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Uniform resolvent estimates for the discrete Schrödinger operator in dimension three

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Abstract. In this note, we prove the uniform resolvent estimate of the discrete Schrödinger operator with dimension three. To do this, we show a Fourier decay of the surface measure on the Fermi surface.

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1. Introduction

We consider the three-dimensional discrete Laplacian

$$H_0u(x) = -\sum_{|x-y|=1}(u(y) - u(x)).$$

We denote the Fourier expansion by $\mathcal{F}_d$:

$$\hat{u}(\xi) = \mathcal{F}u(\xi) = \sum_{x\in\mathbb{Z}^3} e^{-2\pi i x\cdot\xi}u(x), \quad \xi \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3.$$

Then it follows that

$$\mathcal{F}_d H_0 u(\xi) = h_0(\xi)\mathcal{F}_d u(\xi), \quad h_0(\xi) = 4\sum_{j=1}^{3}\sin^2(\pi\xi_j). \tag{1}$$

We denote the set of the critical points of h_0 by $\mathrm{Cr}(h_0)$:

$$\mathrm{Cr}(h_0) = \{\xi \in \mathbb{T}^3 \mid \nabla h_0(\xi) = 0\} = \{\xi \in \mathbb{T}^3 \mid \xi_j \in \{0, 1/2\},\ j = 1, 2, 3\}. \tag{2}$$

We call $\xi \in \mathrm{Cr}(h_0)$ an *elliptic threshold* if ξ attains maximum or minimum of h_0 and a hyperbolic threshold otherwise. We set $M_\lambda = h_0^{-1}(\{\lambda\})$ for $\lambda \in [0, 12]$. The set M_λ is called the *Fermi surface*.

In this note, we show the uniform resolvent estimates for discrete Schrödinger operator with dimension three. In case of the continuous Laplacian $-\Delta$ on $\mathbb{R}^d$, the following uniform resolvent estimates are known [9, 6]:

$$\|(-\Delta - z)^{-1} f\|_{L^q(\mathbb{R}^d)} \leq C|z|^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-1} \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for } z \in \mathbb{C} \setminus [0, \infty), \tag{3}$$

where $d \geq 3$, and

$$\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d}, \quad \frac{2d}{d+3} < \frac{1}{p} < \frac{2d}{d+1}, \quad \frac{2d}{d-1} < \frac{1}{q} < \frac{2d}{d-3}.$$

It turns out that $(-\Delta - z)^{-1}$ is uniformly bounded in $B(L^p(\mathbb{R}^d), L^{p'}(\mathbb{R}^d))$ with respect to $z \in \mathbb{C} \setminus [0, \infty)$ if and only if $p = 2d/(d+2)$. On the other hand, in [13, Theorem 1.7 (iii)] (see also Lemma A.1), it is shown that the resolvent $R_0(z) = (H_0 - z)^{-1}$ for the discrete Schrödinger operator is not bounded from $l^p(\mathbb{Z}^d)$ to $l^{p'}(\mathbb{Z}^d)$ with $p = 2d/(d+2)$, $p' = p/(p-1)$ and with $d \geq 5$. The result in [13, Proposition 3.3] shows that the resolvent $R_0(z)$ satisfies

$$\|R_0(z) f\|_{l^{p'}(\mathbb{Z}^d)} \leq C\|f\|_{l^p(\mathbb{Z}^d)} \quad \text{for } z \in \mathbb{C} \setminus [0, 4d], 1 \leq p \leq \frac{2d}{d+3}, d \geq 4. \tag{4}$$

The natural questions are the following:

- is the estimate (4) optimal?
- what about the case of $d = 3$?

For the latter, the authors in [8] showed that (4) hold for $p \in \left[1, \frac{12}{11}\right)$ and for $d = 3$ (see also Lemma A.1). In this paper, we improve their results and give the resolvent estimates which is sharp away from the threshold energies.

The proof of (4) in [13] depends on the endpoint Strichartz estimates ([11]). We point out that the endpoint Strichartz estimates for discrete Schrödinger operators might not be used for the sharp resolvent estimate with dimension three since the Strichartz estimates in [11] are sharp. This is different from the case of the continuous Laplacian $-\Delta$ (in this case, the endpoint Strichartz estimates implies the sharp resolvent estimate (3) with $p = 2d/d + 2$ and with $q = p'$). Instead, we use the strategy in [1] and calculate the Fourier decay of the surface measure for the Fermi surface. In [2], the Fourier decay away from the umbilic points (the points where all principal curvatures vanish) are studied. In this paper, we improve this result and also deal with the Fourier decay near the umbilic point. For its application to the random Schrödinger operators, see [2] and references therein.

The main result of this paper is the following theorem.

Theorem 1.1. (i) ***Resolvent estimates away from the thresholds.*** *Let* $p \in \left[1, \frac{5}{4}\right]$ *and* $r \in \left[1, \frac{10}{3}\right]$. *For* $\varepsilon > 0$, *set*

$$D_\varepsilon = \bigcap_{k=0}^{3} \{z \in \mathbb{C} \mid |z - 4k| \geq \varepsilon\}.$$

Then the resolvent $R_0(z) = (H_0 - z)^{-1}$ *satisfies*

$$\sup_{z \in D_\varepsilon \setminus \mathbb{R}} \|R_0(z)\|_{B(l^p(\mathbb{Z}^3), l^{p'}(\mathbb{Z}^3))} < \infty,$$

$$\sup_{z \in D_\varepsilon \setminus \mathbb{R},\ \|W_j\|_{l^r(\mathbb{Z}^3)} = 1} \|W_1 R_0(z) W_2\|_{B(l^2(\mathbb{Z}^3))} < \infty.$$

(ii) ***Resolvent estimates near the thresholds.*** *Let* $p \in \left[1, \frac{6}{5}\right]$ *and* $r \in [1, 3]$. *Then we have*

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|R_0(z)\|_{B(l^p(\mathbb{Z}^3), l^{p'}(\mathbb{Z}^3))} < \infty,$$

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R},\ \|W_j\|_{l^r(\mathbb{Z}^3)} = 1} \|W_1 R_0(z) W_2\|_{B(l^2(\mathbb{Z}^3))} < \infty.$$

Remark 1.2. [13, Theorem 1.11 (iii)] shows that the range of p and r in (i) are optimal.

Remark 1.3. The above results are proved in [8] for $p \in \left[1, \frac{12}{11}\right)$ and $r \in \left[1, \frac{12}{5}\right)$ (see Lemma A.1).

Remark 1.4. More generally, it follows from [14, Theorem 1.2 (i)] that the uniform resolvent estimates away form the diagonal line hold, that is,

$$\sup_{z \in D_\varepsilon \setminus \mathbb{R}} \|R_0(z)\|_{B(l^p(\mathbb{Z}^3), l^q(\mathbb{Z}^3))} < \infty,$$

for

$$\frac{3}{5} \leq \frac{1}{p} - \frac{1}{q}, \quad \frac{5}{7} < \frac{1}{p}, \quad \frac{1}{q} < \frac{2}{7}.$$

Moreover, [14, Theorem 1.2 (ii)] implies that $R_0(z)$ is Hölder continuous on $B(l^p(\mathbb{Z}^3), l^{p'}(\mathbb{Z}^3))$ for $1 \leq p < 5/4$. From this result and the proof of [13, Theorem 1.9], it is expected that the wave operators $W_\pm = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$ exist and is complete for $H = H_0 + V$ with $V \in l^{\frac{5}{3}}(\mathbb{Z}^3)$. We omit the detail.

Finally, we state a possible conjecture on the resolvent estimates near the threshold energies. The author expects that for $p \in [1, \frac{5}{4}]$, the following estimates hold:

$$\|R_0(z)f\|_{l^{p'}(\mathbb{Z}^3)} \le C\Big(\prod_{k=0}^{3} |z-4k|^{\frac{3}{2}(\frac{1}{p}-\frac{1}{p'})-1}\Big)\|f\|_{l^p(\mathbb{Z}^3)}, \quad \text{for } z \in \mathbb{C}\setminus\mathbb{R}. \tag{5}$$

By virtue of Proposition 2.1 and [1, Proposition A.11], in order to prove (5), we only need to prove

$$\|\chi(D)R_0(z)f\|_{l^{p'}(\mathbb{Z}^3)} \le C\Big(\prod_{k=0}^{3} |z-4k|^{\frac{3}{2}(\frac{1}{p}-\frac{1}{p'})-1}\Big)\|f\|_{l^p(\mathbb{Z}^3)}, \quad \text{for } z \in \mathbb{C},$$

where $\chi \in C^\infty(\mathbb{T}^3)$ is supported around ξ_0 with $\xi_0 \in (M_4 \cup M_8) \setminus \mathrm{Cr}(h_0)$. The estimates (5) can be applied with the Keller type eigenvalue bounds for three-dimensional discrete Schrödinger operators with complex potentials (see [3] for the continuous Laplacian).

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2. Preliminary, reduction to the Fourier decay of the surface measure

2.1. Uniform resolvent estimates near thresholds. To obtain uniform resolvent estimates near thresholds, we only need to the argument in [13, Proposition 3.3] slightly.

Proposition 2.1. *Let $d \ge 3$ and $T\colon \mathbb{T}^d \to \mathbb{R}$ be a smooth function with a non-degenerate critical point ξ_0 with corresponding energy λ_0. Then there exists $\delta > 0$ such that for $\chi \in C_c^\infty(B_\delta(\xi_0))$ and for $r \in [1, d]$, we have*

$$\|W_1\chi(D)^2(T(D)-z)^{-1}W_2\|_{B(l^2(\mathbb{Z}^d))} \le C\|W_1\|_{l^r(\mathbb{Z}^d)}\|W_2\|_{l^r(\mathbb{Z}^d)} \tag{6}$$

with a constant independent of $W_1, W_2 \in l^r(\mathbb{Z}^d)$ and $z \in \mathbb{C}\setminus\mathbb{R}$.

Remark 2.2. In [1, Proposition A.11], it is proved that

$$\|W_1\chi(D)^2(T(D)-z)^{-1}W_2\|_{B(L^2)} \le C|z-\lambda_0|^{\frac{d}{r}-1}G_r(z)\|W_1\|_{L^r}\|W_2\|_{L^r},$$

for $r \in [d, d+1]$, where

$$G_r(z) = \begin{cases} |\log|z-\lambda_0|| & \text{if } \xi_0 \text{ is a saddle point and if } r = d, \\ 1 & \text{otherwise.} \end{cases}$$

Proposition 2.1 improves this result when ξ_0 is a saddle point and when $r = d$, although in [1, Proposition A.11], the Shatten norm estimates are shown. For the results on exact ultrahyperbolic operators, see [5].

Proof. We shall slightly modify the argument in [13, Proposition 3.3]. We may assume $\operatorname{Im} z < 0$. Since $l^{p_1}(\mathbb{Z}^d) \subset l^{p_2}(\mathbb{Z}^d)$ for $p_1 \le p_2$, we may assume $r = d$. We take $\delta > 0$ small enough such that the Hessian of $T(\xi)$ does not vanish on $B_{2\delta}(\xi_0)$. Then for $\chi \in C_c^\infty(B_\delta(\xi_0))$, the stationary phase theorem implies

$$\|\chi(D)e^{-itT(D)}\|_{B(l^1(\mathbb{Z}^d),l^\infty(\mathbb{Z}^d))} \le C\langle t\rangle^{-\frac{d}{2}},$$

where we note that the singularity at $t = 0$ does not occur by virtue of the compactness of supp χ (see the proof in [11, Theorem 3]). Applying [7, Theorem 1.2] with $U(t) = 1_{[0,T)}(t)\chi(D)e^{-itT(D)}$, it follows that the unique solution $u(t,x)$ to

$$i\partial_t u(t,x) - T(D)u(t,x) = g(t,x), \quad u(0,x) = u_0(x) \in l^2(\mathbb{Z}^d) \tag{7}$$

satisfies

$$\|\chi(D)^2 u\|_{L^2([0,T),l^{2^*}(\mathbb{Z}^d))} \le C\|u_0\|_{L^2(\mathbb{Z}^d)} + C\|g\|_{L^2([0,T),l^{2_*}(\mathbb{Z}^d))},$$

where $2^* = 2d/(d-2)$ and $2_* = 2d/(d+2)$.

Let f be a finitely supported function. Set

$$g(t,x) = e^{itz}f(x), \quad u_0(x) = (T(D)-z)^{-1}f(x), \quad u(t,x) = e^{itz}u(x).$$

Since $u(t,x)$ and $g(t,x)$ satisfy (7), we have

$$\gamma(T)\|\chi(D)^2 u_0\|_{l^{2^*}(\mathbb{Z}^d))} \le C\|u_0\|_{L^2(\mathbb{Z}^d)} + C\gamma(T)\|f\|_{l^{2_*}(\mathbb{Z}^d)},$$

where $\gamma(T) = (\int_0^T |e^{itz}|^2 dt)^{1/2}$. Since $\operatorname{Im} z < 0$, we have $\gamma(T) \ge \sqrt{T}$. By letting $T \to \infty$, we obtain

$$\|\chi(D)^2(T(D)-z)^{-1}f\|_{l^{2^*}(\mathbb{Z}^d))} \le C\|f\|_{l^{2_*}(\mathbb{Z}^d)}.$$

Now Lemma A.1 implies (6) for $\operatorname{Im} z < 0$. □

2.2. Uniform resolvent estimates away form thresholds. We use the following propositions essentially due to the arguments in [1, Proposition A.5] and [14, Theorem 1.2]. Although [1, Proposition A.5] is stated only for a hypersurface in $\mathbb{R}^d$, its proof there can be applied with a hypersurface on $\mathbb{T}^d$.

Proposition 2.3. *Let $d \geq 1$ and $M \subset \mathbb{T}^d$ be a hypersurface with normalized defining function $\rho\colon \mathbb{T}^d \to \mathbb{R}$. For $\chi \in C^\infty(\mathbb{T}^d)$ and $k > 0$, assume that*

$$\sup_{x\in\mathbb{Z}^d} (1+|x|)^k \widehat{\chi d\sigma_M(x)} < \infty, \tag{8}$$

where $d\sigma_M$ denotes the canonical surface measure on M. Then for $r \in [1, 2+2k]$, we have

$$\|W_1\chi(D)(\rho(D)-z)^{-1}W_2\|_{B(l^2(\mathbb{Z}^d))} \leq C\|W_1\|_{l^r(\mathbb{Z}^d)}\|W_2\|_{l^r(\mathbb{Z}^d)}.$$

with a constant independent of $W_1, W_2 \in l^r(\mathbb{Z}^d)$ and $z \in \mathbb{C}\setminus\mathbb{R}$.

By using a partition of unity, to prove Theorem 1.1, it suffices to prove the following theorem.

Theorem 2.4. *Let $\lambda \in (0, 12)$. We denote $M = M_\lambda$ and $\rho(\xi) = h_0(\xi) - \lambda$.*

- (i) *Let $\lambda \in (0,4) \cup (8,12)$ and $\xi \in M_\lambda$. Then for any $\chi \in C^\infty(\mathbb{T}^3)$ supported close to ξ, (8) holds for $k = 1$.*
- (ii) *Let $\lambda = 6$ and $\xi \in M_\lambda$. Then for any $\chi \in C^\infty(\mathbb{T}^3)$ supported close to ξ, (8) holds for $k = \frac{2}{3}$.*
- (iii) *Let $\lambda \in (4,8)\setminus\{6\}$ and $\xi \in M_\lambda$. Then for any $\chi \in C^\infty(\mathbb{T}^3)$ supported close to ξ, (8) holds for $k = \frac{3}{4}$.*
- (iv) *Let $\lambda \in \{4,8\}$ and $\xi \in M_\lambda \setminus \mathrm{Cr}(h_0)$. Then for any $\chi \in C^\infty(\mathbb{T}^3)$ supported close to ξ, (8) holds for $k = \frac{1}{2}$.*

Remark 2.5. In [2, Theorem 2.1], (iii) is proved for $r = \frac{3}{4} - \varepsilon$ for any $\varepsilon > 0$ (more precisely, the estimates with a logarithmic loss). Our result (iii) improves the result in [2].

Proof of Theorem 1.1. Proposition 2.3, Theorem 2.4 (i), (ii), and (iii) imply Theorem 1.1 (i). Moreover, Proposition 2.1 and Theorem 2.4 imply Theorem 1.1 (ii). □

In the rest of this paper, we will prove Theorem 2.4.

3. Some oscillatory integrals

In this section, we collect the results on the decay rate of some oscillatory integrals. It is regarded as generalization of the Van der Corput lemma in higher dimensions. Oscillatory integrals of the following forms are studied in [15]:

$$\int_{\mathbb{R}^2} \chi(\eta) e^{i\lambda f(\eta)} d\eta \quad \text{as } \lambda \to \infty.$$

For our purpose, we need the decay rates for the Fourier transform of the surface measure. To do this, we use the recent result by Ikromov and Müller [4]. To prove the decay of such integrals, we need the following elementary lemma.

Lemma 3.1. *Let $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and define*

$$f_1, f_2, f_3 : \mathbb{S} = \{\eta \in \mathbb{R}^2 \mid |\eta| = 1\} \to \mathbb{C}$$

by

$$f_1(\eta) = \alpha\eta_1^2\eta_2 + \beta\eta_1\eta_2^2, \quad f_2(\eta) = \alpha\eta_1^3 + \beta\eta_2^2, \quad f_3(\eta) = \alpha\eta_1^2\eta_2 + \beta\eta_2^2.$$

Then any zeros of the functions f_1, f_2, f_3 are simple.

Proof. We denote $\eta_1 = \cos\theta$ and $\eta_2 = \sin\theta$. Let $\varphi \in [0, 2\pi) \setminus \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ be satisfying $\cos\varphi = \frac{\beta}{\alpha^2+\beta^2}$ and $\sin\varphi = \frac{\alpha}{\alpha^2+\beta^2}$. Then we write

$$\begin{aligned}
f_1 &= \cos\theta\sin\theta(\alpha\cos\theta + \beta\sin\theta) = \sqrt{\alpha^2+\beta^2}\cos\theta\sin\theta\sin(\theta+\varphi), \\
f_2 &= \alpha\cos^3\theta + \beta(1-\cos^2\theta), \\
f_3 &= -\alpha\sin^3\theta + \beta\sin^2\theta + \alpha.
\end{aligned}$$

Since the zeros of $\cos\theta$, $\sin\theta$ and $\sin(\theta+\varphi)$ are simple and since these zeros are distinct, it follows that the zeros of f_1 are simple. A simple calculation gives

$$\frac{df_2}{d\theta} = -\sin\theta\cos\theta(3\alpha\cos\theta - 2\beta).$$

Thus,

$$f_2(\theta) = \frac{df_2}{d\theta}(\theta) = 0 \implies \cos\theta = \pm\sqrt{3}.$$

Since $|\cos\theta| \leq 1$, then the zeros of f_2 are simple. Finally, we have

$$f_3 = -\alpha\sin\theta\Big(\sin\theta - \frac{\beta}{2\alpha} - \sqrt{\frac{\beta^2}{4\alpha^2}+1}\Big)\Big(\sin\theta - \frac{\beta}{2\alpha} + \sqrt{\frac{\beta^2}{4\alpha^2}+1}\Big)$$

which only has simple zeros. □

The next proposition is a consequence of [4, Theorem1.1].

Proposition 3.2. *Let f be a real-valued smooth function near $0 \in \mathbb{R}^2$. For $\chi \in C_c^\infty(\mathbb{R}^d)$ supported near 0, define*

$$I(x) = \int_{\mathbb{R}^2} \chi(\eta) e^{2\pi i(x_1\eta_1 + x_2\eta_2 + x_3 f(\eta))} d\eta, \quad x \in \mathbb{R}^3.$$

If the support of χ is close to 0, the following holds.

(i) *Suppose that f can be written as*

$$f(\eta) = f(0) + \sum_{j=1}^{2} \partial_{\eta_j} f(0)\eta_j + \alpha_{12}\eta_1^2\eta_2 + \alpha_{21}\eta_1\eta_2^2 + O(|\eta|^4) \quad \text{as } |\eta| \to 0$$

with $\alpha_{12}, \alpha_{21} \in \mathbb{R} \setminus \{0\}$. Then we have $I(x) = O(|x|^{-\frac{2}{3}})$ as $|x| \to \infty$.

(ii) *Suppose that f can be written as*

$$f(\eta) = f(0) + \sum_{j=1}^{2} \partial_{\eta_j} f(0)\eta_j + \alpha_2\eta_2^2 + \sum_{\substack{i+j+k=3 \\ i \le j \le k}} \alpha_{ijk}\eta_i\eta_j\eta_k$$
$$+ \sum_{\substack{i+j+k+m=4 \\ i \le j \le k \le m}} \alpha_{ijkm}\eta_i\eta_j\eta_k\eta_m + O(|\eta|^5) \quad \text{as } |\eta| \to 0$$

with $\alpha_2 \in \mathbb{R} \setminus \{0\}$ and $\alpha_{ijk}, \alpha_{ijkm} \in \mathbb{R}$. We assume

$$\alpha_{111} = 0 \implies \alpha_{112} \neq 0 \quad \text{and} \quad \alpha_{1111} = 0.$$

Then we have

$$I(x) = \begin{cases} O(|x|^{-\frac{5}{6}}) & \text{as } |x| \to \infty \quad \text{if } \alpha_{111} \neq 0, \\ O(|x|^{-\frac{3}{4}}) & \text{as } |x| \to \infty \quad \text{otherwise.} \end{cases}$$

Remark 3.3. The results in [15] imply that the above estimates are sharp for $x_1 = x_2 = 0$ and $|x_3| \to \infty$ at least if f is analytic.

Proof. We may assume $f(0) = 0$. Moreover, changing of the variable $x'_j = x_j + \partial_{\eta_j} f(0)x_3$ $(j = 1, 2)$ and $x'_3 = x_3$, we may also assume $\partial_{\eta_j} f(0) = 0$ for $j = 1, 2$.

(i) We use some notations and definitions from [4, before Theorem 1.1]. Let f_{pr} be the principal part of f, $\pi(f)$ be the principal face, $d(f)$ be the Newton distance, $h(f)$ be the height of f and $\nu(f)$ be Varchenko's exponent of f. By a simple calculation, we have

$$\pi(f) = \{\eta \in \mathbb{R}^2 \mid \eta_1 \geq 1, \eta_2 \geq 2, \eta_2 = -\eta_1 + 3\},$$
$$f_{\mathrm{pr}}(\eta) = \alpha_{12}\eta_1^2\eta_2 + \alpha_{21}\eta_1\eta_2^2, d(f) = \frac{3}{2}.$$

Moreover, it turns out that the coordinate η is adapted in the sense of [4]. In fact, it follows that $\pi(f)$ is the compact edge and $m(f_{\mathrm{pr}}) = 1 < \frac{3}{2} = d(f)$ (this follows from Lemma 3.1), where $m(f_{\mathrm{pr}})$ is the vanishing order of $f_{\mathrm{pr}}|_{\mathbb{S}^1}$. This implies that f satisfies the condition (a) in [4, before Lemma 1.5] and hence the coordinate η is adapted. This implies $h(f) = d(f) = \frac{3}{2}$. Since $h(f) < 2$, we have $\nu(f) = 0$ by its definition. Now our claim follows from [4, Theorem 1.1].

(ii) First, we assume $\alpha_{111} \neq 0$. By Lemma 3.1, we have

$$\begin{aligned}
\pi(f) &= \left\{\eta \in \mathbb{R}^2 \,\middle|\, \eta_1 \geq 0, \eta_2 \geq 0, \eta_2 = -\frac{2}{3}\eta_1 + 2\right\},\\
f_{\mathrm{pr}}(\eta) &= \alpha_2\eta_2^2 + \alpha_{111}\eta_1^3,\\
d(f) &= h(f) = \frac{6}{5},\\
m(f_{\mathrm{pr}}) &= 1.
\end{aligned}$$

Since $h(f) < 2$, we obtain $\nu(f) = 0$ and $I(x) = O(|x|^{-\frac{5}{6}})$. Next, we assume $\alpha_{111} = 0$. Since $\alpha_{1111} = 0$, we have

$$\begin{aligned}
\pi(f) &= \left\{\eta \in \mathbb{R}^2 \,\middle|\, \eta_1 \geq 0, \eta_2 \geq 1, \eta_2 = -\frac{1}{2}\eta_1 + 2\right\},\\
f_{\mathrm{pr}}(\eta) &= \alpha_2\eta_2^2 + 2\alpha_{112}\eta_1^2\eta_2,\\
d(f) &= h(f) = \frac{4}{3},\\
m(f_{\mathrm{pr}}) &= 1.
\end{aligned}$$

Since $h(f) < 2$, we obtain $\nu(f) = 0$ and $I(x) = O(|x|^{-\frac{3}{4}})$. □

Remark 3.4. When $\alpha_{111} = 0$ in (ii), the condition $\alpha_{1111} = 0$ is necessary since in general, the principal part is written as $f_{\mathrm{pr}}(\eta) = \alpha_2\eta_2^2 + 2\alpha_{112}\eta_1^2\eta_2 + \alpha_{1111}\eta_1^4$. As is pointed out by J. C. Cuenin and I. A. Ikromov, the optimal decay of I is $O(|x|^{-\frac{1}{2}})$ for the phase function $f(\eta) = (\eta_2 - \eta_1^2)^2 = \eta_2^2 - 2\eta_1^2\eta_2 + \eta_2^4$.

4. Geometry of hypersurfaces

In this section, we study the geometry of the Fermi surface M_λ for $\lambda \in [0, 4d]$.

4.1. General theory. Let $M \subset \mathbb{T}^3$ or $M \subset \mathbb{R}^3$ be an embedded hypersurface of codimension 1. Let $q \in M$. We may assume that there exist an open neighborhood $U \subset \mathbb{T}^3$ or $U \subset \mathbb{R}^3$ of q, an open set $V \subset \mathbb{R}^2$ and a smooth function $f\colon V \to \mathbb{R}$ such that $M \cap U = \{(\xi', f(\xi')) \mid \xi' \in V\}$. We compute the induced Riemannian metric g on M, the unit normal ν, the second fundamental form $A(\xi') = (A(\xi'))_{i,j=1}^2$ and the Gaussian curvature $K_1(\xi')$:

$$g = \sum_{j=1}^{2}(1 + \partial_{\xi_j} f(\xi')^2) d\xi_j^2 + 2\partial_{\xi_1} f(\xi')\partial_{\xi_2} f(\xi') d\xi_1 d\xi_2,$$

$$\nu(\xi') = \frac{1}{\sqrt{1 + |\nabla_{\xi'} f(\xi')|^2}} \begin{pmatrix} -\nabla_{\xi'} f(\xi') \\ 1 \end{pmatrix}, \quad A_{ij}(\xi') = \frac{\partial_{\xi_i}\partial_{\xi_j} f(\xi')}{\sqrt{1 + |\nabla_{\xi'} f(\xi')|^2}}, \tag{9}$$

$$K_1(\xi') = \frac{\det(A_{ij}(\xi'))}{\det g(\xi')} = \frac{\det \partial_{\xi_i}\partial_{\xi_j} f(\xi')}{(1 + |\nabla_{\xi'} f(\xi')|^2)^2}. \tag{10}$$

Lemma 4.1. *We denote the Gaussian curvature at $\xi \in M \cap U \subset \mathbb{T}^3$ by $K(\xi)$, that is $K_1(\xi') = K(\xi', f(\xi'))$ for $\xi = (\xi', f(\xi')) \in M \cap U$. Then it follows that $\nabla_{\xi'} K_1(\xi') \neq 0$ if and only if*

$$(\nabla_\xi h_0 \times \nabla_\xi K)(\xi', f(\xi')) \neq 0.$$

Proof. We recall ∇h_0 is the unit normal of M_λ and is parallel to the vector

$$(-\partial_{\xi_1} f, -\partial_{\xi_2} f, 1).$$

We learn

$$\nabla_{\xi'} K_1(\xi') = (\nabla_{\xi'} K)(\xi', f(\xi')) + (\partial_{\xi_3} K)(\xi', f(\xi'))\nabla_{\xi'} f(\xi')$$

and

$$\begin{pmatrix} -\partial_{\xi_1} f \\ -\partial_{\xi_2} f \\ 1 \end{pmatrix} \times \nabla_\xi K(\xi', f(\xi')) = \begin{pmatrix} -\partial_{\xi_3} K \partial_{\xi_2} f - \partial_{\xi_2} K \\ \partial_{\xi_3} K \partial_{\xi_1} f + \partial_{\xi_1} K \\ -\partial_{\xi_1} f \partial_{\xi_2} K + \partial_{\xi_2} f \partial_{\xi_1} K \end{pmatrix}.$$

It follows that $(\nabla_\xi h_0 \times \nabla_\xi K)(\xi', f(\xi')) = 0$ implies $\nabla_{\xi'} K_1(\xi') = 0$. A simple calculation implies that $\nabla_{\xi'} K_1(\xi') = 0$ gives $-\partial_{\xi_1} f \partial_{\xi_2} K + \partial_{\xi_2} f \partial_{\xi_1} K = 0$ at $\xi = (\xi', f(\xi'))$. This completes the proof. $\square$

It is useful to calculate the Taylor expansion of f in terms of information about the Hessian of f:

Lemma 4.2. *Let $V \subset \mathbb{R}^2$ be an open set and $f \in C^\infty(V;\mathbb{R})$. Moreover, the 2×2-matrix $B(\xi')$ is defined by $B(\xi') = (\partial_{\xi_i}\partial_{\xi_j} f(\xi'))_{j,k=1}^2$. Suppose that there exist smooth functions $\lambda_\pm(\xi') \in C^\infty(V;\mathbb{R})$ and a orthogonal matrix*

$$U(\xi') = \begin{pmatrix} u_+(\xi') & u_-(\xi') \end{pmatrix}, \quad u_\pm(\xi') \in C^\infty(V;\mathbb{R}^2)$$

with

$$u_a(\xi') \cdot u_b(\xi') = \delta_{ab}, \quad a, b \in \{\pm\}$$

such that

$$U(\xi')^{-1} B(\xi') U(\xi') = \begin{pmatrix} \lambda_+(\xi') & 0 \\ 0 & \lambda_-(\xi') \end{pmatrix}.$$

Let $p \in V$. Set $U = U(p)$ and introduce the variable

$$\eta = U^{-1}(\xi' - p).$$

Then we have

$$\begin{aligned} f(\xi') = {} & f(p) + \partial_{\xi'} f(p) \cdot U\eta + \frac{1}{2}(\lambda_+(p)\eta_1^2 + \lambda_-(p)\eta_2^2) \\ & + \frac{1}{3!}(u_+(p) \cdot (\nabla_{\xi'}\lambda_+)(p)\eta_1^3 + u_-(p) \cdot (\nabla_{\xi'}\lambda_-)(p)\eta_2^3) \\ & + \frac{1}{3!}(3u_-(p) \cdot (\nabla_{\xi'}\lambda_+)(p)\eta_1^2\eta_2 + 3u_+(p) \cdot (\nabla_{\xi'}\lambda_-)(p)\eta_1\eta_2^2) \\ & + O(|\eta|^4) \end{aligned}$$

as $\eta \to 0$.

Proof. We note

$$(\xi' - p) \cdot \partial_{\xi'}^2 f(p)(\xi' - p) = \eta \cdot {}^t UB(p)U\eta = \lambda_+(p)^2\eta_1^2 + \lambda_-(p)^2\eta_2^2.$$

Thus, it suffices to prove

$$\begin{aligned} \partial_{\eta_1}^3 f(p) = u_+(p) \cdot (\nabla_{\xi'}\lambda_+)(p), \ \partial_{\eta_1}^2\partial_{\eta_2} f(p) = u_-(p) \cdot (\nabla_{\xi'}\lambda_+)(p), \\ \partial_{\eta_1}\partial_{\eta_2}^2 f(p) = u_+(p) \cdot (\nabla_{\xi'}\lambda_-)(p), \ \partial_{\eta_2}^3 f(p) = u_-(p) \cdot (\nabla_{\xi'}\lambda_-)(p). \end{aligned}$$

To see this, we observe

$$U(p)^{-1}\partial_{\xi'}^2 f(\xi')U(p) = \partial_\eta^2 f(\xi').$$

This implies

$$\begin{pmatrix} \lambda_+(\xi') & 0 \\ 0 & \lambda_-(\xi') \end{pmatrix} = U(\xi')^{-1}U(p)\partial_\eta^2 f(\xi')U(p)^{-1}U(\xi')$$

Differentiating in ξ' and substituting $\xi' = p$, we have

$$\begin{aligned}
&\begin{pmatrix} \partial_{\xi'}\lambda_+(p) & 0 \\ 0 & \partial_{\xi'}\lambda_-(p) \end{pmatrix} \\
&\quad = \partial_{\xi'}\partial_\eta^2 f(\xi') + U(p)^{-1}U(p)\partial_\eta^2 f(p)U(p)^{-1}(\partial_{\xi'}U)(p) \\
&\quad - U(p)^{-1}(\partial_{\xi'}U)(p)U(p)^{-1}U(p)\partial_\eta^2 f(p)U(p)^{-1}U(p) \\
&\quad = \partial_{\xi'}\partial_\eta^2 f(\xi') + \partial_\eta^2 f(p)U(p)^{-1}(\partial_{\xi'}U)(p) \\
&\quad - U(p)^{-1}(\partial_{\xi'}U)(p)\partial_\eta^2 f(p).
\end{aligned} \tag{11}$$

Using

$$(\partial_{\xi'}|u_\pm(\xi')|^2)|_{\xi'=p} = \partial_{\xi'}1 = 0$$

and

$$(\partial_{\xi'}u_+(\xi')\cdot u_-(\xi'))|_{\xi'=p} = \partial_{\xi'}0 = 0,$$

we have

$$\begin{aligned}
u_+(p)\cdot\partial_{\xi'}u_+(p) = u_-(p)\cdot\partial_{\xi'}u_-(p) = 0, \\
u_+(p)\cdot\partial_{\xi'}u_-(p) + \partial_{\xi'}u_+(p)\cdot u_-(p) = 0.
\end{aligned}$$

This implies

$$\begin{aligned}
U(p)^{-1}(\partial_{\xi'}U)(p) &= \begin{pmatrix} u_+(p)\cdot\partial_{\xi'}u_+(p) & u_+(p)\cdot\partial_{\xi'}u_-(p) \\ u_-(p)\cdot\partial_{\xi'}u_+(p) & u_-(p)\cdot\partial_{\xi'}u_-(p) \end{pmatrix} \\
&= \begin{pmatrix} 0 & u_+(p)\cdot\partial_{\xi'}u_-(p) \\ u_-(p)\cdot\partial_{\xi'}u_+(p) & 0 \end{pmatrix}.
\end{aligned}$$

Setting $a = u_+(p)\cdot\partial_{\xi'}u_-(p) = -u_-(p)\cdot\partial_{\xi'}u_+(p)$, $A = \partial_{\eta_1}^2 f(p)$ and $B = \partial_{\eta_2}^2 f(p)$, we have

$$\begin{aligned}
&\partial_\eta^2 f(p)U(p)^{-1}(\partial_{\xi'}U)(p) - U(p)^{-1}(\partial_{\xi'}U)(p)\partial_\eta^2 f(p) \\
&= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & a(A-B) \\ a(A-B) & 0 \end{pmatrix}.
\end{aligned} \tag{12}$$

It follows from (11) and (12) that

$$\partial_{\xi'}\lambda_+(p) = (\partial_{\xi'}\partial_{\eta_1}^2)f(p),\ \ \partial_{\xi'}\lambda_-(p) = (\partial_{\xi'}\partial_{\eta_2}^2)f(p)$$

Using $\partial_{\eta_1} = u_+(p)\cdot\partial_{\xi'}$ and $\partial_{\eta_2} = u_-(p)\cdot\partial_{\xi'}$, we complete the proof. □

4.2. Geometry of the Fermi surface. In the following, we consider the Fermi surface $M = M_\lambda = h_0^{-1}(\{\lambda\})$. We fix some notations. For $j = 1, 2, 3$, we set

$$a_j = a_j(\xi) = \cos 2\pi\xi_j, \quad b_j = b_j(\xi) = \sin 2\pi\xi_j.$$

Set

$$E_\lambda = 3 - \lambda/2 \in (-3, 3) \quad \text{for } 0 < \lambda < 12.$$

From the expression (1), we have

$$\begin{aligned} M_\lambda &= \{\xi \in \mathbb{T}^3 \mid a_1 + a_2 + a_3 = E_\lambda\}, \\ \mathrm{Cr}(h_0) &= \{\xi \in \mathbb{T}^3 \mid b_j = 0, j = 1, 2, 3\}, \end{aligned} \tag{13}$$

where we recall that $\mathrm{Cr}(h_0)$ is defined in (2). We define

$$\begin{aligned} &K(\xi) \in C^\infty(\mathbb{T}^3 \setminus \mathrm{Cr}(h_0); \mathbb{R}): && \text{the Gaussian curvature of } M_\lambda \text{ at } \xi, \\ &\nu(\xi) \in C^\infty(\mathbb{T}^3 \setminus \mathrm{Cr}(h_0); \mathbb{R}^3): && \text{the unit normal of } M_\lambda \text{ at } \xi, \end{aligned}$$

where the smoothness of K follows from the implicit function theorem. For

$$\xi \in M_\lambda \cap \{\partial_{\xi_3} h_0(\xi) \neq 0\} = \{b_3 \neq 0\},$$

we write

$$\xi = (\xi', f_\lambda(\xi')), \quad K(\xi', f_\lambda(\xi')) = K_1(\xi').$$

We note that the map $(\xi', \lambda) \mapsto f_\lambda(\xi')$ is smooth by virtue of the implicit function theorem.

We can calculate K and ν explicitly:

Lemma 4.3. *We have*

$$\begin{aligned} K(\xi) &= \frac{4\pi^2(a_1a_2b_3^2 + a_2a_3b_1^2 + a_3a_1b_2^2)}{(b_1^2 + b_2^2 + b_3^2)^2}, \\ \nu(\xi) &= \frac{1}{\sqrt{b_1^2 + b_2^2 + b_3^2}}(b_1, b_2, b_3). \end{aligned} \tag{14}$$

Proof. Let $\xi_0 \in \mathbb{T}^3 \setminus \mathrm{Cr}(h_0)$ and U be a small neighborhood of ξ_0. We prove (14) at ξ_0. Set $\lambda = h_0(\xi_0)$. By permutating the coordinate, we may assume $b_3(\xi) \neq 0$. By the implicit function theorem, $U \cap M_\lambda$ has a graph representation:

$$U \cap M_\lambda = \{(\xi', f_\lambda(\xi'))\}.$$

Differentiating $h_0(\xi', f_\lambda(\xi')) = \lambda$ twice, for $j = 1, 2$, we have

$$\partial_{\xi_j} f_\lambda(\xi') = -\frac{b_j}{b_3}, \tag{15a}$$

$$\partial_{\xi_j}^2 f_\lambda(\xi') = -\frac{2\pi}{b_3^3}(a_j b_3^2 + a_3 b_j^2), \tag{15b}$$

$$\partial_{\xi_1}\partial_{\xi_2} f_\lambda(\xi') = -\frac{2\pi b_1 b_2 a_3}{b_3^3}. \tag{15c}$$

This implies

$$\begin{aligned}(1 + |\partial_{\xi'} f(\xi')|^2)^2 &= \frac{(b_1^2 + b_2^2 + b_3^2)^2}{b_3^4}, \\ \det \partial_{\xi'}^2 f_\lambda(\xi') &:= \partial_{\xi_1}^2 f(\xi')\partial_{\xi_2}^2 f(\xi') - (\partial_{\xi_1}\partial_{\xi_2} f(\xi'))^2 \\ &= \frac{4\pi^2}{b_3^4}(a_1 a_2 b_3^2 + a_2 a_3 b_1^2 + a_3 a_1 b_2^2).\end{aligned}$$

Substituting these relations into (9) and (10), we obtain (14). □

Now we determine all points where the Gaussian curvature vanishes.

Proposition 4.4. *Let* $0 < \lambda < 12$.

(i) *We have*

$$\begin{aligned}K^{-1}(0) \cap M_\lambda &= \{a_1 a_2 + a_2 a_3 + a_3 a_1 = a_1 a_2 a_3(a_1 + a_2 + a_3)\} \\ &= \{a_1 = a_2 = 0\} \cup \{a_2 = a_3 = 0\} \cup \{a_3 = a_1 = 0\} \\ &\quad \cup \{a_1 + a_2 + a_3 = 1/a_1 + 1/a_2 + 1/a_3,\ a_1, a_2, a_3 \neq 0\}.\end{aligned} \tag{16}$$

Moreover, if $K(\xi) = 0$ *with* $\xi \in M_\lambda \setminus \mathrm{Cr}(h_0)$, *then* $4 \leq \lambda \leq 8$ *holds.*

(ii) *All principal curvatures of* $M_\lambda \setminus \mathrm{Cr}(h_0)$ *at* ξ *vanish if and only if* $\xi_j \in \{1/4, 3/4\}$ *for* $j = 1, 2, 3$ *and* $\lambda = 6$.

(iii) *The Gaussian curvature* $K(\xi)$ *on* M_6 *vanishes if and only if* $\xi_j \in \{1/4, 3/4\}$ *for* $j = 1, 2, 3$.

Proof. (i) The first part immediately follows from the representation (14) and the relations $a_j^2 + b_j^2 = 1$ for $j = 1, 2, 3$.

Next, we prove that $K(\xi) = 0$ with $\xi \in M_\lambda$ implies $4 \leq \lambda \leq 8$. Let $\xi \in K^{-1}(0) \cap M_\lambda$ and set

$$f(t) = t^3 - E_\lambda t^2 + (E_\lambda a_1 a_2 a_3)t - a_1 a_2 a_3.$$

Then (13) and (16) imply that a_1, a_2 and a_3 are all zeros of f. Since

$$\lim_{t\to\pm\infty} f(t) = \pm\infty$$

and $a_j \in [-1, 1]$, we have $f(1) \geq 0$ and $f(-1) \leq 0$, which implies

$$(1 - E_\lambda)(1 - a_1a_2a_3) \geq 0, \quad (1 + E_\lambda)(1 + a_1a_2a_3) \leq 0.$$

These inequalities with $|a_1a_2a_3| \leq 1$ gives $-1 \leq E_\lambda \leq 1$, which is equivalent to $4 \leq \lambda \leq 8$.

(ii) Let $\xi \in M_\lambda \setminus \mathrm{Cr}(h_0)$ with $4 \leq \lambda \leq 8$. By permutating the coordinate, we may assume that $b_3(\xi) \neq 0$ and that we can write $U \cap M_\lambda = \{(\xi', f_\lambda(\xi'))\}$. Then all principal curvatures of M_λ at ξ vanishes if and only if $\partial_{\xi_k}\partial_{\xi_l} f(\xi') = 0$ for each $k, l = 1, 2$. This is also equivalent to

$$\delta_{kl}(1 - a_3^2)a_k + b_kb_l\sqrt{1 - a_k^2}\sqrt{1 - a_l^2}a_3 = 0, \quad k, l = 1, 2. \tag{17}$$

Since it is easy to see that $\xi_j \in \{1/4, 3/4\}$ for $j = 1, 2, 3$ imply (17), then we prove that (17) implies $\xi_j \in \{1/4, 3/4\}$ for $j = 1, 2, 3$. Recall that $|a_3| \neq 1$ since we assume $\partial_{\xi_3} h_0 \neq 0$ on M_λ.

If we suppose $a_3 = 0$, then (17) with $k = l = 1, 2$ imply that $a_k = 0$ for $k = 1, 2$ and hence $\xi_k \in \{1/4, 3/4\}$ for $k = 1, 2, 3$.

If we suppose $a_3 \neq 0$, then (17) with $k = 1$ and $l = 2$ imply that either $|a_1|$ or $|a_2|$ is equal to 1. Then it follows from $a_3 \neq 0$ and from (17) with $k = l = 1$ or $k = l = 2$ that $|a_3| = 1$. Thus we obtain $|a_k| = 1$ for $k = 1, 2$ by (17) with $k = l = 1, 2$. However, this contradicts to $\nabla h_0(\xi) \neq 0$ and hence $a_3 = 0$.

(iii) We note that $\lambda = 6$ is equivalent to $E_\lambda = 0$. (13) and (16) implies

$$a_1 + a_2 + a_3 = a_1a_2 + a_2a_3 + a_3a_1 = 0 \tag{18}$$

at $\xi \in M_6$. Then, it follows that a_1, a_2, a_3 are the solutions to the equation

$$t^3 - a_1a_2a_3 = 0. \tag{19}$$

If $a_1a_2a_3 = 0$, then we have $a_1 = a_2 = a_3 = 0$ and hence $\xi_j \in \{1/4, 3/4\}$ holds for $j = 1, 2, 3$. We suppose $a_1a_2a_3 \neq 0$ and deduce a contradiction. Substituting (19) into $t = a_1$, a_2 and a_3, we have $a_1^3 = a_2^3 = a_3^3 = a_1a_2a_3$. This gives $a_1^2 = a_2a_3$, $a_2^2 = a_3a_1$ and $a_3^2 = a_1a_2$. Combining these relations with (19), we obtain $a_1 = a_2 = a_3$. Thus (18) implies $a_1 = a_2 = a_3 = 0$. This is a contradiction. $\square$

Lemma 4.5. *Let* $4 < \lambda < 8$ *with* $\lambda \neq 6$ *and* $\xi_* \in M_\lambda \cap K^{-1}(0)$. *Then we obtain*

$$\nabla_\xi h_0(\xi_*) \times \nabla_\xi K(\xi_*) \neq 0. \tag{20}$$

In particular, from Lemma 4.1, we have $(\nabla_{\xi'} K_1)(\xi'_*) \neq 0$, *where* $\xi_* = (\xi'_*, f_\lambda(\xi'))$.

Proof. We set

$$\nu(\xi) = (b_1, b_2, b_3), \quad \widetilde{K}(\xi) = a_1a_2b_3^2 + a_2a_3b_1^2 + a_3a_1b_2^2.$$

Using $\widetilde{K}(\xi_*) = 0$, $\nabla h_0 \parallel \nu$ and (14), we see that (20) is equivalent to

$$\nu(\xi_*) \times \nabla_\xi \widetilde{K}(\xi_*) \neq 0.$$

A direct computation gives

$$\nu(\xi) \times \nabla_\xi \widetilde{K}(\xi) = -2\pi \begin{pmatrix} b_2b_3(a_2 - a_3)(1 - a_1(a_1 + a_2 + a_3)) \\ b_3b_1(a_3 - a_1)(1 - a_2(a_1 + a_2 + a_3)) \\ b_1b_2(a_1 - a_2)(1 - a_3(a_1 + a_2 + a_3)) \end{pmatrix}.$$

We note that $(a_1 + a_2 + a_3)(\xi_*) = E_\lambda$. Moreover, it follows that $\lambda \in (4, 8)$ is equivalent to $E_\lambda \in (-1, 1)$. These relations with $-1 \leq a_j \leq 1$ imply that for $\xi \in M_\lambda$, $\nu(\xi) \times \nabla_\xi \widetilde{K}(\xi) = 0$ is equivalent to

$$b_2b_3(a_2 - a_3) = b_3b_1(a_3 - a_1) = b_1b_2(a_1 - a_2) = 0 \quad \text{at } \xi. \tag{21}$$

Since $\xi_* \in M_\lambda \cap K^{-1}(0)$ with $\lambda \in (4, 8) \setminus \{6\}$, (14) implies that (21) does not hold at ξ_*. This completes the proof. $\square$

4.3. Concrete description of the Fourier transform of the surface measure.

Now we set

$$\partial^2_{\xi'} f(\xi') = -2\pi \begin{pmatrix} \dfrac{a_1b_3^2 + a_3b_1^2}{b_3^3} & \dfrac{b_1b_2a_3}{b_3^3} \\ \dfrac{b_1b_2a_3}{b_3^3} & \dfrac{a_2b_3^2 + a_3b_2^2}{b_3^3} \end{pmatrix} =: B(\xi'). \tag{22}$$

Proposition 4.6. *Let* $\xi_* \in M_\lambda \cap K^{-1}(0)$ *with* $\lambda \in (4, 8)$ *with* $b_3(\xi_*) \neq 0$ *and* $U \subset \mathbb{T}^3$ *be a small neighborhood of* ξ *such that* $U \cap M_\lambda$ *has a graph representation*: $U \cap M_\lambda = \{(\xi', f_\lambda(\xi'))\}$.

(i) *If* $\lambda = 6$, *then* f_λ *has the following Taylor expansion near* $\xi_* = (\xi'_*, f_\lambda(\xi'_*))$:

$$f_\lambda(\xi') = f_\lambda(\xi'_*) + (\partial_{\xi'} f_\lambda)(\xi'_*) \cdot \eta + \alpha_{12}\eta_1^2\eta_2 + \alpha_{21}\eta_1\eta_2^2 + R(\eta), \tag{23}$$

where $\eta = (\eta_1, \eta_2) = \xi' - \xi'_*$ *and a real-valued function* R *satisfies* $|\partial_\eta^\gamma R(\eta)| \leq C|\eta|^{\max(4-|\gamma|,0)}$. *Here* $\alpha_{12}, \alpha_{21} \in \mathbb{R} \setminus \{0\}$.

(ii) *Suppose $\lambda \neq 6$. We regard $\xi' - \xi'_*$ as a vector in $\mathbb{R}^2$. Then there exists a 2×2 unitary matrix U such that*

$$
\begin{aligned}
f_\lambda(\xi') = f_\lambda(\xi'_*) &+ (\partial_{\xi'} f_\lambda)(\xi'_*) \cdot U\eta + \alpha_1 \eta_1^2 + \alpha_2 \eta_2^2 \\
&+ (\alpha_{111}\eta_1^3 + 3\alpha_{112}\eta_1^2\eta_2 + 3\alpha_{122}\eta_1\eta_2^2 + \alpha_{222}\eta_2^3) \\
&+ \sum_{\substack{i+j+k+m=4 \\ i \le j \le k \le m}} \alpha_{ijkm}\eta_i\eta_j\eta_k\eta_m + R(\eta),
\end{aligned}
$$

where we set

$$\eta = (\eta_1, \eta_2) = U^{-1}(\xi' - \xi'_*)$$

and a real-valued function R satisfies $|\partial_\eta^\gamma R(\eta)| \le C|\eta|^{\max(5-|\gamma|,0)}$. Here $\alpha_1, \alpha_2, \alpha_{ij} \in \mathbb{R}$ for $i, j = 1, 2$ satisfies $(\alpha_1, \alpha_2) \neq (0,0)$ and

$$
\begin{aligned}
\alpha_1 \neq 0 &\implies (\alpha_{122}, \alpha_{222}) \neq (0,0), \\
\alpha_2 \neq 0 &\implies (\alpha_{111}, \alpha_{112}) \neq (0,0).
\end{aligned}
$$

Moreover, it follows that if $(\alpha_1, \alpha_{111}) = (0,0)$ or $(\alpha_2, \alpha_{222}) = (0,0)$ hold, then we have

$$(a_1(\xi_*), a_2(\xi_*), a_3(\xi_*)) \in \{(0,0,E_\lambda), (0, E_\lambda, 0), (E_\lambda, 0, 0)\}. \tag{24}$$

(iii) *Suppose $\lambda \neq 6$. Let ξ_* satisfying*

$$a_1(\xi_*) = a_3(\xi_*) = 0, \quad a_2(\xi_*) = E_\lambda.$$

Then we have

$$
\begin{aligned}
f_\lambda(\xi') = f_\lambda(\xi'_*) &+ (\partial_{\xi'} f_\lambda)(\xi'_*) \cdot \xi' + \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 \\
&+ (\alpha_{111}\xi_1^3 + 3\alpha_{112}\xi_1^2\xi_2 + 3\alpha_{122}\xi_1\xi_2^2 + \alpha_{222}\xi_2^3) \\
&+ \sum_{\substack{i+j+k+m=4 \\ i \le j \le k \le m}} \alpha_{ijkm}\xi_i\xi_j\xi_k\xi_m + R(\xi'),
\end{aligned}
$$

where

$$\alpha_1 = \alpha_{111} = \alpha_{1111} = 0, \quad \alpha_{112} \neq 0,$$

hold and a real-valued function R satisfies $|\partial_{\xi'}^\gamma R(\xi')| \le C|\xi'|^{\max(5-|\gamma|,0)}$.

Proof. (i) Let $\xi_* \in M_6 \cap K^{-1}(0)$. Proposition 4.4 implies that $(\xi_*)_j \in \{1/4, 1/3\}$ for each $j = 1, 2, 3$ (which automatically implies $b_3(\xi_*) \neq 0$). We prove (23) only for $(\xi_*)_j = 1/4$, $j = 1, 2, 3$. The other cases are similarly proved. Differentiating $h_0(\xi', f_\lambda(\xi')) = 6$ three times, we have (15) and

$$\begin{aligned}\partial_{\xi_m}\partial_{\xi_k}\partial_{\xi_l} f_\lambda(\xi') = \frac{4\pi^2}{b_3}\Big(\Big(\delta_{kl}\delta_{lm}b_k - \frac{b_m b_l b_k}{b_3^2}\Big) \\ - \frac{a_3}{b_3^4}(\delta_{kl}a_k b_m b_3^2 + \delta_{lm}a_l b_k b_3^2 \\ + \delta_{mk}a_m b_l b_3^2 + 3b_m b_k b_l a_3)\Big)\end{aligned} \tag{25}$$

for $j, k, l = 1, 2$. Substituting this into $\xi_* = (1/4, 1/4, 1/4)$, we obtain

$$\partial^3_{\xi_j} f_\lambda(\xi'_*) = 0, \quad \partial^2_{\xi_1}\partial_{\xi_2} f_\lambda(\xi'_*) = \partial^2_{\xi_1}\partial_{\xi_2} f_\lambda(\xi'_*) = -4\pi^2.$$

for $j = 1, 2$. Taylor expanding f_λ, we obtain (23).

(ii) We recall B is the matrix defined in (22). We denote the eigenvalues of B at ξ' by $\lambda_+(\xi')$ and $\lambda_-(\xi')$. Now (10), (22), and Lemma 4.5 imply

$$\nabla_{\xi'} \det B(\xi'_*) \neq 0.$$

Thus,

$$(\nabla_{\xi'}\lambda_+)(\xi'_*)\lambda_-(\xi'_*) + (\nabla_{\xi'}\lambda_-)(\xi'_*)\lambda_+(\xi'_*) \neq 0.$$

Since $\lambda_+(\xi'_*)\lambda_-(\xi'_*) = 0$ and $(\lambda_+(\xi'_*), \lambda_-(\xi'_*)) \neq (0, 0)$, we have

$$\lambda_+(\xi'_*) = 0 \implies \nabla_{\xi'}\lambda_-(\xi'_*) \neq 0, \tag{26}$$

$$\lambda_-(\xi'_*) = 0 \implies \nabla_{\xi'}\lambda_+(\xi'_*) \neq 0. \tag{27}$$

Note that $\lambda_+(\xi'_*)$ and $\lambda_-(\xi'_*)$ are distinct by virtue of Proposition 4.4. Then [10, Theorem XII.4] implies that $\lambda_+(\xi')$ and $\lambda_-(\xi')$ are analytic near ξ'_* and the corresponding unit eigenvectors $u_+(\xi')$ and $u_-(\xi')$ can be chosen to be analytic near ξ'_*. Now our claim follows from Lemma 4.2, (26), and (27). (24) will be proved in Appendix B.

(iii) It suffices to prove $\alpha_{112} \neq 0$ and $\alpha_1 = \alpha_{111} = \alpha_{1111} = 0$, that is,

$$\partial^2_{\xi_1}\partial_{\xi_2} f_\lambda(\xi'_*) \neq 0, \quad \partial^2_{\xi_1} f_\lambda(\xi'_*) = \partial^3_{\xi_1} f_\lambda(\xi'_*) = \partial^4_{\xi_1} f_\lambda(\xi'_*) = 0.$$

The relations $\partial^2_{\xi_1}\partial_{\xi_2} f_\lambda(\xi'_*) \neq 0$ and $\partial^2_{\xi_1} f_\lambda(\xi'_*) = \partial^3_{\xi_1} f_\lambda(\xi'_*) = 0$ directly follow from (15) and (25). Differentiating $h_0(\xi', f_\lambda(\xi')) = \lambda$ four times in ξ_1-variable, we have

$$\begin{aligned}&-8\pi^3 a_1 + (\partial^4_{\xi_1} f_\lambda)b_3 + 8\pi(\partial^3_{\xi_1} f_\lambda)(\partial_{\xi_1} f_\lambda)a_3 + 6\pi(\partial^2_{\xi_1} f_\lambda)^2 a_3 \\ &-24\pi^2(\partial^2_{\xi_1} f_\lambda)(\partial_{\xi_1} f_\lambda)^2 b_3 - 12\pi^3(\partial_{\xi_1} f_\lambda)^4 a_3 = 0.\end{aligned}$$

Substituting

$$a_1(\xi_*) = a_3(\xi_*) = \partial_{\xi_1}^2 f_\lambda(\xi_*') = \partial_{\xi_1}^3 f_\lambda(\xi_*') = 0,$$

we obtain $\partial_{\xi_1}^4 f_\lambda(\xi_*') = 0$. □

5. Proof of Theorem 2.4

By permutating the coordinate, we may assume $\partial_{\xi_3} h_0(\xi) \neq 0$ on supp χ. We use the following representation:

$$\widehat{\chi d\sigma_{M_\lambda}}(\xi) = \int_{\mathbb{T}^3} \chi(\xi', f_\lambda(\xi')) e^{2\pi i (x_1\xi_1 + x_2\xi_2 + x_3 f_\lambda(\xi'))} \frac{d\xi'}{|\nabla h_0(\xi', f_\lambda(\xi'))|},$$

where $\xi' = (\xi_1, \xi_2)$ and we write $M_\lambda = \{(\xi', f_\lambda(\xi'))\}$ locally.

Theorem 2.4 (i) and (iv) directly follows from Proposition 4.4 (i), (ii) and the stationary phase theorem. See [12, Chapter VIII, §3, Theorem 1]. Theorem 2.4 (ii) follows from Proposition 3.2 (i) and Proposition 4.6 (i). Moreover, Proposition 3.2 (ii) and Proposition 4.6 (ii), (iii) imply Theorem 2.4 (ii) (if necessary, permuting the coordinate). We finish the proof.

Appendices

A. Equivalence of uniform resolvent estimates

Next elementary lemma follows from the Hölder inequality and the duality argument.

Lemma A.1. *Let $p \in [1, 2]$ and $r \in [2, \infty]$ satisfying*

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{r}.$$

Set $p' = p/(p-1)$. Then

$$\|A\|_{B(l^p(\mathbb{Z}^d), l^{p'}(\mathbb{Z}^d))} \leq C \tag{28}$$

is equivalent to

$$\|W_1 A W_2\|_{B(l^2(\mathbb{Z}^d))} \leq C \|W_1\|_{l^r(\mathbb{Z}^d)} \|W_2\|_{l^r(\mathbb{Z}^d)} \quad \text{for } W_1, W_2 \in l^r(\mathbb{Z}^d) \tag{29}$$

with a same constant $C > 0$.

Proof. Let $u \in l^2(\mathbb{Z}^d)$. The Hölder inequality with (28) implies

$$\begin{aligned}\|W_1 A W_2 u\|_{l^2(\mathbb{Z}^d)} &\leq \|W_1\|_{l^r(\mathbb{Z}^d)} \|A W_2 u\|_{l^{p'}(\mathbb{Z}^d)} \\ &\leq C\|W_1\|_{l^r(\mathbb{Z}^d)} \|W_2 u\|_{l^p(\mathbb{Z}^d)} \\ &\leq C\|W_1\|_{l^r(\mathbb{Z}^d)} \|W_2\|_{l^r(\mathbb{Z}^d)} \|u\|_{l^2(\mathbb{Z}^d)}.\end{aligned}$$

Thus we have (29). Conversely, assume (29) and fix $W_2 \in l^r(\mathbb{Z}^d)$. First, we prove $\|AW_2\|_{B(l^2(\mathbb{Z}^d),l^{p'}(\mathbb{Z}^d))} \leq C\|W_2\|_{l^r(\mathbb{Z}^d)}$. Let u, w be finitely supported functions. Then (28) implies

$$\begin{aligned}|(w, AW_2u)_{l^2(\mathbb{Z}^d)}| =& |(|w|^{\frac{p}{2}} \mathrm{sgn} w, |w|^{1-\frac{p}{2}} AW_2 u)_{l^2(\mathbb{Z}^d)}| \\ \leq& C\||w|^{\frac{p}{2}}\|_{l^2(\mathbb{Z}^d)} \||w|^{1-\frac{p}{2}}\|_{l^r(\mathbb{Z}^d)} \|W_2\|_{l^r(\mathbb{Z}^d)} \|u\|_{l^2(\mathbb{Z}^d)} \\ =& C\|W_2\|_{l^r(\mathbb{Z}^d)} \|w\|_{l^p(\mathbb{Z}^d)} \|u\|_{l^2(\mathbb{Z}^d)}.\end{aligned}$$

Thus we have $\|AW_2\|_{B(l^2(\mathbb{Z}^d),l^{p'}(\mathbb{Z}^d))} \leq C\|W_2\|_{l^r(\mathbb{Z}^d)}$. Similar argument also implies (28). □

B. Proof of (24)

In this appendix, we prove (24). We introduce the notation

$$a_j = a_j(\xi) = \cos 2\pi\xi_j, \quad b_j = b_j(\xi) = \sin 2\pi\xi_j, \quad c_j = c_j(\xi) = \tan 2\pi\xi_j = \frac{b_j}{a_j}$$

and recall $M_\lambda = h_0^{-1}(\{\lambda\})$ and $E_\lambda = 3 - \lambda/2$. We note

$$E_\lambda \in (-1, 1) \iff \lambda \in (4, 8).$$

For $\xi \in M_\lambda$ with $b_3(\xi) \neq 0$, we write

$$\xi = (\xi', f_\lambda(\xi)).$$

We recall

$$B(\xi') = \partial_{\xi'}^2 f_\lambda(\xi') = -2\pi \begin{pmatrix} \dfrac{a_1 b_3^2 + a_3 b_1^2}{b_3^3} & \dfrac{b_1 b_2 a_3}{b_3^3} \\ \dfrac{b_1 b_2 a_3}{b_3^3} & \dfrac{a_2 b_3^2 + a_3 b_2^2}{b_3^3} \end{pmatrix}$$

and denote the eigenvalues of $B(\xi')$ by $\lambda_+(\xi')$ and $\lambda_-(\xi')$ and the corresponding eigenvectors by $u_+(\xi')$ and $u_-(\xi')$.

In order to prove (24), by permuting the coordinate, it suffices to prove

$$\begin{aligned}&\lambda_+(\xi'_*) = u_+(\xi'_*)\cdot\partial_{\xi'}\lambda_+(\xi'_*) = 0\\ &\quad\Longrightarrow\ (a_1(\xi_*), a_2(\xi_*), a_3(\xi_*)) \in \{(0,0,E_\lambda), (0,E_\lambda,0), (E_\lambda,0,0)\}\end{aligned}$$

if we suppose $E_\lambda \in (-1,1)\setminus\{0\}$. We recall $\lambda_-(\xi'_*) \neq 0$ if $\lambda_+(\xi'_*) = 0$ by Proposition 4.4 and by the condition $E_\lambda \neq 0$. Since

$$\begin{aligned}&a_j = 0 \quad \text{for some } j = 1,2,3\\ &\quad\Longrightarrow\ (a_1,a_2,a_3) \in \{(0,0,E_\lambda), (0,E_\lambda,0), (E_\lambda,0,0)\},\end{aligned}$$

at $\xi \in K^{-1}(\{0\}) \cap M_\lambda$, we only need to prove

$$\begin{aligned}&a_j \neq 0 \quad \text{for all } j = 1,2,3 \quad \text{and} \quad \lambda_+(\xi_*) = 0\\ &\quad\Longrightarrow\ u_+(\xi'_*)\cdot\partial_{\xi'}\lambda_+(\xi'_*) \neq 0.\end{aligned} \tag{30}$$

First, we compute the null eigenvector of $B(\xi'_*)$.

Proposition B.1. *Suppose $E_\lambda \in (-1,1)\setminus\{0\}$. If*

$$\xi_* = (\xi'_*, f_\lambda(\xi'_*)) \in M_\lambda \cap K^{-1}(\{0\}) \setminus (\{a_1 = 0\} \cup \{a_2 = 0\} \cup \{b_3 = 0\})$$

satisfies $\lambda_+(\xi'_*) = 0$, then we have

$$u_+(\xi'_*) \parallel \begin{pmatrix} c_1(\xi_*)\\ c_2(\xi_*)\end{pmatrix}.$$

We will prove this proposition in the next subsection. It follows from this proposition that under the condition $\lambda_+(\xi'_*) = 0$, the equation

$$u_+(\xi'_*)\cdot\partial_{\xi'}\lambda_+(\xi'_*) = 0$$

is equivalent to

$$\begin{pmatrix} c_1(\xi_*)\\ c_2(\xi_*)\end{pmatrix}\cdot(\partial_\xi \det B)(\xi'_*) = 0$$

if $a_1(\xi_*) \neq 0$ and $a_2(\xi_*) \neq 0$ are satisfied. Since $\det B(\xi'_*) = 0$, this equation is also equivalent to

$$\begin{pmatrix} c_1(\xi_*)\\ c_2(\xi_*)\end{pmatrix}\cdot\left(\partial_\xi \det\left(\frac{-b_3^4}{2\pi}B\right)\right)(\xi'_*) = 0. \tag{31}$$

Lemma B.2. *Suppose $E_\lambda \in (-1,1)\setminus\{0\}$. For $\xi = (\xi', f_\lambda(\xi')) \in M_\lambda$, we have*

$$\partial_\xi \det(\frac{-b_3^4}{2\pi}B) = -2\pi\begin{pmatrix} b_1(a_3-a_1)(1-E_\lambda a_2) \\ b_2(a_3-a_2)(1-E_\lambda a_1)\end{pmatrix}.$$

Proof. We set

$$M = \frac{-b_3^4}{2\pi}B.$$

A direct calculation gives

$$M = a_1a_2b_3^2 + a_1a_3b_2^2 + a_2a_3b_1^2.$$

Since

$$\partial_{\xi_j} a_3 = -2\pi b_3(\partial_{\xi_j} f_\lambda) = 2\pi b_j$$

and

$$\partial_{\xi_j} b_3 = 2\pi a_3(\partial_{\xi_j} f_\lambda) = -2\pi b_j a_3/b_3$$

(recall (15)), we obtain

$$\begin{aligned}\partial_{\xi_1} M(\xi) =& 2\pi b_1(a_1-a_3)(1-a_2(a_1+a_2+a_3))\\ \partial_{\xi_2} M(\xi) =& 2\pi b_2(a_2-a_3)(1-a_1(a_1+a_2+a_3)).\end{aligned}$$

This completes the proof. $\square$

Now we compute the left hand side of (31). We assume $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Using the relations $a_jb_jc_j = b_j^2$, we have

$$\begin{aligned}&\frac{1}{2\pi}\begin{pmatrix}c_1\\c_2\end{pmatrix}\cdot\left(\partial_\xi \det\left(\frac{-b_3^4}{2\pi}B\right)\right)\\ &= (b_1c_1+b_2c_2)a_3 - b_1^2 - b_2^2\\ &\quad + E_\lambda(a_1a_2(b_1c_1+b_2c_2) - a_3(b_1c_1a_2 + a_1b_2c_2)).\end{aligned}$$

Since $\sum_{j=1}^3 b_jc_j = 0$ which is proved in Lemma B.3 below, we obtain

$$\begin{aligned}&\frac{1}{2\pi}\begin{pmatrix}c_1\\c_2\end{pmatrix}\cdot\left(\partial_\xi \det\left(\frac{-b_3^4}{2\pi}B\right)\right)\\ &= -a_3b_3c_3 - b_1^2 - b_2^2\\ &\quad - E_\lambda(a_1a_2b_3c_3 + a_3(b_1c_1a_2 + a_1b_2c_2))\\ &= -b_1^2 - b_2^2 - b_3^2 - E_\lambda\left(\frac{a_1a_2b_3^2}{a_3} + \frac{a_2a_3b_1^2}{a_1} + \frac{a_3a_1b_2^2}{a_2}\right)\\ &= -b_1^2 - b_2^2 - b_3^2 - E_\lambda\left(\frac{a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2}{a_1a_2a_3} - 3a_1a_2a_3\right).\end{aligned}$$

From the relations (see (16))

$$a_1 + a_2 + a_3 = E_\lambda, \quad a_1a_2 + a_2a_3 + a_3a_1 = E_\lambda a_1a_2a_3,$$

we obtain

$$\begin{aligned} a_1^2a_2^2 + a_2^2a_3^2 + a_3^2a_1^2 &= E_\lambda a_1a_2a_3(E_\lambda a_1a_2a_3 - 2),\\ b_1^2 + b_2^2 + b_3^2 &= 3 - E_\lambda^2 + 2E_\lambda a_1a_2a_3. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{1}{2\pi}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cdot \left(\partial_\xi \det\left(\frac{-b_3^4}{2\pi}B\right)\right)\\ &= E_\lambda^2 - 3 - 2E_\lambda a_1a_2a_3 - E_\lambda(E_\lambda(E_\lambda a_1a_2a_3 - 2) - 3a_1a_2a_3)\\ &= -a_1a_2a_3E_\lambda^3 + 3E_\lambda^2 + a_1a_2a_3E_\lambda - 3\\ &= (E_\lambda^2 - 1)(-a_1a_2a_3E_\lambda + 3) \neq 0 \end{aligned}$$

since $E_\lambda \in (-1, 1)$. This proves (30).

B.1. Proof of Proposition B.1. We need the following lemmas.

Lemma B.3. *Suppose $E_\lambda \in [-1, 1]$. Then, for*

$$\xi \in M_\lambda \setminus \{a_1 = 0\} \cup \{a_2 = 0\} \cup \{a_3 = 0\},$$

$K(\xi) = 0$ holds if and only if

$$b(\xi) = (b_1(\xi), b_2(\xi), b_3(\xi)) \perp c(\xi) = (c_1(\xi), c_2(\xi), c_3(\xi)),$$

where we recall that $K(\xi)$ is the Gaussian curvature at $\xi \in M_\lambda$ which is defined in Lemma 4.1.

Proof. By virtue of (14), $K(\xi) = 0$ if and only if

$$a_1a_2b_3^2 + a_2a_3b_1^2 + a_3a_1b_2^2 = 0 \quad \text{at } \xi.$$

Since $b_j^2 = a_jb_jc_j$, this equation is equivalent to

$$a_1a_2a_3(b_1c_1 + b_2c_2 + b_3c_3) = 0 \quad \text{at } \xi.$$

This is also equivalent to $b(\xi) \perp c(\xi)$ under the conditions $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. □

Lemma B.4. *Suppose* $E_\lambda \in (-1,1)$. *If* $\xi \in K^{-1}(\{0\}) \cap M_\lambda$, *then we have* $(b_1(\xi), b_2(\xi)) \neq (0,0)$. *In particular, we obtain* $(c_1(\xi), c_2(\xi)) \neq (0,0)$.

Proof. If $b_1(\xi) = b_2(\xi) = 0$, then we have $|a_1(\xi)| = |a_2(\xi)| = 1$. It follows that $a_1(\xi) = a_2(\xi) = \pm 1$ does not hold since these imply $a_3(\xi) = E_\lambda \mp 2 \notin [-1,1]$, which is a contradiction. Thus we have $(a_1(\xi), a_2(\xi)) = (\pm 1, \mp 1)$ and $a_3(\xi) = E_\lambda$ by (13). Using (16), we conclude $E_\lambda^2 = 1$, which is contradicts to $E_\lambda \in (-1,1)$. □

Lemma B.5. *Suppose* $E_\lambda \in (-1,1)$. *For* $\xi \in K^{-1}(\{0\}) \cap M_\lambda \backslash \{a_1 = 0\} \cup \{a_2 = 0\}$ *with* $b_3(\xi) \neq 0$, *we have*

$$B(\xi') \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \tag{32}$$

where we write $\xi = (\xi', f_\lambda(\xi')) \in M_\lambda$. *In particular, the vector* (c_1, c_2) *is the eigenvector of the matrix* B *with* 0*-eigenvalue at* ξ', *where we note that* $(c_1(\xi'), c_2(\xi')) \neq (0,0)$ *by virtue of Lemma* B.4 (*and by* $E_\lambda \neq 0$).

Proof. By virtue of (22), it suffices to prove

$$c_1(a_1 b_3^2 + a_3 b_1^2) + c_2 b_1 b_2 a_3 = 0, \quad c_1 b_1 b_2 a_3 + c_2(a_2 b_3^2 + a_3 b_2^2) = 0.$$

Using the relation $c_j a_j = b_j$ and Lemma B.3, we have

$$\begin{aligned} c_1(a_1 b_3^2 + a_3 b_1^2) + c_2 b_1 b_2 a_3 &= b_1(b_3^2 + (b_1 c_1 + b_2 c_2) a_3) \\ &= b_1(b_3^2 - a_3 b_3 c_3) = 0. \end{aligned}$$

The equation $c_1 b_1 b_2 a_3 + c_2(a_2 b_3^2 + a_3 b_2^2) = 0$ is similarly proved. □

Now Proposition B.1 immediately follows from Lemma B.5.

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