau)$, heat trace asymptotics as well as invariants deduced from them have also been obtained by Wang and Wang in [22], again in dimension $d \geq 3$. We also refer to the works of Jollivet and Sharafutdinov [13, 14] where they find invariants for simply connected domains from the zeta function associated with DN_0 .

Our main theorem shows that for non-zero potential, one can hear more information from polynomial eigenvalue asymptotics.

The spectral inverse problem for the Dirichlet-to-Neumann map consists in extracting information about Ω , g , τ , and λ (or a subset of these parameters) from the eigenvalues $\{\sigma_j: j \in \mathbb{N}\}$. As an application of our methods, we will find spectral invariants when $\tau \equiv 1$, and show that we can recover λ as well as geometric data on Ω . For $\lambda = 0$, the problem has been studied already and is referred to as the Steklov problem. Lee and Uhlmann have shown in [16] that the map DN_0 (but not necessarily its spectrum) determines the Taylor series for g close to the boundary. Girouard, Parnowski, Polterovich and Sher show in [8] that from polynomial order spectral asymptotics, one can determine the number of boundary components and each of their lengths, but nothing more. Our goal is to obtain more information from the spectrum when $\lambda \neq 0$.

Theorem 1.5. *For any $\lambda \in (\mathbb{R} \cap \mathcal{V}) \setminus \{0\}$, the spectrum of DN_λ determines the following quantities:*

- *the number of connected components of the boundary, and their respective perimeters;*
- *each coefficient $s_n^{(m)}$ in (5), and in particular. if $\tau \equiv 1$,*
 - *the parameter λ and*
 - *the total geodesic curvature on each boundary component.*

The previous theorem along with the Gauss–Bonnet theorem also yield the following theorem. We denote by K the Gaussian curvature on Ω .

Theorem 1.6. *Let Ω be a compact orientable Riemannian surface with smooth boundary Σ and genus γ , and let $\lambda \in (\mathbb{R} \cap \mathcal{V}) \setminus \{0\}$. Then, the quantity*

$$4\pi\gamma + \int_{\Omega} K_g \, dA_g$$

is a spectral invariant of $\text{DN}_\lambda(\Omega; 1)$. In particular, if the Gaussian curvature is assumed to be a constant $K(\Omega)$, then the quantity

$$4\pi\gamma + K(\Omega) \text{area}(\Omega) \tag{7}$$

is determined by the spectrum of $\text{DN}_\lambda(\Omega; 1)$.

Here, the genus γ of Ω corresponds to the minimal genus of a closed surface in which Ω can be topologically embedded. Equivalently, it is the genus of the closed surface obtained from Ω by gluing topological disk onto each boundary

component. By restricting the choice of Ω , we can gain more interior geometric information from the spectrum. Note that while the Steklov spectrum is not known to determine interior information in general, for planar domains it is already known from the work of Edward [6, Theorem 4] that we can get lower bounds for the area.

Corollary 1.7. *If Ω is a domain of the standard sphere \mathbb{S}^2 , then its area is a spectral invariant of $\text{DN}_\lambda(\Omega; 1)$ for all $\lambda \in (\mathbb{R} \cap \mathcal{V}) \setminus \{0\}$.*

Proof. If $\Omega \subset \mathbb{S}^2$, then $\gamma = 0$ and $K(\Omega) = 1$. This leaves only $\text{area}(\Omega)$ in (7). \square

Corollary 1.8. *If Ω is a domain in a flat orientable space form, then its genus is a spectral invariant of $\text{DN}_\lambda(\Omega; 1)$ for all $\lambda \in (\mathbb{R} \cap \mathcal{V}) \setminus \{0\}$.*

Proof. If Ω is a domain in a flat orientable space form, then $K(\Omega) = 0$ and only $4\pi\gamma$ remains in (7). \square

The inverse problem for $\text{DN}_\lambda(\Omega; \tau)$ has a concrete interpretation in terms of the inverse scattering problem. In this context, $\Omega \subset \mathbb{R}^2$ has anisotropic refraction index τ . Non-destructive testing is the process of using the far-field data to measure the scattering of an incoming wave at frequency $\sqrt{\lambda}$ by the obstacle Ω . The inverse scattering problem consists in recovering then the refraction index τ , as well as the geometry of Ω . In [3], it is shown that the far-field data determines the spectrum of $\text{DN}_\lambda(\Omega; \tau)$, so that any spectral invariant of DN_λ can be obtained from the far-field data. We have explicit expressions for geometric quantities related to the boundary of Ω when the refraction index is isotropic, i.e., constant. When it is not, we do not give an explicit value of the coefficients s_n , however the algorithmic procedure to compute them in Sections 3 and 6 applies. Similarly, Theorem 4.2 is also valid in that context, giving an exact expression for the first few invariant quantities. Note that the coefficients s_n are polynomials of order at most n in λ with vanishing constant coefficient. This means that it is possible to decouple the coefficients of this polynomial by knowing the asymptotics for $\lambda_1, \dots, \lambda_n$. Physically, this simply means measuring the scattered far-field data for incoming waves at n different frequencies.

1.4. Sketch of the proof of Theorem 1.2. Let us introduce a slightly more general version of problem (1). For $\rho: \Sigma \rightarrow \mathbb{R}_+$ a strictly positive smooth function, we consider the eigenvalue problem

$$\begin{cases} (\Delta_g + \lambda\tau)u = 0 & \text{in } \Omega, \\ \partial_\nu u = \sigma\rho u & \text{on } \Sigma. \end{cases} \quad (8)$$

Our first step will be to show that we can reduce Theorems 1.2 and 1.3 for the problem (8) to proving them for

$$\begin{cases} -\Delta u = \lambda \tau u & \text{in } \mathbb{D}, \\ \partial_\nu u = \sigma \rho u & \text{on } \mathbb{S}^1. \end{cases} \quad (9)$$

In other words, by introducing this extra parameter ρ they only need to be proved in the case where Ω is a disk, and g is the flat metric g_0 .

This reduction will be done by following the strategy set out in [8], where they glue a disk to a collar neighbourhood of every boundary component, and discard the rest of the surface. Since the symbol of DN_λ depends solely on data obtained from a neighbourhood of the boundary, this doesn't change the symbol of the Dirichlet-to-Neumann map. Mapping these topological disks conformally to the unit disk in \mathbb{R}^2 will multiply the factors τ and ρ by a conformal factor, in other words it doesn't change the structure of the problem.

We then follow the general theory set out by Rozenblum in [19] to obtain a complete asymptotic expansion of the eigenvalues of a pseudodifferential operator on a circle in terms of integrals of its symbol. Note that in [19], an abstract algorithm is given to do so, but as is often the case with pseudodifferential symbolic calculus the expressions become unwieldy very quickly, and the difficulty resides in extracting actual geometric information out of it. The symbol is easy to compute for $\rho = 1$, $\lambda = 0$, where it is simply $|\xi|$, with no lower order terms. However, when $\lambda \neq 0$, this is no longer the case, and it will lead to the full asymptotic expansion that we obtain.

We obtain the following theorem for the disk.

Theorem 1.9. *The eigenvalues of the problem (9) satisfy the asymptotic expansion*

$$\sigma_{2j} = \sigma_{2j-1} + O(j^{-\infty}) \sim \frac{j}{\int_{\mathbb{S}^1} \rho \, dx} + \sum_{n=1}^{\infty} b_n j^{-n},$$

where the coefficients b_n depend only on ρ , λ and the values of τ in a neighbourhood of \mathbb{S}^1 , as well as their derivatives.

We will then specialise the previous theorem to the values of τ and ρ coming from the conformal mapping between the disk and Ω . We obtain explicit values of the coefficients b_n in that situation.

1.5. Plan of the paper. In Section 2, we make clear our reduction to the disk and compute the full symbol of the Dirichlet-to-Neumann map. In Section 3 and

Section 4, using the method laid out in [19], we transform the symbol of a general Dirichlet-to-Neumann map on a circle to extract the asymptotic expansion of its eigenvalues. In Section 5, we specify our results to the case of the parametric Steklov problem in order to show Theorem 1.2. Finally, in Section 6, we prove Theorem 1.5. There, we use Diophantine approximation to decouple the sequences obtained in Theorem 1.3 recursively.

1.6. Asymptotic notation. In addition to the asymptotic equivalence introduced in Definition 1.1, we also make use of Landau's asymptotic notation.

- For two sequences $\{a_j\}, \{b_j\}$, we write $a_j = O(b_j)$ if there exist a constant $C > 0$ and $N \in \mathbb{N}$ such that for all $j \geq N$, $|a_j| \leq Cb_j$. We note that if the sequence $\{b_j\}$ is strictly positive, this implies the existence of a (potentially larger) constant C' such that $|a_j| \leq C'b_j$ for all $j \in \mathbb{N}$.
- For two functions $f, g: \mathbb{R}^d \rightarrow \mathbb{R}$, we say that $f = O(g)$ if there exist $C > 0$ and $R > 0$ such that $|f(x)| \leq Cg(x)$ for all $|x| \geq R$.
- We write $a_j = O(j^{-\infty})$ to indicate that for all $N \in \mathbb{N}$, $a_j = O(j^{-N})$.

We can observe that the asymptotic equivalence $a_j \sim b_j$ is equivalent to $a_j - b_j = O(j^{-\infty})$.

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2. The symbol of the Dirichlet-to-Neumann map on surfaces

This section will be split into two parts. First, we follow Melrose's factorisation method, as described in [16]. We will see that the symbol of $\frac{1}{\rho} \text{DN}_\lambda(\Omega; \tau)$ depends only on λ, ρ , as well as on the restriction of τ and the metric g in a neighbourhood of the boundary Σ . This will allow us to show that we can reduce the problem at hand to the situation where Ω is the unit disk \mathbb{D} . In the second part of this section, we explicitly compute the value of the symbol for the disk.

We recall the definition of the rolling radius (see [12]) and the construction of Fermi coordinates. Given $x \in \Sigma$, the exponential map defines a normal geodesic curve $\gamma_x: \mathbb{R}_+ \rightarrow \Omega$,

$$\gamma_x(t) = \exp_x(t\nu).$$

The cut point of x is the point $\text{cut}_\Sigma(x) = \exp_x(t_0\nu)$, where t_0 is the minimal t such that $\exp(t\nu)$ does not minimise the distance to Σ . Smoothness and compactness of the boundary ensures that such a $t_0 > 0$ exists. The *cut locus* of Σ is the set

$$\text{cut}(\Sigma) := \{\text{cut}_\Sigma(x) : x \in \Sigma\}.$$

The *rolling radius* of Ω is defined as

$$\text{Roll}(\Omega) := \text{dist}(\Sigma, \text{cut}(\Sigma));$$

compactness of Σ ensures that $\text{Roll}(\Omega) > 0$. It is called the ‘‘rolling radius’’ because any open ball of radius $r \leq \text{Roll}(\Omega)$ can roll along Σ while always remaining a subset of Ω . For some $0 < \varepsilon < \text{Roll}(\Omega)$, let Υ be a collar ε -neighbourhood of the boundary,

$$\Upsilon := \{x' \in \Omega : \text{dist}(x', \Sigma) < \varepsilon\}.$$

Since $\varepsilon < \text{Roll}(\Omega)$, for every $x' \in \Upsilon$, there is a unique $x \in \Sigma$ and $t < \varepsilon$ such that $x' = \exp_x(t\nu)$; set $x' = (x, t)$. The boundary Σ is characterised by $\{t = 0\}$, and the outward normal derivative is given by $\partial_\nu = -\partial_t$. In these coordinates, the metric has a much simpler form as

$$g(x') = \tilde{g}(x')(dx)^2 + (dt)^2,$$

for some positive function \tilde{g} . The Laplacian reads

$$-\Delta_g = D_t^2 - \frac{i}{2}(\partial_t \log \tilde{g})D_t + \tilde{g}^{-1}D_x^2 - \frac{i}{2}\frac{(\partial_x \tilde{g})}{\tilde{g}^2}D_x,$$

where $D_x = -i\partial_x$ and x now denotes an arc length parameter along Σ .

2.1. Reduction to the disk. We start by observing that Propositions 1.1 and 1.2 of [16] apply to the Schrödinger operator $H = -\Delta - \lambda\tau$.

Proposition 2.1. *There is a family $A(x, t, D_x)$ of pseudodifferential operators depending smoothly on t such that*

$$-\Delta_g - \lambda\tau(x) \equiv (D_t + iE - iA(x, t, D_x))(D_t + iA(x, t, D_x)) \pmod{\Psi^{-\infty}},$$

where

$$E := -\frac{i}{2}(\partial_t \log \tilde{g}).$$

Proof. The proof follows that of [16, Proposition 1.1] in computing the symbol of A recursively. Their construction only relies on ellipticity of H , and the fact that the only derivatives in t are in Δ_g . \square

Remark 2.2. In Subsection 2.2, we make this recursive computation of the symbol explicit for the disk, as we need to obtain concrete values of the coefficients in that case. The reader interested in a more detailed proof of Proposition 2.1 can see that this recursive computation also works for a general Ω .

Proposition 2.1 admits the same corollary as in [16].

Corollary 2.3. *Let $r(x, \xi)$ be the symbol of $\frac{1}{\rho} \text{DN}_\lambda(\Omega; \tau)$ and $a(x, t, \xi)$ be the symbol of A . Then,*

$$r(x, \xi) = -\frac{a(x, 0, \xi)}{\rho(x)}.$$

In other words,

$$\frac{1}{\rho} \text{DN}_\lambda(\Omega; \tau) \equiv \frac{-1}{\rho} A \Big|_\Sigma \pmod{\Psi^{-\infty}}.$$

In particular, the symbol of $\frac{1}{\rho} \text{DN}_\lambda(\Omega; \tau)$ depends only on λ , ρ and the boundary values of g , τ and of their derivatives.

We denote by $\sigma_j(\Omega; \tau; \rho; \lambda)$ the j^{th} eigenvalue of $\frac{1}{\rho} \text{DN}_\lambda(\Omega, \tau)$.

Lemma 2.4. *Let Ω_1, Ω_2 be compact Riemannian surfaces with smooth boundary Σ_1, Σ_2 . Suppose there exists an isometry φ between collar neighbourhoods Υ_1 of Σ_1 and Υ_2 of Σ_2 . Let $\tau \in C^\infty(\Omega_2)$ and $\rho \in C^\infty(\Sigma_2)$. Then,*

$$\sigma_j(\Omega_1; \varphi^* \tau; \varphi^* \rho; \lambda) \sim \sigma_j(\Omega_2; \tau; \rho; \lambda)$$

where φ^* denotes the pullback by φ .

Proof. By Corollary 2.3, the operators $\frac{1}{\varphi^* \rho} \text{DN}_\lambda(\Omega_1; \varphi^* \tau)$ and $\frac{1}{\rho} \text{DN}_\lambda(\Omega_2; \tau)$ have the same symbol, or in other words they are equivalent up to smoothing operators:

$$\frac{1}{\varphi^* \rho} \text{DN}_\lambda(\Omega_1; \varphi^* \tau) \equiv \frac{1}{\rho} \text{DN}_\lambda(\Omega_2; \tau) \pmod{\Psi^{-\infty}}.$$

In [8, Lemma 2.1], it is shown that whenever two elliptic selfadjoint pseudodifferential operators are equivalent up to smoothing operators, then their eigenvalues are asymptotically equivalent. \square

Lemma 2.5. *Let Ω be a compact simply connected surface with smooth boundary Σ . Let $\varphi: \bar{\mathbb{D}} \rightarrow \Omega$ be conformal. Then, the Steklov problem (8) on Ω is isospectral to the problem*

$$\begin{cases} -\Delta u = \lambda e^{2f} (\varphi^* \tau) u & \text{in } \mathbb{D}, \\ \partial_{\nu} u = \sigma e^f (\varphi^* \rho) u & \text{on } \mathbb{S}^1, \end{cases}$$

where $f: \bar{\mathbb{D}} \rightarrow \mathbb{R}$ is such that $\varphi^* g = e^{2f} g_0$.

Proof. It follows directly from the observation, see [13], that the Laplacian and normal derivatives transform under a conformal mapping $\varphi: (\bar{\mathbb{D}}, g_0) \rightarrow (\Omega; g)$ as

$$\Delta_{g_0}(\varphi^* u) = e^{2f} \varphi^*(\Delta_g u)$$

and

$$\partial_{\nu, g_0}(\varphi^* u) = e^f \varphi^*(\partial_{\nu, g} u)$$

respectively. □

Lemma 2.6. *Let (Ω, g) be a compact Riemannian surface whose smooth boundary Σ has ℓ connected components $\Sigma_1, \dots, \Sigma_\ell$. For every $1 \leq m \leq \ell$, there exist a metric g_m on the unit disk, a collar neighbourhood Υ_m of Σ_m , and a collar neighbourhood $\tilde{\Upsilon}_m$ of \mathbb{S}^1 such that $g_m|_{\tilde{\Upsilon}_m}$ is isometric to $g|_{\Upsilon_m}$.*

Proof. Since Σ_m is a closed curve, we can assume without loss of generality that it is parameterised as $f: [0, 2\pi] \rightarrow \Sigma_m$. Let $0 < \varepsilon < \min\{\text{Roll}(\Omega), 1/2\}$ and $\Upsilon_m(\varepsilon)$ be an ε -collar neighbourhood of Σ_m . Let $\tilde{\Upsilon}_m(\varepsilon) \subset \mathbb{D}$ be defined as

$$\tilde{\Upsilon}_m(\varepsilon) := \{(r, \theta) \in \mathbb{D}: r > 1 - \varepsilon\}.$$

Endow Υ_m with Fermi coordinates and define $\varphi_m: \tilde{\Upsilon}_m(\varepsilon) \rightarrow \Upsilon_m(\varepsilon)$ by

$$\varphi(1 - t, \theta) = (f(\theta), t),$$

and let $\tilde{g}_m = \varphi_m^* g$ so that \tilde{g}_m is isometric to g . Let

$$U := \left\{ (r, \theta) \in \mathbb{D}: r < 1 - \frac{\varepsilon}{2} \right\}$$

and let $\{\psi_\Upsilon, \psi_U\}$ be a partition of unity subordinated to $(\Upsilon_m(\varepsilon), U)$. Define the metric g on \mathbb{D} as

$$g_m = \psi_\Upsilon \tilde{g}_m + \psi_U g_0,$$

where g_0 is the flat metric. Then, g_m is the desired metric with $\Upsilon_m = \Upsilon_m(\varepsilon/2)$ and $\tilde{\Upsilon}_m = \tilde{\Upsilon}_m(\varepsilon/2)$. □

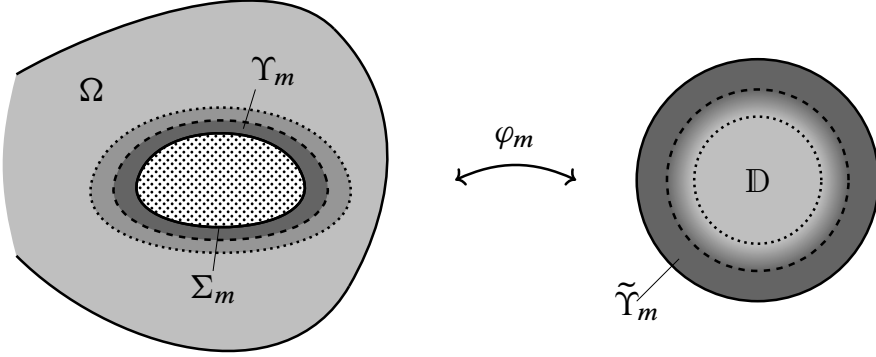


Figure 2. Isometric collar neighbourhoods of Σ_m and S^1 . In the outer, darker region $\tilde{\Upsilon}_m$ of \mathbb{D} , the metric is isometric to the metric on Υ_m . In the inner, lighter region, it is the Euclidean metric. In the intermediate region, it is a convex combination of both the pullback of a metric in the intermediate region on Ω , and the Euclidean metric.

This leads us to the main theorem of this subsection, reducing the problem to the one on the unit disk.

Theorem 2.7. *Let (Ω, g) be a compact Riemannian surface whose smooth boundary Σ has ℓ connected components $\Sigma_1, \dots, \Sigma_\ell$, and let Ω' be the disjoint union of ℓ identical unit disks $\mathbb{D}_1, \dots, \mathbb{D}_\ell$ with boundary $\Sigma' = S^1_1 \sqcup \dots \sqcup S^1_\ell$. There exist*

$$\tau_0: \Omega' \longrightarrow \mathbb{C} \quad \text{and} \quad \rho_0: \Sigma' \longrightarrow \mathbb{C}$$

such that

$$\sigma_j(\Omega; \tau; \rho; \lambda) \sim \sigma_j(\Omega'; \tau_0; \rho_0; \lambda).$$

Proof. The proof follows that of [8, Theorem 1.4]. For $1 \leq m \leq \ell$, let Ω_m be a topological disk with a Riemannian metric that is isometric to a collar neighbourhood Υ_m of Σ_m , and denote by $\Omega_\#$ the union of the disks Ω_m . We abuse notation and denote also by τ any smooth function on $\Omega_\#$ whose value on Υ_m coincides with τ on Ω . This is justified since only its value in a neighbourhood of the boundary affects eigenvalue asymptotics. It follows from Lemma 2.4 that

$$|\sigma_j(\Omega; \tau; \rho; \lambda) - \sigma_j(\Omega_\#; \tau; \rho; \lambda)| = O(j^{-\infty}).$$

For every m , the Riemann mapping theorem implies the existence of a conformal diffeomorphism $\varphi_m: (\overline{\mathbb{D}_m}, g_0) \rightarrow (\Omega_m, g_m)$. Given that $\varphi_m^* g_m = e^{2f_m} g_0$, define

τ_0 and ρ_0 , for $x \in \mathbb{D}_m$ and \mathbb{S}_m^1 respectively, as

$$\begin{cases} \tau_0(x) := e^{2f_m} \tau(\varphi_m(x)), \\ \rho_0(x) := e^{f_m} \rho(\varphi_m(x)). \end{cases}$$

It follows from Lemma 2.5 that $\frac{1}{\rho} \text{DN}_\lambda(\Omega_m; \tau)$ is isospectral to $\frac{1}{\rho_0} \text{DN}_\lambda(\overline{\mathbb{D}_m}; \tau_0)$. The conclusion then follows from the fact that the spectrum of the Dirichlet-to-Neumann map defined on a disjoint union of domains is the union of their respective spectra. \square

2.2. The symbol of the Dirichlet-to-Neumann map on the disk. We now compute the full symbol of $\Lambda := \frac{1}{\rho} \text{DN}_\lambda(\mathbb{D}; \tau)$ on $\mathbb{S}^1 = \partial\mathbb{D}$ from the factorisation obtained in Proposition 2.1. Let us introduce boundary normal coordinates (x, t) for the collar neighbourhood $\mathbb{S}^1 \times [0, \delta)$, for some small but fixed δ . The flat metric in these coordinates reads

$$g(x, t) = (1-t)^2(dx)^2 + (dt)^2,$$

and the Laplacian reads as

$$-\Delta = D_t^2 + \frac{i}{1-t} D_t + \frac{1}{(1-t)^2} D_x^2.$$

We are therefore looking for a factorisation of the form

$$-\Delta_g - \lambda \tau(x) \equiv (D_t + iE(t) - iA(x, t, D_x))(D_t + iA(x, t, D_x)) \pmod{\Psi^{-\infty}},$$

where $E(t) = (1-t)^{-1}$.

Rearranging, this implies finding A such that

$$A^2(x, t, D_x) - \frac{1}{(1-t)^2} D_x^2 + i[D_t, A] - E(t)A(x, t, D_x) + \lambda \tau(x) \equiv 0 \pmod{\Psi^{-\infty}},$$

which at the level of symbols is tantamount to finding $a(x, t, \xi)$ such that

$$\sum_{K \geq 0} \frac{1}{K!} (\partial_\xi^K a)(D_x^K a) - \frac{\xi^2}{(1-t)^2} + \partial_t a - \frac{a}{1-t} + \lambda \tau = 0,$$

where

$$a(x, t, \xi) \sim \sum_{m \leq 1} a_m(x, t, \xi)$$

is the symbol of A and the coefficients a_m are positively homogeneous of degree m in ξ .

By gathering the terms of degree two, we obtain

$$a_1 = -\frac{|\xi|}{1-t},$$

while gathering the terms of degree one yields

$$a_0(x, t, \xi) = \frac{-1}{2a_1} \left(\partial_t a_1 - \frac{a_1}{1-t} \right) = 0.$$

One can observe that neither a_1 nor a_0 depend on $\lambda\tau$. However, by gathering the terms of order 0, we get

$$a_{-1}(x, t, \xi) = \frac{-\lambda\tau}{2a_1} = \frac{\lambda(1-t)\tau}{2|\xi|}.$$

For $m \leq -1$, a_{m-1} is found recursively by gathering the terms of order m and is given by

$$a_{m-1}(x, t, \xi) = -\frac{1}{2a_1} \left(\sum_{\substack{j,k \\ m \leq j, k \leq 1 \\ \gamma = j+k-m}} \frac{1}{\gamma!} D_\xi^\gamma(a_j) \partial_x^\gamma(a_k) + \partial_t a_m - \frac{a_m}{1-t} \right). \quad (10)$$

Note that this is the same recurrence relation as the one appearing in [16] as soon as $m < -1$. For the sequel, we will require explicit knowledge of the term of order -2 . From the previous equation we deduce that

$$a_{-2}(x, t, \xi) = \frac{(1-t)\lambda}{4|\xi|^2} (i\tau_x \operatorname{sgn}(\xi) - 2\tau + (1-t)\tau_t).$$

As indicated by Corollary 2.3, the symbol of Λ is given by

$$r(x, \xi) = -\rho(x)^{-1} a(x, 0, \xi)$$

where the sign is chosen so that Λ is a positive operator. Note that ∂_t is the interior normal derivative hence $\partial_t = -\partial_\nu$. Writing $f(x) := f(x, 0)$ for the restriction of any function to the boundary, the first few terms of the symbol of Λ read as

$$r(x, \xi) = \frac{|\xi|}{\rho(x)} - \frac{\lambda\tau(x)}{2\rho(x)|\xi|} + r_{-2}(x, \xi) + O(|\xi|^{-3}),$$

with

$$r_{-2}(x, \xi) = \frac{-\lambda}{4\rho(x)|\xi|^2} (i\tau_x(x) \operatorname{sgn}(\xi) - 2\tau(x) - \partial_\nu \tau(x)).$$

2.3. Symmetries of the symbol. When λ and τ are real, we see from these first expressions, that the real part of the symbol is an even function of ξ , while its imaginary part is an odd function of ξ . This is equivalent to the following definition.

Definition 2.8. A symbol $a(x, \xi)$ is *Hermitian* if $a(x, -\xi) = \overline{a(x, \xi)}$ for all $x, \xi \in \mathbb{R}$.

We now show recursively that the symbol of Λ is Hermitian.

Proposition 2.9. For $\lambda \in \mathbb{R}$, $\tau \in C^\infty(\mathbb{D}; \mathbb{R})$, the symbol r_m is Hermitian for all $m \leq 1$.

The proposition follows from (10) and the following lemma whose proof is straightforward.

Lemma 2.10. Let a and b be two Hermitian symbols corresponding to operators A and B . Then,

- (1) $\partial_x a$ and $D_\xi a$ are Hermitian;
- (2) $a + b$ and ab are Hermitian;
- (3) the symbol of AB is Hermitian.

Proof. The first two claims are a trivial computation. The third claim follows from the fact that the symbol of AB is obtained from a and b using the operations described by the first two claims. \square

3. Transformation of the symbol

In this section, we follow and make explicit the strategy laid out in [19], [1, Section 2], and [7] in the specific case of the parametric Dirichlet-to-Neumann map.

Specifically, we want to find a sequence $P_N \in \Psi^1$ such that

- $\Lambda U_N = U_N P_N \pmod{\Psi^{1-N}}$ for a bounded operator U_N ;
- the symbol of P_N depends only on the cotangent variable ξ up to order $1 - N$.

Such a procedure (making the symbol dependent solely on ξ) will be referred to as a *diagonalisation* of the symbol. It is motivated by the following proposition resulting from [19, Theorem 9].

Proposition 3.1. *Let A be an elliptic, self-adjoint pseudodifferential operator of order 1 and let P be the operator with symbol*

$$p(x, \xi) = \sum_{m=0}^N p_{1-m}(\xi),$$

where p_{1-m} depends only on ξ and is positively homogeneous of order $1-m$. Suppose that $AU - UP \in \Psi^{-N}$ for some bounded operator U . Then, the eigenvalues of A are given by the union of two sequences, $\{\sigma_j^+\}$ and $\{\sigma_j^-\}$ that satisfy

$$\sigma_j^\pm = \sum_{m=0}^N p_{1-m}(\pm j) + O(j^{-N}).$$

3.1. Diagonalisation of the principal symbol. We start by diagonalising the principal symbol of $\Lambda = \frac{1}{\rho} \text{DN}_\lambda(\mathbb{D}; \tau)$. Let

$$L = \frac{1}{2\pi} \int_0^{2\pi} \rho(x) dx \tag{11}$$

and

$$S(x, \eta) = \frac{\eta}{L} \int_0^x \rho(t) dt.$$

The function S is a generating function for the canonical transformation $(y, \xi) = T(x, \eta)$ given by the relations

$$\xi = \frac{\partial S}{\partial x}, \quad y = \frac{\partial S}{\partial \eta}.$$

We define the Fourier integral operator Φ with phase function S as

$$\Phi u(x) = \int_{\mathbb{R}} e^{iS(x, \xi)} \hat{u}(\xi) d\xi,$$

where \hat{u} is the Fourier transform of u . We use Φ to diagonalise the principal symbol of Λ in the following proposition.

Proposition 3.2. *For any N , there is an operator $B_N \in \Psi^1$ such that its principal symbol depends only on ξ and such that*

$$\Lambda \Phi - \Phi B_N \in \Psi^{1-N}.$$

Proof. We are looking for the symbol of B in the form

$$b(x, \xi) = b_1(\xi) + \sum_{m \leq 0} b_m(x, \xi),$$

with $b_j(x, \xi)$ positively homogeneous of order j in ξ . Let us first study the operator $\Lambda\Phi$. It acts on smooth functions as

$$\begin{aligned} \Lambda\Phi u(x) &= \frac{1}{2\pi} \iiint r(x, \eta) e^{i(x-y)\eta} e^{iS(y, \xi)} \hat{u}(\xi) \, d\eta \, dy \, d\xi \\ &= \frac{1}{2\pi} \int k(x, \xi) e^{iS(x, \xi)} \hat{u}(\xi) \, d\xi, \end{aligned}$$

where

$$k(x, \xi) = \iint r(x, \eta) e^{i(x-y)\eta} e^{i(S(y, \xi) - S(x, \xi))} \, dy \, d\eta.$$

We now look for the asymptotic expansion of k as a symbol on S^1 , up to symbols of order $-\infty$. Note that the expressions here have sense in terms of distributions, see [7, Section 2.2.2]. By following the method of proof in [7, Theorem 6.5], we can localise the symbol by finding smooth cut-off functions $h_1(x, y)$ and $h_2(\xi, \eta)$ supported in suitable neighbourhoods of $x = y$ and $\xi = \eta$ such that if

$$k'(x, \xi) = \iint r(x, \eta) e^{i(x-y)\eta} e^{i(S(y, \xi) - S(x, \eta))} h_1(x, y) h_2(\xi, \eta) \, dy \, d\eta,$$

then $\text{Op}(k - k') \in \Psi^{-\infty}$. By Taylor's theorem, we can write

$$S(y, \xi) - S(x, \xi) = \frac{\partial S(x, \xi)}{\partial x} (y - x) + R(x, y, \xi) (y - x)^2$$

with

$$R(x, y, \xi) = \int_0^1 (1-t) \frac{\partial^2 S(x + t(y-x), \xi)}{\partial x^2} \, dt. \quad (12)$$

We can rewrite k' as

$$k'(x, \xi) = \iint r(x, \eta) e^{i(x-y)(\eta - R(x, y, \xi)(y-x) - \frac{\partial S}{\partial x})} h_1(x, y) h_2(\xi, \eta) \, dy \, d\eta.$$

Changing variables as $\tilde{\eta} = \eta - R(x, y, \xi)(y-x)$ and $\tilde{\xi} = \frac{\partial S(x, \xi)}{\partial x} = \frac{\xi \rho}{L}$, we obtain that k' is of the form

$$k'(x, \xi) = \iint K(x, y, \tilde{\xi}, \tilde{\eta}) e^{i(x-y)(\tilde{\eta} - \tilde{\xi})} \, dy \, d\tilde{\eta}$$

where

$$K(x, y, \tilde{\xi}, \tilde{\eta}) = r(x, \tilde{\eta} + R(x, y, \xi)(y-x))h_1(x, y)h_2(\xi, \tilde{\eta} + R(x, y, \xi)(y-x)).$$

From [7, Lemma 2.13], we know that $k'(x, \xi)$ is a symbol given by

$$k'(x, \xi) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_{\tilde{\eta}}^{\alpha} D_y^{\alpha} K(x, y, \tilde{\xi}, \tilde{\eta}) \Big|_{\substack{\tilde{\eta}=\tilde{\xi} \\ y=x}}.$$

By the choice of cut-off functions, when x is close to y and $\tilde{\eta}$ is close to $\tilde{\xi}$, we have that h_1 and h_2 are constant and equal to one. Hence, they don't intervene in the symbol's calculation and

$$k'(x, \xi) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_{\tilde{\eta}}^{\alpha} D_y^{\alpha} r(x, \tilde{\eta} + R(x, y, \xi)(y-x)) \Big|_{\substack{\tilde{\eta}=\tilde{\xi} \\ y=x}}.$$

We now make the following observation: if $r(x, \tilde{\eta} + R(x, y, \xi)(y-x))$ is a symbol of order m , then applying $\partial_{\tilde{\eta}}^{\alpha} D_y^{\alpha}$ results in a symbol of order $m - \alpha$. In fact, for $\alpha = 1$, and denoting by ∂_2 the derivative with respect to the second argument, we have

$$\begin{aligned} \partial_{\tilde{\eta}} D_y r(x, \tilde{\eta} + R(x, y, \xi)(y-x)) \Big|_{\substack{\tilde{\eta}=\tilde{\xi} \\ y=x}} &= -i[\partial_2^2 r(x, \tilde{\xi})]R(x, x, \xi) \\ &= -i \left[\partial_2^2 r \left(x, \frac{\rho(x)\xi}{L} \right) \right] \frac{\xi \rho'(x)}{2L}. \end{aligned}$$

It is clear from this last equation that it is a symbol of order $m - 1$. Induction on α is then straightforward. This yields the asymptotic symbolic expansion $k'(x, \xi) = \sum_{m \leq 1} \tilde{a}_m(x, \xi)$, where

$$\tilde{a}_m(x, \xi) = \sum_{0 \leq \alpha \leq 1-m} \frac{1}{\alpha!} \partial_{\tilde{\eta}}^{\alpha} D_y^{\alpha} r_{m+\alpha}(x, \tilde{\eta} + R(x, y, \xi)(y-x)) \Big|_{\substack{\tilde{\eta}=\frac{\xi \rho(x)}{L} \\ y=x}}. \quad (13)$$

We can compute the first few terms of the symbolic expansion, using the fact that in $\mathbb{R} \setminus \{0\}$ the second derivative of a_1 in the second variable vanishes identically. This gives

$$\begin{aligned} \tilde{a}_1(x, \xi) &= \frac{|\xi|}{L}; \\ \tilde{a}_0(x, \xi) &= 0; \\ \tilde{a}_{-1}(x, \xi) &= -\frac{\lambda L \tau}{2\rho^2 |\xi|}; \\ \tilde{a}_{-2}(x, \xi) &= \frac{\lambda L^2}{4\xi^2 \rho^3} (\tau_r - i \operatorname{sgn}(\xi) \tau_x + 2\tau) + \frac{i \lambda L^2 \tau \operatorname{sgn}(\xi) \rho'}{2\xi^2 \rho^4}. \end{aligned}$$

Let us now compute the symbol of ΦB . We have

$$\begin{aligned}\Phi B u(x) &= \frac{1}{2\pi} \iiint e^{iy(\xi-\eta)} e^{iS(x,\eta)} b(y, \xi) \hat{u}(\xi) \, dy \, d\eta \, d\xi \\ &= \frac{1}{2\pi} \int f(x, \xi) e^{iS(x,\xi)} \hat{u}(\xi) \, d\xi,\end{aligned}$$

where

$$f(x, \xi) = \iint b(y, \xi) e^{i(S(x,\eta)-S(x,\xi))} e^{iy(\xi-\eta)} \, dy \, d\eta.$$

As above, this integral only converges in the sense of distributions. As in [7], we can find a smooth cut-off function $h(\xi, \eta)$ supported in a neighbourhood of $\xi = \eta$ such that the symbol

$$f'(x, \xi) = \iint b(y, \xi) e^{i(S(x,\eta)-S(x,\xi))} e^{iy(\xi-\eta)} h(\xi, \eta) \, dy \, d\eta$$

satisfies $\text{Op}(f - f') \in \Psi^{-\infty}$.

Let us observe that

$$S(x, \eta) - S(x, \xi) = \frac{(\eta - \xi)}{L} \int_0^x \rho(x) \, dx = (\eta - \xi) F(x)$$

and that $F(x) = \frac{\partial S}{\partial \xi}(x, \xi)$. After the change of variable $\tilde{y} = y + x - F(x)$, the equation for f' becomes

$$\begin{aligned}f'(x, \xi) &= \iint b(\tilde{y} - x + F(x), \xi) h(\xi, \eta) e^{i(x-\tilde{y})(\eta-\xi)} \, d\tilde{y} \, d\eta \\ &= \iint Q(x, \tilde{y}, \eta, \xi) e^{i(x-\tilde{y})(\eta-\xi)} \, d\tilde{y} \, d\eta.\end{aligned}$$

Once again from [7, Lemma 2.13], we have that f' is a symbol in S^1 and

$$f'(x, \xi) = \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_\eta^\alpha D_{\tilde{y}}^\alpha Q(x, \tilde{y}, \xi, \eta) \Big|_{\substack{\tilde{y}=x \\ \eta=\xi}}.$$

Since Q is constant in η close to ξ , the derivatives in η always vanish. Hence, the symbol of B_N is given by

$$f'_N(x, \xi) = \sum_{-N \leq m \leq 1} b_m \left(\frac{1}{L} \int_0^x \rho(t) \, dt, \xi \right).$$

To have the terms of the same order of homogeneity cancel out, we need to choose

$$b_m(x, \xi) = \tilde{a}_m(s(x), \xi), \quad (14)$$

where $s(x)$ is the number s such that

$$x = \frac{1}{L} \int_0^s \rho(t) dt.$$

This concludes the proof. \square

3.2. Diagonalisation of the full symbol. Let us denote by P_1 the operator with symbol

$$p^{(1)}(x, \xi) = b_1(\xi) + \sum_{m \leq -1} b_m(x, \xi). \quad (15)$$

The diagonalisation of the full symbol is based on the following lemma inspired by the methods laid out by Rozenblum [19] and Agranovich [1]. We include it for completeness.

Lemma 3.3. *Let $N \geq 0$ and suppose that there exists a bounded operator U_N such that $\Lambda U_N - U_N P_N \in \Psi^{-\infty}$ where P_N is a pseudodifferential operator whose symbol is given by*

$$p^{(N)}(x, \xi) = \sum_{m=0}^N p_{1-m}^{(N)}(\xi) + p_{-N}^{(N)}(x, \xi) + O(|\xi|^{-(N+1)}).$$

Then if

$$p_{-N}^{(N+1)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} p_{-N}^{(N)}(x, \xi) dx \quad (16)$$

and K is the pseudodifferential operator with symbol

$$k(x, \xi) = 1 - iL \operatorname{sgn} \xi \int_0^x p_{-N}^{(N)}(t, \xi) - p_{-N}^{(N+1)}(\xi) dt, \quad (17)$$

there exists an operator P_{N+1} with symbol

$$p^{(N+1)}(x, \xi) = \sum_{m=0}^N p_{1-m}^{(N)}(\xi) + p_{-N}^{(N+1)}(\xi) + O(|\xi|^{-(N+1)})$$

satisfying $\Lambda(U_N K) - (U_N K)P_{N+1} \in \Psi^{-\infty}$.

Proof. Starting off with the pseudodifferential operator P_N , we would like to find a bounded operator K and a pseudodifferential operator P_{N+1} whose symbol $p^{(N+1)}$ satisfies

$$p^{(N+1)}(x, \xi) = \sum_{m=0}^N p_{1-m}^{(N)}(\xi) + p_{-N}^{(N+1)}(\xi) + O(|\xi|^{-(N+1)})$$

such that $P_N K - K P_{N+1} \in \Psi^{-\infty}$. We choose K to have symbol $1 + k_{-N}(x, \xi)$ with k_{-N} positively homogeneous of order $-N$ in ξ . The symbol of $P_N K - K P_{N+1}$ is then given by

$$p_{-N}^{(N)}(x, \xi) - p_{-N}^{(N+1)}(\xi) - i(\partial_{\xi} p_1^{(N)})(\partial_x k_{-N}) + O(|\xi|^{-N-1}).$$

The symbol $p_1^{(N)}$ comes from the diagonalisation of the principal symbol and is given by $p_1^{(N)}(\xi) = p_1^{(1)}(\xi) = b_1(\xi) = \frac{|\xi|}{L}$. Hence, we see that the terms of order $-N$ cancel if the symbol of K is given by (17) and since $0 = k_{-N}(0, \xi) = k_{-N}(2\pi, \xi)$, we must take $p_{-N}^{(N+1)}$ as in (16). In order to get that $P_N K - K P_{N+1} \in \Psi^{-\infty}$ knowing that the symbol of P is given by

$$p^{(N)}(x, \xi) = \sum_{m=0}^N p_{1-m}^{(N)}(\xi) + \sum_{m \geq N+1} p_{1-m}^{(N)}(x, \xi),$$

we need to take P_{N+1} with symbol

$$p^{(N+1)}(x, \xi) = \sum_{m=0}^N p_{1-m}^{(N+1)}(\xi) + p_{-N}^{(N+1)}(\xi) + \sum_{m \geq N+2} p_{1-m}^{(N+1)}(x, \xi),$$

which is calculated inductively as

$$p_m^{(N+1)} = p_m^{(N)} + \sum_{\alpha=0}^{1-m-N} \frac{1}{\alpha!} [(\partial_x^{\alpha} k_{-N})(D_{\xi}^{\alpha} p_{m+\alpha+N}^{(N)}) - (\partial_x^{\alpha} p_{m+\alpha+N}^{(N+1)})(D_{\xi}^{\alpha} k_{-N})] \quad (18)$$

for $m \leq -N - 1$. It follows that $\Lambda(U_N K) - (U_N K)P_{N+1}$ is smoothing. \square

The previous lemma gives us a family of operators P_N that diagonalise Λ down to any desired order. By applying it $N - 1$ times starting from P_1 , we get that there exists P_N with symbol

$$p^{(N)}(x, \xi) = \frac{|\xi|}{L} + \sum_{m=1}^{N-1} \frac{1}{2\pi} \int_0^{2\pi} p_{-m}^{(m)}(x, \xi) dx + O(|\xi|^{-N})$$

such that $\Lambda U_N - U_N P_N$ is smoothing for some bounded operator U_N . We summarise the properties of the operators P_N that were proved along the discussion above in the following proposition.

Proposition 3.4. *The symbols $p^{(N)}$ of P_N possess the following properties.*

- (1) *The first symbol $p^{(1)} = b_1(\xi) + \sum_{m \leq -1} b_m(x, \xi)$, see (15).*
- (2) *For $m \geq 1 - N$, $p_m^{(N+1)} = p_m^{(N)}$ and $\partial_x p_m^{(N)} = 0$. In other words, for every $m \leq -1$ the sequence stabilises and eventually becomes diagonal with respect to ξ .*
- (3) *For $m \leq -N - 1$, $p_m^{(N+1)}$ is given recursively by equation (18).*
- (4) *When the sequence stabilises, the diagonalised symbol can be explicitly computed as $p_{-N}^{(N+1)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} p_{-N}^{(N)}(x, \xi) dx$.*

One can see that $p_m^{(N)}$ is a polynomial in λ with coefficients that are functions of x and ξ . From this point of view, we observe the following.

Lemma 3.5. *For each $m \leq -1$ and for each $N \geq 1$, the function $p_m^{(N)}$ is a polynomial in λ of degree at most $-m$ whose constant coefficient vanishes.*

Proof. We denote by $\deg(p)$ the degree of a function $p(x, \xi)$ as a polynomial in λ . We proceed by induction on both N and m .

It is easily seen from (10) and the expressions for a_1 and a_{-1} that the functions a_m (and hence r_m) are polynomials of order $\lceil \frac{-m}{2} \rceil \leq -m$ whenever $m \leq -1$. It then follows from equations (13) and (14) that $\deg(p_m^{(1)}) = \deg(b_m) \leq -m$ for all $m \leq -1$.

Let $N \geq 1$ be arbitrary and suppose that $\deg(p_m^{(N')}) \leq -m$ for all $1 \leq N' \leq N$ and $m \leq -1$. From Proposition 3.4, we know that

$$\deg(p_{-1}^{(N+1)}) = \deg(p_{-1}^{(1)}) = 1.$$

Let $m_0 \leq -1$ and suppose that $\deg(p_m^{(N+1)}) \leq -m$ for all $-1 \geq m \geq m_0$. We want to estimate the degree of $p_{m_0-1}^{(N+1)}$. Its expression is given by (18) and we can see that the term of highest degree in λ in the sum is obtained when $\alpha = 0$. Hence,

$$\deg(p_{m_0-1}^{(N+1)}) \leq \deg(k_{-N}) + \deg(p_{m_0-1+N}^{(N+1)}) \quad (19)$$

From the definition of k_{-N} , we have

$$\deg(k_{-N}) = \deg(p_{-N}^{(N)}) \leq N \quad (20)$$

by the induction hypothesis. Since $m_0 - 1 + N \geq m_0$, the induction hypothesis yields

$$\deg(p_{m_0-1+N}^{(N+1)}) \leq -m_0 + 1 - N. \quad (21)$$

Therefore, by combining (19), (20), and (21), $\deg(p_{m_0-1}^{(N+1)}) \leq -m_0 + 1$ and the claim follows by induction.

Finally, to show that the constant coefficient of $p_m^{(N)}$ vanishes, it suffices to show that it is the case for a_m . Proceeding inductively, since $a_0 = 0$, notice from (10) that the only term in a_{m-1} that could be constant in λ is $\frac{1}{\gamma!} D_\xi^\gamma(a_1) \partial_x^\gamma(a_1)$ with $\gamma = 2 - m$. However, $D_\xi^\gamma(a_1) = 0$ for $\gamma \geq 2$. \square

Remark 3.6. That $p_m^{(N)} = 0$ whenever $\lambda = 0$ is not surprising. Indeed, this corresponds to the classic Dirichlet-to-Neumann operator whose symbol is precisely $|\xi|$.

If one is interested in computing the symbols explicitly in a given example the calculations quickly become very involved. The following lemma allows us to reduce the number of computations to obtain the k -th term in the diagonalised symbol.

Lemma 3.7. For all $N \geq \lceil \frac{-m}{2} \rceil$,

$$\int_0^{2\pi} p_m^{(-m)} dx = \int_0^{2\pi} p_m^{(N)} dx.$$

Proof. If $m \geq 1 - 2N$, then $\partial_x^\alpha p_{m+\alpha+N}^{(N)} = \partial_x^\alpha p_{m+\alpha+N}^{(N+1)} = 0$ for all $\alpha > 0$. We also have $p_{m+N}^{(N)} = p_{m+N}^{(N+1)}$ and hence

$$p_m^{(N+1)} = p_m^{(N)} + \sum_{\alpha=1}^{1-m-N} \frac{1}{\alpha!} (\partial_x^\alpha k_{-N}) (D_\xi^\alpha p_{m+\alpha+N}^{(N)}).$$

Therefore, since $p_{m+\alpha+N}^{(N)}$ doesn't depend on x , integrating both sides yields

$$\int_0^{2\pi} p_m^{(N+1)} dx = \int_0^{2\pi} p_m^{(N)} dx + \sum_{\alpha=1}^{1-m-N} \frac{1}{\alpha!} (D_\xi^\alpha p_{m+\alpha+N}^{(N)}) \int_0^{2\pi} (\partial_x^\alpha k_{-N}) dx.$$

The rightmost integral vanishes for all α since k_{-N} is periodic and thus

$$\int_0^{2\pi} p_m^{(N+1)} dx = \int_0^{2\pi} p_m^{(N)} dx.$$

Finally, if $m = -2N$, we have

$$\int_0^{2\pi} p_{-2N}^{(N+1)} dx = \int_0^{2\pi} p_{-2N}^{(N)} dx + \int_0^{2\pi} k_{-N} (p_{-N}^{(N)} - p_{-N}^{(N+1)}) dx$$

and since $\partial_x k_{-N} = -iL \operatorname{sgn} \xi (p_{-N}^{(N)} - p_{-N}^{(N+1)})$ the rightmost integral vanishes. The result then follows since $m \geq -2N$ is equivalent to $N \geq \lceil \frac{-m}{2} \rceil$. \square

The previous lemma simplifies calculations. Indeed, in order to get the diagonalised term of order $-m$, it suffices to apply the diagonalisation lemma $\lceil \frac{m}{2} \rceil$ rather than m times. In particular, we get

$$\int_0^{2\pi} p_{-2}^{(2)} dx = \int_0^{2\pi} p_{-2}^{(1)} dx = \int_0^{2\pi} b_{-2}(x, \xi) dx = \int_0^{2\pi} \tilde{a}_{-2}(s(x), \xi) dx.$$

Using that $s'(x) = \frac{L}{\rho(s(x))}$, we get

$$\begin{aligned} \int_0^{2\pi} p_{-2}^{(2)} dx &= \frac{1}{L} \int_0^{2\pi} \rho(x) \tilde{a}_{-2}(x, \xi) dx \\ &= \frac{\lambda L}{4|\xi|^2} \int_0^{2\pi} \frac{\tau_r + 2\tau}{\rho^2} dx \end{aligned}$$

where the terms containing $i \operatorname{sgn} \xi$ vanish from the fact that

$$\int_0^{2\pi} \frac{\tau_x}{\rho^2} dx = 2 \int_0^{2\pi} \frac{\tau \rho'}{\rho^3} dx,$$

this equality being obtained by integrating by parts. Therefore, by doing a similar calculation for $\int_0^{2\pi} b_{-1}(x, \xi) dx$, we see that the symbol of P_2 is given by

$$p^{(2)}(x, \xi) = \frac{\xi}{L} - \frac{\lambda}{4\pi|\xi|} \int_0^{2\pi} \frac{\tau}{\rho} dx + \frac{\lambda L}{8\pi|\xi|^2} \int_0^{2\pi} \frac{\tau_r + 2\tau}{\rho^2} dx + O(|\xi|^{-3}). \quad (22)$$

4. General eigenvalue asymptotics from the symbol

4.1. Self-adjointness. For $\lambda \in \mathbb{R} \cap \mathcal{V}$ and τ real-valued, the operator $\Lambda := \frac{1}{\rho} \operatorname{DN}_\lambda(\mathbb{D}; \tau)$ is self-adjoint and therefore has real spectrum. This follows from the fact that $\operatorname{DN}_\lambda(\mathbb{D}; \tau)$ is self-adjoint and the following lemma applied to $P = \operatorname{DN}_\lambda(\mathbb{D}; \tau)$.

Lemma 4.1. *Let P be a self-adjoint pseudodifferential operator on $L^2(\mathbb{S}^1; dx)$ and $\rho > 0$ be a positive function on \mathbb{S}^1 and denote $M_{1/\rho}$ the operator of multiplication by ρ^{-1} . For $f \in \text{Diff}(\mathbb{S}^1)$, define by K_f the composition operator $K_f u = u \circ f$. Defining*

$$g(x) = \frac{1}{L} \int_0^x \rho(t) dt \in \text{Diff}(\mathbb{S}^1),$$

the operator

$$Q = K_g^{-1} M_{1/\rho} P K_g$$

is self-adjoint on $L^2(\mathbb{S}^1; dx)$.

Proof. The operator K_g is an invertible isometry

$$L^2(\mathbb{S}^1; dx) \longrightarrow L^2(\mathbb{S}^1; \rho(x)/L dx).$$

Indeed, for $u, v \in L^2(\mathbb{S}^1; dx)$, we have

$$\begin{aligned} (K_g u, K_g v)_{L^2(\rho(x)/L dx)} &= \int_0^{2\pi} u(g(x)) v(g(x)) g'(x) dx \\ &= \int_0^{2\pi} u(x) v(x) dx \\ &= (u, v)_{L^2(dx)}. \end{aligned}$$

The operator $M_{1/\rho} P$ is self adjoint on $L^2(\mathbb{S}^1; \rho(x)/L dx)$, hence we have

$$\begin{aligned} (u, Qv)_{L^2(dx)} &= (u, K_g^{-1} M_{1/\rho} P K_g v)_{L^2(dx)} \\ &= (K_g u, M_{1/\rho} P K_g v)_{L^2(\rho(x)/L dx)} \\ &= (M_{1/\rho} P K_g u, K_g v)_{L^2(\rho(x)/L dx)} \\ &= (K_g^{-1} M_{1/\rho} P K_g u, v)_{L^2(dx)} \\ &= (Qu, v)_{L^2(dx)}, \end{aligned}$$

proving that Q is self adjoint. \square

4.2. General eigenvalue asymptotics. We have shown how to diagonalise the symbol down to any order. We can now deduce the spectral asymptotics of Λ from Proposition 3.1. Eigenvalue asymptotics for an elliptic pseudodifferential operator on a circle are discussed also in [1, Theorem 3.1].

Theorem 4.2. *The eigenvalues of Λ are asymptotically double and admit a full asymptotic expansion given by*

$$\sigma_{2j} \sim \sigma_{2j-1} \sim \frac{j}{L} + \sum_{n=1}^{\infty} \frac{1}{2\pi j^n} \int_0^{2\pi} p_{-n}^{(n)}(x, 1) dx$$

for all $N \geq 0$. Truncating the series to its first two terms, this yields

$$\sigma_{2j} = \frac{j}{L} - \frac{\lambda}{4\pi j} \int_{\mathbb{S}^1} \frac{\tau}{\rho} dx + \frac{\lambda L}{8\pi j^2} \int_{\mathbb{S}^1} \frac{\tau_r + 2\tau}{\rho^2} dx + O(j^{-3}). \quad (23)$$

Proof. The fact that the eigenvalues admit a complete asymptotic expansion follows from Proposition 3.1 and Lemma 3.3. Moreover, (23) follows from equation (22) and Proposition 3.1. It remains to show that the eigenvalues are asymptotically double. This will follow from Proposition 3.1 if we can show that, for all $N \in \mathbb{N}$, there exist a bounded operator U_N and a pseudodifferential operator P_N with symbol

$$p(x, \xi) = \sum_{m=0}^N p_{1-m}(\xi) + O(|\xi|^{-N})$$

such that p_{1-m} is an even function of ξ (since then $p_{1-m}(j) = p_{1-m}(-j)$) and such that $\Lambda U_N - U_N P_N$ is smoothing. To do so, it is sufficient to show that a symbol being Hermitian is an invariant property of the diagonalisation procedure, see Definition (2.8). The claim will then follow since Λ is self-adjoint and hence all its eigenvalues must be real.

We know from Proposition 2.9 that the symbol of Λ is Hermitian. In order to diagonalise the principal symbol, we conjugated by the Fourier integral operator Φ . The resulting symbol is given by

$$b(x, \xi) \sim \sum_{m \leq 1} \tilde{a}_m(s(x), \xi),$$

where \tilde{a}_m is given by (13). It suffices to show that \tilde{a}_m is Hermitian for all m . This is a consequence of the fact that

$$D_{\tilde{\eta}}^{\alpha} \partial_y^{\alpha} r_{m+\alpha}(x, \tilde{\eta} + R(x, y, \xi)(y-x)) \Big|_{\substack{\tilde{\eta} = \frac{\xi \rho(x)}{L} \\ y=x}} \quad (24)$$

is Hermitian for all $\alpha \geq 0$. Indeed, by Leibniz's formula and (12) we have

$$\partial_y^{\beta} [R(x, y, \xi)(y-x)] \Big|_{y=x} = \frac{\xi}{(\beta+1)L} \rho^{(\beta)}(x)$$

for all $\beta \geq 0$. Hermiticity of (24) then follows from Faà di Bruno's formula since each derivative in the second argument will come with a power of ξ , thus preserving the parity in the real and imaginary parts.

Let $N \geq 0$ and suppose that $\Lambda U_N - U_N P_N \in \Psi^{-\infty}$ as in the notation of Lemma 3.3 is such that the symbol $p^{(N)}$ of P_N is Hermitian. From (17), (18), and Lemma 2.10, we see that the symbol $p^{(N+1)}$ of P_{N+1} is also Hermitian. The fact that the spectrum is asymptotically double then follows from the previous discussion. \square

5. Eigenvalue asymptotics

Let (Ω, g) be a simply connected compact Riemannian surface with smooth boundary Σ . We are now interested in finding the spectral asymptotic for the operator $\text{DN}_\lambda(\Omega; \tau)$ corresponding to the problem

$$\begin{cases} -\Delta_g u = \lambda \tau u & \text{in } \Omega, \\ \partial_\nu u = \sigma u & \text{on } \Sigma; \end{cases}$$

the parametric Steklov problem on Ω . By the Riemann mapping theorem, there exists a conformal diffeomorphism φ which maps (\mathbb{D}, g_0) onto Ω such that $\varphi^* g = e^{2f} g_0$ for some smooth function $f: \mathbb{D} \rightarrow \mathbb{R}$. Therefore, the parametric Steklov problem on (Ω, g) is isospectral to the problem

$$\begin{cases} -\Delta u = \lambda e^{2f} \varphi^* \tau u & \text{in } \mathbb{D}, \\ \partial_\nu u = \sigma e^f u & \text{on } \mathbb{S}^1. \end{cases}$$

In the notation of (11), we have

$$L = \frac{1}{2\pi} \int_0^{2\pi} e^f dx = \frac{\text{per}_g(\Sigma)}{2\pi}.$$

We are now in a position to prove our main results about eigenvalue asymptotics.

Proof of Theorem 1.2. The theorem follows directly from Theorem 4.2 for the existence of the complete asymptotic expansion. The fact that s_n is a polynomial in λ of degree at most n follows directly from Lemma 3.5. For the explicit values of s_{-1} and s_{-2} when $\tau \equiv 1$, we replace in (23) the values of τ and ρ by the conformal

factor. The second term in (23) is given by

$$\frac{\lambda}{4\pi j} \int_{\mathbb{S}^1} e^f dx = \frac{\lambda L}{2j}.$$

Finally, the third term is given by $\frac{\lambda L}{8\pi j^2}(G + 4\pi)$, where

$$G := \int_{\mathbb{S}^1} \frac{(e^{2f})_r}{e^{2f}} dx = \int_{\mathbb{S}^1} \partial_v \log e^{2f} dx = 2 \int_{\mathbb{S}^1} \partial_v f dx.$$

By Green's theorem, we have

$$G = 2 \int_{\mathbb{D}} \Delta f dA.$$

Recall that the Gaussian curvature of (\mathbb{D}, φ^*g) is given by

$$K_{\varphi^*g} = -e^{-2f} \Delta f.$$

Hence, since $\varphi^*K_g = K_{\varphi^*g}$ and $\varphi^*dA_g = e^{2f} dA$,

$$G = -2 \int_{\mathbb{D}} K_{\varphi^*g} e^{2f} dA = -2 \int_{\mathbb{D}} \varphi^*(K_g dA_g) = -2 \int_{\Omega} K_g dA_g.$$

Combining everything and using the Gauss–Bonnet theorem yields

$$\frac{\lambda L}{8\pi j^2}(G + 4\pi) = \frac{\lambda L}{4\pi j^2} \left(2\pi - \int_{\Omega} K_g dA_g \right) = \frac{\lambda L}{4\pi j^2} \int_{\Sigma} k_g ds$$

since Ω is simply connected, and hence its Euler characteristic is 1. \square

Proof of Theorem 1.3. Let (Ω, g) now be any Riemannian surface whose smooth boundary Σ has ℓ connected components $\Sigma_1, \dots, \Sigma_\ell$. For $1 \leq m \leq \ell$, let Ω_m be a smooth topological disk with a Riemannian metric g_m such that there is an isometry $\varphi_m: \tilde{\Upsilon}_m \rightarrow \Upsilon_m$, for collar neighbourhoods $\tilde{\Upsilon}_m$ of $\partial\Omega_m$ and Υ_m of Σ_m . The existence of (Ω_m, g_m) is guaranteed by Lemma 2.6. Define $\tau_m: \tilde{\Upsilon}_m \rightarrow \mathbb{R}$ by $\tau_m = \varphi_m^* \tau|_{\Upsilon_m}$. Extend τ_m to a smooth function on Ω_m , which we still denote τ_m . This can be done, say, with an harmonic extension and then smoothing it with a mollifier, maybe reducing a little bit the size of the collar neighbourhoods. Denote by $\Omega_{\#}$ the disjoint union of the disks Ω_m , $g_{\#}$ the metric which restricts to g_m on every Ω_m and $\tau_{\#} \in C^\infty(\Omega)$ to be the function which restricts to τ_m on every Ω_m .

From Lemma 2.4, we know that

$$\sigma_j(\lambda, \tau, \Omega) \sim \sigma_j(\lambda, \tau_{\#}, \Omega_{\#}).$$

This concludes the proof of statement (A), and also implies statement (B) since every other metric and function τ' satisfying the conclusion of statement (A) is isometric to $\tau_{\#}$ and $g_{\#}$ in a collar neighbourhood of $\Omega_{\#}$.

Since $\Omega_{\#}$ is a union of disks, its spectrum is given by the union of each disk's spectrum. Applying Theorem 1.2 to each Ω_m , and using that the parametric Steklov spectrum of a disjoint union of surfaces is the union of their spectra we see that the spectrum of Ω is the union of ℓ different sequences taking the form of equation (2). This is statement (C) of Theorem 1.3, which concludes its proof. \square

6. Geometric spectral invariants

In this section we obtain spectral invariants that have a geometric interpretation in the case where the potential is a constant, $\tau \equiv 1$. When the surface Ω is simply connected, the search for spectral invariants is easier. From the first two terms of the eigenvalue asymptotic expansion, we can deduce uniquely the values of both L and λ . Hence, from the third term, we can deduce uniquely the value of $\int_{\Sigma} k_g \, ds$ and it is a spectral invariant. From the Gauss–Bonnet theorem, we get the following result.

Corollary 6.1. *Let (Ω, g) be a simply connected compact Riemannian surface with smooth boundary Σ . Then, the total curvature*

$$\int_{\Omega} K_g \, dA_g$$

is a spectral invariant of the constant potential parametric Steklov problem on Ω . In particular, if the Gaussian curvature is assumed to be a constant $K(\Omega)$, the quantity

$$K(\Omega) \text{ area}(\Omega)$$

is a spectral invariant of the constant potential parametric Steklov problem on Ω .

In the multiply connected case, we need to introduce some definitions to talk about functions between two multisets. To determine the number of boundary components and the lengths of them, we will use methods from Diophantine

approximation. This is in the spirit of [8], where they obtained those quantities as invariants of the Steklov problem with $\lambda = 0$. There, they had an asymptotic expansion of the form (5)–(6), where all the coefficients s_n were 0. However in order to obtain the number of boundary components and their lengths as spectral invariants, they need only that the second term is $o(1)$, which we do have.

Recovering λ as well as the total geodesic curvature of the boundary is more complicated and requires an algorithmic procedure to recover subsequences (which can be explicitly constructed) once we know the number of boundary components and the length of the largest one. We start by introducing terminology found in [8, Section 2.3]

Definition 6.2. Let A, B be two multiset of positive real numbers. We say that $F: A \rightarrow B$ is *close* if it has the property that for every $\varepsilon > 0$, there are only finitely many $x \in A$ with $|F(x) - x| \geq \varepsilon$. We say that F is an *almost-bijection* if for all but finitely many $y \in B$, the pre-image $F^{-1}(y)$ consists in a single point.

For a finite set of positive real numbers $M = \{\alpha_1, \dots, \alpha_\ell\}$, we denote by $R(M)$ the multiset

$$R(M) := \{0, \dots, 0\} \cup \alpha_1 \mathbb{N} \cup \alpha_2 \mathbb{N} \cup \dots \cup \alpha_\ell \mathbb{N} \cup \alpha_\ell \mathbb{N},$$

where 0 is repeated ℓ times and the union is understood in the sense of multisets, i.e., multiplicity is conserved.

Proposition 6.3. Let $M = \{\alpha_1, \dots, \alpha_\ell\}$ be a finite multi-set of positive numbers. For $N \in \mathbb{N}$, let

$$\Xi^{(N)} = \{\{\xi_j^{(m,N)}: j \in \mathbb{N}\}: 1 \leq m \leq \ell\}$$

be a set of sequences given by $\xi_0^{(m,N)} = 0$ and such that for $j \geq 1$,

$$\xi_{2j}^{(m,N)} = \xi_{2j-1}^{(m,N)} + O(j^{-N-1}) = j\alpha_m + \sum_{n=1}^N s_n^{(m)} j^{-n} + O(j^{-N-1}). \quad (25)$$

Then, M and the quantities $s_n^{(m)}$ for $1 \leq n \leq N$ are uniquely determined by the sequence $S(\Xi^{(N)})$ defined as the reordering of the union of the sequences $\xi^{(m,N)}$ in increasing order.

Proof. Let us first describe heuristically how the proof goes. In the first step, we simply show that Lemmas 2.6 and 2.8 of [8] apply to this situation. This will allow us to recover M from $S(\Xi^{(N)})$, and we assume from then on that M , and therefore $R(M)$, are already known to be spectral invariants.

In the second step, we show that for any $\alpha_m \in M$ which is not an integer multiple of another strictly smaller element of M , we can identify a subsequence along which $S(\Xi^{(N)})_j = \xi_{k(j)}^{(m,N)}$ where $k: \mathbb{N} \rightarrow \mathbb{N}$ is a function that can be computed explicitly. For this, we use Dirichlet's simultaneous approximation theorem.

In the third step, we obtain the coefficients of those sequences α_m that we decoupled in the previous step. Obviously, if α_m appears only once in M this is trivial, the difficulty comes when α_m has multiplicity.

In the fourth step, we proceed inductively and show that if α_m is an integer multiple of some other $\alpha_n \in M$, but we already know the coefficients of the relevant sequences for α_n , then we can apply the same procedures as in steps 2 and 3 to recover the coefficients of $\xi^{(m,N)}$ for any N .

Step 1. We obtain M from $S(\Xi^{(N)})$. Combining Lemmas 2.6 and 2.8 of [8], as soon as A is a multiset such that there exists a close almost-bijection

$$F: R(M) \longrightarrow A,$$

then we can recover M from A . Let us describe how this is done.

The close almost-bijection gives us

$$\alpha_1 = \limsup_{j \rightarrow \infty} A_{j+1} - A_j,$$

this is the content of [8, Lemma 2.6]. We then write $A^{(1)} = A$, and $R_1 = R(M)$. Assuming that for $2 \leq m \leq \ell + 1$ we have found α_{m-1} , we write

$$R_m = R_{m-1} \setminus (\alpha_{m-1}\mathbb{N} \cup \alpha_{m-1}\mathbb{N}).$$

Assuming that there is a close almost-bijection $F_{m-1}: R_{m-1} \rightarrow A^{(m-1)}$, there is $K \in \mathbb{N}$ such that for all $k \geq N$, there are at least two elements of $A^{(m-1)}$ at distance less than $\alpha_1/10$, say, from $\alpha_{m-1}k$. Construct $A^{(m)}$ by removing from $A^{(m-1)}$ the two closest such elements (in case of ties choose the largest).

Lemma 2.8 of [8] states that the existence of a close almost-bijection

$$F_{m-1}: R_{m-1} \longrightarrow A^{(m-1)}$$

implies that there is a close almost-bijection

$$F_m: R_m \longrightarrow A^{(m)}.$$

Applying recursively [8, Lemma 2.6] gives us

$$\alpha_m = \limsup_{j \rightarrow \infty} A_{j+1}^{(m)} - A_j^{(m)},$$

which is the necessary ingredient for this recursion to continue. This is done until $A^{(\ell+1)}$ and $R_{\ell+1}$ are finite, at which point M is exhausted and we have recovered M from A .

Now, it is not hard to see that the map $F: R(M) \rightarrow S(\Xi^{(N)})$ that maps $R(M)_j$ to $S(\Xi^{(N)})_j$ is a close almost-bijection. Indeed, it follows from the definition of the sequences $\xi^{(m,N)}$ that

$$S(\Xi^{(N)})_j = R(M)_j + O(j^{-1})$$

which implies that F is a close almost-bijection. Our previous analysis tells us that we can recover M from $S(\Xi^{(N)})$, for any $N \geq 1$.

Step 2. Suppose without loss of generality that the smallest element of M is 1. Define on positive real numbers the strict partial order $x < y$ if there is an integer $n \geq 2$ such that $y = nx$, and denote by $x \leq y$ the non-strict version of this partial order, i.e., if $n = 1$ is allowed. For any multiset U of positive real numbers, we say that $x \in U$ is *minimal in U* if for all $y \in U$, either $x \leq y$, or x and y are incomparable. Let $I \subset \{1, \dots, \ell\}$ be defined as

$$I = \{1 \leq m \leq \ell: \alpha_m \text{ is minimal in } M\}.$$

We claim that there exist $\delta > 0$ and subsets $E_m \subset \mathbb{N}$ of infinite cardinality for each $m \in I$ such that for all $j \in E_m$,

$$[j\alpha_m - \delta, j\alpha_m + \delta] \cap R(M) = \underbrace{\{j\alpha_m, \dots, j\alpha_m\}}_{2\mu(m) \text{ times}}, \quad (26)$$

where $\mu(m)$ is the multiplicity of α_m in M .

Split M into $M_1 \cup M_2$, where $M_1 \subset \mathbb{Q}$ and $M_2 \subset \mathbb{R} \setminus \mathbb{Q}$. Let Q be the smallest common integer multiple of elements in M_1 . Dirichlet's simultaneous approximation theorem states that there is an infinite subset $E \subset \mathbb{N}$ such that for all $q \in E$ and $\alpha_m \in M_2$ there exists $p_{q,m} \in \mathbb{N}$ such that

$$\left| \frac{Q}{\alpha_m} - \frac{p_{q,m}}{q} \right| < \frac{1}{q^{1+1/\ell}}$$

or, equivalently,

$$|Qq - p_{q,m}\alpha_m| < \alpha_m q^{-1/\ell}.$$

This means that for all $q \in E$, there is an integer multiple of α_m within $q^{-1/\ell}$ of qQ . Note that for $\alpha_m \in M_1$, the integer multiple is actually exactly qQ . In that case we put $p_{q,m} = Qq\alpha_m^{-1}$. Set

$$\delta = \frac{1}{2} \min\{|\alpha_m - n\alpha_k|: m \in I, \alpha_k \neq \alpha_m, n \in \mathbb{N}\},$$

and observe that $\delta > 0$ from the assumption that α_m is minimal in M for all $m \in I$. Assume that α_ℓ is the largest element of M and for $m \in I$, set

$$E_m := \left\{ p_{q,m} + 1 : q \in E, q^{-1/\ell} < \frac{\delta}{2\alpha_\ell} \right\}. \quad (27)$$

We claim that for all $j \in E_m$, (26) holds. Indeed, if $\alpha_k \neq \alpha_m$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} |j\alpha_m - n\alpha_k| &= |(p_{q,m} + 1)\alpha_m - (p_{q,k} + n')\alpha_k| \\ &\geq |\alpha_m - n'\alpha_k| - |p_{q,m}\alpha_m - p_{q,k}\alpha_k| \\ &\geq 2\delta - (\alpha_m + \alpha_k)q^{-1/\ell} \\ &> \delta. \end{aligned}$$

It follows that no integer multiple of $\alpha_k \neq \alpha_m$ is within distance δ of $j\alpha_m$, when $j \in E_m$. On the other hand, by definition of $R(M)$, and assuming without loss of generality that $\delta < 1$, $j\alpha_m$ is the only integer multiple of α_m in the interval $[j\alpha_m - \delta, j\alpha_m + \delta]$, and this happens with multiplicity $2\mu(m)$.

Step 3. For $m \in I$, we recover the quantities $s_n^{(k)}$ from $S(\Xi^{(N)})$ for any $n \leq N$ and for all k such that $\alpha_k = \alpha_m$.

Let $j \in E_m$ and observe that for any $N \in \mathbb{N}$, the indices in the sequence $S(\Xi^{(N)})$ for the elements in the interval $[j\alpha_m - \delta, j\alpha_m + \delta]$ can be uniquely determined from $R(M)$, which is determined by $S(\Xi^{(N)})$ as seen in the first step of this proof. It follows from (25), that for all k such that $\alpha_m = \alpha_k$ and $j \in E_m$ large enough, we have

$$\{\xi_p^{(k,N)} : p \in \mathbb{N}\} \cap [j\alpha_m - \delta, j\alpha_m + \delta] = \{\xi_{2j-1}^{(k,N)}, \xi_{2j}^{(k,N)}\}.$$

For any $N \geq 1$, consider the set

$$X_1^{(m,N)} = \{(x - j\alpha_m)j : j \in E_m, x \in S(\Xi^{(N)}) \cap [j\alpha_m - \delta, j\alpha_m + \delta]\}.$$

From the definition of E_m , we have

$$X_1^{(m,N)} = \bigcup_{k:\alpha_k=\alpha_m} \{(\xi_{2j-1}^{(k,N)} - j\alpha_m)j, (\xi_{2j}^{(k,N)} - j\alpha_m)j\}_{j \in E_m}.$$

Consider the limit points of $X_1^{(m,N)}$. We claim that those points are exactly the values of $s_1^{(k)}$ for which $\alpha_k = \alpha_m$. In fact, from the previous equation, $X_1^{(m,N)}$ is a union of sequences and the claim follows from the fact that

$$\lim_{\substack{j \rightarrow \infty \\ j \in E_m}} (\xi_{2j-1}^{(k,N)} - j\alpha_m)j = \lim_{\substack{j \rightarrow \infty \\ j \in E_m}} (\xi_{2j}^{(k,N)} - j\alpha_m)j = s_1^{(k)}.$$

Moreover, we can know the number of k' such that $s_1^{(k')} = s_1^{(k)}$, which we denote by $\text{mult}(s_1^{(k)})$. Indeed, by setting

$$\varepsilon = \frac{1}{2} \min\{|s_1^{(k)} - s_1^{(k')}| : s_1^{(k)} \neq s_1^{(k')}, \alpha_k = \alpha_m\},$$

we have that $\frac{\text{mult}(s_1^{(k)})}{\mu(m)}$ is given by

$$\lim_{J \rightarrow \infty} \frac{|\{(x - j\alpha_m)j \in X_1^{(m,N)} \cap (s_1^{(k)} - \varepsilon, s_1^{(k)} + \varepsilon) : j \in E_m, j \leq J\}|}{2|\{j \in E_m : j \leq J\}|}.$$

Note that from the construction, we cannot directly know which k is associated to each $s_1^{(k)}$, but without loss of generality we can label them in any way we choose since we know their multiplicity. For k with $\alpha_k = \alpha_m$, we construct the sequences

$$\eta_j^{(k,1)} = j\alpha_m + s_1^{(k)} j^{-1}$$

taking into account the multiplicity of $s_1^{(k)}$. We let $\text{mult}(\eta_j^{(k,1)})$ be the number of such sequences identical to $\eta_j^{(k,1)}$. In this case, $\text{mult}(\eta_j^{(k,1)}) = \text{mult}(s_1^{(k)})$. Moreover, $\eta_j^{(k,1)}$ is determined by $S(\Xi^N)$ for any $N \geq 1$.

Suppose now that we know $s_1^{(k)}, \dots, s_T^{(k)}$ from $S(\Xi^N)$ for all $N \geq T$ and k for which $\alpha_k = \alpha_m$, and consider the sequences

$$\eta_j^{(k,T)} = j\alpha_m + \sum_{n=1}^T s_n^{(k)} j^{-n}.$$

As previously, for $N \geq T + 1$, consider the set

$$X_{T+1}^{(k,N)} = \{(x - \eta_j^{(k,T)})j^{T+1} : j \in E_m, x \in S(\Xi^N) \cap [j\alpha_m - \delta, j\alpha_m + \delta]\}$$

which we can rewrite as

$$X_{T+1}^{(k,N)} = \bigcup_{k': \alpha_{k'} = \alpha_m} \{(\xi_{2j-1}^{(k',N)} - \eta_j^{(k,T)})j^{T+1}, (\xi_{2j}^{(k',N)} - \eta_j^{(k,T)})j^{T+1}\}_{j \in E_m}.$$

We claim that the limit points of $X_{T+1}^{(k,N)}$ are precisely the coefficients $s_{T+1}^{(k')}$ such that $\eta_j^{(k',T)} = \eta_j^{(k,T)}$ for all $j \in \mathbb{N}$. This follows from the fact that

$$\begin{aligned} \lim_{\substack{j \rightarrow \infty \\ j \in E_m}} (\xi_{2j-1}^{(k',N)} - \eta_j^{(k,T)})j^{T+1} &= \lim_{\substack{j \rightarrow \infty \\ j \in E_m}} (\xi_{2j}^{(k',N)} - \eta_j^{(k,T)})j^{T+1} \\ &= \begin{cases} s_{T+1}^{(k')} & \text{if } \eta_j^{(k',T)} = \eta_j^{(k,T)}, \\ \pm\infty & \text{otherwise.} \end{cases} \end{aligned}$$

We can also deduce the multiplicity of each $s_{T+1}^{(k)}$ in a similar fashion as before. It follows that we can construct the sequences

$$\eta_j^{(k, T+1)} = j\alpha_m + \sum_{n=1}^{T+1} s_n^{(k)} j^{-n}$$

and we know the multiplicity of each such sequence. By induction, we can then deduce any coefficient $s_n^{(k)}$ from $S(\Xi^{(N)})$ as long as $N \geq n$.

Step 4. We now turn our attention to $m \notin I$, and assume that we have already proved the proposition for all k such that $\alpha_k < \alpha_m$. Defining this time

$$\delta = \frac{1}{2} \min\{|\alpha_m - n\alpha_k| : \alpha_k \not\leq \alpha_m, n \in \mathbb{N}\}$$

and E_m as in (27), it follows from the same construction as in Step 2 that

$$[j\alpha_m - \delta, j\alpha_m + \delta] \cap R(M) = \underbrace{\{j\alpha_m, \dots, j\alpha_m\}}_{\mu \text{ times}},$$

where $\mu = 2 \sum_{\alpha_k \leq \alpha_m} \mu(k)$. We observe that once again, the indices in the sequence $S(\Xi^{(N)})$ of those elements are uniquely determined by $R(M)$ for any $N \geq 1$. For every k such that $\alpha_k \leq \alpha_m$, write $r(k)$ to be the integer such that $\alpha_m = r(k)\alpha_k$. Defining $X_1^{(m, N)}$ as in step 3, its limit points are now given by the values of $\frac{s_1^{(k)}}{r(k)}$ for which $\alpha_k \leq \alpha_m$. From the induction hypothesis, we know those values whenever $r(k) > 1$, so we can disregard them. What is left are the values of $s_1^{(k)}$ for which $\alpha_k = \alpha_m$. Proceeding in a similar manner as in step 3, but with

$$\eta_j^{(k, T+1)} = j\alpha_m + \sum_{n=1}^{T+1} s_n^{(k)} (r(k)j)^{-n}$$

and disregarding the values we already know, we are then able to recover recursively the values of $s_n^{(k)}$ from $S(\Xi^{(N)})$ for any $n \leq N$ for each k with $\alpha_k = \alpha_m$. The set M is finite, hence our inductive procedure necessarily terminates, finishing the proof. \square

Theorem 1.5 follows directly from Proposition 6.3.

Proof of Theorem 1.5. Let $N \in \mathbb{N}$ and for $1 \leq m \leq \ell$, let $\xi^{(m, N)}$ and $\Xi^{(N)}$ be as defined in the statement of Theorem 1.3. Since $\sigma_j \sim S(\Xi^{(\infty)})_j$, there exists a constant C_N such that for all $j \geq 0$,

$$|\sigma_j - S(\Xi^{(N)})_j| \leq C_N j^{-N-1}. \quad (28)$$

We define a new set of sequences $\tilde{\xi}^{(m,N)}$ by

$$\tilde{\xi}_j^{(m,N)} = \xi_j^{(m,N)} + \sigma_{\iota(m,j)} - \xi_j^{(m,N)} \quad (29)$$

where $\iota : \{1, \dots, \ell\} \times \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is a bijective map such that

$$S(\Xi^{(N)})_{\iota(m,j)} = \xi_j^{(m,N)}.$$

By (28), we have

$$\tilde{\xi}_j^{(m,N)} = \xi_j^{(m,N)} + O_N(j^{-N-1}).$$

Therefore, the sequences $\tilde{\xi}_j^{(m,N)}$ satisfy the hypotheses of Proposition 6.3 with $M = \{\frac{2\pi}{\text{per}(\Sigma_1)}, \dots, \frac{2\pi}{\text{per}(\Sigma_\ell)}\}$. Letting $\tilde{\Xi}^{(N)} = \{\tilde{\xi}^{(1,N)}, \dots, \tilde{\xi}^{(\ell,N)}\}$, it follows that $S(\tilde{\Xi}^{(N)})$ determines M and the coefficients $s_n^{(m)}$ for $1 \leq m \leq \ell$ and $n \leq N$. By taking N arbitrarily large, we can recover any coefficient $s_n^{(m)}$. On the other hand, by (29), every $\tilde{\xi}_j^{(m,N)}$ is simply an eigenvalue σ_j , and since ι is a bijection,

$$S(\tilde{\Xi}^{(N)})_j = \sigma_j$$

for all $j \geq 0$. It follows that the spectrum of DN_λ is given by $S(\tilde{\Xi}^{(N)})$ for any $N \in \mathbb{N}$ and therefore determines the number of connected components of the boundary and their respective perimeters (from M), as well as all the coefficients $s_n^{(m)}$.

In particular, from (3) we have

$$s_{-1}^{(m)}(\lambda; \Omega) = -\frac{\lambda \text{per}(\Sigma_m)}{4\pi},$$

allowing us to recover λ , and

$$s_{-2}^{(m)}(\lambda; \Omega) = \frac{\lambda \text{per}(\Sigma_m)}{8\pi} \int_{\Sigma_m} k_g \, ds,$$

allowing us to recover the total geodesic curvature on each boundary component. \square

We can now prove Theorem 1.6 as well.

Proof of Theorem 1.6. Since the total geodesic curvature on each boundary component is a spectral invariant, the total integral

$$\int_{\Sigma} k_g \, ds = \sum_{m=1}^{\ell} \int_{\Sigma_m} k_g \, ds$$

is a spectral invariant. Applying the Gauss–Bonnet theorem, we get

$$\int_{\Sigma} k_g \, ds = 2\pi(2 - 2\gamma - \ell) - \int_{\Omega} K_g \, dA_g$$

where γ is the genus of Ω . Since the number of boundary components ℓ is a spectral invariant, we can deduce that the quantity

$$4\pi\gamma + \int_{\Omega} K_g \, dA_g$$

is also a spectral invariant of the constant potential parametric Steklov problem. \square

Remark 6.4. It is impossible to completely decouple the genus and the average of the Gaussian curvature as spectral invariants from the eigenvalue asymptotic expansion since the addition of a handle far from the boundary changes the genus of Ω but leaves the symbol of the Dirichlet-to-Neumann operator unchanged. However, a priori information on Ω , such as being a domain of a specific space form of constant Gaussian curvature, can yield additional information, as in Corollaries 1.7 and 1.8.

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