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Spectrum of the semi-relativistic Pauli–Fierz model II

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Abstract. We consider the ground state of the semi-relativistic Pauli–Fierz Hamiltonian

$$H = |\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| + H_{\mathrm{f}} + V(\boldsymbol{x}).$$

Here $\boldsymbol{A}(\boldsymbol{x})$ denotes the quantized radiation field with an ultraviolet cutoff function and H_{f} the free field Hamiltonian with dispersion relation $|\boldsymbol{k}|$. The Hamiltonian H describes the dynamics of a *massless* and semi-relativistic charged particle interacting with the quantized radiation field with an ultraviolet cutoff function. In 2016, the first two authors proved the existence of the ground state Φ_m of the massive Hamiltonian H_m is proven. Here, the massive Hamiltonian H_m is defined by H with dispersion relation $\sqrt{\boldsymbol{k}^2 + m^2}$ ($m > 0$). In this paper, the existence of the ground state of H is proven. To this aim, we estimate a singular and non-local pull-through formula and show the equicontinuity of $\{a(k)\Phi_m\}_{0<m<m_0}$ with some m_0, where $a(k)$ denotes the formal kernel of the annihilation operator. Showing the compactness of the set $\{\Phi_m\}_{0<m<m_0}$, the existence of the ground state of H is shown.

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1. Introduction

1.1. Semi-relativistic Pauli–Fierz model. In this paper we are concerned with the existence of the ground state of the so-called "semi-relativistic Pauli–Fierz model" (abbreviated as SRPF model), which describes an interaction between a semi-relativistic charged particle and the quantized radiation field. The existence of a ground state of a model in quantum field theory is a fascinating problem: the existence of the ground state of typical models including the non-relativistic Pauli–Fierz model [29], the SRPF model with a massive particle, the Nelson model [28] and spin-boson model has been proven. As far as we know, however, that of the SRPF model with a massless particle has been left open so far.

The non-relativistic Pauli–Fierz Hamiltonian is given by

$$H_{\mathrm{PF}} = \frac{1}{2M}(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))^2 + H_{\mathrm{f},m} + V(\boldsymbol{x}),$$

where M denotes the mass of a charged particle, $\boldsymbol{p}$ the three-dimensional momentum operator, $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_{\hat{\varphi}}(\boldsymbol{x})$ the quantized radiation field with an ultraviolet cutoff function $\hat{\varphi}$, $H_{\mathrm{f},m}$ the free field Hamiltonian with dispersion relation $\omega_m(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + m^2}$ with an artificial photon mass $m \geq 0$ and photon momentum $\boldsymbol{k} \in \mathbb{R}^3$, and $V(\boldsymbol{x})$ an external potential. The spectrum of H_{PF} has been studied, e.g., in [5, 8, 24]; the Nelson model has been studied, e.g., in [4, 3, 32, 6, 7]; finally, the spin-boson model has been studied, e.g., in [31, 2]. The existence and uniqueness of the ground state of H_{PF} are established for $m \geq 0$ under some conditions on V and $\hat{\varphi}$. In particular, in the case of $m = 0$ (which is a physically reasonable case) the bottom of the spectrum of H_{PF} lies at the bottom of its essential spectrum, and then it is not discrete. See [1, 9, 17, 33] as a review for ground states of models in quantum field theory.

The SRPF Hamiltonian is defined by H_{PF} with kinetic term $\frac{1}{2M}(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))^2$ replaced by a semi-relativistic version

$$\sqrt{(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))^2 + M^2}.$$

It is of the form

$$H_{M,m} = \sqrt{(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))^2 + M^2} + H_{\mathrm{f},m} + V(\boldsymbol{x}). \tag{1.1}$$

It may also be further generalized to a model with N-charged particles for some $N \geq 2$. In the specific model studied here, we fix the number of the charged particle to one. The SRPF Hamiltonian has two singularities:

zero photon mass: $m = 0$;

zero particle mass: $M = 0$.

So far, the SRPF Hamiltonian with $(M, m) \neq (0, 0)$ has been studied in several works. The Hamiltonian $H_{M,0}$ with $M > 0$ is investigated in the series of papers [21, 22, 20, 23, 26]. The SRPF Hamiltonian with a massless particle

$$H_m = H_{0,m} = |\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| + H_{\mathrm{f},m} + V(\boldsymbol{x})$$

is studied in [12] for $m > 0$. However, the analysis of the SRPF Hamiltonian with $(M, m) = (0, 0)$ has been left open. Thus, we focus on studying the Hamiltonian with $(M, m) = (0, 0)$:

$$H = |\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| + H_{\mathrm{f}} + V(\boldsymbol{x}). \tag{1.2}$$

The kinetic energy term $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$ is a non-local operator and has a singularity in low energy part. In the next section we explain the details of technical improvements needed to investigate H.

1.2. Technical improvements

1.2.1. Compactness arguments. In [12] it is shown that H_m ($m > 0$) has the normalized ground state Φ_m if the external potential satisfies that $V(\boldsymbol{x}) \to \infty$ as $|\boldsymbol{x}| \to \infty$. Take a subsequence m_j such that Φ_{m_j} weakly converges to some vector Φ_0 as $m_j \to 0$ with $j \to \infty$. It is known that if $\Phi_0 \neq 0$, then Φ_0 is the ground state of H. See [2, Lemma 4.9].

In order to establish $\Phi_0 \neq 0$, we improve methods developed by [6, 7, 8]. We shall construct a compact operator C such that

$$\operatorname*{s-lim}_{m_j \to 0} C\Phi_{m_j} = C\Phi_0 \neq 0.$$

Let $j \in C_0^\infty([0, \infty))$ be a function such that $0 \leq j(s) \leq 1$ and

$$j(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1, \\ 0 & \text{if } s \geq 2. \end{cases} \tag{1.3}$$

For $R > 0$, let

$$\chi_1 = j(|\boldsymbol{x}|/R), \quad \chi_2 = j(|\boldsymbol{p}|/R), \quad \chi_3 = j(\mathrm{N}/R), \quad \chi_4 = j(H_\mathrm{f}/R)$$

and

$$\chi_5 = \Gamma(j(|i\nabla_{\boldsymbol{k}}/R|)).$$

Here N denotes the number operator and $\Gamma(j(|i\nabla_{\boldsymbol{k}}/R|))$ is the second quantization of $j(|i\nabla_{\boldsymbol{k}}/R|)$. We can see that $C = \chi_1\chi_2\chi_3\chi_4\chi_5$ is compact and

$$\sup_{j \in \mathbb{N}} \|(1 - \chi_\ell)\Phi_{m_j}\| = o(R^0), \quad \ell = 1, \ldots, 5 \tag{1.4}$$

as $R \to \infty$. From this, we shall show that $C\Phi_{m_j} \to C\Phi_0 \neq 0$ as $m_j \to 0$, and we conclude that H has the ground state. It is crucial to show (1.4) for $\ell = 3, 5$;

$$\lim_{R \to \infty} \sup_{j \in \mathbb{N}} \|(1 - j(\mathrm{N}/R))\Phi_{m_j}\| = 0, \tag{1.5}$$

$$\lim_{R \to \infty} \sup_{j \in \mathbb{N}} \|(1 - \Gamma(j(|i\nabla_{\boldsymbol{k}}/R|))\Phi_{m_j}\| = 0. \tag{1.6}$$

We explain where the crucial part is and how to overcome the difficulties when studying H.

1.2.2. Non-local pull-through formula and infrared divergence. To prove equation (1.5), we apply the pull-through formula and we have to reduce the infrared divergence. The unperturbed Hamiltonian associated with H_m is given by

$$H(0) = |\boldsymbol{p}| + H_{\mathrm{f},m} + V(\boldsymbol{x}).$$

Hence, the interaction of H_m is the non-local operator of the form

$$H_{\mathrm{I}} = |\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| - |\boldsymbol{p}|$$

and we have

$$H_m = H(0) + H_{\mathrm{I}}.$$

It is standard to apply the so-called "pull-through formula" to show (1.5):

$$a(k)\Phi_m = (H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}[a(k), H_{\mathrm{I}}]\Phi_m,$$

where $E_m = \inf \sigma(H_m)$. It is however hard to estimate $[a(k), H_{\mathrm{I}}]$, since H_{I} is singular and non-local. It is also unclear how to specify the domains of both kinetic term $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$ and commutator $[a(k), H_{\mathrm{I}}]$.

To reduce the infrared divergence, we combine several methods: Hirokawa's trick (5.4), functional integration (Proposition 3.3), diamagnetic inequality (Lemma 3.4), Hardy's inequality (3.6) and Hardy–Kato's inequality (6.3) ([25, Lemma 8.2] and [10]):

$$\||\boldsymbol{p}|^{-\frac{1}{2}}|\Psi|\|^2 \le \frac{\pi}{2}\||\boldsymbol{x}|^{\frac{1}{2}}\Psi\|^2.$$

We give a comment on the reduction of the infrared divergence. The Pauli transformation $U(\boldsymbol{x}) = \exp(i\boldsymbol{x}\cdot\boldsymbol{A}(0))$ was useful to reduce the infrared divergence of the non-relativistic Pauli–Fierz Hamiltonian H_{PF} (see, e.g., [5]). The Pauli transformation may be also applied to H, and on a certain domain we have

$$U^{-1}(\boldsymbol{x})H_m U(\boldsymbol{x}) = |\boldsymbol{p} + \boldsymbol{A}(0) - \boldsymbol{A}(\boldsymbol{x})| + H_{\mathrm{f},m} + h(\boldsymbol{x}) + V(\boldsymbol{x}), \tag{1.7}$$

where

$$h(\boldsymbol{x}) = -i\int \boldsymbol{x}\cdot\boldsymbol{e}(k)\frac{\hat{\varphi}(\boldsymbol{k})}{\sqrt{|\boldsymbol{k}|}}(a^\dagger(k) - a(k))dk + \sum_{j=1,2}\|\hat{\varphi}\boldsymbol{e}(\cdot, j)\cdot\boldsymbol{x}\|^2, \tag{1.8}$$

and $\boldsymbol{e}(k) = (\boldsymbol{e}(\boldsymbol{k},1), \boldsymbol{e}(\boldsymbol{k},2))$ are polarization vectors. Since we need delicate arguments to signify the domains of both sides of (1.7), it takes effort to justify operator identity (1.7). Therefore, we apply an alternative method to reduce the infrared divergence of the SRPF Hamiltonian H.

1.2.3. Equicontinuity. To prove (1.6), we show that

$$\{a(k)\Phi_m\}_{0<m<m_0}$$

(for some $m_0 > 0$) is equicontinuous in Theorem 6.6. This is a Fock-space-version of the Kolmogorov–Riesz–Fréchet theorem which proves that an equicontinuous set $D \subset L^p(\mathbb{R}^d)$ is compact under some condition. See, e.g., [17, Theorem 2.13 and Corollary 2.14]. As far as we know, this result is new, and we do not require extra regularity conditions on $\hat{\varphi}$.

1.3. Previous results and organizations. The SRPF Hamiltonian is studied, for instance, in [27, 26, 18, 12, 21, 22, 20, 23]. The existence of the ground state for the SRPF Hamiltonian was first proven by Könenberg, Matte, and Stockmeyer [21] for $M > 0$ and $m = 0$. In the non-relativistic Pauli–Fierz Hamiltonian, the bottom of the spectrum of $H_{M,0}$ coincides with that of its essential spectrum. The case of $M = 0$, but $m > 0$, is investigated by Hidaka and Hiroshima [12], where $V(\boldsymbol{x}) \to \infty(|\boldsymbol{x}| \to \infty)$ is assumed and HVZ type theorem is shown. In particular, for $m > 0$, there exists a strictly positive gap between the ground state energy and the bottom of the essential spectrum of H_m, and hence the ground state Φ_m of H_m exists for each $m > 0$. The decaying potential $V(\boldsymbol{x})$ is not investigated in [12]. The binding condition for the decaying potential is however proven in Hiroshima and Sasaki [18]. Finally, the uniqueness of the ground state is shown in [16] for arbitrary $m \geq 0$ and $M \geq 0$ by a functional integration.

This paper is organized as follows. In Section 2, we give the definition of the SRPF Hamiltonian and state the main theorem. In Section 3, we discuss the bound and domain of $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$. In Section 4, we establish a singular and non-local pull-through formula. In Section 5, we estimate $\|\mathrm{N}^{\frac{1}{2}}\Phi_m\|$ by the singular and non-local pull-through formula. In Section 6, we prove the spatial localization of Φ_m by showing that $\{a(k)\Phi_m\}_{0<m<m_0}$ is equicontinuous. In Section 7, we prove the main theorem by compactness argument.

2. Definition of SRPF model and main results

2.1. Definition of SRPF model. We define the Hamiltonian of the SRPF model as a self-adjoint operator acting in a Hilbert space over the complex field. The operator consists of a particle part and a quantum field part. We firstly introduce the quantum field part.

The single photon Hilbert space is defined by

$$W = L^2(\mathbb{R}^3 \times \{1, 2\})$$

endowed with the inner product

$$\langle f, g\rangle = \int \overline{f(k)} g(k) dk,$$

where $\int \dots dk = \sum_{j=1,2} \int_{\mathbb{R}^3} \dots d\boldsymbol{k}$ with $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1,2\}$. The boson Fock space over W is given by

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} (\otimes_{\rm s}^0 W),$$

where $\otimes_{\rm s}^n W$ denotes the symmetric tensor product of W and $\otimes_{\rm s}^0 W = \mathbb{C}$. The inner product on $\mathscr{F}$ is defined by

$$\langle \Phi, \Psi\rangle = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \Psi^{(n)}\rangle_{\otimes_{\rm s}^n W}.$$

Thus, $\Psi \in \mathscr{F}$ can be identified with an ℓ^2-sequence $(\Psi^{(n)})_{n=0}^{\infty}$ such that

$$\sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2_{\otimes_{\rm s}^n W} < \infty.$$

The Fock vacuum is the sequence defined by

$$\Omega = (1, 0, 0, \dots) \in \mathscr{F}.$$

Let T be a densely-defined closable operator in W. The second quantization of T is a closed operator in $\mathscr{F}$, which is defined by

$$d\Gamma(T) = \bigoplus_{n=0}^{\infty} \overline{T^{(n)}},$$

where $T^{(n)} = \sum_{j=1}^n \mathbf{1} \otimes \cdots \mathbf{1} \otimes \overset{j\text{th}}{T} \otimes \mathbf{1} \cdots \otimes \mathbf{1}$ with $T^{(0)} = 0$ and $\overline{S}$ denotes the closure of closable operator S. If T is a non-negative self-adjoint operator in W, then $d\Gamma(T)$ turns to be also non-negative and self-adjoint. We denote the spectrum (resp. point spectrum) of T by $\sigma(T)$ (resp. $\sigma_{\rm p}(T)$). The Fock vacuum Ω is an eigenvector of $d\Gamma(T)$ with associated eigenvalue 0, i.e., $d\Gamma(T)\Omega = 0$. The number operator is defined by $\mathrm{N} = d\Gamma(\mathbf{1})$. Note that $\sigma(\mathrm{N}) = \mathbb{N} \cup \{0\}$. Let

$$\omega_m(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + m^2}, \quad \boldsymbol{k} \in \mathbb{R}^3,$$

be a dispersion relation. It can be regarded as a multiplication operator in W. Here m describes the mass of a single boson. Furthermore, the free field Hamiltonian $H_{\mathrm{f},m}$ is given by the second quantization of ω_m:

$$H_{\mathrm{f},m} = d\Gamma(\omega_m).$$

We notice that $H_{\mathrm{f},m}$ is a non-negative self-adjoint operator in $\mathscr{F}$, and the spectrum of $H_{\mathrm{f},m}$ is given by

$$\sigma(H_{\mathrm{f},m}) = \{0\} \cup [m, \infty), \quad \sigma_{\mathrm{p}}(H_{\mathrm{f},m}) = \{0\}.$$

For $m = 0$, we write $\omega(\boldsymbol{k}) = \omega_0(\boldsymbol{k}) = |\boldsymbol{k}|$ and $H_{\mathrm{f}} = d\Gamma(\omega)$. The creation operator $a^\dagger(f)$ smeared by $f \in W$ is given by

$$(a^\dagger(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)}), \quad n \geq 1,$$

and

$$(a^\dagger(f)\Psi)^{(0)} = 0,$$

with domain

$$\mathrm{D}(a^\dagger(f)) = \left\{ \Psi \in \mathscr{F} \;\middle|\; \sum_{n=1}^{\infty} \| \sqrt{n} S_n(f \otimes \Psi^{(n-1)}) \|^2_{\otimes_{\mathrm{s}}^n W} < \infty \right\}.$$

Here S_n is the symmetrization operator on $\otimes^n W$. The annihilation operator smeared by $f = f(k) = f(\boldsymbol{k}, j) \in W$ is defined by the adjoint of $a^\dagger(\bar{f})$: $a(f) = (a^\dagger(\bar{f}))^*$. Both $a(f)$ and $a^\dagger(f)$ are linear in f, and satisfy the canonical commutation relations

$$[a(f), a^\dagger(g)] = \langle \bar{f}, g \rangle_W, \quad [a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)].$$

We informally write

$$a^\sharp(f) = \int a^\sharp(k) f(k) dk = \sum_{j=1,2} \int_{\mathbb{R}^3} a^\sharp(\boldsymbol{k}, j) f(\boldsymbol{k}, j) d\boldsymbol{k}.$$

Let us introduce the finite particle subspace $\mathscr{F}_{\mathrm{fin}}$ by

$$\mathscr{F}_{\mathrm{fin}} = \mathrm{L.H.}\{\Omega, a^\dagger(h_1) \cdots a^\dagger(h_n)\Omega \mid h_j \in C_0^\infty(\mathbb{R}^3 \times \{1,2\}), j = 1, \ldots, n, n \geq 1\},$$

where $C_0^\infty(\mathbb{R}^3 \times \{1,2\}) = C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$. Note that $\mathscr{F}_{\mathrm{fin}}$ is dense in $\mathscr{F}$. Next, we shall define the quantized radiation field $\boldsymbol{A}(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^3$. Let $\boldsymbol{e}(\boldsymbol{k}, j)$ be the polarization vectors defined by

$$\boldsymbol{e}(\boldsymbol{k}, 1) = \frac{(k_2, k_1, 0)}{\sqrt{k_1^2 + k_2^2}}, \quad \boldsymbol{e}(\boldsymbol{k}, 2) = \frac{\boldsymbol{k}}{|\boldsymbol{k}|} \times \boldsymbol{e}(\boldsymbol{k}, 1).$$

Note that $\boldsymbol{e}(\boldsymbol{k}, j)$, $j = 1, 2$, satisfy

$$\boldsymbol{k} \cdot \boldsymbol{e}(\boldsymbol{k}, j) = 0, \quad \boldsymbol{e}(\boldsymbol{k}, j) \cdot \boldsymbol{e}(\boldsymbol{k}, j') = \delta_{jj'}, \quad j, j' = 1, 2.$$

We write

$$\boldsymbol{e}(\cdot) = (e_1(\cdot), e_2(\cdot), e_3(\cdot)).$$

Note that $e_\mu(\cdot, j) \in C^\infty(\mathbb{R}^3 \backslash L_{12})$, where

$$L_{12} = \{\boldsymbol{k} = (k_1, k_2, k_3) \in \mathbb{R}^3 \mid k_1 = k_2 = 0\}.$$

The quantized radiation field $\boldsymbol{A}(\boldsymbol{x}) = (A_1(\boldsymbol{x}), A_2(\boldsymbol{x}), A_3(\boldsymbol{x}))$ is defined by

$$A_\mu(\boldsymbol{x}) = \frac{1}{\sqrt{2}} \sum_{j=1,2} \int_{\mathbb{R}^3} e_\mu(\boldsymbol{k}, j)\big(a^\dagger(\boldsymbol{k}, j)\phi_\omega(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + a(\boldsymbol{k}, j)\phi_\omega(-\boldsymbol{k})e^{+i\boldsymbol{k}\cdot\boldsymbol{x}}\big)d\boldsymbol{k},$$

where the function ϕ_ω has the form

$$\phi_\omega(\boldsymbol{k}) = \frac{\hat{\varphi}(\boldsymbol{k})}{\sqrt{\omega(\boldsymbol{k})}},$$

and $\hat{\varphi}(\boldsymbol{k})$ is called an *ultraviolet cutoff function.* Let us introduce assumptions on $\hat{\varphi}$:

(A1) $\hat{\varphi}(\boldsymbol{k}) = \overline{\hat{\varphi}(-\boldsymbol{k})}$ and $\omega^{-\frac{1}{2}}\hat{\varphi} \in L^2(\mathbb{R}^3)$;

(A2) $\omega^{-1}\hat{\varphi} \in L^2(\mathbb{R}^3)$ and $\omega^{\frac{3}{2}}\hat{\varphi} \in L^2(\mathbb{R}^3)$.

Remark 2.1. A physically relevant choice $\hat{\varphi}(\boldsymbol{k}) = \mathbf{1}_{\{\omega \le \Lambda\}}(\boldsymbol{k})$ satisfies assumptions (A1) and (A2), where $\mathbf{1}_{\{\omega \le \Lambda\}}$ is the indicator function of

$$\{\boldsymbol{k} \in \mathbb{R}^3 \mid \omega(\boldsymbol{k}) \le \Lambda\}.$$

By assumption (A1), $A_\mu(\boldsymbol{x})$ is essentially self-adjoint on $\mathscr{F}_{\mathrm{fin}}$ for each $\boldsymbol{x} \in \mathbb{R}^3$. We denote the closure of $A_\mu(\boldsymbol{x})$ by the same symbol. Assumption (A2) will be used for the self-adjointness of the total Hamiltonian.

Next, we explain the particle part. The Hilbert space for the particle is

$$L^2(\mathbb{R}^3_{\boldsymbol{x}}) = L^2(\mathbb{R}^3, d\boldsymbol{x}),$$

where $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ denotes the position of the particle. Let $\boldsymbol{p} = (p_1, p_2, p_3) = -i(\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ be the momentum operator of the particle. The massless particle Hamiltonian under consideration is a semi-relativistic Schrödinger operator given by

$$H_{\mathrm{p}} = |\boldsymbol{p}| + V(\boldsymbol{x}) = \sqrt{-\Delta} + V(\boldsymbol{x}),$$

The Hilbert space for the SRPF model is defined by

$$\mathscr{H} = L^2(\mathbb{R}^3_{\boldsymbol{x}}) \otimes \mathscr{F}.$$

If no confusion may arise, we use the following identification:

$$\mathcal{H} \cong L^2(\mathbb{R}^3_{\boldsymbol{x}}; \mathcal{F}) \cong \int_{\mathbb{R}^3}^{\oplus} \mathcal{F}\, d\boldsymbol{x}.$$

Under this identification, we can define the constant fiber direct integral

$$\int_{\mathbb{R}^3}^{\oplus} A_\mu(\boldsymbol{x}) d\boldsymbol{x},$$

which is also denoted by $A_\mu(\boldsymbol{x})$ for simplicity. Then, $A_\mu(\boldsymbol{x}), \mu = 1,2,3$, are self-adjoint operators in $\mathcal{H}$. The interaction between the particle and quantized radiation field is described by the minimal coupling, i.e., the interacting Hamiltonian is obtained by replacing $\boldsymbol{p}$ by $\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})$. Thus, the total Hamiltonian of the massless SRPF model is formally defined by

$$H = |\boldsymbol{p} \otimes \mathbf{1} - \boldsymbol{A}(\boldsymbol{x})| + \mathbf{1} \otimes H_{\mathrm{f}} + V(\boldsymbol{x}) \otimes \mathbf{1}.$$

For notational convenience, in the sequel we will omit the symbol $\otimes$. Thus, H can be simply written as

$$H = |\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| + H_{\mathrm{f}} + V(\boldsymbol{x}).$$

Note that the definition of H is currently unclear, and we have to specify the definition of $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$ and the conditions for $V(\boldsymbol{x})$. We use the notation

$$C^\infty(T) = \bigcap_{n=1}^{\infty} \mathrm{D}(T^n)$$

for the operator T. By assumption (A2), the non-relativistic kinetic energy

$$T_{\boldsymbol{A}} = (\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))^2$$

is well defined on $\mathrm{D}(\boldsymbol{p}^2) \cap C^\infty(\mathrm{N})$, and the next proposition has been established.

Proposition 2.2 ([16, Proposition 3.4]). *Assume* (A1) *and* (A2). *Then $T_{\boldsymbol{A}}$ is essentially self-adjoint on* $\mathrm{D}(\boldsymbol{p}^2) \cap C^\infty(\mathrm{N})$.

We set

$$\mathcal{H}_{\mathrm{fin}} = C_0^\infty(\mathbb{R}^3) \hat{\otimes} \mathcal{F}_{\mathrm{fin}},$$

where $\hat{\otimes}$ denotes the algebraic tensor product. Proposition 2.2 can be extended:

Proposition 2.3. *Assume* (A1) *and* (A2). *Then* $T_{\boldsymbol{A}}$ *is essentially self-adjoint on* $\mathcal{H}_{\mathrm{fin}}$.

Proof. Set $\mathcal{D}_1 = \mathrm{D}(\boldsymbol{p}^2) \cap C^\infty(\mathrm{N})$. Then, by Proposition 2.2, $\overline{T_{\boldsymbol{A}} \lceil \mathcal{D}_1}$ is self-adjoint. We use the fact that $\mathcal{H}_{\mathrm{fin}}$ is a core for $\boldsymbol{p}^2 + \mathrm{N}$. Let $\Psi \in \mathcal{D}_1$. Then $\Psi \in \mathrm{D}(\boldsymbol{p}^2 + \mathrm{N})$, and hence there exists a sequence $\{\Psi_n\}_n \subset \mathcal{H}_{\mathrm{fin}}$ such that $\Psi_n \to \Psi$ and $(\boldsymbol{p}^2 + \mathrm{N})\Psi_n \to (\boldsymbol{p}^2 + \mathrm{N})\Psi$ as $n \to \infty$. On the other hand, for $\Phi \in \mathcal{H}_{\mathrm{fin}}$, we have

$$\|T_{\boldsymbol{A}}\Phi\| = \|(\boldsymbol{p}^2 - 2\boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{p} + \boldsymbol{A}(\boldsymbol{x})^2)\Phi\| \le a\|(\boldsymbol{p}^2 + \mathrm{N})\Phi\| + b\|\Phi\| \tag{2.1}$$

for some $a, b > 0$. From (2.1), we know that $\{T_{\boldsymbol{A}}\Psi_n\}_n$ is a convergent sequence. Therefore, we have $\Psi \in \mathrm{D}(\overline{T_{\boldsymbol{A}} \lceil \mathcal{H}_{\mathrm{fin}}})$, which means that $T_{\boldsymbol{A}} \lceil \mathcal{D}_1 \subset \overline{T_{\boldsymbol{A}} \lceil \mathcal{H}_{\mathrm{fin}}}$. Since the self-adjoint extension is unique, we have $\overline{T_{\boldsymbol{A}} \lceil \mathcal{H}_{\mathrm{fin}}} = \overline{T_{\boldsymbol{A}} \lceil \mathcal{D}_1}$ which is self-adjoint. □

We denote the closure of $T_{\boldsymbol{A}}$ by the same symbol. The semi-relativistic kinetic energy $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$ is defined through the spectral measure of $T_{\boldsymbol{A}}$, i.e.,

$$|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| = \sqrt{T_{\boldsymbol{A}}}.$$

Definition 2.4. The massless SRPF Hamiltonian is defined by

$$H = \sqrt{T_{\boldsymbol{A}}} + V + H_{\mathrm{f}}. \tag{2.2}$$

The Hamiltonian with a photon mass m is also defined by

$$H_m = \sqrt{T_{\boldsymbol{A}}} + V + H_{\mathrm{f},m}. \tag{2.3}$$

Obviously, $H_m\big|_{m=0} = H$.

2.2. The main results.

We define two classes of external potentials.

Definition 2.5. (1) $V \in V_{\mathrm{rel}}$ if and only if $\mathrm{D}(|\boldsymbol{p}|) \subset \mathrm{D}(V)$ and there exist $0 \le a < 1$ and $0 \le b$ such that $\|Vf\| \le a\||\boldsymbol{p}|f\| + b\|f\|$ for any $f \in \mathrm{D}(|\boldsymbol{p}|)$.

(2) $V \in V_{\mathrm{conf}}$ if and only if $\lim_{|x|\to\infty} V(x) = \infty$, $\mathrm{D}(V) \subset \mathrm{D}(|\boldsymbol{x}|)$, and $V \in C^2(\mathbb{R}^3)$ with $\partial_\mu V, \partial_\mu^2 V \in L^\infty(\mathbb{R}^3)$ for $\mu = 1, 2, 3$.

Examples of V_{rel} and V_{conf} are $-Z/|\boldsymbol{x}| \in V_{\mathrm{rel}}$ and $\langle \boldsymbol{x} \rangle = \sqrt{1 + \boldsymbol{x}^2} \in V_{\mathrm{conf}}$.

Proposition 2.6 ([11, Theorem 1.9]). *Assume* (A1) *and* (A2). *Suppose that* $V \in V_{\mathrm{conf}} \cup V_{\mathrm{rel}}$. *Then, for any* $m \ge 0$, H_m *is self-adjoint on* $\mathrm{D}(|\boldsymbol{p}|) \cap \mathrm{D}(V) \cap \mathrm{D}(H_{\mathrm{f},m})$ *and essentially self-adjoint on* $\mathcal{H}_{\mathrm{fin}}$.

We remark the following. Thought H_m depends on the choice of polarization vectors, it can be shown that all H_m with measurable polarization vectors are unitary equivalent. Hence, the spectrum of H_m is independent of the choice of measurable polarization vectors. See [30, Appendix A].

If T is self-adjoint and bounded from below, then an eigenvector f such that $Tf = Ef$ with $E = \inf\sigma(T)$ is called a *ground state* of T. The existence and the uniqueness of the ground state of the massive Hamiltonian H_m has been established:

Proposition 2.7 ([12, Theorem 2.8] and [16, Theorem 5.12 (2)]). *Assume* (A1) *and* (A2). *Suppose that* $V \in V_{\mathrm{conf}}$. *Then* H_m *has the normalized ground state* Φ_m *for each* $m > 0$, *and there exist* C *and* c *such that*

$$\sup_{m>0} \|\Phi_m(\boldsymbol{x})\|_{\mathscr{F}} \le Ce^{-c|\boldsymbol{x}|}, \quad \boldsymbol{x} \in \mathbb{R}^3. \tag{2.4}$$

Remark 2.8. In Proposition 2.7 it is assumed that V is a confining potential. However, in [16, Theorem 5.12 (1)] a spatial decay of bound states of H_m with a decaying potential is shown for $m \ge 0$. Let $H_m\Psi = E_m\Psi$. Suppose that V is negative and $\lim_{|\boldsymbol{x}|\to\infty} E_m - V(x) < 0$. Then

$$\|\Psi(x)\|_{\mathscr{F}} \le \begin{cases} C\,\langle \boldsymbol{x}\rangle^{-3-1} & \text{if } m = 0, \\ C_m e^{-c_m|\boldsymbol{x}|} & \text{if } m > 0, \end{cases}$$

with some constants c_m, C_m, and C.

One common method to prove the existence of the ground state of H is to show that the weak limit of Φ_m as $m \to 0$ is a non-zero vector Φ_0. In Proposition 2.7, under some condition on V and cutoff, it is shown that H_m has the ground state Φ_m for each $m > 0$. Thus, we investigate the limit of Φ_m under the following general conditions:

(A3) for any $m > 0$, H_m has a normalized ground state Φ_m;

(A4) there exists $m_0 > 0$ such that $\sup_{0<m<m_0} \|\langle \boldsymbol{x}\rangle^2\, \Phi_m\| < \infty$.

The main result in this paper is the following:

Theorem 2.9. *Assume* (A1)–(A4) *and* $V \in V_{\mathrm{conf}} \cup V_{\mathrm{rel}}$. *Then* H *has the unique ground state.*

3. Domains and bounds of $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$

In this section, we discuss domains and bounds of operators related to $(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x}))^2$. In the spectral analysis of H, we need to compute and estimate commutators related to $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$. Since $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$ is non-local, it is not obvious that $\mathrm{N}^{\frac{1}{2}}|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|$ is well defined on a dense domain.

Let $\Omega(x) = \pi^{-\frac{1}{4}}e^{-\frac{1}{2}x^2}$. Obviously, in the case of one mode annihilation operator and creation operator $a = (x + d/dx)/\sqrt{2}$ and $a^\dagger = (x - d/dx)/\sqrt{2}$ in $L^2(\mathbb{R})$, we have

$$|a + a^\dagger|\Omega = \sqrt{2}\pi^{-\frac{1}{4}}|x|e^{-\frac{1}{2}x^2},$$

which is not twice differentiable, because of the singularity at $x = 0$. Namely,

$$|a + a^\dagger|\Omega \notin \mathrm{D}(a^\dagger a) = \mathrm{D}\Big(-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 - \frac{1}{2}\Big).$$

From this observation, $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|\Psi \in \mathrm{D}(\mathrm{N})$ may not be expected for $\Psi \in \mathscr{H}_{\mathrm{fin}}$. However, since we can see that

$$|a + a^\dagger|\Omega \in \mathrm{D}((a^\dagger a)^{\frac{1}{2}}) = \mathrm{D}\Big(\frac{d}{dx}\Big) \cap \mathrm{D}(x),$$

we may expect that $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|\Psi \in \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$ for $\Psi \in \mathscr{H}_{\mathrm{fin}}$. We can indeed show the proposition below:

Proposition 3.1. *Suppose* (A1) *and* (A2). *Then* $|\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|\Psi \in \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$ *for any* $\Psi \in \mathscr{H}_{\mathrm{fin}}$.

The proof will be given later in this section. The next lemma is a basic fact about the domains related to $T_{\boldsymbol{A}}$ and N.

Lemma 3.2. *Assume* (A1) *and* (A2). *If* $\Psi \in \mathscr{H}_{\mathrm{fin}}$, *then* $\Psi \in \mathrm{D}(T_{\boldsymbol{A}}^2)$ *and* $T_{\boldsymbol{A}}^2\Psi \in C^\infty(\mathrm{N})$.

Proof. Note that $\mathscr{H}_{\mathrm{fin}} \subset \mathrm{D}(\boldsymbol{p}^2) \cap C^\infty(\mathrm{N}) \subset \mathrm{D}(T_{\boldsymbol{A}})$. By the properties of polarization vectors, we know that $\boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{p} = \boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{x})$, so

$$T_{\boldsymbol{A}}\Psi = (\boldsymbol{p}^2 - 2\boldsymbol{A}(\boldsymbol{x}) \cdot \boldsymbol{p} + \boldsymbol{A}(\boldsymbol{x})^2)\Psi$$

for $\Psi \in \mathscr{H}_{\mathrm{fin}}$. By (A2), we have $\boldsymbol{k}^2\phi_\omega \in L^2(\mathbb{R}^3)$, which means that $A_\mu(\boldsymbol{x})\Phi \in \mathrm{D}(\boldsymbol{p}^2)$ if $\Phi \in \mathrm{D}(\boldsymbol{p}^2) \cap \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$. Hence, $\boldsymbol{p}^2\Psi$, $\boldsymbol{A}(\boldsymbol{x})\cdot\boldsymbol{p}\Psi$, $\boldsymbol{A}(\boldsymbol{x})^2\Psi \in \mathrm{D}(\boldsymbol{p}^2)$. Clearly, each vectors have finite photon number. Thus $T_{\boldsymbol{A}}\Psi \in \mathrm{D}(\boldsymbol{p}^2) \cap C^\infty(\mathrm{N}) \subset \mathrm{D}(T_{\boldsymbol{A}})$, and $T_{\boldsymbol{A}}\Psi \in \mathrm{D}(T_{\boldsymbol{A}})$. It is clear that $T_{\boldsymbol{A}}^2\Psi \in C^\infty(\mathrm{N})$. $\square$

In order to prove Proposition 3.1, we need some inequalities derived by the functional integral representation. We consider the probabilistic representation. Let $(B_t)_{t\geq 0}$ be the three-dimensional Brownian motion on a probability space $(\mathcal{W}, B(\mathcal{W}), P^{\boldsymbol{x}})$. Here $P^{\boldsymbol{x}}$ is the Wiener measure starting from $\boldsymbol{x} \in \mathbb{R}^3$. Then we can consider the partial isometry

$$\begin{aligned} L^2(\mathbb{R}^3, d\boldsymbol{x}) &\longrightarrow \int_{\mathbb{R}^3}^{\oplus} L^2(\mathcal{W}, dP^{\boldsymbol{x}}) d\boldsymbol{x}, \\ f(\boldsymbol{x}) &\longmapsto f(B_0(w)), \quad (\boldsymbol{x}, w) \in \mathbb{R}^3 \times \mathcal{W}. \end{aligned} \tag{3.1}$$

Since $B_0(w) = \boldsymbol{x}$ a.s., the above identification is trivial. However, the semigroup for the free particle can be described as

$$(e^{-\frac{t}{2}\boldsymbol{p}^2} f)(\boldsymbol{x}) \longmapsto f(\boldsymbol{x} + B_t(w)), \quad (\boldsymbol{x}, w) \in \mathbb{R}^3 \times \mathcal{W}.$$

The expectation with respect to $P^{\boldsymbol{x}}$ is simply denoted by $\mathbb{E}^{\boldsymbol{x}}[\ldots]$. In the following, we use this embedding (3.1) as an identification, and we simply use $L^2(\mathbb{R}^3 \times \mathcal{W})$ to denote $\int_{\mathbb{R}^3}^{\oplus} L^2(\mathcal{W}, dP^{\boldsymbol{x}}) d\boldsymbol{x}$. Next, we introduce a probabilistic description for the field. Let $\mathcal{A}(F)$ be the Gaussian random process indexed by $F \in \oplus^3 L^2(\mathbb{R}^3)$ on a probability space (Q, Σ, μ) such that $\mathbb{E}_\mu[\mathcal{A}(F)] = 0$. The covariance is given by

$$\mathbb{E}_\mu[\mathcal{A}(F)\mathcal{A}(G)] = \frac{1}{2} \sum_{\mu,\nu=1}^{3} \langle \widehat{F}_\mu, d_{\mu\nu} \widehat{G}_\nu \rangle,$$

where $d_{\mu\nu} = \delta_{\mu\nu} - k_\mu k_\nu / |\boldsymbol{k}|^2$ and $\widehat{F}_\mu$ denotes the Fourier transform of F_μ. The unitary equivalence between $L^2(Q)$ and $\mathcal{F}$ is established, and under this equivalence it follows that, for $F = F_1 \oplus F_2 \oplus F_3 \in \oplus^3 L^2(\mathbb{R}^3)$,

$$\begin{aligned} \mathcal{A}(F) &\cong A(F) \\ &= \frac{1}{\sqrt{2}} \sum_{\mu=1}^{3} \sum_{j=1,2} \int_{\mathbb{R}^3} e_\mu(\boldsymbol{k}, j) \big(a^\dagger(\boldsymbol{k}, j) \widehat{F}_\mu(\boldsymbol{k}) + a(\boldsymbol{k}, j) \widehat{F}_\mu(-\boldsymbol{k})\big) d\boldsymbol{k}. \end{aligned} \tag{3.2}$$

Namely, each Segal's field operator can be considered as a Gaussian random process. In the following, we use the identifications $L^2(\mathbb{R}^3, d\boldsymbol{x}) \to L^2(\mathbb{R}^3 \times \mathcal{W})$ and $\mathcal{F} \cong L^2(Q)$.

Proposition 3.3 ([15]). *The Feynman–Kac formula of $e^{-\frac{t}{2}T_{\boldsymbol{A}}}$ is given by*

$$\langle \Phi, e^{-\frac{t}{2}T_{\boldsymbol{A}}} \Psi \rangle = \int_{\mathbb{R}^3} \mathbb{E}^{\boldsymbol{x}}[\langle \Phi(B_0), e^{-i\mathcal{A}(K)} \Psi(B_t) \rangle_{L^2(Q)}] d\boldsymbol{x}, \quad \Psi, \Phi \in \mathcal{H}.$$

Here

$$K(\cdot) = \bigoplus_{\mu=1}^{3} \int_0^t \tilde{\varphi}(\cdot - B_s) dB_s^\mu \tag{3.3}$$

with $\tilde{\varphi} = (\phi_\omega)\check{} = (\hat{\varphi}/\sqrt{\omega})\check{}$.

Let $\mathcal{N}$ be the number operator in $L^2(Q)$. For $F \in \oplus^3 L^2(\mathbb{R}^3)$, the conjugate momentum of $\mathcal{A}(F)$ is denoted by $\Pi(F)$, namely, $\Pi(F) = i[\mathcal{N}, \mathcal{A}(F)]$ and the corresponding field operator is

$$\pi(F) = \frac{i}{\sqrt{2}} \sum_{\mu=1}^{3} \sum_{j=1,2} \int_{\mathbb{R}^3} e_\mu(\boldsymbol{k}, j)(a^\dagger(\boldsymbol{k}, j)\widehat{F}_\mu(\boldsymbol{k}) - a(\boldsymbol{k}, j)\widehat{F}_\mu(-\boldsymbol{k}))d\boldsymbol{k}.$$

Then the identity

$$\mathcal{N} e^{-i\mathcal{A}(K)} = e^{-i\mathcal{A}(K)}(\mathcal{N} - \Pi(K) - \xi_K) \tag{3.4}$$

holds, where ξ_K is a stochastic process defined by

$$\xi_K = \frac{1}{2} \sum_{\mu,\nu=1}^{3} \langle \widehat{K}_\mu, d_{\mu\nu} \widehat{K}_\mu \rangle_{L^2(\mathbb{R}^3)}.$$

Note that $\widehat{K}_\mu = \int_0^t \phi_\omega(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot B_s} dB_s^\mu$ is an $L^2(\mathbb{R}^3_{\boldsymbol{k}})$-valued stochastic integral, and hence $\pi(K)$ is an operator-valued stochastic integral in $L^2(\mathbb{R}^3 \times \mathcal{W}) \otimes \mathcal{F}$. Let

$$P_\mu = p_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_{\mathrm{f}\mu}, \quad \mu = 1, 2, 3$$

be the total momentum, where $P_{\mathrm{f}\mu} = d\Gamma(k_\mu)$ is the field momentum. The corresponding filed momentum in $L(Q)$ is denoted by $\mathcal{P}_{\mathrm{f}\mu}$. The commutation relation between $\mathcal{P}_{\mathrm{f}\nu}$ and $e^{-i\mathcal{A}(K)}$ is given by

$$\mathcal{P}_{\mathrm{f}\nu} e^{-i\mathcal{A}(K)} = e^{-i\mathcal{A}(K)}(\mathcal{P}_{\mathrm{f}\nu} - \mathcal{A}(\partial_\nu K)),$$

where the last term is obtained from $\mathcal{A}(\partial_\nu K) = i[\mathcal{P}_{\mathrm{f}\nu}, \mathcal{A}(K)]$, and the corresponding field operator is

$$\begin{aligned} \mathcal{A}(\partial_\nu K) \cong \frac{1}{\sqrt{2}} \sum_{\mu=1}^{3} \sum_{j=1,2} \int_{\mathbb{R}^3} e_\mu(\boldsymbol{k}, j)\big(&a^\dagger(\boldsymbol{k}, j)(ik_\nu \widehat{F}_\mu)(\boldsymbol{k}) \\ &+ a(\boldsymbol{k}, j)(ik_\nu \widehat{F}_\mu)(-\boldsymbol{k})\big) d\boldsymbol{k}. \end{aligned}$$

Note that ∂_ν in the above expression means the derivative for the photon coordinate.

Let $U_{\mathcal{F}}\colon \mathcal{F} \to L^2(Q)$ be the unitary operator implementing the identification $\mathcal{F} \cong L^2(Q)$. Then $(\mathbf{1} \otimes U_{\mathcal{F}})\Psi$ ($\Psi \in \mathcal{H}$) is a function in $L^2(\mathbb{R}^3_{\boldsymbol{x}} \times Q)$ and the absolute value of Ψ is defined under this identification. The following is a variation of diamagnetic inequalities.

Lemma 3.4. *Assume* (A1) *and* (A2).

(1) *For any* $\Psi \in \mathcal{H}$,

$$\|(T_{\boldsymbol{A}} + s)^{-\frac{1}{2}}\Psi\| \le \|(\boldsymbol{p}^2 + s)^{-\frac{1}{2}}|\Psi|\|, \quad s > 0.$$

(2) *If* $\Psi \in \mathrm{D}(|\boldsymbol{x}|)$, *then* $\Psi \in \mathrm{D}(T_{\boldsymbol{A}}^{-\frac{1}{2}})$ *and it holds that*

$$\|T_{\boldsymbol{A}}^{-\frac{1}{2}}\Psi\| \le 2\||\boldsymbol{x}|\Psi\|.$$

(3) *Let* $\varrho = \varrho(\boldsymbol{x})$ *be a measurable function such that* $|\varrho(\boldsymbol{x})| < \infty$ *a.e. and* $s > 0$. *Suppose that* $\|\varrho(\boldsymbol{p}^2 + s)^{-1}|\Psi|\| < \infty$. *Then* $(T_{\boldsymbol{A}} + s)^{-1}\Psi \in \mathrm{D}(\varrho)$ *and it holds that*

$$\|\varrho(T_{\boldsymbol{A}} + s)^{-1}\Psi\| \le \|\varrho(\boldsymbol{p}^2 + s)^{-1}|\Psi|\|. \tag{3.5}$$

Proof. By Proposition 3.3, we have

$$\begin{aligned}
\|(T_{\boldsymbol{A}} + s)^{-\frac{1}{2}}\Psi\|^2 &= \frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}}\langle \Psi, e^{-\frac{t}{2}T_{\boldsymbol{A}}}\Psi\rangle dt \\
&= \frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}} dt \int_{\mathbb{R}^3} \mathbb{E}^{\boldsymbol{x}}[\langle \Psi(B_0), e^{-i\mathcal{A}(K)}\Psi(B_t)\rangle_{L^2(Q)}] d\boldsymbol{x} \\
&\le \frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}} dt \int_{\mathbb{R}^3} \mathbb{E}^{\boldsymbol{x}}[\langle |\Psi(B_0)|, |\Psi(B_t)|\rangle_{L^2(Q)}] d\boldsymbol{x} \\
&= \frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}}\langle |\Psi|, e^{-\frac{t}{2}\boldsymbol{p}^2}|\Psi|\rangle dt \\
&= \|(\boldsymbol{p}^2 + s)^{-\frac{1}{2}}|\Psi|\|^2.
\end{aligned}$$

Thus (1) follows. Next we assume that $\Psi \in \mathrm{D}(|\boldsymbol{x}|)$. Clearly $|\Psi| \in \mathrm{D}(|\boldsymbol{x}|)$ and by Hardy's inequality, we have $|\Psi| \in \mathrm{D}(|\boldsymbol{p}|^{-1})$ and

$$\||\boldsymbol{p}|^{-1}|\Psi|\| \le 2\||\boldsymbol{x}||\Psi|\| = 2\||\boldsymbol{x}|\Psi\|. \tag{3.6}$$

By (1) and the monotone convergence theorem, we have $\Psi \in \mathrm{D}(T_{\boldsymbol{A}}^{-\frac{1}{2}})$ and

$$\|T_{\boldsymbol{A}}^{-\frac{1}{2}}\Psi\| = \lim_{s\to+0}\|(T_{\boldsymbol{A}}+s)^{-\frac{1}{2}}\Psi\| \le \lim_{s\to+0}\|(\boldsymbol{p}^2+s)^{-\frac{1}{2}}|\Psi|\| \le 2\||\boldsymbol{x}|\Psi\|,$$

which proves (2). Next we prove (3). By the Feynman–Kac formula (Proposition 3.3), we have

$$\begin{aligned}
&\|\varrho(\boldsymbol{x})(T_{\boldsymbol{A}}+s)^{-1}\Psi\| \\
&= \sup_{\Phi\in\mathrm{D}(\varrho^*),\|\Phi\|=1} |\langle \varrho^*\Phi, (T_{\boldsymbol{A}}+s)^{-1}\Psi\rangle| \\
&= \sup_{\Phi\in\mathrm{D}(\varrho^*),\|\Phi\|=1} \left|\frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}}dt\int_{\mathbb{R}^3}\mathbb{E}^{\boldsymbol{x}}[\langle(\varrho^*\Phi)(B_0), e^{-i\mathcal{A}(K)}\Psi(B_t)\rangle_{L^2(Q)}]d\boldsymbol{x}\right| \\
&\le \sup_{\Phi\in\mathrm{D}(\varrho^*),\|\Phi\|=1} \frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}}dt\int_{\mathbb{R}^3}\mathbb{E}^{\boldsymbol{x}}[\langle|(\varrho^*\Phi)(B_0)|, |\Psi(B_t)|\rangle_{L^2(Q)}]d\boldsymbol{x} \\
&= \sup_{\Phi\in\mathrm{D}(\varrho^*),\|\Phi\|=1} \frac{1}{2}\int_0^\infty e^{-\frac{ts}{2}}\langle|\varrho||\Phi|, e^{-\frac{t}{2}\boldsymbol{p}^2}|\Psi|\rangle dt \\
&= \sup_{\Phi\in\mathrm{D}(\varrho^*),\|\Phi\|=1} \langle|\varrho||\Phi|, (\boldsymbol{p}^2+s)^{-1}|\Psi|\rangle \\
&\le \||\varrho|(\boldsymbol{p}^2+s)^{-1}|\Psi|\|,
\end{aligned}$$

which proves (3). □

Lemma 3.5. *Assume* (A1) *and* (A2). *Let K be $\oplus^3 L^2(\mathbb{R}^3)$-valued stochastic integral given by* (3.3). *Suppose that $\Phi \in \mathrm{D}(\mathrm{N}^k)$. Then, for $k \in \mathbb{N}$, there exists a polynomial $P_k = P_k(\tau)$ of degree k such that*

$$\|(\mathrm{N}-\pi(K)-\xi_K)^k\Phi\|_{\mathscr{F}} \le P_k(|\xi_K|)\|(\mathrm{N}+\mathbf{1})^k\Phi\|_{\mathscr{F}}. \tag{3.7}$$

Proof. The proof is due to an induction with respect to k. In this proof, the symbol $\|\cdot\|$ means the norm of $\mathscr{F}$.

For $k=1$, it can be seen that

$$\|(\mathrm{N}-\pi(K)-\xi_K)\Phi\| \le \|\mathrm{N}\Phi\| + \|\pi(K)\Phi\| + |\xi_K|\|\Phi\|.$$

Since

$$\|\pi(K)\Phi\| \le C|\xi_K|^{\frac{1}{2}}\|(\mathrm{N}+\mathbf{1})^{\frac{1}{2}}\Phi\|,$$

(3.7) follows with $P_1(\tau) = 1+(C^2+\tau)+\tau$.

Next, we suppose that (3.7) is true for $k = 1, \dots, n$. Then we have

$$\begin{aligned}\|(\mathrm{N} - \pi(K) - \xi_K)^{n+1}\Phi\| \le \|&(\mathrm{N} - \pi(K) - \xi_K)^n \mathrm{N}\Phi\| \\ &+ \|(\mathrm{N} - \pi(K) - \xi_K)^n \pi(K)\Phi\| \\ &+ \|(\mathrm{N} - \pi(K) - \xi_K)^n \xi_K \Phi\|.\end{aligned}$$

By the induction hypothesis, it can be seen that

$$\begin{aligned}\|(\mathrm{N} - \pi(K) - \xi_K)^n \mathrm{N}\Phi\| &\le P_n(|\xi_K|)\|(\mathrm{N} + \mathbf{1})^{n+1}\Phi\|, \\ \|(\mathrm{N} - \pi(K) - \xi_K)^n \xi_K \Phi\| &\le P_n(|\xi_K|)|\xi_K|\|(\mathrm{N} + \mathbf{1})^n \Phi\|, \\ \|(\mathrm{N} - \pi(K) - \xi_K)^n \pi(K)\Phi\| &\le P_n(|\xi_K|)\|(\mathrm{N} + \mathbf{1})^n \pi(K)\Phi\|.\end{aligned}$$

By a simple computation, we have

$$\begin{aligned}(\mathrm{N} + 1)\pi(K)(\mathrm{N} + \mathbf{1})^{-1} &= \pi(K) + [\mathrm{N}, \pi(K)](\mathrm{N} + \mathbf{1})^{-1} \\ &= \pi(K) + iA(K)(\mathrm{N} + \mathbf{1})^{-1},\end{aligned}$$

and hence the operator norm of $(\mathrm{N} + \mathbf{1})^n \pi(K)(\mathrm{N} + \mathbf{1})^{-(n+1)}$ can be estimated as

$$\begin{aligned}&\|(\mathrm{N} + \mathbf{1})^n \pi(K)(\mathrm{N} + \mathbf{1})^{-(n+1)}\| \\ &\quad \le \|(\mathrm{N} + \mathbf{1})^{n-1}\pi(K)(\mathrm{N} + \mathbf{1})^{-n}\| + \|(\mathrm{N} + \mathbf{1})^{n-1}A(K)(\mathrm{N} + \mathbf{1})^{-(n+1)}\| \\ &\quad \le \|(\mathrm{N} + \mathbf{1})^{n-1}\pi(K)(\mathrm{N} + \mathbf{1})^{-n}\| + \|(\mathrm{N} + \mathbf{1})^{n-1}A(K)(\mathrm{N} + \mathbf{1})^{-n}\| \\ &\quad \vdots \\ &\quad \le 2^{n-1}C\|\pi(K)(\mathrm{N} + \mathbf{1})^{-1}\| + 2^{n-1}C\|A(K)(\mathrm{N} + \mathbf{1})^{-1}\| \le 2^n C|\xi_K|^{\frac{1}{2}}.\end{aligned}$$

Thus,

$$\|(\mathrm{N} - \pi(K) - \xi_K)^{n+1}\Phi\| \le P_n(|\xi_K|)(1 + |\xi_K| + 2^n(C^2 + |\xi_K|))\|(\mathrm{N} + \mathbf{1})^{n+1}\Phi\|$$

and inequality (3.7) follows with $P_{n+1}(\tau) = P_n(\tau)(1 + \tau + 2^n(C^2 + \tau))$. □

Lemma 3.6. *Assume* (A1) *and* (A2). *Let* $n \in \mathbb{N}$ *be arbitrary. Then, for any* $\Psi \in \mathrm{D}(\mathrm{N}^n)$ *and* $t \ge 0$, *we have* $e^{-tT_A}\Psi \in \mathrm{D}(\mathrm{N}^n)$ *and*

$$\|\mathrm{N}^n e^{-tT_A}(\mathrm{N} + \mathbf{1})^{-n}\| \le C_n(t^n + 1)$$

for some constant $C_n > 0$.

Proof. It is enough to show that

$$|\langle \mathrm{N}^n \Phi, e^{-\frac{t}{2}T_A}\Psi\rangle| \le C\|\Phi\|, \quad \Phi \in \mathscr{H}_{\mathrm{fin}}, \tag{3.8}$$

with

$$C = C_n(t^n+1)\|(\mathrm{N}+\mathbf{1})^n\Psi\|.$$

By the Feynman–Kac formula (Proposition 3.3), the equivalence $\Pi(K) \cong \pi(K)$ and (3.4), we have

$$\begin{aligned}
&|\langle \mathrm{N}^n \Phi, e^{-\frac{t}{2}T_A}\Psi\rangle| \\
&= \left| \int_{\mathbb{R}^3} \mathbb{E}^{\boldsymbol{x}}[\langle \mathcal{N}^n \Phi(B_0), e^{-i\mathcal{A}(K)}\Psi(B_t)\rangle_{L^2(Q)}]d\boldsymbol{x} \right| \\
&= \left| \int_{\mathbb{R}^3} \mathbb{E}^{\boldsymbol{x}}[\langle \Phi(B_0), e^{-i\mathcal{A}(K)}(\mathcal{N}-\Pi(K)-\xi_K)^n\Psi(B_t)\rangle_{L^2(Q)}]d\boldsymbol{x} \right|.
\end{aligned}$$

By Lemma 3.5, we have

$$\begin{aligned}
&|\langle \mathrm{N}^n \Phi, e^{-\frac{t}{2}T_A}\Psi\rangle| \\
&\le \int_{\mathbb{R}^3} \|\Phi(\boldsymbol{x})\|_{L^2(Q)} \mathbb{E}^{\boldsymbol{x}}[P_n(|\xi_K|)^2]^{\frac12} \mathbb{E}^{\boldsymbol{x}}[\|(\mathcal{N}+\mathbf{1})^n\Psi(B_t)\|^2_{L^2(Q)}]^{\frac12} d\boldsymbol{x}.
\end{aligned} \tag{3.9}$$

By the Burkholder–Davis–Gundy inequality [15, Theorem 4.6]

$$\mathbb{E}^{\boldsymbol{x}}[|\xi_K|^m] \le c_m t^m \|\phi_\omega\|^m, \quad m \in \mathbb{N},$$

holds with some constant c_m independent of $\boldsymbol{x}$. Then we get

$$\mathbb{E}^{\boldsymbol{x}}[P_n(|\xi_K|)^2]^{\frac12} < C_n(t^n+1)$$

for some $C_n > 0$, and so the right-hand side of (3.9) is bounded by

$$\begin{aligned}
&C_n(t^n+1)\int_{\mathbb{R}^3} \|\Phi(\boldsymbol{x})\|_{L^2(Q)} \mathbb{E}^{\boldsymbol{x}}[\|(\mathcal{N}+\mathbf{1})^n\Psi(B_t)\|^2_{L^2(Q)}]^{\frac12} d\boldsymbol{x} \\
&\le C_n(t^n+1)\|\Phi\| \int_{\mathbb{R}^3} \mathbb{E}^{\boldsymbol{x}}[\|(\mathcal{N}+\mathbf{1})^n\Psi(B_t)\|^2_{L^2(Q)}]^{\frac12} d\boldsymbol{x} \\
&= C_n(t^n+1)\|\Phi\|\|(\mathrm{N}+\mathbf{1})^n\Psi\|.
\end{aligned}$$

Hence, the proof is complete. □

Set

$$R_s = (T_{\boldsymbol{A}} + s)^{-1}.$$

Lemma 3.7. *Assume* (A1) *and* (A2). *Let* $n \in \mathbb{N}$ *and* $s > 0$. *Then it follows that* $\mathrm{Ran}(R_s(\mathrm{N}^n + \mathbf{1})^{-1}) \subset \mathrm{D}(\mathrm{N}^n)$, *and*

$$\|\mathrm{N}^n R_s(\mathrm{N}^n + \mathbf{1})^{-1}\| \le C_n(s^{-n-1} + s^{-1}) \tag{3.10}$$

holds for some $C_n > 0$.

Proof. Using the formula $(A+s)^{-1} = \int_0^\infty e^{-t(A+s)}dt$, we have, for any $\Phi \in \mathscr{H}_{\mathrm{fin}}$ and $\Psi \in \mathrm{D}(\mathrm{N})$,

$$|\langle \mathrm{N}^n \Phi, R_s \Psi\rangle| \le \int_0^\infty e^{-ts}\|\Phi\|\|\mathrm{N}^n e^{-tT_{\boldsymbol{A}}}(\mathrm{N}^n + \mathbf{1})^{-1}\|\|(\mathrm{N}^n + \mathbf{1})\Psi\| dt.$$

By Lemma 3.6, we have

$$|\langle \mathrm{N}^n \Phi, R_s \Psi\rangle| \le \int_0^\infty e^{-ts} C_n(t^n + 1)\|\Phi\|\|(\mathrm{N}^n + \mathbf{1})\Psi\| dt.$$

Thus, (3.10) follows. □

We set

$$T_{\boldsymbol{A},M} = T_{\boldsymbol{A}} + M^2.$$

Note that $\mathrm{D}(\sqrt{T_{\boldsymbol{A},M}}) = \mathrm{D}(\sqrt{T_{\boldsymbol{A}}})$, since $\sqrt{T_{\boldsymbol{A},M}} - \sqrt{T_{\boldsymbol{A}}}$ is bounded.

Lemma 3.8. *Assume* (A1) *and* (A2). *Let* $M > 0$. *Then* $T_{\boldsymbol{A},M}^{-\frac{1}{2}}\Psi \in \mathrm{D}(\mathrm{N})$ *for any* $\Psi \in \mathrm{D}(\mathrm{N})$, *and*

$$\|\mathrm{N} T_{\boldsymbol{A},M}^{-\frac{1}{2}}(\mathrm{N} + \mathbf{1})^{-1}\| \le C_1 \frac{1 + 2M^2}{2M^3}, \tag{3.11}$$

where C_1 *is the constant in Lemma* 3.7.

Proof. By the integral expression of $T_{\boldsymbol{A},M}^{-\frac{1}{2}}$,

$$T_{\boldsymbol{A},M}^{-\frac{1}{2}} = \frac{2}{\pi}\int_0^\infty R_{\lambda^2 + M^2} d\lambda,$$

we have

$$
\begin{aligned}
|\langle \mathrm{N}\Phi, T_{\boldsymbol{A},M}^{-\frac{1}{2}}\Psi\rangle| &\le \frac{2}{\pi}\int_0^\infty \|\Phi\|\|\mathrm{N}R_{\lambda^2+M^2}\Psi\|d\lambda \\
&\le \frac{2C_1}{\pi}\|\Phi\|\|(\mathrm{N}+\mathbf{1})\Psi\|\int_0^\infty((\lambda^2+M^2)^{-2}+(\lambda^2+M^2)^{-1})d\lambda
\end{aligned}
$$

by Lemma 3.7. Therefore $T_{\boldsymbol{A},M}^{-\frac{1}{2}}\Psi \in \mathrm{D}(\mathrm{N})$ and (3.11) hold. □

Lemma 3.9. *Assume* (A1) *and* (A2).

(1) *For all* $\Psi \in \mathrm{D}(\mathrm{N}T_{\boldsymbol{A}}) \cap \mathrm{D}(\mathrm{N}) \cap \mathrm{D}(\mathrm{N}T_{\boldsymbol{A}}^2)$, $T_{\boldsymbol{A}}^{\frac{3}{2}}\Psi \in \mathrm{D}(\mathrm{N})$ *and the bound*

$$\|\mathrm{N}T_{\boldsymbol{A}}^{\frac{3}{2}}\Psi\| \le C(\|\mathrm{N}T_{\boldsymbol{A}}\Psi\| + \|(\mathrm{N}+\mathbf{1})\Psi\| + \|(\mathrm{N}+\mathbf{1})T_{\boldsymbol{A}}^2\Psi\|)$$

holds for some C *independent of* Ψ.

(2) *For any* $\Psi \in \mathscr{H}_{\mathrm{fin}}$,

$$\limsup_{M\to+0}\|\mathrm{N}T_{\boldsymbol{A}}^2 T_{\boldsymbol{A},M}^{-\frac{1}{2}}\Psi\| < \infty.$$

Proof. By the integral expression of $T_{\boldsymbol{A}}^{\frac{1}{2}}$, we have, for any $\Phi \in \mathscr{H}_{\mathrm{fin}}$,

$$|\langle \mathrm{N}\Phi, T_{\boldsymbol{A}}^{\frac{3}{2}}\Psi\rangle| \le \frac{2}{\pi}\int_0^1 |\langle \mathrm{N}\Phi, R_{\lambda^2}T_{\boldsymbol{A}}^2\Psi\rangle|d\lambda + \frac{2}{\pi}\int_1^\infty |\langle \mathrm{N}\Phi, R_{\lambda^2}T_{\boldsymbol{A}}^2\Psi\rangle|d\lambda.$$

First, we estimate the integral $\int_0^1 \ldots d\lambda$. Since $T_{\boldsymbol{A}}R_{\lambda^2} = \mathbf{1} - \lambda^2 R_{\lambda^2}$, we have

$$
\begin{aligned}
\langle \mathrm{N}\Phi, R_{\lambda^2}T_{\boldsymbol{A}}^2\Psi\rangle &= \langle \mathrm{N}\Phi, T_{\boldsymbol{A}}\Psi\rangle - \lambda^2\langle \mathrm{N}\Phi, R_{\lambda^2}T_{\boldsymbol{A}}\Psi\rangle \\
&= \langle \mathrm{N}\Phi, T_{\boldsymbol{A}}\Psi\rangle - \lambda^2\langle \mathrm{N}\Phi, (\mathbf{1}-\lambda^2 R_{\lambda^2})\Psi\rangle,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_0^1 |\langle \mathrm{N}\Phi, R_{\lambda^2}T_{\boldsymbol{A}}^2\Psi\rangle|d\lambda \\
&\quad\le \|\Phi\|\|\mathrm{N}T_{\boldsymbol{A}}\Psi\| + \int_0^1 \lambda^2\|\Phi\|\|\mathrm{N}\Psi\|d\lambda + \int_0^1 \lambda^4\|\Phi\|\|\mathrm{N}R_{\lambda^2}\Psi\|d\lambda.
\end{aligned}
$$

By Lemma 3.7, we see that the last integral becomes finite and the bound

$$\int_0^1 |\langle \mathrm{N}\Phi, R_{\lambda^2} T_{\boldsymbol{A}}\Psi\rangle| d\lambda \le C\|\Phi\|\big(\|\mathrm{N}T_{\boldsymbol{A}}\Psi\| + \|(\mathrm{N}+\mathbf{1})\Psi\|\big)$$

holds for some $C > 0$. Next, we consider the second part $\int_1^\infty d\lambda$. By Lemma 3.7 again, we get the bound

$$\begin{aligned}\frac{2}{\pi}\int_1^\infty \big|\langle \mathrm{N}\Phi, R_{\lambda^2} T_{\boldsymbol{A}}^2\Psi\rangle\big| d\lambda &\le \frac{2}{\pi}\|\Phi\| \int_1^\infty C_1(\lambda^{-4}+\lambda^{-2})\|(\mathrm{N}+\mathbf{1})T_{\boldsymbol{A}}^2\Psi\| d\lambda \\ &= C\|\Phi\|\|(\mathrm{N}+\mathbf{1})T_{\boldsymbol{A}}^2\Psi\|\end{aligned}$$

for some $C > 0$.

Since $\Phi \in \mathscr{H}_{\mathrm{fin}}$ is arbitrary, these inequalities imply that $T_{\boldsymbol{A}}^{\frac{3}{2}}\Psi \in \mathrm{D}(\mathrm{N})$ and

$$\|\mathrm{N}T_{\boldsymbol{A}}^{\frac{3}{2}}\Psi\| \le C(\|\mathrm{N}T_{\boldsymbol{A}}\Psi\| + \|(\mathrm{N}+\mathbf{1})\Psi\| + \|(\mathrm{N}+\mathbf{1})T_{\boldsymbol{A}}^2\Psi\|)$$

for some $C > 0$. This shows (1). The proof of (2) is similar to the proof of (1). By Lemma 3.2, $\mathscr{H}_{\mathrm{fin}} \subset \mathrm{D}(\mathrm{N}T_{\boldsymbol{A}}) \cap \mathrm{D}(\mathrm{N}) \cap \mathrm{D}(\mathrm{N}T_{\boldsymbol{A}}^2)$. Thus, as above, one can similarly show that

$$\sup_{0<M<1} \|\mathrm{N}T_{\boldsymbol{A}}^2 T_{\boldsymbol{A},M}^{-\frac{1}{2}}\Psi\| \le C(\|\mathrm{N}T_{\boldsymbol{A}}\Psi\| + \|(\mathrm{N}+\mathbf{1})\Psi\| + \|(\mathrm{N}+\mathbf{1})T_{\boldsymbol{A}}^2\Psi\|),$$

where C is a constant independent of Ψ and M. Thus (2) holds. □

We are in the position to prove Proposition 3.1.

Proof of Proposition 3.1: Let $\Psi \in \mathscr{H}_{\mathrm{fin}}$. Set

$$T = T_{\boldsymbol{A}} \quad \text{and} \quad T_M = T_{\boldsymbol{A},M}$$

for simplicity. We will show that

$$\limsup_{M\to 0} \|\mathrm{N}^{\frac{1}{2}} T_M^{-\frac{1}{2}} T\Psi\| < \infty. \tag{3.12}$$

By Lemma 3.2, we have $\Psi \in \mathrm{D}(T^2)$, in particular $\Psi \in \mathrm{D}(T^{\frac{3}{2}})$. Since $TT_M^{-\frac{1}{2}}\Psi \in \mathrm{D}(T)$, there exists a sequence $\{\Phi_j\}_j \subset \mathscr{H}_{\mathrm{fin}}$, such that

$$\Phi_j \longrightarrow TT_M^{-\frac{1}{2}}\Psi \quad \text{and} \quad T\Phi_j \longrightarrow T^2 T_M^{-\frac{1}{2}}\Psi \quad \text{as } j \to \infty.$$

Then we have

$$\begin{aligned}\|\mathrm{N}^{\frac{1}{2}} T_M^{-\frac{1}{2}} T\Psi\|^2 &= \langle T T_M^{-\frac{1}{2}}\Psi, \mathrm{N} T T_M^{-\frac{1}{2}}\Psi\rangle = \lim_{j\to\infty}\langle \Phi_j, \mathrm{N} T T_M^{-\frac{1}{2}}\Psi\rangle \\ &= \lim_{j\to\infty}\langle([T,\mathrm{N}]+\mathrm{N}T)\Phi_j, T_M^{-\frac{1}{2}}\Psi\rangle.\end{aligned} \tag{3.13}$$

The commutator $[\mathrm{N}, T]$ can be computed as follows

$$[\mathrm{N}, T] = i(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x}))\cdot\boldsymbol{\pi} + i\boldsymbol{\pi}\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})),$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is defined by

$$\pi_\mu = i[\mathrm{N}, A_\mu(\boldsymbol{x})] = \frac{i}{\sqrt{2}}(-a(\overline{g_\mu(\boldsymbol{x})}) + a^\dagger(g_\mu(\boldsymbol{x}))),$$

with $g_\mu(\boldsymbol{x}) = e_\mu\phi_\omega e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \in W$. Since $\sum_{\mu=1}^3[A_\mu(\boldsymbol{x}), \pi_\mu] = 2i\|\phi_\omega\|^2$, we have

$$[\mathrm{N}, T] = 2i\boldsymbol{\pi}\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})) + 2\|\phi_\omega\|^2.$$

Thus, (3.13) becomes

$$\begin{aligned}&\lim_{j\to\infty}(-2i\langle\Phi_j, \boldsymbol{\pi}\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x}))T_M^{-\frac{1}{2}}\Psi\rangle - 2\|\phi_\omega\|^2\langle\Phi_j, T_M^{-\frac{1}{2}}\Psi\rangle + \langle T\Phi_j, \mathrm{N}T_M^{-\frac{1}{2}}\Psi\rangle) \\ &\quad= -2i\langle T T_M^{-\frac{1}{2}}\Psi, \boldsymbol{\pi}\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x}))T_M^{-\frac{1}{2}}\Psi\rangle - 2\|\phi_\omega\|^2\langle T T_M^{-\frac{1}{2}}\Psi, T_M^{-\frac{1}{2}}\Psi\rangle \\ &\qquad+ \langle T^2 T_M^{-\frac{1}{2}}\Psi, \mathrm{N}T_M^{-\frac{1}{2}}\Psi\rangle \\ &\quad\le -2i\langle T T_M^{-\frac{1}{2}}\Psi, \boldsymbol{\pi}\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x}))T_M^{-\frac{1}{2}}\Psi\rangle + \langle T^2 T_M^{-\frac{1}{2}}\Psi, \mathrm{N}T_M^{-\frac{1}{2}}\Psi\rangle.\end{aligned}$$

Hence, by the Schwarz inequality, we have

$$\begin{aligned}\|\mathrm{N}^{\frac{1}{2}} T_M^{-\frac{1}{2}} T\Psi\|^2 \le 2\Big(\sum_{\mu=1}^3\|\pi_\mu T T_M^{-\frac{1}{2}}\Psi\|^2\Big)^{\frac{1}{2}}\Big(\sum_{\mu=1}^3\|(p_\mu - A_\mu(\boldsymbol{x}))T_M^{-\frac{1}{2}}\Psi\|^2\Big)^{\frac{1}{2}} \\ + \|\mathrm{N}T^2 T_M^{-\frac{1}{2}}\Psi\|\|T_M^{-\frac{1}{2}}\Psi\|.\end{aligned}$$

Noting $\sum_{\mu=1}^3(p_\mu - A_\mu(\boldsymbol{x}))^2 = T$ and

$$\sum_{\mu=1}^3\|\pi_\mu\Phi\|^2 \le 4\|\phi_\omega\|^2\|(\mathrm{N}+\mathbb{1})^{\frac{1}{2}}\Phi\|^2$$

for $\Phi \in \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$, we have the bound

$$\|\mathrm{N}^{\frac{1}{2}} T_M^{-\frac{1}{2}} T\Psi\|^2 \le 4\|\phi_\omega\|\|(\mathrm{N}+\mathbb{1})^{\frac{1}{2}} T T_M^{-\frac{1}{2}}\Psi\|\|\Psi\| + \|\mathrm{N}T^2 T_M^{-\frac{1}{2}}\Psi\|\|T_M^{-\frac{1}{2}}\Psi\|. \tag{3.14}$$

By Lemma 3.9, we have

$$\limsup_{M\to 0} \|\mathrm{N} T^2 T_M^{-\frac{1}{2}} \Psi\| < \infty, \quad \limsup_{M\to 0} \|(\mathrm{N}+\mathbb{1})^{\frac{1}{2}} T^2 T_M^{-\frac{1}{2}} \Psi\| < \infty. \tag{3.15}$$

On the other hand, since $\Psi \in \mathrm{D}(|\boldsymbol{x}|)$, by Lemma 3.4, we have $\Psi \in \mathrm{D}(T^{-\frac{1}{2}})$ and

$$\lim_{M\to 0} \|T_M^{-\frac{1}{2}} \Psi\| = \|T^{-\frac{1}{2}} \Psi\| \le 2\| |\boldsymbol{x}| \Psi\| < \infty. \tag{3.16}$$

Therefore, from (3.14)–(3.16), we conclude that (3.12) holds. By Lemma 3.2 $T\Psi \in \mathrm{D}(\mathrm{N})$, and hence

$$T T_M^{-\frac{1}{2}} \Psi = T_M^{-\frac{1}{2}} T\Psi \in \mathrm{D}(\mathrm{N})$$

by Lemma 3.8. Thus, $\mathrm{N}^{\frac{1}{2}} T_M^{-\frac{1}{2}} T\Psi \in \mathscr{H}$. By (3.12), for any $\Phi \in \mathscr{H}_{\mathrm{fin}}$, we see that

$$\begin{aligned}
\left|\langle T^{\frac{1}{2}}\Psi, \mathrm{N}^{\frac{1}{2}}\Phi\rangle\right| &= \lim_{M\to 0} |\langle T T_M^{-\frac{1}{2}}\Psi, \mathrm{N}^{\frac{1}{2}}\Phi\rangle| = \lim_{M\to 0} |\langle \mathrm{N}^{\frac{1}{2}} T T_M^{-\frac{1}{2}}\Psi, \Phi\rangle| \\
&\le (\limsup_{M\to 0} \|\mathrm{N}^{\frac{1}{2}} T T_M^{-\frac{1}{2}}\Psi\|)\|\Phi\|.
\end{aligned}$$

Since $\mathscr{H}_{\mathrm{fin}}$ is a core for $\mathrm{N}^{\frac{1}{2}}$, the above bound implies $T^{\frac{1}{2}}\Psi \in \mathrm{D}((\mathrm{N}^{\frac{1}{2}})^*) = \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$, which completes the proof of Proposition 3.1. □

4. Singular and non-local pull-through formulae

Throughout we assume that (A1)–(A4) hold. For $0 < m < m_0$, recall that Φ_m is the normalized ground state of H_m. For each function $\Psi^{(n+1)} \in \otimes_{\mathrm{s}}^{n+1} W$, the map $\mathbb{R}^3 \times \{1,2\} \ni k \mapsto \Psi^{(n+1)}(k,\ldots)$ is a $\otimes_{\mathrm{s}}^n W$-valued function, and

$$\int \|\Psi^{(n+1)}(k,\ldots)\|^2_{\otimes_{\mathrm{s}}^n W} dk = \|\Psi^{(n+1)}\|^2_{\otimes_{\mathrm{s}}^{n+1} W}$$

holds. Thus, for each $\Psi \in \mathscr{F}$ and almost every k, one can define the function

$$(a\Psi)(k) = (\sqrt{n+1}\Psi^{(n+1)}(k,\cdot))_{n=0}^{\infty} \in \mathop{\times}_{n=0}^{\infty} (\otimes_{\mathrm{s}}^n W),$$

where $\times$ denotes the Cartesian product. We write $a(k)\Psi$ for $(a\Psi)(k)$. We can check that $\Psi \in \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$ if and only if

(1) $a(k)\Psi \in \mathscr{F}$ a.e. k and

(2) $\int \|a(k)\Psi\|^2_{\mathscr{F}} dk < \infty$.

If $\Psi \in D(\mathrm{N}^{\frac{1}{2}})$, then

$$\|\mathrm{N}^{\frac{1}{2}}\Psi\|_{\mathcal{F}}^2 = \int \|a(k)\Psi\|_{\mathcal{F}}^2 dk,$$
$$\langle \Phi, a(f)\Psi\rangle_{\mathcal{F}} = \int f(k)\langle \Phi, a(k)\Psi\rangle_{\mathcal{F}} dk$$

hold for all $\Phi \in \mathcal{F}$ and $f \in W$. For $\Psi \in \mathcal{H} = L^2(\mathbb{R}^3_{\boldsymbol{x}}) \otimes \mathcal{F}$, we can define $a(k)\Psi$ by $a(k)\Psi = \Psi(\boldsymbol{x}, k, \ldots)$. In this section, we will establish the pull-through formula

$$a(k)\Phi_m = \phi_\omega(\boldsymbol{k})(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1} J(k)\Phi_m, \tag{4.1}$$

where $J(k)$ is an operator valued function. In the case of $M = 0$, it is crucial to consider the operator domain in the derivation of (4.1).

Let $f \in C_0^\infty(\mathbb{R}^3 \times \{1,2\})$ and $\Psi \in \mathscr{H}_{\mathrm{fin}}$. By Proposition 3.1, we have

$$T_{\boldsymbol{A}}^{\frac{1}{2}}\Psi \in \mathrm{D}(\mathrm{N}^{\frac{1}{2}}) \subset \mathrm{D}(a^\dagger(f)) \quad \text{and} \quad a^\dagger(f)\Psi \in \mathscr{H}_{\mathrm{fin}} \subset \mathrm{D}(H_m)$$

follows. From these facts, we can verify the following calculations:

$$\begin{aligned}
&\langle (H_m - E_m)\Psi, a(\bar{f})\Phi_m\rangle \\
&\quad = \langle a^\dagger(f)(H_m - E_m)\Psi, \Phi_m\rangle \\
&\quad = \langle [a^\dagger(f), H_m - E_m]\Psi, \Phi_m\rangle + \langle (H_m - E_m)a^\dagger(f)\Psi, \Phi_m\rangle \\
&\quad = \langle [a^\dagger(f), H_m]\Psi, \Phi_m\rangle.
\end{aligned}$$

Since

$$[a^\dagger(f), H_m] = [a^\dagger(f), \sqrt{T_{\boldsymbol{A}}}] - a^\dagger(\omega_m f)$$

holds on $\mathscr{H}_{\mathrm{fin}}$, we have

$$\begin{aligned}
&\langle (H_m - E_m)\Psi, a(\bar{f})\Phi_m\rangle \\
&\quad = \langle [a^\dagger(f), \sqrt{T_{\boldsymbol{A}}}]\Psi, \Phi_m\rangle - \langle \Psi, a(\omega_m \bar{f})\Phi_m\rangle \\
&\quad = \langle \sqrt{T_{\boldsymbol{A}}}\Psi, a(\bar{f})\Phi_m\rangle - \langle a^\dagger(f)\Psi, \sqrt{T_{\boldsymbol{A}}}\Phi_m\rangle - \langle \Psi, a(\omega_m \bar{f})\Phi_m\rangle \\
&\quad = \frac{2}{\pi}\int_0^\infty \big(\langle T_{\boldsymbol{A}} R_{t^2}\Psi, a(\bar{f})\Phi_m\rangle - \langle a^\dagger(f)\Psi, T_{\boldsymbol{A}} R_{t^2}\Phi_m\rangle\big)dt - \langle \Psi, a(\omega_m \bar{f})\Phi_m\rangle \\
&\quad = \frac{2}{\pi}\int_0^\infty \langle [a^\dagger(f), T_{\boldsymbol{A}} R_{t^2}]\Psi, \Phi_m\rangle dt - \langle \Psi, a(\omega_m \bar{f})\Phi_m\rangle,
\end{aligned} \tag{4.2}$$

where we used the formula

$$\sqrt{S} = \frac{2}{\pi} \int_0^\infty \frac{S}{S+t^2} dt, \quad S \geq 0. \tag{4.3}$$

We shall compute the commutator in the integrand of (4.2). It is enough to consider the case $t > 0$. Note that $R_{t^2}\Psi \in \mathrm{D(N)}$ by Lemma 3.7. By $S/(S+t^2) = \mathbf{1} - t^2/(S+t^2)$ and the resolvent equation, we have

$$\begin{aligned} \langle [a^\dagger(f), T_{\boldsymbol{A}} R_{t^2}]\Psi, \Phi_m\rangle &= -t^2 \langle [a^\dagger(f), R_{t^2}]\Psi, \Phi_m\rangle \\ &= -t^2 \langle [T_{\boldsymbol{A}}, a^\dagger(f)] R_{t^2}\Psi, R_{t^2}\Phi_m\rangle. \end{aligned}$$

The commutator above is easily seen to be

$$\begin{aligned} &[T_{\boldsymbol{A}}, a^\dagger(f)] \\ &\quad = (\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))\cdot[\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}), a^\dagger(f)] + [\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}), a^\dagger(f)]\cdot(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})) \\ &\quad = -\sqrt{2}(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))\cdot\langle e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}\phi_\omega, f\rangle_W, \end{aligned}$$

where

$$(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}\phi_\omega)(\boldsymbol{k}, j) = e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi_\omega(\boldsymbol{k})(e_1(k), e_2(k), e_3(k)).$$

Thus,

$$\begin{aligned} &\langle [a^\dagger(f), T_{\boldsymbol{A}} R_{t^2}]\Psi, \Phi_m\rangle \\ &\quad = \sqrt{2}t^2 \langle \langle e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}\phi_\omega, f\rangle R_{t^2}\Psi, (\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))R_{t^2}\Phi_m\rangle \\ &\quad = \sqrt{2}t^2 \int \overline{f(k)}\phi_\omega(\boldsymbol{k})\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Psi, \boldsymbol{e}(k)\cdot(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))R_{t^2}\Phi_m\rangle dk \\ &\quad = \sqrt{2}t^2 \int \overline{f(k)}\phi_\omega(\boldsymbol{k})\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Psi, V_{\boldsymbol{e}(k)} R_{t^2}\Phi_m\rangle dk, \end{aligned} \tag{4.4}$$

where, for $\boldsymbol{w} \in \mathbb{R}^3$, we introduced the operator

$$V_{\boldsymbol{w}} = \boldsymbol{w}\cdot(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})). \tag{4.5}$$

Therefore, the first term in (4.2) becomes

$$\begin{aligned} &\frac{2}{\pi}\int_0^\infty \langle [a^\dagger(f), T_{\boldsymbol{A}} R_{t^2}]\Psi, \Phi_m\rangle dt \\ &\quad = \frac{2\sqrt{2}}{\pi}\int_0^\infty t^2 dt \int \overline{f(k)}\phi_\omega(\boldsymbol{k})\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Psi, V_{\boldsymbol{e}(k)} R_{t^2}\Phi_m\rangle dk. \end{aligned} \tag{4.6}$$

Although the iterated integral in (4.6) converges, the total integrability is not clear, especially around $t = 0$. In order to use Fubini's lemma, we have to show the total integrability of (4.6).

We show several properties on $V_{\boldsymbol{w}}$ in the next lemma.

Lemma 4.1. *Assume* (A1) *and* (A2). *Then, for any* $\boldsymbol{w} \in \mathbb{R}^3$, $V_{\boldsymbol{w}}$ *is essentially self-adjoint on* $\mathscr{H}_{\text{fin}}$. *We use the same symbol for its closure. Moreover, the following hold*:

(1) *if* $\Psi \in \mathrm{D}(T_{\boldsymbol{A}}^{\frac{1}{2}})$, *then* $\Psi \in \mathrm{D}(V_{\boldsymbol{w}})$ *and* $\|V_{\boldsymbol{w}}\Psi\| \leq |\boldsymbol{w}|\|T_{\boldsymbol{A}}^{\frac{1}{2}}\Psi\|$;

(2) *if* $\Psi \in \mathrm{D}(T_{\boldsymbol{A}}^{\frac{1}{4}})$, *then* $\Psi \in \mathrm{D}(|V_{\boldsymbol{w}}|^{\frac{1}{2}})$ *and* $\||V_{\boldsymbol{w}}|^{\frac{1}{2}}\Psi\| \leq |\boldsymbol{w}|^{\frac{1}{2}}\|T_{\boldsymbol{A}}^{\frac{1}{4}}\Psi\|$;

(3) *for all* $\boldsymbol{k} \in \mathbb{R}^3$ *with* $(k_1, k_2) \neq (0,0)$, $V_{\boldsymbol{e}(k)}$ *strongly commutes with* $e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$.

Proof. The essential self-adjointness follows from Nelson's commutator theorem with auxiliary operator $\boldsymbol{p}^2 + \mathrm{N} + \mathbf{1}$. For $\Psi \in \mathrm{D}(T_{\boldsymbol{A}}^{\frac{1}{2}})$, by the Schwarz inequality,

$$
\begin{aligned}
\|V_{\boldsymbol{w}}\Psi\|^2 &\leq \sum_{\mu,\nu=1}^{3} |w_\mu w_\nu||\langle (p_\mu - A_\mu(\boldsymbol{x}))\Psi, (p_\nu - A_\nu(\boldsymbol{x}))\Psi\rangle| \\
&\leq \Big(\sum_{\mu=1}^{3} |w_\mu|\|(p_\mu - A_\mu(\boldsymbol{x}))\Psi\| \Big)^2 \\
&\leq |\boldsymbol{w}|^2 \sum_{\mu=1}^{3} \|(p_\mu - A_\mu(\boldsymbol{x}))\Psi\|^2,
\end{aligned}
$$

which implies (1). Statement (2) can be derived from the Löwner–Heinz inequality [19, Theorem 2]. Finally, we prove (3). Note that $e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$ is a unitary operator. Noting $\boldsymbol{k}\cdot\boldsymbol{e}(k) = 0$, we can show that

$$
e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}(k)\cdot(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = \boldsymbol{e}(k)\cdot(\boldsymbol{p} - \boldsymbol{k} - \boldsymbol{A}(\boldsymbol{x})) = \boldsymbol{e}(k)\cdot(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}))
$$

on $\mathscr{H}_{\text{fin}}$. Taking the closure on both sides, we have $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}V_{\boldsymbol{e}(k)}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} = V_{\boldsymbol{e}(k)}$. Thus (3) is proven. □

The next lemma shows that the integral in (4.6) is absolutely convergent.

Lemma 4.2. *For* $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1,2\}$ *with* $(k_1, k_2) \neq (0,0)$ *and* $\Psi, \Phi \in \mathscr{H}$, *the bound*

$$
\int_0^\infty |\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Psi, V_{\boldsymbol{e}(k)} R_{t^2}\Phi\rangle| t^2 dt \leq \frac{\pi}{4}\|\Psi\|\|\Phi\| \tag{4.7}
$$

holds, and

$$t^2|f(k)\phi_\omega(\boldsymbol{k})\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}\Psi, V_{\boldsymbol{e}(k)}R_{t^2}\Phi\rangle|$$

is integrable in $(k,t)\in(\mathbb{R}^3\times\{1,2\})\times[0,\infty)$.

Proof. Note that $R_{t^2}\Phi, R_{t^2}\Psi\in \mathrm{D}(V_{\boldsymbol{e}(k)})$ for all $t>0$ and $\Psi,\Phi\in\mathcal{H}$. For $t>0$ and $\boldsymbol{k}\in\mathbb{R}^3\backslash L_{12}$, we have

$$\begin{aligned}
&|\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}\Psi, V_{\boldsymbol{e}(k)}R_{t^2}\Phi\rangle|\\
&\quad=|\langle |V_{\boldsymbol{e}(k)}|^{\frac12}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}\Psi, \mathrm{sgn}(V_{\boldsymbol{e}(k)})|V_{\boldsymbol{e}(k)}|^{\frac12}R_{t^2}\Phi\rangle|\\
&\quad\le \||V_{\boldsymbol{e}(k)}|^{\frac12}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}\Psi\|\||V_{\boldsymbol{e}(k)}|^{\frac12}R_{t^2}\Phi\| \qquad (4.8)\\
&\quad\le \||V_{\boldsymbol{e}(k)}|^{\frac12}R_{t^2}\Psi\|\||V_{\boldsymbol{e}(k)}|^{\frac12}R_{t^2}\Phi\|\\
&\quad\le \|T_{\boldsymbol{A}}^{\frac14}R_{t^2}\Psi\|\|T_{\boldsymbol{A}}^{\frac14}R_{t^2}\Phi\|,
\end{aligned}$$

where we used Lemma 4.1 and the fact that $\boldsymbol{e}(k)$ is a normalized vector. Thus,

$$\begin{aligned}
&\int_0^\infty \left|\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}\Psi, V_{\boldsymbol{e}(k)}R_{t^2}\Phi_m\rangle\right|t^2dt\\
&\quad\le\Big(\int_0^\infty \|T_{\boldsymbol{A}}^{\frac14}R_{t^2}\Phi\|^2dt\Big)^{\frac12}\Big(\int_0^\infty \|T_{\boldsymbol{A}}^{\frac14}R_{t^2}\Psi\|^2t^2dt\Big)^{\frac12}.
\end{aligned}$$

Since $\int_0^\infty \|T_{\boldsymbol{A}}^{\frac14}R_{t^2}\Psi\|^2t^2dt=(\pi/4)\|\Psi\|^2$, (4.7) follows. □

As a consequence of Lemma 4.2, we can apply Fubini's lemma to (4.6), and we have

$$(4.6)=\frac{2\sqrt2}{\pi}\int\overline{f(k)}\phi_\omega(\boldsymbol{k})dk\int_0^\infty\langle e^{i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}\Psi, V_{\boldsymbol{e}(k)}R_{t^2}\Phi_m\rangle t^2dt. \qquad (4.9)$$

Thus, we obtain the following result.

Corollary 4.3. *For each* $k=(\boldsymbol{k},j)\in\mathbb{R}^3\times\{1,2\}$ *with* $(k_1,k_2)\neq(0,0)$*, the integral*

$$J(k)=\frac{2\sqrt2}{\pi}\int_0^\infty R_{t^2}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}V_{\boldsymbol{e}(k)}R_{t^2}t^2dt \qquad (4.10)$$

defines a bounded operator on $\mathcal{H}$ *with the operator norm*

$$\|J(k)\|\le\frac{1}{\sqrt2}.$$

Proof. This is a direct consequence of Lemma 4.2. □

Now we can state the main proposition in this section.

Proposition 4.4 (singular and non-local pull-through formula). *Assume conditions* (A1)–(A4). *For all $m > 0$ and a.e. $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1,2\}$, it follows that*

$$a(k)\Phi_m = \phi_\omega(\boldsymbol{k})(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1} J(k)\Phi_m. \tag{4.11}$$

Proof. Combining (4.9) and Corollary 4.3, we have the identity

$$\int \overline{f(k)} \langle (H_m - E_m)\Psi, a(k)\Phi_m\rangle dk + \int \overline{f(k)}\omega_m(\boldsymbol{k})\langle \Psi, a(k)\Phi_m\rangle dk$$
$$= \int \overline{f(k)}\phi_\omega(\boldsymbol{k})\langle \Psi, J(k)\Phi_m\rangle dk$$

for all $f \in C_0^\infty(\mathbb{R}^3 \times \{1,2\})$ and $\Psi \in \mathcal{H}_{\mathrm{fin}}$. Thus

$$\langle (H_m - E_m + \omega_m(\boldsymbol{k}))\Psi, a(k)\Phi_m\rangle = \phi_\omega(\boldsymbol{k})\langle \Psi, J(k)\Phi_m\rangle \tag{4.12}$$

holds for all $\Psi \in \mathcal{H}_{\mathrm{fin}}$ and $k = (\boldsymbol{k}, j) \in (\mathbb{R}^3 \times \{1,2\}) \setminus N_\Psi$ with some null sets N_Ψ. Since $\mathcal{H}_{\mathrm{fin}}$ is dense and we can take a countable dense subset $\mathcal{D}$ of $\mathcal{H}_{\mathrm{fin}}$, (4.12) holds for $\Psi \in \mathcal{D}$ for $k \in (\mathbb{R}^3 \times \{1,2\}) \setminus \left(\bigcup_{\Phi \in \mathcal{D}} N_\Phi\right)$:

$$(H_m - E_m + \omega_m(\boldsymbol{k}))a(k)\Phi_m = \phi_\omega(\boldsymbol{k})J(k)\Phi_m$$

for $k \in (\mathbb{R}^3 \times \{1,2\}) \setminus \left(\bigcup_{\Phi \in \mathcal{D}} N_\Phi\right)$. Therefore (4.11) follows. □

5. Photon number localization

Our goal in this section is to prove the following result.

Proposition 5.1. *Assume* (A1)–(A4). *Let $0 < m < m_0$. Then, there exists a constant $C > 0$ independent of m such that*

$$\|a(k)\Phi_m\|^2 \leq C\frac{|\hat{\varphi}(\boldsymbol{k})|^2}{\omega(\boldsymbol{k})}(1 + |\boldsymbol{k}|)^2 \tag{5.1}$$

for a.e. $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1,2\}$.

We can show the uniform photon number localization of Φ_m as a corollary of Proposition 5.1:

Corollary 5.2. *Assume* (A1)–(A4). *Then*

$$\sup_{0<m<m_0} \|\mathrm{N}^{\frac{1}{2}}\Phi_m\| < \infty.$$

Proof. By Corollary 4.3, we have the bound

$$\|a(k)\Phi_m\|^2 \leq |\phi_\omega(\boldsymbol{k})|^2 \|(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}\|^2 \|J(k)\|^2 \leq \frac{|\hat{\varphi}(\boldsymbol{k})|^2}{2\omega(\boldsymbol{k})^{\frac{3}{2}}}. \tag{5.2}$$

Combining (5.1) and (5.2), we get the bound

$$\|a(k)\Phi_m\|^2 \leq \frac{|\hat{\varphi}(\boldsymbol{k})|^2}{2\omega(\boldsymbol{k})} \min\{2C(1+\boldsymbol{k}^2), \omega(\boldsymbol{k})^{-\frac{1}{2}}\}. \tag{5.3}$$

By (5.3), we get

$$\|\mathrm{N}^{\frac{1}{2}}\Phi_m\|^2 \leq \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(\boldsymbol{k})|^2}{2\omega(\boldsymbol{k})} \min\{2C(1+\boldsymbol{k}^2), \omega(\boldsymbol{k})^{-\frac{1}{2}}\} d\boldsymbol{k} < \infty.$$

Take $\sup_{0<m<m_0}$ on both sides above. Thus the corollary follows. □

Remark 5.3. The right-hand side of (5.2) has a singularity at $\boldsymbol{k} = 0$, and then the right-hand side of (5.2) is not integrable if $\hat{\varphi}(0) \neq 0$. This type of singularity is often referred to as an infrared divergence.

To derive (5.1) we use a method due to [13, p. 214] and [14, (7.7)]. We decompose $J(k)$ into three terms:

$$J(k) = \frac{2\sqrt{2}}{\pi}(L_1(k)\langle \boldsymbol{x}\rangle^2 + L_2(k)\langle \boldsymbol{x}\rangle + L_3(k)), \tag{5.4}$$

where

$$L_1 = L_1(k) = \int_0^1 R_{t^2} V_{\boldsymbol{e}(k)}(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1) R_{t^2} \langle \boldsymbol{x}\rangle^{-2} t^2 dt,$$

$$L_2 = L_2(k) = \int_1^\infty R_{t^2} V_{\boldsymbol{e}(k)}(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1) R_{t^2} \langle \boldsymbol{x}\rangle^{-1} t^2 dt,$$

$$L_3 = L_3(k) = \int_0^\infty R_{t^2} V_{\boldsymbol{e}(k)} R_{t^2} t^2 dt.$$

Note that the velocity operator $V_{\boldsymbol{e}(k)}$ commutes with $e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}$.

5.1. Estimate on L_1. In order to prove that $L_1(k)$ is bounded, we introduce an operator Z by

$$Z = \int_0^1 \langle \boldsymbol{x} \rangle^{-2} (t^2 + \boldsymbol{p}^2)^{-1} \boldsymbol{x}^2 (t^2 + \boldsymbol{p}^2)^{-1} \langle \boldsymbol{x} \rangle^{-2} t^3 dt. \tag{5.5}$$

Lemma 5.4. *The operator Z is non-negative, bounded and $\|Z\| \leq 6$.*

Proof. Since Z is symmetric and non-negative, it is enough to show that $|\langle u, Zu \rangle| \leq C\|u\|^2, u \in L^2(\mathbb{R}^3)$ for some $C > 0$. We use the commutation relation:

$$x_\mu (t^2 + \boldsymbol{p}^2)^{-1} = (t^2 + \boldsymbol{p}^2)^{-1} x_\mu + \frac{-2i p_\mu}{(t^2 + \boldsymbol{p}^2)^2}.$$

For $u \in L^2(\mathbb{R}^3)$, we have

$$\begin{aligned}
|\langle u, Zu \rangle| &= \sum_{\mu=1}^{3} \int_0^1 \|x_\mu (t^2 + \boldsymbol{p}^2)^{-1} \langle \boldsymbol{x} \rangle^{-2} u\|^2 t^3 dt \\
&= \sum_{\mu=1}^{3} \int_0^1 \|(t^2 + \boldsymbol{p}^2)^{-1} x_\mu \langle \boldsymbol{x} \rangle^{-2} u\|^2 t^3 dt \\
&\quad + 4 \operatorname{Im} \int_0^1 \langle \boldsymbol{p} \cdot \boldsymbol{x} \langle \boldsymbol{x} \rangle^{-2} u, (t^2 + \boldsymbol{p}^2)^{-3} \langle \boldsymbol{x} \rangle^{-2} u \rangle t^3 dt \\
&\quad + \sum_{\mu=1}^{3} \int_0^1 \|-2i p_\mu (t^2 + \boldsymbol{p}^2)^{-2} \langle \boldsymbol{x} \rangle^{-2} u\|^2 t^3 dt.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_0^1 \frac{t^3}{(t^2 + \boldsymbol{p}^2)^2} dt = \frac{1}{2} \left(\log \left(1 + \frac{1}{\boldsymbol{p}^2} \right) - \frac{1}{1 + \boldsymbol{p}^2} \right) < \frac{1}{2\boldsymbol{p}^2}, \\
&\int_0^1 \frac{t^3}{(t^2 + \boldsymbol{p}^2)^3} dt = \frac{1}{4\boldsymbol{p}^2 (1 + \boldsymbol{p}^2)^2} \leq \frac{1}{4\boldsymbol{p}^2}, \\
&\int_0^1 \frac{t^3}{(t^2 + \boldsymbol{p}^2)^4} dt = \frac{1}{12|\boldsymbol{p}|^4 (1 + \boldsymbol{p}^2)^2} + \frac{1}{6\boldsymbol{p}^2 (1 + \boldsymbol{p}^2)^3} \leq \frac{1}{12|\boldsymbol{p}|^4}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
|\langle u, Zu\rangle| &\leq \frac{1}{2}\sum_{\mu=1}^{3} \||\boldsymbol{p}|^{-1}x_\mu \langle \boldsymbol{x}\rangle^{-2}u\|^2 + \sum_{\mu=1}^{3} \|x_\mu \langle \boldsymbol{x}\rangle^{-2}u\| \left\| \frac{p_\mu}{\boldsymbol{p}^2} \langle \boldsymbol{x}\rangle^{-2} u \right\| \\
&\quad + \frac{1}{3}\||\boldsymbol{p}|^{-1} \langle \boldsymbol{x}\rangle^{-2}u\|^2 \\
&\leq \frac{1}{2}\sum_{\mu=1}^{3} \||\boldsymbol{p}|^{-1}x_\mu \langle \boldsymbol{x}\rangle^{-2}u\|^2 + \|\langle \boldsymbol{x}\rangle^{-1} u\| \||\boldsymbol{p}|^{-1}\langle \boldsymbol{x}\rangle^{-2} u\| \\
&\quad + \frac{1}{3}\||\boldsymbol{p}|^{-1} \langle \boldsymbol{x}\rangle^{-2} u\|^2.
\end{aligned}
$$

By Hardy's inequality, we have

$$
\begin{aligned}
|\langle u, Zu\rangle| &\leq 2\|\boldsymbol{x}^2 \langle \boldsymbol{x}\rangle^{-2} u\|^2 + 2\|\langle \boldsymbol{x}\rangle^{-1} u\| \||\boldsymbol{x}| \langle \boldsymbol{x}\rangle^{-2} u\| + \frac{4}{3}\||\boldsymbol{x}| \langle \boldsymbol{x}\rangle^{-2} u\|^2 \\
&\leq \frac{16}{3}\|u\|^2 \leq 6\|u\|^2
\end{aligned}
$$

for all $u \in L^2(\mathbb{R}^3)$. Then the proof is complete. □

Lemma 5.5. *For every $k \in \mathbb{R}^3 \times \{1, 2\}$, operator $L_1(k)$ is bounded and*

$$
\|L_1(k)\| \leq 2|\boldsymbol{k}|. \tag{5.6}
$$

Proof. For any $\Psi, \Phi \in \mathcal{H}$, we have

$$
\begin{aligned}
&|\langle \Psi, L_1(k)\Phi\rangle| \\
&\quad \leq \int_0^1 \|V_{\boldsymbol{e}(k)} R_{t^2}\Psi\| \|\boldsymbol{k}\cdot \boldsymbol{x} R_{t^2} \langle \boldsymbol{x}\rangle^{-2} \Phi\| t^2 dt \\
&\quad \leq |\boldsymbol{k}| \int_0^1 \|T_{\boldsymbol{A}}^{\frac{1}{2}} R_{t^2}\Psi\| \||\boldsymbol{x}|(\boldsymbol{p}^2 + t^2)^{-1} \langle \boldsymbol{x}\rangle^{-2} |\Phi|\| t^2 dt \\
&\quad \leq |\boldsymbol{k}| \left(\int_0^1 \|T_{\boldsymbol{A}}^{\frac{1}{2}} R_{t^2}\Psi\|^2 t dt\right)^{\frac{1}{2}} \left(\int_0^1 \||\boldsymbol{x}|(\boldsymbol{p}^2 + t^2)^{-1} \langle \boldsymbol{x}\rangle^{-2} |\Phi|\|^2 t^3 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Here we used Lemma 4.1 and the diamagnetic inequality (Lemma 3.4 (3)) for the second inequality, and the Schwarz inequality for the third inequality. Since

$$
\left\| \int_0^1 \frac{T_{\boldsymbol{A}}}{(T_{\boldsymbol{A}} + t^2)^2} t dt \right\| = \left\| \frac{1}{2(T_{\boldsymbol{A}} + 1)} \right\| \leq \frac{1}{2},
$$

we have

$$|\langle \Psi, L_1(k)\Phi\rangle| \leq \frac{1}{\sqrt{2}}|\boldsymbol{k}|\|\Psi\|\langle|\Phi|, Z|\Phi|\rangle^{\frac{1}{2}}.$$

This estimate and Lemma 5.4 imply (5.6). $\square$

5.2. Estimate on L_2. We shall estimate $L_2(k)$. Set

$$T_{\boldsymbol{A}-\boldsymbol{k}} = (\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{A}(\boldsymbol{x}))^2, \quad R_{t^2}(\boldsymbol{k}) = (T_{\boldsymbol{A}-\boldsymbol{k}} + t^2)^{-1}.$$

We have the identities

$$\begin{aligned}(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)R_{t^2} &= R_{t^2}(\boldsymbol{k})(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1) + R_{t^2}(T_{\boldsymbol{A}} - T_{\boldsymbol{A}-\boldsymbol{k}})R_{t^2}(\boldsymbol{k}),\\ T_{\boldsymbol{A}} - T_{\boldsymbol{A}-\boldsymbol{k}} &= -2V_{\boldsymbol{k}} - \boldsymbol{k}^2.\end{aligned}$$

We then have

$$(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)R_{t^2} = R_{t^2}(\boldsymbol{k})(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1) - 2R_{t^2}V_{\boldsymbol{k}}R_{t^2}(\boldsymbol{k}) - \boldsymbol{k}^2R_{t^2}R_{t^2}(\boldsymbol{k}).$$

According to above identity, we decompose $L_2(k)$ into three terms:

$$L_2(k) = L_{21}(k) + L_{22}(k) + L_{23}(k), \tag{5.7}$$

where

$$\begin{aligned}L_{21}(k) &= \int_1^\infty R_{t^2}V_{\boldsymbol{e}(k)}R_{t^2}(\boldsymbol{k})(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)\,\langle\boldsymbol{x}\rangle^{-1}\,t^2dt,\\ L_{22}(k) &= -2\int_1^\infty R_{t^2}V_{\boldsymbol{e}(k)}R_{t^2}V_{\boldsymbol{k}}R_{t^2}(\boldsymbol{k})\,\langle\boldsymbol{x}\rangle^{-1}\,t^2dt,\\ L_{23}(k) &= -\boldsymbol{k}^2\int_1^\infty R_{t^2}V_{\boldsymbol{e}(k)}R_{t^2}R_{t^2}(\boldsymbol{k})\,\langle\boldsymbol{x}\rangle^{-1}\,t^2dt.\end{aligned}$$

In order to estimate L_{21}, we show the next lemma.

Lemma 5.6. *If* $\Psi \in \mathrm{D}(T_{\boldsymbol{A}-\boldsymbol{k}}^{\frac{1}{2}})$ *and* $\Phi \in \mathrm{D}(T_{\boldsymbol{A}-\boldsymbol{k}}^{\frac{1}{4}})$, *then*

$$\|V_{\boldsymbol{e}(k)}\Psi\| \leq \|T_{\boldsymbol{A}-\boldsymbol{k}}^{\frac{1}{2}}\Psi\| \tag{5.8}$$

and

$$\||V_{\boldsymbol{e}(k)}|^{\frac{1}{2}}\Phi\| \leq \|T_{\boldsymbol{A}-\boldsymbol{k}}^{\frac{1}{4}}\Phi\| \tag{5.9}$$

hold for $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1, 2\}$.

Proof. Note that $\boldsymbol{e}(k) \perp \boldsymbol{k}$ and $V_{\boldsymbol{e}(k)} = \boldsymbol{e}(k) \cdot (\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{A}(\boldsymbol{x}))$ hold. Thus, the proof is the same as that of Lemma 4.1. □

Lemma 5.7. *For $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1,2\}$, we have*

$$\|L_{21}(k)\| \le |\boldsymbol{k}|. \tag{5.10}$$

Proof. Write $V_{\boldsymbol{e}(k)} = \operatorname{sgn}(V_{\boldsymbol{e}(k)})|V_{\boldsymbol{e}(k)}|$. By the Schwarz inequality, Lemmas 4.1, and 5.6, we have

$$\begin{aligned}
&|\langle \Psi, L_{21}(k)\Phi\rangle| \\
&\quad\le \int_1^\infty \|\operatorname{sgn}(V_{\boldsymbol{e}(k)})|V_{\boldsymbol{e}(k)}|^{\frac{1}{2}} R_{t^2}\Psi\| \||V_{\boldsymbol{e}(k)}|^{\frac{1}{2}} R_{t^2}(\boldsymbol{k})(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)\langle \boldsymbol{x}\rangle^{-1}\Phi\| t^2 dt \\
&\quad\le \left(\int_0^\infty \|T_{\boldsymbol{A}}^{\frac{1}{4}} R_{t^2}\Psi\|^2 t^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty \|T_{\boldsymbol{A}-\boldsymbol{k}}^{\frac{1}{4}} R_{t^2}(\boldsymbol{k})(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)\langle \boldsymbol{x}\rangle^{-1}\Phi\|^2 t^2 dt\right)^{\frac{1}{2}} \\
&\quad= \left(\frac{\pi}{4}\|\Psi\|^2\right)^{\frac{1}{2}} \left(\frac{\pi}{4}\|(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)\langle \boldsymbol{x}\rangle^{-1}\Phi\|^2\right)^{\frac{1}{2}},
\end{aligned}$$

where we used

$$\int_0^\infty a t^2/(a^2 + t^2)^2 dt = \pi/4 \quad \text{for } a > 0.$$

Since $|(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - 1)\langle \boldsymbol{x}\rangle^{-1}| \le |\boldsymbol{k}|$ and $\pi/4 < 1$, (5.10) follows. □

Bounds for $L_{22}(k)$ and $L_{23}(k)$ are given in the following.

Lemma 5.8. *For $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1,2\}$,*

$$\|L_{22}(k)\| \le 2|\boldsymbol{k}| \quad \textit{and} \quad \|L_{23}(k)\| \le \boldsymbol{k}^2.$$

Proof. We have

$$\|L_{22}(k)\| \le 2\int_1^\infty \|t^2 R_{t^2}\| \|V_{\boldsymbol{e}(k)} R_{t^2}^{\frac{1}{2}}\| \|R_{t^2}^{\frac{1}{2}} V_{\boldsymbol{k}}\| \|R_{t^2}(\boldsymbol{k})\langle \boldsymbol{x}\rangle^{-1}\| dt.$$

By Lemma 5.6, $\|V_{\boldsymbol{e}(k)} R_{t^2}^{\frac{1}{2}}\| \le 1$ and $\|R_{t^2}^{\frac{1}{2}} V_{\boldsymbol{k}}\| = \|V_{\boldsymbol{k}} R_{t^2}^{\frac{1}{2}}\| \le |\boldsymbol{k}|$. Thus,

$$\|L_{22}(k)\| \le 2\int_1^\infty |\boldsymbol{k}| \cdot t^{-2} dt = 2|\boldsymbol{k}|.$$

Similarly,

$$\|L_{23}(k)\| \leq \boldsymbol{k}^2 \int_1^\infty \|t^2 R_{t^2}\| \|V_{\boldsymbol{e}(k)} R_{t^2}\| \|R_{t^2}(\boldsymbol{k}) \langle \boldsymbol{x} \rangle^{-1}\| dt$$
$$\leq \boldsymbol{k}^2 \int_1^\infty t^{-3} dt \leq \boldsymbol{k}^2. \qquad \square$$

5.3. Estimate on L_3. We shall estimate $L_3(k)$. A crucial property of $L_3(k)$ is the identity

$$L_3(k) = \frac{i\pi}{4}[T_{\boldsymbol{A}}^{\frac{1}{2}}, \boldsymbol{e}(k)\cdot\boldsymbol{x}] = \frac{i\pi}{4}[H_m - E_m, \boldsymbol{e}(k)\cdot\boldsymbol{x}],$$

which will enable us to obtain an infrared regular bound for $L_3(k)$. This is due to [13, p. 214] and [14, (7.7)]. Given two operators A and B, we define the quadratic form $[A, B]_{\mathrm{w}}$ as

$$[A, B]_{\mathrm{w}}(u, v) = \langle Au, Bv\rangle - \langle Bu, Av\rangle, \quad u, v \in \mathrm{D}(A) \cap \mathrm{D}(B).$$

We also write this as $\langle u, [A, B]_{\mathrm{w}} v\rangle$.

Lemma 5.9. *For $\Psi \in \mathcal{H}_{\mathrm{fin}}$ and $\Phi \in \mathrm{D}(H_m) \cap \mathrm{D}(|\boldsymbol{x}|)$,*

$$\langle \Psi, L_3(k)\Phi\rangle = \frac{i\pi}{4}\langle \Psi, [H_m - E_m, \boldsymbol{e}(k)\cdot\boldsymbol{x}]_{\mathrm{w}}\Phi\rangle.$$

In particular, $\boldsymbol{e}(k)\cdot\boldsymbol{x}\Phi_m \in \mathrm{D}(H_m)$ and it holds that

$$L_3(k)\Phi_m = \frac{i\pi}{4}(H_m - E_m)(\boldsymbol{e}(k)\cdot\boldsymbol{x})\Phi_m.$$

Proof. By the definition of L_3,

$$\langle \Psi, L_3(k)\Phi\rangle = \int_0^\infty \langle R_{t^2}\Psi, V_{\boldsymbol{e}(k)} R_{t^2}\Phi\rangle t^2 dt.$$

We note that, by Lemma 3.4, $R_{t^2}\Psi, R_{t^2}\Phi \in \mathrm{D}(|\boldsymbol{x}|)$ for $t > 0$. Since $T_{\boldsymbol{A}} R_{t^2} = \mathbf{1} - t^2 R_{t^2}$, we have

$$T_{\boldsymbol{A}} R_{t^2}\Psi, T_{\boldsymbol{A}} R_{t^2}\Phi \in \mathrm{D}(|\boldsymbol{x}|).$$

For any $\psi \in \mathcal{H}_{\mathrm{fin}}$, we have

$$V_{\boldsymbol{e}(k)}\psi = \frac{i}{2}[T_{\boldsymbol{A}}, \boldsymbol{e}(k)\cdot\boldsymbol{x}]\psi.$$

Thus, for $\varphi \in \mathrm{D}((\boldsymbol{e}(k) \cdot \boldsymbol{x})T_{\boldsymbol{A}})$, it follows that

$$\langle \psi, V_{\boldsymbol{e}(k)}\varphi \rangle = \frac{i}{2}(\langle T_{\boldsymbol{A}}\psi, \boldsymbol{e} \cdot \boldsymbol{x}\varphi \rangle - \langle \psi, (\boldsymbol{e} \cdot \boldsymbol{x})T_{\boldsymbol{A}}\varphi \rangle). \tag{5.11}$$

Since $\mathscr{H}_{\mathrm{fin}}$ is a core for $T_{\boldsymbol{A}}$, (5.11) can be extended for all $\psi \in \mathrm{D}(T_{\boldsymbol{A}}) \cap \mathrm{D}(|\boldsymbol{x}|)$. Hence,

$$\begin{aligned}
\langle R_{t^2}\Psi, V_{\boldsymbol{e}(k)}R_{t^2}\Phi \rangle &= \frac{i}{2}(\langle T_{\boldsymbol{A}}R_{t^2}\Psi, (\boldsymbol{e} \cdot \boldsymbol{x})R_{t^2}\Phi \rangle - \langle R_{t^2}\Psi, (\boldsymbol{e} \cdot \boldsymbol{x})T_{\boldsymbol{A}}R_{t^2}\Phi \rangle) \\
&= \frac{i}{2}(\langle \boldsymbol{e} \cdot \boldsymbol{x}\Psi, R_{t^2}\Phi \rangle - \langle R_{t^2}\Psi, \boldsymbol{e} \cdot \boldsymbol{x}\Phi \rangle) \\
&= \frac{i}{2t^2}(-\langle \boldsymbol{e} \cdot \boldsymbol{x}\Psi, T_{\boldsymbol{A}}R_{t^2}\Phi \rangle + \langle T_{\boldsymbol{A}}R_{t^2}\Psi, \boldsymbol{e} \cdot \boldsymbol{x}\Phi \rangle).
\end{aligned}$$

By the formula (4.3),

$$\langle \Psi, L_3(k)\Phi \rangle = \frac{i\pi}{4}\langle \Psi, [T_{\boldsymbol{A}}^{\frac{1}{2}}, \boldsymbol{e} \cdot \boldsymbol{x}]_{\mathrm{w}}\Phi \rangle = \frac{i\pi}{4}\langle \Psi, [H_m - E_m, \boldsymbol{e} \cdot \boldsymbol{x}]_{\mathrm{w}}\Phi \rangle. \qquad \square$$

5.4. Proof of Proposition 5.1

Proof of Proposition 5.1: By the singular and non-local pull-through formula (4.11) and the decomposition (5.4), we have

$$\begin{aligned}
\|a(k)\Phi_m\| \leq |\phi_\omega(\boldsymbol{k})|\frac{2\sqrt{2}}{\pi}\Big(&\omega_m(\boldsymbol{k})^{-1}\|L_1(k)\|\|\langle \boldsymbol{x}\rangle^2\,\Phi_m\| \\
&+ \omega_m(\boldsymbol{k})^{-1}\|L_2(k)\|\|\langle \boldsymbol{x}\rangle\,\Phi_m\| \\
&+ \|(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}L_3(k)\Phi_m\|\Big),
\end{aligned}$$

where we used the inequality

$$\|(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}\| \leq \omega_m(\boldsymbol{k})^{-1}.$$

By Lemmas 5.5, 5.7, 5.8 and (5.7), we have

$$\|L_1(k)\| \leq 2|\boldsymbol{k}| \quad \text{and} \quad \|L_2(k)\| \leq |\boldsymbol{k}| + 2|\boldsymbol{k}| + \boldsymbol{k}^2 \tag{5.12}$$

Moreover, by Lemma 5.9, we have

$$\begin{aligned}
&\|(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}L_3(k)\Phi_m\| \\
&\quad \leq \frac{\pi}{4}\|(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}(H_m - E_m)(\boldsymbol{e}(k) \cdot \boldsymbol{x})\Phi_m\| \\
&\quad \leq \frac{\pi}{4}\||\boldsymbol{x}|\Phi_m\|.
\end{aligned} \tag{5.13}$$

By assumption (A4), the bounds

$$\sup_{0<m<m_0} \||\boldsymbol{x}|\Phi_m\| < \infty, \qquad \sup_{0<m<m_0} \|\langle \boldsymbol{x}\rangle^2 \Phi_m\| < \infty \tag{5.14}$$

hold. Therefore, by (5.12)–(5.14), we have

$$\|a(k)\Phi_m\| \leq C|\phi_\omega(\boldsymbol{k})|\Big(\frac{|\boldsymbol{k}|+\boldsymbol{k}^2}{\omega_m(\boldsymbol{k})}+1\Big) \leq C\frac{|\hat{\varphi}(\boldsymbol{k})|}{\omega(\boldsymbol{k})^{\frac{1}{2}}}(2+|\boldsymbol{k}|), \quad 0<m<m_0,$$

for some $C>0$. This immediately implies (5.1). The integrability of $\|a(k)\Phi_m\|^2$ follows from the assumption (A2). □

6. Equicontinuity and spatial localization of photon

In this section we show that the photons of the massive ground state Φ_m are spatially localized uniformly in $0<m<m_0$. Throughout this section, we assume (A1)–(A4).

6.1. Continuity of $J(k)$. We shall show the continuity of $k \mapsto J(k)$ in this section. We decompose $J(k)-J(k')$ as follows

$$J(k)-J(k') = \Delta J_1 + \Delta J_2,$$

with

$$\Delta J_1 = \int_0^\infty R_{t^2}(V_{\boldsymbol{e}(k)} - V_{\boldsymbol{e}(k')})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}R_{t^2}t^2dt,$$

$$\Delta J_2 = \int_0^\infty R_{t^2}V_{\boldsymbol{e}(k')}(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}})R_{t^2}t^2dt.$$

Lemma 6.1. *Let $k=(\boldsymbol{k},j)$ and $k'=(\boldsymbol{k}',j)$. For any $\Phi \in \mathrm{D}(|\boldsymbol{x}|^{\frac{1}{2}})$ it follows that*

$$\|\Delta J_1\Phi\| \leq |\boldsymbol{e}(k)-\boldsymbol{e}(k')|(\|\Phi\| + |\boldsymbol{k}|^{\frac{1}{2}}\||\boldsymbol{x}|^{\frac{1}{2}}\Phi\|). \tag{6.1}$$

Proof. Set $\boldsymbol{e}=\boldsymbol{e}(k)$ and $\boldsymbol{e}'=\boldsymbol{e}(k')$. Since

$$V_{\boldsymbol{e}} - V_{\boldsymbol{e}'} = (\boldsymbol{e}-\boldsymbol{e}')\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})) = V_{\boldsymbol{e}-\boldsymbol{e}'} = \operatorname{sgn}(V_{\boldsymbol{e}-\boldsymbol{e}'})|V_{\boldsymbol{e}-\boldsymbol{e}'}|,$$

for any $\Psi \in \mathscr{H}$, we have

$$\begin{aligned}|\langle \Psi, \Delta J_1 \Phi\rangle| &\leq \int_0^\infty \||V_{\boldsymbol{e}-\boldsymbol{e}'}|^{\frac{1}{2}} R_{t^2}\Psi\| \||V_{\boldsymbol{e}-\boldsymbol{e}'}|^{\frac{1}{2}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Phi\| t^2 dt \\ &\leq |\boldsymbol{e}-\boldsymbol{e}'| \int_0^\infty \|T_{\boldsymbol{A}}^{\frac{1}{4}} R_{t^2}\Psi\| \|T_{\boldsymbol{A}}^{\frac{1}{4}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Phi\| t^2 dt,\end{aligned} \tag{6.2}$$

where we used Lemma 4.1. We note that

$$\begin{aligned}\|T_{\boldsymbol{A}}^{\frac{1}{4}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2}\Phi\|^2 &= \|T_{\boldsymbol{A}+\boldsymbol{k}}^{\frac{1}{4}} R_{t^2}\Phi\|^2 \\ &= \langle R_{t^2}\Phi, |\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})-\boldsymbol{k}| R_{t^2}\Phi\rangle \\ &\leq \langle R_{t^2}\Phi, |\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})| R_{t^2}\Phi\rangle + |\boldsymbol{k}|\langle R_{t^2}\Phi, R_{t^2}\Phi\rangle \\ &= \|T_{\boldsymbol{A}}^{\frac{1}{4}} R_{t^2}\Phi\|^2 + |\boldsymbol{k}|\|R_{t^2}\Phi\|^2.\end{aligned}$$

Thus, (6.2) is bounded by

$$\begin{aligned}&|\boldsymbol{e}-\boldsymbol{e}'|\Big(\int_0^\infty \|T_{\boldsymbol{A}}^{\frac{1}{4}} R_{t^2}\Psi\|^2 t^2 dt\Big)^{\frac{1}{2}} \Big(\int_0^\infty (\|T_{\boldsymbol{A}}^{\frac{1}{4}} R_{t^2}\Phi\|^2 + |\boldsymbol{k}|\|R_{t^2}\Phi\|^2) t^2 dt\Big)^{\frac{1}{2}} \\ &= |\boldsymbol{e}-\boldsymbol{e}'|\Big(\frac{\pi}{4}\|\Psi\|^2\Big)^{\frac{1}{2}} \Big(\frac{\pi}{4}\|\Phi\|^2 + \frac{\pi}{4}|\boldsymbol{k}|\|T_{\boldsymbol{A}}^{-\frac{1}{4}}\Phi\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

From the diamagnetic inequality and Hardy–Kato's inequality we have

$$\|T_{\boldsymbol{A}}^{-\frac{1}{4}}\Phi\|^2 \leq \||\boldsymbol{p}|^{-\frac{1}{2}}\Phi\|^2 \leq \frac{\pi}{2}\||\boldsymbol{x}|^{\frac{1}{2}}\Phi\|^2. \tag{6.3}$$

Therefore, we have the bound

$$\begin{aligned}\|\Delta J_1\Phi\| = \sup_{\|\Psi\|=1} |\langle\Psi, \Delta J_1\Phi\rangle| &\leq \frac{\pi}{4}|\boldsymbol{e}-\boldsymbol{e}'|\big(\|\Phi\|^2 + \frac{\pi}{2}|\boldsymbol{k}|\||\boldsymbol{x}|^{\frac{1}{2}}\Phi\|^2\big)^{\frac{1}{2}} \\ &\leq |\boldsymbol{e}-\boldsymbol{e}'|(\|\Phi\| + |\boldsymbol{k}|^{\frac{1}{2}}\||\boldsymbol{x}|^{\frac{1}{2}}\Phi\|),\end{aligned}$$

which implies (6.1). □

We decompose ΔJ_2 into two terms:

$$\Delta J_2 = \Delta J_{21} + \Delta J_{22},$$

with

$$\Delta J_{21} = \int_0^1 R_{t^2} V_{e(k')} (e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}) R_{t^2} t^2 dt,$$

$$\Delta J_{22} = \int_1^\infty R_{t^2} V_{e(k')} (e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}) R_{t^2} t^2 dt.$$

Lemma 6.2. *For any* $\Phi \in \mathrm{D}(\boldsymbol{x}^2)$,

$$\|\Delta J_{21}\Phi\| \le 2|\boldsymbol{k} - \boldsymbol{k}'| \|\langle \boldsymbol{x}\rangle^2 \Phi\|.$$

Proof. The proof is similar to that of Lemma 5.5. □

Lemma 6.3. *Let* $k = (\boldsymbol{k}, j)$ *and* $k' = (\boldsymbol{k}', j)$*. For any* $\Phi \in \mathrm{D}(|\boldsymbol{x}|^{\frac{1}{2}})$ *it holds that*

$$\|\Delta J_{22}\Phi\| \le 2|\boldsymbol{k} - \boldsymbol{k}'|(1 + |\boldsymbol{k}'|)\|\Phi\| + |\boldsymbol{k}'^2 - \boldsymbol{k}^2|\|\Phi\| + |\boldsymbol{k} - \boldsymbol{k}'|\||\boldsymbol{x}|\Phi\|. \tag{6.4}$$

Proof. Recall that

$$R_{t^2}(\boldsymbol{k}) = e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} R_{t^2} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = ((\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{k})^2 + t^2)^{-1}.$$

Then,

$$\begin{aligned}
&(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}) R_{t^2} \\
&\quad = (R_{t^2}(\boldsymbol{k}) - R_{t^2}(\boldsymbol{k}')) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + R_{t^2}(\boldsymbol{k}')(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}) \\
&\quad = R_{t^2}(\boldsymbol{k}')(T_{\boldsymbol{A}-\boldsymbol{k}'} - T_{\boldsymbol{A}-\boldsymbol{k}}) R_{t^2}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + R_{t^2}(\boldsymbol{k}')(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}) \\
&\quad = 2R_{t^2}(\boldsymbol{k}') V_{\boldsymbol{k}'-\boldsymbol{k}} R_{t^2}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} + (\boldsymbol{k}'^2 - \boldsymbol{k}^2) R_{t^2}(\boldsymbol{k}') R_{t^2}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} \\
&\qquad + R_{t^2}(\boldsymbol{k}')(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}}).
\end{aligned}$$

According to this decomposition, ΔJ_{22} can be furthermore decomposed into three terms:

$$\Delta J_{22} = \Delta J_{221} + \Delta J_{222} + \Delta J_{223} \tag{6.5}$$

with

$$\Delta J_{221} = \int_1^\infty R_{t^2} V_{e(k')} 2R_{t^2}(\boldsymbol{k}') V_{\boldsymbol{k}'-\boldsymbol{k}} R_{t^2}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} t^2 dt,$$

$$\Delta J_{222} = \int_1^\infty R_{t^2} V_{e(k')} (\boldsymbol{k}'^2 - \boldsymbol{k}^2) R_{t^2}(\boldsymbol{k}') R_{t^2}(\boldsymbol{k}) e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} t^2 dt,$$

$$\Delta J_{223} = \int_1^\infty R_{t^2} V_{\boldsymbol{e}(k')} R_{t^2}(\boldsymbol{k}')(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}})t^2 dt.$$

We can estimate ΔJ_{221} as

$$\begin{aligned} &\|\Delta J_{221}\Phi\| \\ &\quad \le 2\int_1^\infty \|t^2 R_{t^2}\| \|V_{\boldsymbol{e}(k')} R_{t^2}(\boldsymbol{k}')^{\frac{1}{2}}\| \|R_{t^2}(\boldsymbol{k}')^{\frac{1}{2}} V_{\boldsymbol{k}'-\boldsymbol{k}}\| \|R_{t^2}(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\Phi\| dt \\ &\quad \le 2\int_1^\infty |\boldsymbol{k}'-\boldsymbol{k}|(1+|\boldsymbol{k}'|)t^{-2}\|\Phi\| dt = 2|\boldsymbol{k}'-\boldsymbol{k}|(1+|\boldsymbol{k}'|)\|\Phi\|, \end{aligned} \tag{6.6}$$

where we used bounds below:

$$\begin{aligned} \|t^2 R_{t^2}\| &\le 1, \\ \|R_{t^2}(\boldsymbol{k})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\Phi\| &\le t^{-2}\|\Phi\|, \\ \|V_{\boldsymbol{e}(k')} R_{t^2}(\boldsymbol{k}')^{\frac{1}{2}}\| &= \|V_{\boldsymbol{e}(k')} e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}} R_{t^2}^{\frac{1}{2}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\| = \|V_{\boldsymbol{e}(k')} R_{t^2}^{\frac{1}{2}}\| \le 1, \\ \|R_{t^2}(\boldsymbol{k}')^{\frac{1}{2}} V_{\boldsymbol{k}'-\boldsymbol{k}}\| &= \|V_{\boldsymbol{k}'-\boldsymbol{k}} R_{t^2}(\boldsymbol{k}')^{\frac{1}{2}}\| \\ &= \|(\boldsymbol{k}'-\boldsymbol{k})\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x}))e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}} R_{t^2}^{\frac{1}{2}} e^{i\boldsymbol{k}'\cdot\boldsymbol{x}}\| \\ &= \|(\boldsymbol{k}'-\boldsymbol{k})\cdot(\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})-\boldsymbol{k}')R_{t^2}^{\frac{1}{2}}\| \\ &\le |\boldsymbol{k}'-\boldsymbol{k}|\big(\||\boldsymbol{p}-\boldsymbol{A}(\boldsymbol{x})|R_{t^2}^{\frac{1}{2}}\| + |\boldsymbol{k}'|\big) \\ &\le |\boldsymbol{k}'-\boldsymbol{k}|(1+|\boldsymbol{k}'|). \end{aligned}$$

Next, we estimate ΔJ_{222} as

$$\begin{aligned} \|\Delta J_{222}\Phi\| &\le |\boldsymbol{k}'^2 - \boldsymbol{k}^2| \int_1^\infty \|t^2 R_{t^2}\| \|V_{\boldsymbol{e}(k')} R_{t^2}(\boldsymbol{k}')\| \|R_{t^2}(\boldsymbol{k})\| \|\Phi\| dt \\ &\le |\boldsymbol{k}'^2 - \boldsymbol{k}^2| \|\Phi\|. \end{aligned} \tag{6.7}$$

Finally, we estimate ΔJ_{223}. We see that

$$\begin{aligned} \|\Delta J_{223}\Phi\| &= \sup_{\|\Psi\|=1} |\langle\Psi, \Delta J_{223}\Phi\rangle| \\ &\le \sup_{\|\Psi\|=1} \int_1^\infty \||V_{\boldsymbol{e}'}|^{\frac{1}{2}} R_{t^2}\Psi\| \||V_{\boldsymbol{e}'}|^{\frac{1}{2}} R_{t^2}(\boldsymbol{k}')(e^{-i\boldsymbol{k}\cdot\boldsymbol{x}} - e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}})\Phi\| t^2 dt \end{aligned}$$

$$\le \sup_{\|\Psi\|=1} \int_1^\infty \|T_A^{\frac{1}{4}} R_{t^2}\Psi\| \| |V_{e'}|^{\frac{1}{2}} R_{t^2}(e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} - 1)\Phi\| t^2 dt$$
$$\le \sup_{\|\Psi\|=1} \int_1^\infty \|T_A^{\frac{1}{4}} R_{t^2}\Psi\| \|T_A^{\frac{1}{4}} R_{t^2}(e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} - 1)\Phi\| t^2 dt$$
$$\le \sup_{\|\Psi\|=1} \left(\int_0^\infty \|T_A^{\frac{1}{4}} R_{t^2}\Psi\|^2 t^2 dt \right)^{\frac{1}{2}}$$
$$\times \left(\int_0^\infty \|T_A^{\frac{1}{4}} R_{t^2}(e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} - 1)\Phi\|^2 t^2 dt \right)^{\frac{1}{2}}$$
$$= \frac{\pi}{4}\|(e^{-i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{x}} - 1)\Phi\| \le |\boldsymbol{k}-\boldsymbol{k}'| \| |\boldsymbol{x}|\Phi\|. \tag{6.8}$$

Combining estimates (6.6), (6.7), and (6.8), we get (6.1). □

Lemma 6.4. *For almost every $k, k' \in \mathbb{R}^3 \times \{1,2\}$, it follows that*

$$\sup_{0<m<m_0} \|(J(k) - J(k'))\Phi_m\| \le |\boldsymbol{e}(k) - \boldsymbol{e}(k')|(1 + |\boldsymbol{k}|^{\frac{1}{2}} D) + 2D|\boldsymbol{k}-\boldsymbol{k}'|$$
$$+ 2|\boldsymbol{k}-\boldsymbol{k}'|(1+|\boldsymbol{k}'|) + |\boldsymbol{k}'^2 - \boldsymbol{k}^2| + |\boldsymbol{k}-\boldsymbol{k}'|D,$$

where D is a constant defined by $D = \sup_{0<m<m_0} \|\langle \boldsymbol{x}\rangle^2 \Phi_m\|$.

Proof. This is a consequence of Lemmas 6.1, 6.2 and 6.3. □

6.2. Equicontinuity of $\{a(k)\Phi_m\}$.

In this section we show that $\{a(k)\Phi_m\}_{0<m<m_0}$ is equicontinuous. In order to investigate a more general setting on equicontinuity we introduce domain D_ϵ. For any $0 < \epsilon \ll 1$, we define a measurable set $D_\epsilon \subset \mathbb{R}^3$ so that, for any $\rho \in L^2(\mathbb{R}^3)$,

$$\lim_{\epsilon\to+0} \int_{D_\epsilon} |\rho(\boldsymbol{k})|^2 d\boldsymbol{k} = 0.$$

Example 6.5. An example of D_ϵ is given by

$$D_\epsilon = \{\boldsymbol{k} \in \mathbb{R}^3 \mid k_1^2 + k_2^2 \le \epsilon\} \cup \{\boldsymbol{k} \in \mathbb{R}^3 \mid |\boldsymbol{k}| \ge 1/\epsilon\} \tag{6.9}$$

For simplicity, the set $\{k = (\boldsymbol{k}, j) \mid \boldsymbol{k} \in D_\epsilon, j = 1, 2\}$ is also denoted by D_ϵ.

Theorem 6.6 (equicontinuity). *Assume* (A1)–(A4). *Then,*

$$\sup_{0<m<m_0} \int_{D_\epsilon^c} \|a(k)\Phi_m - a(k-s)\Phi_m\|^2 dk \longrightarrow 0 \quad (|\boldsymbol{s}| \to 0), \tag{6.10}$$

where D_ϵ is given by (6.9).

Proof. We fix $\epsilon > 0$ arbitrarily. Note that D_ϵ satisfies

(d1) $D_\epsilon \subset D_{\epsilon'}$ for $\epsilon < \epsilon'$,

(d2) $\mathrm{dist}(D_\epsilon^c, D_{\frac{\epsilon}{2}}) \geq \frac{\epsilon}{2}$.

By the definition, $\boldsymbol{e}(\boldsymbol{k}, j)$, $j = 1, 2$ are uniformly continuous in D_ϵ^c. For $k = (\boldsymbol{k}, j) \in D_\epsilon^c$, we set $k' = (\boldsymbol{k} - \boldsymbol{s}, j)$. By (d2), $|\boldsymbol{s}| < \frac{\epsilon}{2}$ implies $k' \in D_{\frac{\epsilon}{2}}^c$, and hence $\omega(\boldsymbol{k}), \omega(\boldsymbol{k}') \geq \frac{\epsilon}{2}$. We decompose $a(k)\Phi_m - a(k')\Phi_m$ into three terms:

$$a(k)\Phi_m - a(k')\Phi_m = A_1 + A_2 + A_3,$$

where

$$\begin{aligned}
A_1 &= \phi_\omega(\boldsymbol{k})(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1}(J(k) - J(k'))\Phi_m, \\
A_2 &= \phi_\omega(\boldsymbol{k})\big\{(H_m - E_m + \omega_m(\boldsymbol{k}))^{-1} - (H_m - E_m + \omega_m(\boldsymbol{k}'))^{-1}\big\}J(k')\Phi_m, \\
A_3 &= (\phi_\omega(\boldsymbol{k}) - \phi_\omega(\boldsymbol{k}'))(H_m - E_m + \omega_m(\boldsymbol{k}'))^{-1}J(k')\Phi_m.
\end{aligned}$$

By Lemma 6.4, we can estimate the norm of A_1 as follows:

$$\begin{aligned}
\|A_1\| &\leq |\phi_\omega(\boldsymbol{k})|\omega_m(\boldsymbol{k})^{-1}\|(J(k) - J(k'))\Phi_m\| \\
&\leq |\phi_\omega(\boldsymbol{k})|\frac{2}{\epsilon}\|(J(k) - J(k'))\Phi_m\| \\
&\leq C|\phi_\omega(\boldsymbol{k})|\big(|\boldsymbol{e}(\boldsymbol{k}, j) - \boldsymbol{e}(\boldsymbol{k} - \boldsymbol{s}, j)| + |\boldsymbol{s}|\big),
\end{aligned}$$

where C is a constant independent of $\boldsymbol{k}, \boldsymbol{s}$ and m. Thus,

$$\lim_{|\boldsymbol{s}| \to 0} \int_{D_\epsilon^c} \|A_1\|^2 dk = 0. \tag{6.11}$$

Next, we consider A_2. By Corollary 4.3,

$$\begin{aligned}
\|A_2\| &\leq |\phi_\omega(\boldsymbol{k})|\omega_m(\boldsymbol{k})^{-1}\omega_m(\boldsymbol{k}')^{-1}|\omega_m(\boldsymbol{k}) - \omega_m(\boldsymbol{k}')|\|J(k')\| \\
&\leq |\phi_\omega(\boldsymbol{k})|\frac{4}{\epsilon^2}|\boldsymbol{k} - \boldsymbol{k}'|\frac{1}{\sqrt{2}} \\
&= \frac{2\sqrt{2}}{\epsilon^2}|\phi_\omega(\boldsymbol{k})||\boldsymbol{s}|.
\end{aligned}$$

Thus,

$$\lim_{|\boldsymbol{s}|\to 0}\int_{D_\epsilon^c}\|A_2\|^2 dk = 0. \tag{6.12}$$

The norm of A_3 can be similarly estimated as follows:

$$\|A_3\| \le |\phi_\omega(\boldsymbol{k}) - \phi_\omega(\boldsymbol{k}-\boldsymbol{s})|\frac{\sqrt{2}}{\epsilon}.$$

Since $\phi_\omega \in L^2(\mathbb{R}^3_{\boldsymbol{k}})$, the shift $\boldsymbol{s} \mapsto \phi_\omega(\cdot - \boldsymbol{s})$ is strongly continuous, and hence

$$\lim_{|\boldsymbol{s}|\to 0}\int_{D_\epsilon^c}\|A_3\|^2 dk = 0. \tag{6.13}$$

Therefore, by (6.11), (6.12) and (6.13), we can show (6.10). □

6.3. Spatial localization of photon. Let $\mathcal{B}(K)$ be the set of bounded operator on K. For $T \in \mathcal{B}(W)$ with $\|T\| \le 1$, we define the second quantization of T, $\Gamma(T) \in \mathcal{B}(\mathcal{F})$, by

$$\Gamma(T) = \bigoplus_{n=0}^{\infty}(\oplus^n T).$$

We set $\oplus^0 T = \mathbf{1}$. Let $j \in C_0^\infty([0,\infty))$ be a function such that $0 \le j(s) \le 1$ and

$$j(s) = \begin{cases} 1 & \text{if } 0 \le s \le 1, \\ 0 & \text{if } s \ge 2. \end{cases}$$

For $R > 0$, we set

$$\chi(\boldsymbol{y}) = j(|\boldsymbol{y}|) \quad \text{and} \quad \chi_R = \chi(i\nabla_{\boldsymbol{k}}/R),$$

and

$$\Gamma_R = \Gamma(\chi_R) = \mathbf{1}_W \otimes \Gamma(\chi_R).$$

In this section we shall prove the proposition below:

Proposition 6.7 (spatial localization of photon). *Assume* (A1)–(A4). *Then it holds that*

$$\lim_{R\to\infty}\sup_{0<m<m_0}\|(\mathbf{1}-\Gamma_R)\Phi_m\| = 0. \tag{6.14}$$

The proof of Proposition 6.7 is given after general lemmas stated below. For $f \in L^2(\mathbb{R}^3)$, it holds that

$$\chi_R f = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \hat{\chi}(\boldsymbol{s}) f(\cdot - R^{-1}\boldsymbol{s}) d\boldsymbol{s}. \tag{6.15}$$

Note that $\hat{\chi}$ is a rapidly decreasing smooth function. We can extend this type formula to the state in $\mathcal{H}$.

Lemma 6.8. *For $\Phi \in \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$, we have*

$$\|d\Gamma(\chi_R)^{\frac{1}{2}}\Phi\|^2 = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} d\boldsymbol{s} \int \hat{\chi}(\boldsymbol{s}) \langle a(k)\Phi, a(k - R^{-1}s)\Phi\rangle dk, \tag{6.16}$$

where $k - R^{-1}s = (\boldsymbol{k} - R^{-1}\boldsymbol{s}, j)$ with $k = (\boldsymbol{k}, j) \in \mathbb{R}^3 \times \{1, 2\}$, and the integral (6.16) is absolutely convergent.

Proof. The particle part is irrelevant to this result, so, for simplicity, we only consider the field part. For each n-particle part $\Phi^{(n)}$, from (6.15), we have

$$\begin{aligned}
&(\chi_R \otimes \mathbf{1}_{\otimes_{\mathrm{s}}^{n-1} W})\Phi^{(n)}(k_1, \ldots, k_n) \\
&\quad = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \hat{\chi}(\boldsymbol{s}) \Phi^{(n)}(k_1 - R^{-1}s, k_2, \ldots, k_n) d\boldsymbol{s},
\end{aligned}$$

which is a strong integral in $\otimes_{\mathrm{s}}^n W$. Thus by the symmetry of the state and the definition of $a(k)$, we have

$$\begin{aligned}
&(\chi_R^{(n)} \Phi^{(n)})(k_1, \ldots, k_n) \\
&\quad = n(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \hat{\chi}(\boldsymbol{s}) \Phi^{(n)}(k_1 - R^{-1}s, k_2, \ldots, k_n) d\boldsymbol{s} \\
&\quad = \sqrt{n}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \hat{\chi}(\boldsymbol{s}) (a(k_1 - R^{-1}s)\Phi)^{(n-1)}(k_2, \ldots, k_n) d\boldsymbol{s}.
\end{aligned}$$

Since $\Phi^{(n)}(k, \cdot) = n^{-\frac{1}{2}}(a(k)\Phi)^{(n-1)}(\cdot)$, we have

$$\begin{aligned}
&\langle \Phi^{(n)}, \chi_R^{(n)} \Phi^{(n)} \rangle \\
&\quad = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} d\boldsymbol{s} \int \hat{\chi}(\boldsymbol{s}) \langle (a(k)\Phi)^{(n-1)}, (a(k - R^{-1}s)\Phi)^{(n-1)} \rangle_{\otimes_{\mathrm{s}}^{n-1} W} dk,
\end{aligned}$$

for $n = 1, 2, \ldots,$ and

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}^3} d\boldsymbol{s} \int |\hat{\chi}(\boldsymbol{s})| \left| \langle (a(k)\Phi)^{(n-1)}, (a(k - R^{-1}s)\Phi)^{(n-1)} \rangle_{\otimes_s^{n-1} W} \right| dk < \infty.$$

Thus, by Fubini's lemma,

$$\begin{aligned}
&\|d\Gamma(\chi_R)^{\frac{1}{2}}\Phi\|^2 \\
&\quad = \sum_{n=1}^{\infty} \langle \Phi^{(n)}, \chi_R^{(n)} \Phi^{(n)} \rangle \\
&\quad = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} d\boldsymbol{s} \int \hat{\chi}(\boldsymbol{s}) \sum_{n=1}^{\infty} \langle (a(k)\Phi)^{(n-1)}, (a(k - R^{-1}s)\Phi)^{(n-1)} \rangle_{\otimes_s^{n-1} W} dk.
\end{aligned}$$

Thus (6.16) follows. □

Lemma 6.9. *Let $\{\Psi_m\}_{0<m<m_0}$ be normalized vectors in $\mathcal{H}$ so that*

(c1) $\{\Psi_m\}_{0<m<m_0} \subset \mathrm{D}(\mathrm{N}^{\frac{1}{2}})$ *and* $\sup_{0<m<m_0} \|\mathrm{N}^{\frac{1}{2}}\Psi_m\| < \infty$,

(c2) *for $s = (\boldsymbol{s}, j)$ and $k - s = (\boldsymbol{k} - \boldsymbol{s}, j)$,*

$$\lim_{|\boldsymbol{s}| \to 0} \sup_{0<m<m_0} \int \|a(k)\Psi_m - a(k - s)\Psi_m\|^2 dk = 0.$$

Then $\{\Psi_m\}_{0<m<m_0}$ satisfies

$$\lim_{R\to\infty} \sup_{0<m<m_0} \|d\Gamma(\mathbf{1} - \chi_R)^{\frac{1}{2}}\Psi_m\| = 0. \tag{6.17}$$

Proof. By Lemma 6.8 and $(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \hat{\chi}(\boldsymbol{s}) d\boldsymbol{s} = \chi(0) = 1$, we have

$$\begin{aligned}
&\|d\Gamma(\mathbf{1} - \chi_R)^{\frac{1}{2}}\Psi_m\|^2 \\
&\quad = \|\mathrm{N}^{\frac{1}{2}}\Psi_m\|^2 - \|d\Gamma(\chi_R)^{\frac{1}{2}}\Psi_m\|^2 \\
&\quad = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} d\boldsymbol{s} \int \hat{\chi}(\boldsymbol{s}) \langle a(k)\Psi_m, a(k)\Psi_m - a(k - R^{-1}s)\Psi_m \rangle dk \\
&\quad \le (2\pi)^{-\frac{3}{2}} \|\hat{\chi}\|_{L^1}^{\frac{1}{2}} \|\mathrm{N}^{\frac{1}{2}}\Psi_m\| \left(\int_{\mathbb{R}^3} d\boldsymbol{s} |\hat{\chi}(\boldsymbol{s})| \int \|a(k)\Psi_m - a(k - R^{-1}s)\Psi_m\|^2 dk \right)^{\frac{1}{2}} \\
&\quad \le (2\pi)^{-\frac{3}{2}} \|\hat{\chi}\|_{L^1}^{\frac{1}{2}} C \left(\int_{\mathbb{R}^3} |\hat{\chi}(\boldsymbol{s})| F_m(R^{-1}\boldsymbol{s}) d\boldsymbol{s} \right)^{\frac{1}{2}},
\end{aligned}$$

where $C = \sup_{0<m<m_0} \|\mathrm{N}^{\frac{1}{2}}\Psi_m\|$ and

$$F_m(R^{-1}\boldsymbol{s}) = \int \|a(k)\Psi_m - a(k - R^{-1}s)\Psi_m\|^2 dk.$$

By condition (c1), we have $F_m(R^{-1}\boldsymbol{s}) \le 4C^2$ for all m. By condition (c2), for any $\varepsilon > 0$, there exists $M > 0$ such that, for all $R > M$ and $|\boldsymbol{s}| < R^{\frac{1}{2}}$, it holds that $\sup_{0<m<m_0} F_m(R^{-1}\boldsymbol{s}) < \varepsilon$. Thus,

$$\begin{aligned}\sup_{0<m<m_0} \int_{\mathbb{R}^3} |\hat{\chi}(\boldsymbol{s})| F_m(R^{-1}\boldsymbol{s}) d\boldsymbol{s} &\le \int_{|\boldsymbol{s}|<R^{\frac{1}{2}}} |\hat{\chi}(\boldsymbol{s})|\varepsilon d\boldsymbol{s} + \int_{|\boldsymbol{s}|>R^{\frac{1}{2}}} |\hat{\chi}(\boldsymbol{s})| 4C^2 d\boldsymbol{s} \\ &\le \varepsilon \|\hat{\chi}\|_{L^1} + 4C^2 \int_{|\boldsymbol{s}|>R^{\frac{1}{2}}} |\hat{\chi}(\boldsymbol{s})| d\boldsymbol{s}.\end{aligned}$$

Therefore,

$$\limsup_{R\to\infty} \left(\sup_{0<m<m_0} \int_{\mathbb{R}^3} |\hat{\chi}(\boldsymbol{s})| F_m(R^{-1}\boldsymbol{s}) d\boldsymbol{s} \right) \le \varepsilon \|\hat{\chi}\|_{L^1}.$$

Since $\varepsilon > 0$ is arbitrary, the lemma follows. □

We extend Lemma 6.9.

Lemma 6.10. *Let $\{\Psi_m\}_{0<m<m_0}$ be normalized vectors in $\mathcal{H}$ so that*

(a) *there exists $g \in W$ such that*

$$\sup_{0<m<m_0} \|a(k)\Psi_m\| \le |g(k)| \quad \textit{for a.e. } k;$$

(b) *for any $0 < \epsilon \ll 1$,*

$$\lim_{|\boldsymbol{s}|\to 0} \sup_{0<m<m_0} \int_{D_\epsilon^c} \|a(k)\Psi_m - a(k - s)\Psi_m\|^2 dk = 0,$$

where $k = (\boldsymbol{k}, j), k - s = (\boldsymbol{k} - \boldsymbol{s}, j)$.

Then (6.17) holds.

Proof. From condition (a), the condition (c1) in Lemma 6.9 follows. We shall show (c2) in Lemma 6.9. By condition (a), we have

$$\begin{aligned}&\sup_{0<m<m_0} \int \|a(k)\Psi_m - a(k - s)\Psi_m\|^2 dk \\ &\le \sup_{0<m<m_0} \int_{D_\epsilon^c} \|a(k)\Psi_m - a(k - s)\Psi_m\|^2 dk + \int_{D_\epsilon} |g(k)|^2 dk.\end{aligned} \tag{6.18}$$

By condition (b), the first term in (6.18) vanishes as $\boldsymbol{s} \to 0$. Thus

$$0 \le \limsup_{|\boldsymbol{s}|\to 0} \sup_{0<m<m_0} \int \|a(k)\Psi_m - a(k-s)\Psi_m\|^2 dk \le \int_{D_\epsilon} |g(k)|^2 dk$$

holds for all $\epsilon > 0$. By the definition of D_ϵ, the right-hand side of this inequality converges to zero as $\epsilon \to +0$. Therefore, the condition (c2) in Lemma 6.9 is satisfied, and (6.17) holds. □

We are in the position to prove Proposition 6.7.

Proof of Proposition 6.7: It is shown that

$$\lim_{R\to\infty} \sup_{0<m<m_0} \|d\Gamma(\mathbf{1} - \chi_R)^{\frac{1}{2}} \Phi_m\|^2 = 0$$

implies (6.14) by [6, equation (IV.13)]. Hence, it is sufficient to show that conditions (a) and (b) in Lemma 6.10 are satisfied with Ψ_m replaced by Φ_m. Proposition 5.1 yields that

$$\sup_{0<m<m_0} \|a(k)\Phi_m\| \le C \frac{|\hat{\varphi}(\boldsymbol{k})|}{\omega(\boldsymbol{k})^{\frac{1}{2}}}(1 + |\boldsymbol{k}|), \qquad \text{a.e. } k,$$

and the right-hand side above is square integrable in k by (A2). Thus condition (a) holds. Condition (b) is shown in Theorem 6.6. □

7. Proof of the main theorem

We show two general lemmas below. For a self-adjoint operator A, we denote the form domain of A by $Q(A)$, and $(\cdot, A\,\cdot)$ denotes the quadratic form associated with A. If A is bounded from below, we set $E_0(A) = \inf\sigma(A)$. For self-adjoint operators A, B, we denote $A \ge B$ if and only if $Q(A) \subset Q(B)$ and $(\Psi, A\Psi) \ge (\Psi, B\Psi)$ for all $\Psi \in Q(A)$. We use the following fact.

Lemma 7.1. *Let $A, A_j, j = 1, 2, \ldots,$ be self-adjoint operators bounded from below such that $A_1 \ge A_2 \ge \cdots \ge A$. Assume that there exists a subspace $D \subset Q(A_1)$ such that D is a form core for A and $\lim_{j\to\infty}(\Phi, A_j\Phi) = (\Phi, A\Phi)$ for $\Phi \in D$. Then $\lim_{j\to\infty} E_0(A_j) = E_0(A)$.*

Proof. By the variational principle, we have $E_0(A) \le E_0(A_j) \le (\Phi, A_j\Phi)$ for any normalized $\Phi \in D$. Since $E_0(A_j)$ is monotone decreasing in j, it has a limit as $j \to \infty$. Since D is a form core for A, we have

$$E_0(A) \le \lim_{j\to\infty} E_0(A_j) \le \inf_{\Phi\in D, \|\Phi\|=1} (\Phi, A\Phi) = E_0(A).$$

Therefore, $E(A_j) \to E(A_0)$ as $j \to \infty$. □

Lemma 7.2. *Let $A, A_j, j = 1, 2, \dots,$ be self-adjoint operators bounded from below such that $A_1 \ge A_2 \ge \dots \ge A$. Assume that* $\lim_{j\to\infty} E_0(A_j) = E_0(A)$. *Let $\Phi_j \in Q(A_j)$, $j = 1, 2, \dots,$ be a normalized sequence such that*

$$\langle \Phi_j, A_j\Phi_j\rangle \le E_0(A_j) + o(j^0),$$

and Φ_j weakly converges to some Φ as $j \to \infty$. Then $\Phi \in \mathrm{D}(A)$ and

$$A\Phi = E_0(A)\Phi$$

holds. In particular, if $\Phi \ne 0$, Φ is a ground state of A.

Proof. Since $\Phi_j \in Q(A_j) \subset Q(A)$, we have

$$\begin{aligned} 0 &\le (\Phi_j, (A - E_0(A))\Phi_j) \\ &\le (\Phi_j, (A_j - E_0(A))\Phi_j) \\ &\le E_0(A_j) - E_0(A) + o(j^0) \longrightarrow 0 \quad \text{as } j \to \infty. \end{aligned}$$

Thus, $\|(A - E_0(A))^{\frac{1}{2}}\Phi_j\| \to 0$ as $j \to \infty$. For any $\Psi \in Q(A)$,

$$\begin{aligned} \langle (A - E_0(A))^{\frac{1}{2}}\Psi, \Phi\rangle &= \lim_{j\to\infty} \langle (A - E_0(A))^{\frac{1}{2}}\Psi, \Phi_j\rangle \\ &= \lim_{j\to\infty} \langle \Psi, (A - E_0(A))^{\frac{1}{2}}\Phi_j\rangle = 0. \end{aligned}$$

This implies that $\Phi \in Q(A)$ and $(A - E_0(A))^{\frac{1}{2}}\Phi = 0$, and therefore $\Phi \in \mathrm{D}(A)$ and $(A - E_0(A))\Phi = 0$. □

We need a bound to show the main theorem.

Lemma 7.3. *Assume* (A1)–(A4) *and $V \in V_{\mathrm{conf}} \cup V_{\mathrm{rel}}$. Then, for all $m \ge 0$,*

$$\||\boldsymbol{p}|\Psi\|^2 + \|H_{\mathrm{f},m}\Psi\|^2 \le C(\|H_m\Psi\|^2 + \|\Psi\|^2), \quad \Psi \in \mathrm{D}(H_m) \tag{7.1}$$

holds for some C independent of $m \ge 0$.

Proof. In the case of $V \in V_{\mathrm{conf}}$, the lemma was proven by [11]. Since the proof for the case of $V \in V_{\mathrm{rel}}$ is similar, we briefly give an outline of the proof. By the definition of V_{rel}, there exist constants $0 < a < 1$ and $0 < b$ such that

$$\|V\Psi\| \leq a\||\boldsymbol{p}|\Psi\| + b\|\Psi\|, \quad \Psi \in \mathrm{D}(H_m). \tag{7.2}$$

Set $H^0 = |\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})| + H_{\mathrm{f},m}$ and take an arbitrary $\Psi \in \mathscr{H}_{\mathrm{fin}}$. It is shown that for an arbitrary $\epsilon > 0$,

$$\begin{aligned}\|H^0\Psi\|^2 &\geq (1-\epsilon)\||\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x})|\Psi\|^2 + (1-\epsilon)\|H_{\mathrm{f},m}\Psi\|^2 - C_\epsilon\|\Psi\|^2 \\ &\geq \frac{1-\epsilon}{1+\epsilon}(\||\boldsymbol{p}|\Psi\|^2 + \|H_{\mathrm{f},m}\Psi\|^2) - C'_\epsilon\|\Psi\|^2,\end{aligned} \tag{7.3}$$

with some constants C_ϵ and C'_ϵ (see [11]). Thus, by (7.2), (7.3), and

$$\|H^0\Psi\| \leq \|H_m\Psi\| + \|V\Psi\|, \tag{7.4}$$

we have (7.1) for all $\Psi \in \mathscr{H}_{\mathrm{fin}}$. Since $\mathscr{H}_{\mathrm{fin}}$ is a core for H_m, the lemma follows by a limiting argument. □

Now we are in the position to prove the main theorem.

Proof of Theorem 2.9. The uniqueness of the ground state is shown in [16, Corollary 6.2]. We shall show the existence of the ground state. We can choose a subsequence $\{\Phi_{m_j}\}_j$ such that $m_j \downarrow 0$ as $j \to \infty$ and Φ_{m_j} weakly converges to some vector $\Phi_0 \in \mathscr{H}$. Applying Lemmas 7.1 and 7.2 under the identifications

$$A = H, \quad A_j = H_{m_j}, \quad \Phi_j = \Phi_{m_j}, \quad D = \mathscr{H}_{\mathrm{fin}}, \quad \Phi = \Phi_0,$$

we can see that $\Phi_0 \in \mathrm{D}(H)$ and

$$H\Phi_0 = E_0\Phi_0, \quad E_0 = \inf\sigma(H). \tag{7.5}$$

Now we shall show that Φ_{m_j} strongly converges to Φ_0. We first claim that the following bounds hold:

$$\sup_{j\in\mathbb{N}} \||\boldsymbol{x}|\Phi_{m_j}\| < \infty, \tag{7.6}$$

$$\sup_{j\in\mathbb{N}} \||\boldsymbol{p}|\Phi_{m_j}\| < \infty, \tag{7.7}$$

$$\sup_{j\in\mathbb{N}} \|H_{\mathrm{f}}\Phi_{m_j}\| < \infty, \tag{7.8}$$

$$\sup_{j\in\mathbb{N}} \|\mathrm{N}^{\frac{1}{2}}\Phi_{m_j}\| < \infty, \tag{7.9}$$

$$\lim_{R\to\infty}\sup_{j\in\mathbb{N}} \|(\mathbf{1} - \Gamma_R)\Phi_{m_j}\| = 0. \tag{7.10}$$

By assumption (A4), bound (7.6) holds. By Lemma 7.3 and $\|H_{\mathrm{f}}\Psi\| \le \|H_{\mathrm{f},m}\Psi\|$, we have both bounds (7.7) and (7.8). Bound (7.9) is shown by Corollary 5.2 and (7.10) by Proposition 6.7. From (7.6)–(7.10), we have

$$\sup_{j\in\mathbb{N}} \|(1-\chi_\ell)\Phi_{m_j}\| = o(R^0), \quad \ell = 1,\dots,5$$

as $R \to \infty$, where

$$\begin{aligned}
&\chi_1 = j(|\boldsymbol{x}|/R), \quad \chi_2 = j(|\boldsymbol{p}|/R),\\
&\chi_3 = j(\mathrm{N}/R), \quad \chi_4 = j(H_{\mathrm{f}}/R),\\
&\chi_5 = \Gamma_R.
\end{aligned}$$

Here $j(\cdot)$ is the smooth function defined by (1.3). This fact implies that

$$\begin{aligned}
&\sup_{j\in\mathbb{N}} \|(1-\chi_1\chi_2\chi_3\chi_4\chi_5)\Phi_{m_j}\|\\
&\quad\le \sup_{j\in\mathbb{N}}\big(\|(1-\chi_1)\Phi_{m_j}\| + \|\chi_1(1-\chi_2)\Phi_{m_j}\| + \|\chi_1\chi_2(1-\chi_3)\Phi_{m_j}\|\\
&\qquad + \|\chi_1\chi_2\chi_3(1-\chi_4)\Phi_{m_j}\| + \|\chi_1\chi_2\chi_3\chi_4(1-\chi_5)\Phi_{m_j}\|\big)\\
&\quad\le \sup_{j\in\mathbb{N}}\sum_{\ell=1}^{5}\|(1-\chi_\ell)\Phi_{m_j}\|\\
&\quad\le o(R^0).
\end{aligned} \tag{7.11}$$

Since $\chi_1\chi_2\chi_3\chi_4\chi_5$ is compact in $\mathscr{H}$ for all $R > 0$, $\chi_1\chi_2\chi_3\chi_4\chi_5\Phi_{m_j}$ strongly converges to $\chi_1\chi_2\chi_3\chi_4\chi_5\Phi_0$ as $j \to \infty$. Thus, by (7.11), we have

$$\begin{aligned}
\|\Phi_0\| &= \lim_{R\to\infty}\|\chi_1\chi_2\chi_3\chi_4\chi_5\Phi_0\|\\
&= \lim_{R\to\infty}\lim_{j\to\infty}\|\chi_1\chi_2\chi_3\chi_4\chi_5\Phi_{m_j}\|\\
&\ge \limsup_{R\to\infty}\limsup_{j\to\infty}(1-\|(1-\chi_1\chi_2\chi_3\chi_4\chi_5)\Phi_{m_j}\|)\\
&\ge \limsup_{R\to\infty}(1-o(R^0)) = 1.
\end{aligned}$$

We conclude that Φ_{m_j} strongly converges to Φ_0. In particular, $\Phi_0 \ne 0$. By (7.5), Φ_0 is a normalized ground state of H. Then the proof is complete. □

We give an example of the existence of the ground state.

Example 7.4. Suppose (A1) and (A2), and $V \in V_{\mathrm{conf}}$. Then H_m has the ground state for each $m > 0$ by [12]. In this case, (A3) and (A4) are satisfied. Then H also has the ground state.

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