J. Spectr. Theory 11 (2021), 1935–1951
DOI 10.4171/JST/390


# A Faber–Krahn inequality for the Riesz potential operator for triangles and quadrilaterals

Rajesh Mahadevan and Franco Olivares-Contador

**Abstract.** We prove an analog of the Faber–Krahn inequality for the Riesz potential operator. The proof is based on Riesz's inequality under Steiner symmetrization and the continuity of the first eigenvalue of the Riesz potential operator with respect to the convergence, in the complementary Hausdorff distance, of a family of uniformly bounded non-empty convex open sets.

**Mathematics Subject Classification (2020).** Primary: 49Q20, 52B60; Secondary: 47A75, 47J10.

**Keywords.** Faber–Krahn inequality, polygonal domains, Steiner symmetrization, Riesz's inequality, Lieb's theorem.

## 1. Introduction and main result

The Faber–Krahn inequality states that among all open sets of a given volume in $\mathbb{R}^d$ the ball minimizes the first eigenvalue of the Dirichlet Laplacian. Rayleigh, in 1894, originally formulated this as an assertion (which cannot be regarded as more than a conjecture at that stage), only for 2 dimensions, which he rather colorfully phrased in physical terms as follows: "if the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle" (see [25, pp. 339–340]). The conjecture was proved independently by Faber [10] and Krahn [17, 18] in the 1920s.

Since then many minimization or maximization problems for the principal eigenvalue of standard elliptic operators with respect to the shape of the domain (with constraints on the volume, perimeter, etc.) have been studied. These results, usually, go under the name of Faber–Krahn inequalities (which we refer to as FKI for short) and we recall a few of these below. The discussion below is by no means exhaustive.

Pólya and Szegő [23], in 1951, conjectured that among all $N$-gons of fixed area, the regular $N$-gon of the same area minimizes the first eigenvalue of the Dirichlet Laplacian. Although known to be true in the class of triangles and in the class of quadrilaterals [23] (see also [12] for a modern exposition), this remains a challenging open problem in the class of polygons with $N$ sides, $N \geq 5$.

FKIs for certain other, local or non-local, elliptic operators are also available. In the local setting, an FKI for the $p$-Laplacian was first obtained in 1999 by Bhattacharya [3] and has been revisited in recent years in a few papers [21, 8]. On the other hand, in the non-local setting, an FKI for fractional Laplacian under Dirichlet boundary conditions has been studied in a paper in 2014 by Brasco, Lindgren, and Parini [5] in the class of bounded open sets and by Olivares-Contador [22], in 2017, in the class of domains which are triangles or quadrilaterals. Another class of non-local operators for which FKIs have been obtained concerns the Riesz potential operator which is an integral operator. Rozenblum, Ruzhansky, and Suragan [26] and Kal'menov and Suragan [15] obtain FKIs for the Riesz potential operator in the class of open bounded domains in Euclidean space whereas analogous results for other geometries have been obtained by Ruzhansky and Suragan [27]. An FKI for a certain other class of fractional elliptic operators has been obtained by Kassymov, Ruzhansky, and Torebev [16] while considering cylindrical domains with triangular or quadrilateral sections.

The Riesz potential operator is only the inverse of a fractional Laplacian operator with a certain non-local boundary condition (see Rozenblum, Ruzhansky, and Suragan [26]). Thus, the FKI for the fractional Laplacian under Dirichlet boundary condition does not imply the FKI for the Riesz potential operator or vice versa. In particular, the attainment of an FKI for the Riesz potential operator (in the class of all bounded domains) by Rozenblum et al. is a result which is independent of the FKI for a fractional Laplacian operator obtained earlier by Brasco et al. In a similar vein, in this paper we prove the FKI for the Riesz operator in the class of triangles or quadrilaterals; the corresponding results for the Dirichlet fractional Laplacian were established in [22] and are distinct from what we do here. Apart from showing the existence of a maximizer, we show also its uniqueness in the classes of triangles and quadrilaterals. We state below the main result of the paper after introducing the Riesz operator and the associated eigenvalue problem.

**Definition 1.1.** Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, $d > 1$. For $0 < \alpha < d$, the Riesz potential operator on $L^2(\Omega)$ is defined by

$$(I_\alpha u)(x) = \pi^{-\frac{d}{2}} 2^{-\alpha} \frac{\Gamma((d-\alpha)/2)}{\Gamma(\alpha/2)} \int_\Omega \frac{u(y)}{|x-y|^{d-\alpha}}\, dy \quad \text{a.e. } x \in \Omega. \tag{1.1}$$

It is known that the Riesz potential operator is a compact, self-adjoint, and non-negative operator (see [26, Section 2]) from which, by the spectral theory for compact self-adjoint operators, it has a discrete spectrum included in the positive real line and with 0 as its only point of accumulation. Moreover, the first (largest) eigenvalue of the Riesz operator admits the following characterization

$$\lambda_1(\Omega) = \max\left\{\int_\Omega\int_\Omega \frac{u(x)u(y)}{|x-y|^{d-\alpha}}\,dx\,dy : u \in L^2(\Omega), \|u\|_{L^2(\Omega)} = 1\right\}. \tag{1.2}$$

The maximizer for (1.2) exists and satisfies the following Euler–Lagrange equation

$$\int_\Omega\int_\Omega \frac{u(x)\phi(y)}{|x-y|^{d-\alpha}}\,dx\,dy = \lambda_1(\Omega)\int_\Omega u(x)\phi(x)\,dx \quad \text{for all } \phi \in L^2(\Omega). \tag{1.3}$$

Moreover, the first eigenvalue is simple and any corresponding eigenfunction is a continuous, non-vanishing function of constant sign on $\Omega$, being facts which sometimes go under the name of Jentzsch's theorem (see [27, Lemma 3.1], [13, p. 256], [14], or [29, Chapter 4, Section 18]). We now state our main theorem.

**Theorem 1.1.** *Let* $0 < \alpha < 2$. *The maximum of* $\lambda_1(\Omega)$ *among all triangles* (*open*) *of given area is obtained when* $\Omega$ *is an equilateral triangle and only when* $\Omega$ *is an equilateral triangle* (*up to a set of Lebesgue measure* 0). *Similarly, the maximum of* $\lambda_1(\Omega)$ *among all quadrilaterals* (*open*) *of given area is obtained when* $\Omega$ *is a square and only when* $\Omega$ *is a square* (*up to a set of Lebesgue measure* 0).

We also prove two secondary results which are used in the proof of the main result. The first of these is a discussion of the equality case in Riesz's inequality for Steiner symmetrization in any dimension (Proposition 2.3). Although, in the one-dimensional case, this was treated in [19] (see also Lieb and Loss [20, Chapter 3]), a discussion of the equality case, in higher dimensions, for the Steiner symmetrization is not easy to find. The equality case in Riesz's inequality for the Schwarz symmetrization is well studied and we refer the interested reader to [20, 6]. The second is the continuity of the first eigenvalue of the Riesz operator with respect to convergence in the Hausdorff complementary distance of a family of uniformly bounded convex open sets (Proposition 2.9).

The plan of the paper is as follows. The basic definitions, notions, and secondary results which will be used in this paper are given in Section 2. Section 3 contains the proof of the main theorem.

## 2. Tools

**2.1. Properties of $\lambda_1$.** We now list several properties of $\lambda_1$ which will be required in the proof of the Proposition 2.9.

**Proposition 2.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^d$.*

1. ***Translation invariance.*** $\lambda_1(\Omega) = \lambda_1(\Omega + x)$ *for all* $x \in \mathbb{R}^d$.
2. ***Invariance under orthonormal transformations.*** $\lambda_1(\Omega) = \lambda_1(T(\Omega))$ *for every orthonormal transformation* $T \in O(n)$.
3. ***Homothety law.*** $\lambda_1(k\Omega) = k^\alpha \lambda_1(\Omega)$ *for* $k > 0$.
4. ***Domain monotonicity*** *Given bounded domains $A$ and $B$ in $\mathbb{R}^d$, if $A \subset B$, then $\lambda_1(A) \leq \lambda_1(B)$.*

*Proof.* We only prove property (3). Properties (1), (2), and (4) can be proved analogously. For this, note that $u \mapsto w$, given by

$$w(x) := u\Big(\frac{x}{k}\Big),$$

is an isomorphism from $L^2(\Omega)$ to $L^2(k\Omega)$ and under this isomorphism, we have

$$\frac{\displaystyle\int_{k\Omega}\int_{k\Omega} \frac{w(x)w(y)}{|x-y|^{d-\alpha}}\,dx\,dy}{\displaystyle\int_{k\Omega} |w|^2(x)\,dx} = \frac{\displaystyle\int_{k\Omega}\int_{k\Omega} \frac{u\left(\frac{x}{k}\right)u\left(\frac{y}{k}\right)}{|x-y|^{d-\alpha}}\,dx\,dy}{\displaystyle\int_{k\Omega} \left|u\left(\frac{x}{k}\right)\right|^2\,dx}$$

$$= k^\alpha \frac{\displaystyle\int_{\Omega}\int_{\Omega} \frac{u(z)u(r)}{|z-r|^{d-\alpha}}\,dz\,dr}{\displaystyle\int_{\Omega} |u(z)|^2\,dz}.$$

Then, property (3) follows from the characterization (1.2) by taking the maximum with respect to $w$ in $L^2(k\Omega)$ which is equivalent, under the isomorphism, to taking the maximum with respect to $u \in L^2(\Omega)$ on the right-hand side. □

**2.2. Steiner symmetrizations of sets and functions.** In this subsection we recall the notion of Steiner symmetrization and some of its properties. The definitions given below follow Lieb and Loss [20] or Brascamp, Lieb, and Luttinger [4].

We also refer to the same texts and Gruber [11] for the main properties. For complementary information we refer the reader to [1, 7, 9, 12].

For any measurable subset $\Omega \subset \mathbb{R}$ with finite Lebesgue measure, we denote by $\Omega^*$ the open interval with centre at 0 having the same measure as $\Omega$. For any nonnegative Borel measurable function $f$ on $\mathbb{R}$ vanishing at infinity (in the sense that all its positive level sets have finite measure), we define the symmetric-decreasing rearrangement $f^*$ of $f$ as follows:

$$f^*(x) = \int_0^\infty \chi_{\{y\in\mathbb{R}: f(y)>t\}^*}(x)\, dt. \tag{2.1}$$

The function $f^*$ so defined is a Borel measurable function.

**Definition 2.1.** Let $f$ be a nonnegative, Borel measurable function on $\mathbb{R}^d$ which vanishes at infinity, and let $H$ be any hyperplane ($(d-1)$-dimensional plane) through the origin of $\mathbb{R}^d$. We set up an orthogonal coordinate system on $\mathbb{R}^d$, with basis vectors $\{e_1, e_2, \dots, e_d\}$, in such a way that if a generic point $x$ is represented as $(x', x_d) = (x_1, x_2, \dots, x_{d-1}, x_d)$, then $H$ is the plane $x_d = 0$.

A nonnegative, Borel measurable function $f^\star$ on $\mathbb{R}^d$ is called the *Steiner symmetrization* with respect to $H$ of $f$, if $f^\star(x_1, x_2, \dots, x_{d-1}, \cdot)$ is the symmetric-decreasing rearrangement of $f(x_1, x_2, \dots, x_{d-1}, \cdot)$ with respect to the $x_d$ variable, for each fixed $(x_1, \dots, x_{d-1})$.

It can be seen that this naturally leads to the following definition for the Steiner symmetrization of a bounded measurable set $\Omega$ with respect to the hyperplane $H$.

**Definition 2.2** (Steiner symmetrization of a set). For any bounded Borel measurable set $\Omega \subset \mathbb{R}^d$, the Steiner symmetrization of $\Omega$ with respect to $H$, to be denoted by $\Omega^\star$, is given by

$$\Omega^\star = \bigcup_{\substack{b\in H \\ \Omega\cap L_b \neq \emptyset}} \Big\{ b + t e_d : |t| \le \frac{1}{2}|\Omega \cap L_b| \Big\}, \tag{2.2}$$

where $|\Omega \cap L_b|$ is the one-dimensional Lebesgue measure of $\Omega \cap L_b$ with $L_b$ being the line with direction $e_d$ passing through the point $b$ for any $b \in H$. $\square$

We recall the following properties of Steiner symmetrization of functions.

**Proposition 2.2.** (1) *The definitions of the Steiner symmetrization of sets and functions are consistent, that is,*

$$\chi_{A^\star} = (\chi_A)^\star \quad \textit{and} \quad \{x: f(x) \ge t\}^\star = \{x: f^\star(x) \ge t\}.$$

*for all Borel measurable sets $A$ with finite Lebesgue measure and for all non-negative Borel measurable functions $f$ which vanish at infinity.*

(2) *Let $f$ be a nonnegative Borel measurable function with $f \in L^2(\mathbb{R}^d)$. Then,*

$$\|f\|_2 = \|f^\star\|_2.$$

*Proof.* For (1) see [1, 183–184]. The proof of (2) follows easily from (1) and the $d$-dimensional layer cake representation (see [20, Theorem 1.13]). □

**Proposition 2.3** (Riesz's inequality). *Let $f$, $g$, and $h$ be non-negative Borel measurable functions that vanish at infinity on $\mathbb{R}^d$, and let $f^\star, g^\star$, and $h^\star$ be their respective Steiner symmetrizations with respect to a given hyperplane $H$ taken, as above, as $\{x \in \mathbb{R}^d | x_d = 0\}$. Then, for*

$$I(f, g, h) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) g(x-y) h(y)\, dx\, dy,$$

*we have*

$$I(f, g, h) \le I(f^\star, g^\star, h^\star). \tag{2.3}$$

*Moreover, if $g$ is symmetric with respect to the hyperplane $H$ and strictly decreasing in the orthogonal direction (moving away) and both $f$ and $h$ are nontrivial, then there is equality in (2.3) if and only if there exists $w \in \mathbb{R}^d$ of the form $(0, 0, \ldots, 0, k)$ for some $k \in \mathbb{R}$, such that $f(x) = f^\star(x-w)$ and $h(x) = h^\star(x-w)$ for almost all $x \in \mathbb{R}^d$. The $k$ for which this holds is unique.*

We refer to Lemma 3.2 of [4] for a proof of inequality (2.3). We only treat the case of equality for which we need the following lemma.

**Lemma 2.4.** *Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set with $|A| > 0$. If $A = A + x$ for some $x \in \mathbb{R} \setminus \{0\}$, then $|A| = \infty$.*

*Proof.* Since $|A| > 0$, necessarily there exists an $n \in \mathbb{Z}$ for which $B := A \cap [n, n+1]$ has positive measure. Notice that $B + x \subseteq A + x = A$ and then, using induction, we also obtain $B + mx \subseteq A$ for every $m \in \mathbb{Z}$. Now, we assume, without loss of generality, that $x > 0$ and then choose $M \in \mathbb{N}$ such that $Mx > 1$. Then it follows that the intervals $[sMx + n, sMx + n + 1]$ are disjoint for distinct $s \in \mathbb{Z}$. Since, $B + sMx \subset [sMx + n, sMx + n + 1]$, we obtain that the sets $B + sMx$ are disjoint for distinct $s \in \mathbb{Z}$. Therefore, necessarily it follows that $|A| = \infty$, since $A$ contains infinitely many disjoint copies of $B$. □

*Proof of Proposition* 2.3. The equality case in the one-dimensional case is discussed in Lieb [19] (see also [20, Theorem 3.9, p. 93]).

The equality case in the $d$-dimensional case, for $d \geq 2$, is discussed below. The proof, unlike the proof of the equality case in Riesz's inequality under Schwarz symmetrization sketched in Theorem 3.9 of Lieb and Loss [20], does not require induction on the dimension. We exploit, directly, the one-dimensional result. Since we have chosen an orthogonal coordinate system wherein $H$ is the hyperplane $\{(x', 0) : x' = (x_1, x_2, \dots, x_{d-1}) \in \mathbb{R}^{d-1}\}$, the hypotheses on $g$ gives us $g(z', \cdot) = g^*(z', \cdot)$ and so the equality $I(f, g, h) = I(f^\star, g, h^\star)$ may be written as

$$\begin{aligned} &\int\limits_{\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}\times\mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d)\, dx_d\, dy_d\, dx'\, dy' \\ &\quad = \int\limits_{\mathbb{R}^{d-1}\times\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}\times\mathbb{R}} f^\star(x', x_d) g(x' - y', x_d - y_d) h^\star(y', y_d)\, dx_d\, dy_d\, dx'\, dy', \end{aligned} \tag{2.4}$$

with $(x', y') \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$.

For any fixed $(x', y') \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$, by definition $f^\star(x', \cdot)$ and $h^\star(y', \cdot)$ are the one-dimensional symmetric-decreasing rearrangements of $f(x', \cdot)$ and $h(y', \cdot)$ respectively. Now for any $x', y' \in \mathbb{R}^{d-1}$, Riesz's inequality applied to the functions $f(x', \cdot), g(x' - y', \cdot)$, and $h(y', \cdot)$ viewed as functions of the final variable gives us

$$\begin{aligned} &\int\limits_{\mathbb{R}\times\mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d)\, dx_d\, dy_d \\ &\quad \leq \int\limits_{\mathbb{R}\times\mathbb{R}} f^\star(x', x_d) g(x' - y', x_d - y_d) h^\star(y', y_d) dx_d dy_d. \end{aligned} \tag{2.5}$$

When we view the left-hand side and right-hand side terms in (2.5) as functions of $(x', y')$, this inequality is an inequality between functions on $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ which holds pointwise. Whereas, (2.4) tells us that their integrals on $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ are equal. From, this it can be deduced that there is in fact equality in the almost everywhere sense in (2.5), that is,

$$\begin{aligned} &\int\limits_{\mathbb{R}\times\mathbb{R}} f(x', x_d) g(x' - y', x_d - y_d) h(y', y_d)\, dx_d\, dy_d \\ &\quad = \int\limits_{\mathbb{R}\times\mathbb{R}} f^\star(x', x_d) g(x' - y', x_d - y_d) h^\star(y', y_d)\, dx_d\, dy_d \quad \text{a.e. } x', y'. \end{aligned} \tag{2.6}$$

Let $S$ be the set of the couples $(x', y')$ for which equality holds in (2.6), so that $(\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}) \setminus S$ has measure 0. Also, let

$$M = \{x' \in \mathbb{R}^{d-1}: f(x', \cdot) \text{ is not the zero function}\},$$
$$N = \{y' \in \mathbb{R}^{d-1}: h(y', \cdot) \text{ is not the zero function}\}.$$

Note that, by our hypothesis, both $M$ and $N$ are of positive measure. Also, for any $(x', y') \in M \times N$, by the definition of $M$ and $N$, both $f(x', \cdot)$ and $h(y', \cdot)$ are non-zero functions. It can be observed that for any $(x', y') \in S \bigcap (M \times N)$ we have equality for Riesz's inequality in one dimension. Therefore, the equality case of the one-dimensional result (as found in [19, Lemma 3]) now tells us that there is a shift, $k(x', y')$, such that

$$f(x', x_d) = f^{\star}(x', x_d - k(x', y')) \quad \text{a.e. } x_d, \tag{2.7}$$

and

$$h(y', y_d) = h^{\star}(y', y_d - k(x', y')) \quad \text{a.e. } y_d. \tag{2.8}$$

Moreover, this shift is unique since, otherwise, $f(x', \cdot)$ (and $h(y', \cdot)$) would be symmetric about two points which means that $f(x', \cdot)$ (similarly, $h(y', \cdot)$) is translation invariant and this is not possible because of the hypotheses on the level sets of $f$ (and $h$).

Now, $S \bigcap (M \times N)$ is of full measure in $(M \times N)$ and so, for almost all $y' \in N$ the section $(S \bigcap (M \times N))_{y'}$ which is the section of $S \bigcap (M \times N)$ at $y'$ is of full measure in $M$. Consider any $y'_0$ in $N$ for which $(S \bigcap (M \times N))_{y'_0}$ has the same measure as $M$. Now, for any $x', z' \in (S \bigcap (M \times N))_{y'_0}$, we observe that

$$\begin{aligned} f(x', x_d) &= f^{\star}(x', x_d - k_{x', y'_0}) \quad \text{a.e. } x_d, \\ h(y'_0, y_d) &= h^{\star}(y'_0, y_d - k_{x', y'_0}) \quad \text{a.e. } y_d, \\ f(z', x_d) &= f^{\star}(z', x_d - k_{z', y'_0}) \quad \text{a.e. } x_d, \\ h(y'_0, y_d) &= h^{\star}(y'_0, y_d - k_{z', y'_0}) \quad \text{a.e. } y_d, \end{aligned}$$

from which we obtain

$$h(y'_0, y_d) = h(y'_0, y_d + k_{z', y'_0} - k_{x', y'_0}) \quad \text{a.e. } y_d \tag{2.9}$$

Since $y'_0$ belongs to $N$ we see that $\{y_d : h(y'_0, y_d) \neq 0\}$ is of positive measure. Also, since $h(y'_0, \cdot)$ vanishes at infinity, it is possible to fix a $t$ with $0 < t < \infty$ such that the measure of $A = \{y_d : h(y'_0, y_d) > t\}$ is finite and positive. Also notice that, by (2.9), we have $A = A + k_{x', y'_0} - k_{z', y'_0}$. Therefore, using Lemma 2.4, we have $k_{z', y'_0} = k_{x', y'_0}$. Let us denote this common value by $k(y'_0)$. This implies that $f(x', \cdot)$ is symmetric-decreasing about $x_d = k(y'_0)$ independently of $x'$ in $(S \bigcap (M \times N))_{y'_0}$. That is, $f(x', \cdot)$ is symmetric-decreasing about $x_d = k(y'_0)$ for almost every $x'$ in $M$. Since, $f(x', \cdot)$ is independent of $y'_0$ we can also conclude that $k(y'_0)$ does not really depend on $y'_0$ since a non-zero function vanishing at infinity cannot be symmetric-decreasing with respect to two distinct points. Therefore, for almost every $x'$ in $M$, $f(x', \cdot)$ is symmetric-decreasing about $x_d = k$ where $k$ now is independent of $x', y'$.

Then, (2.8) allows us to conclude that, for almost all $w' \in M$, $h(w', \cdot)$ is also Steiner symmetric about $y_d = k$. □

**Remark 1.** We note that the hypotheses on $f$ and $h$ requiring their vanishing at infinity, apart from the hypothesis on $g$, are crucial for being able to say that the equality in Riesz's inequality implies that they are Steiner symmetric except for a common shift. These hypotheses exclude, effectively, the possibility that the shift depends on the section or that there are multiple possibilities for the shifts.

To end this subsection, we also recall the following properties of Steiner symmetrization of sets. To start with, we define a few basic concepts. A convex body is a compact convex set with non-empty interior. By $\mathcal{K}^d$ we denote the set of all convex bodies in $\mathbb{R}^d$. For a convex body $A$ in $\mathbb{R}^d$, the inradius $r(A)$ is the supremum of the radii of balls contained in $A$ and the circumradius $R(A)$ is the infimum of the radii of balls containing $A$.

**Proposition 2.5.** *Let $A, B \in \mathcal{K}^d$. Then,*

(1) $A^\star \subseteq B^\star$ *for* $A \subseteq B$;

(2) $r(A) \leq r(A^\star)$;

(3) $R(A^\star) \leq R(A)$;

(4) $V(A) = V(A^\star)$ *where $V(A)$ denotes the volume of $A$.*

*Proof.* See [11, Proposition 9.1, pp. 169–171]. □

### 2.3. The Minkowski addition and the Minkowski difference

**Definition 2.3.** The Minkowski addition of two sets $X, Y \subset \mathbb{R}^d$ is defined by

$$X \oplus Y := \bigcup_{y \in Y} (X + y). \tag{2.10}$$

The Minkowski difference of two sets $X, Y \subset \mathbb{R}^d$ is defined by

$$X \ominus Y := \bigcap_{y \in Y} (X - y). \tag{2.11}$$

If $Y = -Y$, then

$$X \ominus Y = \bigcap_{y \in Y} (X + y). \tag{2.12}$$

**Note.** If $K$ is a convex body the set $K \ominus B(0, \epsilon)$ is called the inner parallel body of $K$ at distance $\epsilon$ (see [28, p. 93 and p. 148]).

**Proposition 2.6.** *Let $X, Z \subset \mathbb{R}^d$ with $X \subset Z$ and let $\epsilon > 0$. Then*

$$X \ominus B(0, \epsilon) \subseteq Z \backslash ((Z \backslash X) \oplus B(0, \epsilon)). \tag{2.13}$$

*Proof.* From (2.11) it follows that $(X \ominus B(0, \epsilon))^{\mathrm{c}} = X^{\mathrm{c}} \oplus B(0, \epsilon)$ from which

$$Z \cap (X \ominus B(0, \epsilon))^{\mathrm{c}} = Z \cap (X^{\mathrm{c}} \oplus B(0, \epsilon)). \tag{2.14}$$

On the other hand, we have trivially

$$Z \cap ((Z \cap X^{\mathrm{c}}) \oplus B(0, \epsilon)) \subseteq Z \cap (X^{\mathrm{c}} \oplus B(0, \epsilon)).$$

So, after taking the complement with respect to $Z$ in the anterior and then using (2.14), we get

$$Z \backslash (Z \cap (X \ominus B(0, \epsilon))^{\mathrm{c}}) \subseteq Z \backslash (Z \cap ((Z \cap X^{\mathrm{c}}) \oplus B(0, \epsilon))).$$

From this, in view of the hypothesis that $X \subseteq Z$, we get

$$X \ominus B(0, \epsilon) \subseteq Z \backslash ((Z \cap X^{\mathrm{c}}) \oplus B(0, \epsilon)). \qquad \square$$

**Proposition 2.7.** *Let $X$ be an open convex set of $\mathbb{R}^d$. Then, the following holds:*

$$X \ominus B(0, \epsilon) = \bar{X} \ominus B(0, \epsilon).$$

*Proof.* On the one hand, it is clear that $X \ominus B(0, \epsilon) \subseteq \bar{X} \ominus B(0, \epsilon)$.

On the other hand, for any $x \in \overline{X} \ominus B(0, \epsilon)$, it follows from the definition (2.11) that $B(x, \epsilon) \subseteq \bar{X}$. Since for an open convex set it is true that $\dot{\overline{X}} = X$ (see [24, Theorem 2.28]) we get $B(x, \epsilon) \subseteq X$, and so, $x \in X \ominus B(0, \epsilon)$. This proves the inclusion which is less obvious. $\square$

The main ingredient in the proof of Proposition 2.9 is the following Lemma.

**Lemma 2.8.** *Let $K$ be a convex body in $\mathbb{R}^d$, with $B(0, r) \subset K \subset B(0, R)$ for some numbers $r > 0$ and $R > 0$. If $0 < \epsilon < \frac{r^2}{4R}$, then*

$$\left(1 - 4\frac{R\epsilon}{r^2}\right) K \subset K \ominus B(0, \epsilon) \subset K. \tag{2.15}$$

*Proof.* See [28, Lemma 2.3.6, p. 93]. $\square$

**2.4. Hausdorff distance.** In this subsection, we quickly recall the definition of the complementary Hausdorff distance for bounded open sets (see [12, 28] for more details) and prove a continuity property of the first eigenvalue of the Riesz operator with respect to the convergence in this metric. The continuity result is similar to that proved in [22] for the eigenvalues of fractional Laplacian with Dirichlet boundary conditions. We are able to remove some of the hypothesis used therein.

**Definition 2.4.** Let $K$ and $C$ be two non-empty compact sets in $\mathbb{R}^d$. Then their Hausdorff distance is defined as

$$d^H(K,C) = \inf\{\epsilon \geq 0\colon K \subseteq C \oplus B(0,\epsilon) \text{ and } C \subseteq K \oplus B(0,\epsilon)\}.$$

Let $O_1$ and $O_2$ be two open subsets of a compact set $B$. Then the so-called *complementary Hausdorff distance* is defined by

$$d_H(O_1,O_2) = d^H(B\backslash O_1, B\backslash O_2). \tag{2.16}$$

**Proposition 2.9.** *Let $B$ be a fixed compact set in $\mathbb{R}^d$ and $\Omega_n$ be a family of non-empty convex open subsets of $B$ which converges, for the complementary Hausdorff distance, to a non-empty convex open set $\Omega$. Then,*

$$\lambda_1(\Omega) = \lim_{n\to\infty} \lambda_1(\Omega_n).$$

*Proof.* Since $\lambda_1$ is invariant under translation, we can assume that $0 \in \Omega$. Since $\Omega$ is an open set, there is an open ball such that $B(0,r) \subseteq \Omega$. We assume, without loss of generality, that $B$ is the closure of the ball $B(0,R)$ for some $R$ large enough.

**Step 1.** Since, $d_H(\Omega_n,\Omega) \to 0$, by the definition of the complementary Hausdorff distance, for any $\epsilon > 0$ there exist $n_\epsilon$ such that

$$B\backslash\Omega \subset (B\backslash\Omega_n) \oplus B(0,\epsilon) \quad \text{for all } n \geq n_\epsilon \tag{2.17}$$

and

$$B\backslash\Omega_n \subset (B\backslash\Omega) \oplus B(0,\epsilon) \quad \text{for all } n \geq n_\epsilon. \tag{2.18}$$

Further, by taking the relative complement in (2.18) with respect to $B$ and thereafter applying Proposition 2.6 with the choices $X = \Omega$ and $Z = B$, we obtain

$$\Omega \ominus B(0,\epsilon) \subseteq \Omega_n \quad \text{for all } n \geq n_\epsilon. \tag{2.19}$$

Therefore, by Proposition 2.7, we also have

$$\bar{\Omega} \ominus B(0,\epsilon) \subseteq \Omega_n \quad \text{for all } n \geq n_\epsilon. \tag{2.20}$$

From the above, by choosing $0 < \epsilon < r$, we get

$$B(0, r-\epsilon) = B(0,r) \ominus B(0,\epsilon) \subseteq \Omega \ominus B(0,\epsilon) \subseteq \Omega_n \quad \text{for all } n \geq n_\epsilon.$$

So, if $0 < \epsilon < \frac{r}{2}$, then we shall also have

$$B(0, r/2) \subseteq \Omega_n \quad \text{for all } n \geq n_\epsilon. \tag{2.21}$$

**Step 2.** Let us now fix $0 < \epsilon < \frac{r^2}{16R}$. For this choice, we also have $0 < \epsilon < \frac{r^2}{4R}$ and so, applying Lemma 2.8 with the compact set $\overline{\Omega}$ in mind, we get

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega \subset \left(1 - 16\frac{R\epsilon}{r^2}\right)\overline{\Omega} \subset \overline{\Omega} \ominus B(0,\epsilon). \tag{2.22}$$

So, using (2.20), it follows that

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega \subset \Omega_n \quad \text{for all } n \geq n_\epsilon. \tag{2.23}$$

Then, by using the domain monotonicity and homothety properties or the Riesz eigenvalues given in Proposition 2.1, we can obtain the inequality

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)^\alpha \lambda_1(\Omega) \leq \lambda_1(\Omega_n).$$

After taking the limit inferior, as $n \to \infty$, we obtain

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)^\alpha \lambda_1(\Omega) \leq \varliminf_{n\to\infty} \lambda_1(\Omega_n). \tag{2.24}$$

If we now take the limit as $\epsilon \to 0$ in (2.24) we get

$$\lambda_1(\Omega) \leq \varliminf_{n\to\infty} \lambda_1(\Omega_n). \tag{2.25}$$

**Step 3.** Arguing similarly as in Step 1, but starting from (2.17), we can also obtain the inclusion

$$\Omega_n \ominus B(0,\epsilon) \subseteq \Omega \text{ for all } n \geq n_\epsilon.$$

In view of (2.21) and since we have chosen $0 < \epsilon < \frac{r^2}{16R}$, by applying Lemma 2.8 with the compact set $\overline{\Omega}_n$ in mind we obtain

$$\left(1 - 16\frac{R\epsilon}{r^2}\right)\Omega_n \subset \left(1 - 16\frac{R\epsilon}{r^2}\right)\overline{\Omega}_n \subset \overline{\Omega}_n \ominus B(0,\epsilon).$$

So, continuing similarly as in Step 2, we conclude that

$$\varlimsup_{n\to\infty} \lambda_1(\Omega_n) \leq \lambda_1(\Omega). \tag{2.26}$$

The desired result follows from (2.25) and (2.26). □

## 3. Proof of the main theorem

*Proof of Theorem* 1.1. Let $\triangle_1$ be an arbitrary open triangle of positive area $a$. We successively define the open triangles $\triangle_{n+1}$ by taking the Steiner symmetrization of $\triangle_n$ with respect to the perpendicular bisector of a side with respect to which there is no symmetry. Then by Proposition 2.5 (4), each of the triangles has area $a$ and also the triangles are uniformly bounded since, by part 3 of the same proposition, the circumradius decreases after successive Steiner symmetrizations. This can be used to show that the sequence $\triangle_n$ converges with respect to the complementary Hausdorff distance to an open equilateral triangle $\triangle$ (see [23, p. 158], [2, Theorem 20.7, pp. 153–154], or [12, Theorem 3.3.3]).

Let $f_n$ be an eigenfunction for $\lambda_1(\triangle_n)$, that is, a function for which the maximum is attained in (1.2) which we can take to be continuous, non-negative and having unit $L^2$ norm. By property 1 of Proposition 2.1, the eigenvalue $\lambda_1(\triangle_n)$ is invariant under translations, and so in this expression we may always assume that the coordinate system has its origin at the circumcenter of the triangle. Furthermore, for the ease of considering Steiner symmetrization of functions with respect to the chosen line of Steiner symmetrization of $\Delta_n$, by the rotation invariance of the eigenvalue, we may that this line is oriented along the $x_1$-axis. Let $f_n^\star$ be the corresponding Steiner symmetrization of $\tilde{f}_n$ (the extension of $f_n$ by zero outside $\triangle_n$) which is a Borel measurable function vanishing at infinity. We note that $f_n^\star$ has to be supported on the closure of $\triangle_{n+1}$. By property 2 in Proposition 2.2, we note that $f_n^\star$ also has norm 1 in the $L^2$ norm. Notice that the Riesz potential $|x|^{-(2-\alpha)}$, is Steiner symmetric with respect to the $x_1$ axis. Since $\alpha < 2$, it is also a strictly decreasing function away from the $x_1$-axis in the $x_2$-direction. Therefore, we can apply Proposition 2.3 to the function $\tilde{f}_n$ (taken twice) and with the function in the middle taken as the Riesz potential to obtain

$$\begin{aligned}\lambda_1(\triangle_n) &= \int\limits_{\triangle_n}\int\limits_{\triangle_n} \frac{f_n(x) f_n(y)}{|x-y|^{2-\alpha}}\,dx\,dy \le \int\limits_{\triangle_{n+1}}\int\limits_{\triangle_{n+1}} \frac{f_n^\star(x) f_n^\star(y)}{|x-y|^{2-\alpha}}\,dx\,dy \\ &\le \max_{w\in L^2(\Delta_{n+1})}\left\{\int\limits_{\triangle_{n+1}}\int\limits_{\triangle_{n+1}} \frac{w(x)w(y)}{|x-y|^{2-\alpha}}\,dx\,dy \colon \|w\|_{L^2(\Delta_{n+1})} = 1\right\} \\ &= \lambda_1(\triangle_{n+1})\end{aligned} \tag{3.1}$$

for each $n$. Therefore,

$$\lambda_1(\triangle_1) \le \lambda_1(\triangle_n) \quad \text{for all } n.$$

We then use the continuity property of the Riesz eigenvalue for the complementary Hausdorff convergence (Proposition 2.9) and get

$$\lambda_1(\triangle_1) \leq \lim_{n\to\infty} \lambda_1(\triangle_n) = \lambda_1(\triangle).$$

This shows that, among all triangles of given area $a > 0$, $\lambda_1$ attains its maximum value for an equilateral triangle.

The proof, in the case of quadrilaterals, that the maximum is attained for a square uses a similar argument as in the case of triangles. Indeed, starting from any quadrilateral (open), it will be enough to construct a sequence of Steiner symmetrizations of the quadrilateral which converge in the complementary Hausdorff distance to a square. To start with, apply Steiner symmetrization with respect to an axis which is perpendicular to a diagonal of the quadrilateral, for which the other two vertices aren't on the same side of this diagonal. The resulting object is a *convex* quadrilateral which is symmetric with respect to this axis. Next, we Steiner symmetrize with respect to a perpendicular axis and thereby get a rhombus. This is to be followed by a Steiner symmetrization with respect to an axis perpendicular to one of the sides to produce a rectangle. The rectangle is then Steiner symmetrized with respect to an axis perpendicular to a diagonal to get, again, a rhombus. By repeating the procedures for the rhombus and rectangle, we end up with an infinite sequence of rhombi and rectangles which converge, ultimately, in the complementary Hausdorff distance, to a square (refer to [23, pp. 158–159] or [2, Theorem 20.8, pp. 154–155]).

Now we address the proof of the uniqueness in the case of triangular domains. Suppose that $\Delta$ is an open triangle of given area for which the maximum is attained. If $\Delta$ is not already an equilateral triangle, then there is at least one axis $m$ (perpendicular to one of its sides) such that $\Delta$ is not Steiner symmetric with respect to $m$. Let $\Delta^\star$ be the Steiner symmetrization of $\Delta$ respect to $m$. Without loss of generality, $m$ passes through the origin. We take $f$ to be a continuous positive eigenfunction of norm 1 (in the $L^2$ norm) associated to $\lambda_1(\Delta)$. Let $f^\star$ be the Steiner symmetrization of $\tilde{f}$ (the extension of $f$ by zero outside $\Delta$) with respect to $m$. We note that $f^\star$ has to be supported on the closure of $\Delta^\star$. By property (2) of Proposition 2.2, we note that $f^\star$ also has norm 1 in the $L^2$ norm. Then, we apply Proposition 2.3 with $\tilde{f}$ and with the function in the middle taken as the Riesz potential on $\mathbb{R}^d$ to obtain

$$\lambda_1(\Delta^\star) \geq \int_{\Delta^\star}\int_{\Delta^\star} \frac{f^\star(x)f^\star(y)}{|x-y|^{2-\alpha}}\,dx\,dy \geq \int_{\Delta}\int_{\Delta} \frac{f(x)f(y)}{|x-y|^{2-\alpha}}\,dx\,dy = \lambda_1(\Delta). \quad (3.2)$$

Now, since we assumed that $\Delta$ maximizes $\lambda_1$ this leads to the observation that $\lambda_1(\Delta^\star) = \lambda_1(\Delta)$ and so, we get the equality case in Riesz's inequality. Then, by Proposition 2.3 it follows that $f$ is a translate of $f^\star$ up to a set of measure 0, that is, $f(\cdot)=f^*(\cdot - y)$ for some $y \in \mathbb{R}^d$ a.e. Furthermore, $f^\star$ is a maximizer for $\lambda_1(\Delta^\star)$ and so, by Jentzsch's theorem, is continuous and positive on $\Delta^\star$. Since, the functions $f$ and $f^\star$ are strictly positive on $\Delta$ and $\Delta^\star$ respectively, we therefore have

$$\Delta = \{x \in \Delta\colon f(x) > 0\} = \{x \in \mathbb{R}^d\colon f^\star(x-y) > 0\}$$

(up to a set of measure 0)

$$= \Delta^\star + y.$$

Thus, $\Delta$ is Steiner symmetric (up to a set of measure 0) with respect to $m$ contrary to our supposition. So, we conclude that the equilateral triangle is the only maximizer (up to a set of Lebesgue measure 0).

The uniqueness in the quadrilateral case is proved similarly. □

**Acknowledgments.** F. Olivares-Contador was supported by CONICYT through the grants CONICYT-PCHA/Doctorado Nacional/2020-21161103. He is grateful to Almut Burchard for useful discussions on the Riesz potential operator and fractional Laplacians. The authors are thankful to the referee for a careful reading of the submitted manuscript and several helpful comments. The authors would also like to thank Mark S. Ashbaugh for his great help in preparing this improved version whereby we could eliminate several inconsistencies.

## References

[1] A. Baernstein, II, *Symmetrization in analysis*. With D. Drasin and R. S. Laugesen. With a foreword by W. Hayman. New Mathematical Monographs, 36. Cambridge University Press, Cambridge, 2019. MR 3929712 Zbl 07035366

[2] R. V. Benson, *Euclidean geometry and convexity*. McGraw-Hill, New York etc., 1966. MR 0209949 Zbl 0187.44103

[3] T. Bhattacharya, A proof of the Faber–Krahn inequality for the first eigenvalue of the $p$-Laplacian. *Ann. Mat. Pura Appl.* (4) **177** (1999), 225–240. MR 1747632 Zbl 0966.35091

[4] H. J. Brascamp, E. H. Lieb, and J. M. Luttinger, A general rearrangement inequality for multiple integrals. *J. Functional Analysis* **17** (1974), 227–237. MR 0346109 Zbl 0286.26005

[5] L. Brasco, E. Lindgren, and E. Parini, The fractional Cheeger problem. *Interfaces Free Bound.* **16** (2014), no. 3, 419–458. MR 3264796 Zbl 1301.49115

[6] A. Burchard, Cases of equality in the Riesz rearrangement inequality. *Ann. of Math.* (2) **143** (1996), no. 3, 499–527. MR 1394967 Zbl 0876.26016

[7] A. Burchard, A short course on rearrangement inequalities. Preprint, 2009. https://www.math.toronto.edu/almut/rearrange.pdf

[8] A. M. H. Chorwadwala, R. Mahadevan, and F. Toledo, On the Faber–Krahn inequality for the Dirichlet $p$-Laplacian. *ESAIM Control Optim. Calc. Var.* **21** (2015), no. 1, 60–72. MR 3348415 Zbl 1319.35145

[9] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. MR 1158660 Zbl 0804.28001

[10] G. Faber, Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. *Münch. Ber.* **1923,** 169–172. JFM 49.0342.03

[11] P. Gruber, *Convex and discrete geometry.* Grundlehren der mathematischen Wissenschaften, 336. Springer, Berlin, 2007. MR 2335496 Zbl 1139.52001

[12] A. Henrot, *Extremum problems for eigenvalues of elliptic operators.* Frontiers in Mathematics. Birkhäuser, Basel, 2006. MR 2251558 Zbl 1109.35081

[13] H. Hochstadt, *Integral equations.* Pure and Applied Mathematics. John Wiley & Sons, New York etc., 1973. MR 0390680 Zbl 0259.45001

[14] K. Jörgens, *Linear integral operators.* Translated from the German by G. F. Roach. Surveys and Reference Works in Mathematics, 7. Pitman, Boston, MA, and London, 1982. MR 0647629 Zbl 0499.47029

[15] T. Sh. Kal'menov and D. Suragan, Inequalities for the eigenvalues of the Riesz potential. *Mat. Zametki* **102** (2017), no. 6, 844–850. In Russian. English translation, *Math. Notes* **102** (2017), no. 5–6, 770–775. MR 3733327 Zbl 1432.31004

[16] A. Kassymov, M. Ruzhansky, and B. T. Torebek, Rayleigh-Faber–Krahn, Lyapunov and Hartman–Wintner inequalities for fractional elliptic problems. Preprint, 2020. arXiv:2006.16672

[17] E. Krahn, Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.* **94** (1925), no. 1, 97–100. MR 1512244 JFM 51.0356.05

[18] E. Krahn, Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen. *Acta Univ. Dorpat* A **9** (1926), 1–44. JFM 52.0510.03

[19] E. H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Studies in Appl. Math.* **57** (1977), no. 2, 93–105. MR 0471785 Zbl 0369.35022

[20] E. H. Lieb and M. Loss, *Analysis.* Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, R.I., 2001. MR 1817225 Zbl 0966.26002

[21] I. Ly, The first eigenvalue for the $p$-Laplacian operator. *JIPAM. J. Inequal. Pure Appl. Math.* **6** (2005), no. 3, Article 91, 12 pp. MR 2164332 Zbl 1127.35019

[22] F. Olivares-Contador, The Faber–Krahn inequality for the first eigenvalue of the fractional Dirichlet $p$-Laplacian for triangles and quadrilaterals. *Pacific J. Math.* **288** (2017), no. 2, 425–434. MR 3646898 Zbl 1366.49059

[23] G. Pólya and G. Szegő, *Isoperimetric inequalities in mathematical physics.* Annals of Mathematics Studies, 27. Princeton University Press, Princeton, N.J., 1951. MR 0043486 Zbl 0044.38301

[24] W. Prenowitz and J. Jantosciak, *Join geometries.* A theory of convex sets and linear geometry. Undergraduate Texts in Mathematics. Springer, Berlin etc., 1979. MR 0528635 Zbl 0421.52001

[25] J. W. S. Rayleigh, *The theory of sound.* Vol. I. Second edition. Revised and enlarged. Macmillan, London, 1894. Reprint, Dover Publications, New York, 1945. MR 0016009 (reprint) Zbl 0061.45904 (reprint) JFM 25.1604.01

[26] G. Rozenblum, M. Ruzhansky, and D. Suragan, Isoperimetric inequalities for Schatten norms of Riesz potentials. *J. Funct. Anal.* **271** (2016), no. 1, 224–239. MR 3494249 Zbl 1381.35111

[27] M. Ruzhansky and D. Suragan, On first and second eigenvalues of Riesz transforms in spherical and hyperbolic geometries. *Bull. Math. Sci.* **6** (2016), no. 2, 325–334. MR 3510695 Zbl 1343.35181

[28] R. Schneider, *Convex bodies: the Brunn-Minkowski theory.* Second expanded edition. Encyclopedia of Mathematics and its Applications, 151. Cambridge University Press, Cambridge, 2014. MR 3155183 Zbl 1287.52001

[29] V. S. Vladimirov, *Equations of mathematical physics.* Translated from the Russian by A. Littlewood. Edited by A. Jeffrey. Pure and Applied Mathematics, 3. Marcel Dekker, New York, 1971. MR 0268497 Zbl 0207.09101

Received September 11, 2020

Rajesh Mahadevan, Departamento de Mathemática, FCFM, Universidad de Concepción, Avenida Esteban Iturra s/n, Barrio Universitario, Casilla 160 C, Concepción, Chile

e-mail: rmahadevan@udec.cl

Franco Olivares-Contador, Departamento de Matemática, FCFM, Universidad de Concepción, Avenida Esteban Iturra s/n, Barrio Universitario, Casilla 160 C, Concepción, Chile

e-mail: folivaresc@udec.cl