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Eigenfunctions growth of $\mathcal{R}$-limits on graphs

Siegfried Beckus and Latif Eliaz

Abstract. A characterization of the essential spectrum of Schrödinger operators on infinite graphs is derived involving the concept of $\mathcal{R}$-limits. This concept, which was introduced previously for operators on $\mathbb{N}$ and $\mathbb{Z}^d$ as "right-limits," captures the behaviour of the operator at infinity. For graphs with sub-exponential growth rate, we show that each point in $\sigma_{\text{ess}}(H)$ corresponds to a bounded generalized eigenfunction of a corresponding $\mathcal{R}$-limit of H. If, additionally, the graph is of uniform sub-exponential growth, also the converse inclusion holds.

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1. Introduction

This work deals with Schrödinger operators $H: \ell^2(G) \to \ell^2(G)$ of the form

$$(H\psi)(v) := \sum_{u \sim v} (\psi(u) - \psi(v)) + W(v)\psi(v) \tag{1.1}$$

where $W: V(G) \to \mathbb{R}$ is a bounded function (the potential) and G is an infinite, connected graph with a uniform bound on the vertex degree.

Weyl's theorem asserts that the essential spectrum of a linear bounded self-adjoint operator is invariant under compact perturbations. In light of this, one naturally expects that the essential spectrum only depends on the geometry at infinity of the underlying space. This relation has been exposed for Schrödinger operators on $\mathbb{N}$ or $\mathbb{Z}^n$. More precisely, the essential spectrum was characterized by the union of the sets $\sigma_\infty(H')$ where H' runs over all right-limits of H and $\sigma_\infty(H')$ denotes the set of bounded generalized eigenvalues, see [13, 45]. Here the term right-limit refers to the study of both the potential and the geometry at infinity. In particular, if G is the Cayley graph of $\mathbb{Z}$ with the usual generators, a right-limit is a strong limit point of a sequence of shifts going to infinity of the original operator.

This notion of right-limits was extended to $\mathbb{Z}^n$ in [27] and recently to general graphs with uniformly bounded vertex degree in [8]. Since the name right-limit is no longer appropriate, these operators are called $\mathcal{R}$-limits.

For infinite connected graphs with uniform bound on the vertex degree, the authors of [8] proved that the union over the spectra of all $\mathcal{R}$-limits of H is contained in the essential spectrum of H, where H is a bounded Jacobi operator. Moreover, they show that the converse inclusion holds on regular trees, and that on the contrary there exists an infinite, connected graph with uniform bound on the vertex degree such that equality does not hold. This is the starting point of the present work. Let $\mathcal{R}(H)$ be the set of all $\mathcal{R}$-limits of H. We show that

$$\sigma_{\text{ess}}(H) \subseteq \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H') \tag{1.2}$$

holds under additional assumptions on the growth rate of balls, see Theorem 2.2. In general this inclusion can be strict, see Proposition 2.3. However, if additionally the graph admits a uniform sub-exponential growth, then this inclusion is an equality and moreover,

$$\sigma_{\text{ess}}(H) = \bigcup_{H' \in \mathcal{R}(H)} \sigma(H') = \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H'). \tag{1.3}$$

see Theorem 2.4.

Such results go back to the concept of limit operators based on Favard [15], Muhamadiev [34, 35], Lange and Rabinovich [26], Rabinovich, Roch, and Silbermann [37, 38, 39], and Chandler-Wilde and Lindner [12, 13], see also [29, 30, 36, 46]. In recent developments [25] pushed these methods forward to metric measure spaces including graphs of property A [48]. Besides other things it is shown there that the essential spectrum coincides with the union of the spectra of so called limit operators. Another approach to tackle such questions comes from C^*-algebras [18, 19, 20, 21, 22, 31] which uses the concept of localization at infinity, which coincides with the concept of $\mathcal{R}$-limits. In this case, similar results for operators on locally compact, non-compact abelian groups are obtained. This was recently extended to groupoid C^*-algebra [2, 10, 11]. For a more comprehensive review and further references on the subject see [13, 27, 45]. Another recent work [23] develops a similar characterization for the essential spectrum of the Laplacian on *Klaus-sparse graphs*. Note that the class of *Klaus-sparse graphs* have a non-trivial intersection with the class of uniform sub-exponential growing graph, and neither of these classes contains the other. Furthermore, [1] takes advantage of the above mentioned result on trees from [8], to calculate the essential spectrum of Jacobi matrices on homogeneous trees, which are generated by an *Angelesco system*.

In order to prove our second main result (Theorem 2.4), the so-called Shnol-type theorem is used. Shnol [42] proved that if a generalized eigenfunction admits at most a polynomial growth rate then the corresponding energy is in the spectrum of the operator. This result was independently discovered by Simon [43]. Since then various remarkable generalizations to the Dirichlet form setting were proven, see e.g. [4, 3, 5] and references therein. In the literature also the converse question is addressed [6, 17, 28]. To be more precise, one seeks to find for μ-almost every element in the spectrum, a generalized eigenfunction that has at most sub-exponential growth, where μ is the spectral measure of the operator H. Such a converse theorem is used in the proof of Theorem 2.2.

1.1. Organisation. The main results of this work are presented in Section 2. In Section 3, two examples are provided where the essential spectrum can be computed with the help of the main results. After introducing the main concepts such as $\mathcal{R}$-limits, the proof of the main Theorem 2.2 as well as of Proposition 2.3 is provided in Section 4. Then the proof of Theorem 2.4, which states the Equality (1.3) for d-bounded graphs of uniform sub-exponential growth, is given in Section 5.

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2. Setting and main results

A *graph* G consists of a countable *vertex set* $V(G)$ and an *edge set* $E(G)$ where an edge is represented by a tuple of vertices. Throughout this work, we deal with undirected graphs and so the edge (u, v) is identified with the edge (v, u) for $u, v \in V(G)$. The tuple (u, u) for some $u \in V(G)$ is called a *loop*. We only consider graphs without loops. Two vertices u and v are called *adjacent* ($u \sim v$) if $(u, v) \in E(G)$. The *vertex degree* $\deg(v)$ of a vertex $v \in V(G)$ is defined by

$$\deg(v) := \sharp\{u \in V(G) : v \sim u\},$$

where $\sharp S$ denotes the cardinality of the set S. The tuple (G, v_0) is called *rooted d-bounded graph* if $v_0 \in V(G)$ is a fixed vertex and $\deg(v) \leq d$ for all $v \in V(G)$. A *path* between two vertices $u, v \in G$ is given by a chain of vertices $(u_1, \dots, u_n)$ satisfying $u_1 = u, u_n = v$ and $u_i \sim u_{i+1}$ for all $1 \leq i \leq n - 1$. A graph is called *connected* if there is a path between any two vertices $u, v \in V(G)$.

Let $\ell^2(G) := \ell^2(V(G))$ denote the Hilbert space of all square summable functions $\psi : V(G) \to \mathbb{C}$. Furthermore, $\ell^\infty(G) := \ell^\infty(V(G))$ is the Banach space of bounded functions $\psi : V(G) \to \mathbb{C}$ equipped with the uniform norm $\|\psi\|_\infty := \sup_{u \in V(G)} |\psi(u)|$.

Throughout this work, we study the self-adjoint, linear and bounded operators acting on the Hilbert space $\ell^2(G)$ of the form (1.1), where G is an infinite (i.e. $\sharp V(G) = \infty$), d-bounded and connected graph. Whenever W is chosen to be identically zero, the operator is denoted by Δ, which is called the *graph Laplacian*. Furthermore, $A = A_G$ denotes the *adjacency operator* on the graph G, which is a Schrödinger operator with $W(v) := \deg(v)$ for $v \in V(G)$. A triple (H, G, v_0) denotes a Schrödinger operator of the form (1.1) defined on the rooted graph (G, v_0).

Let H be a Schrödinger operator on a rooted graph (G, v_0). The *spectrum* of H is denoted by $\sigma(H)$. The *discrete spectrum* $\sigma_{\text{disc}}(H) \subseteq \sigma(H)$ is the set of isolated eigenvalues of finite multiplicity and the *essential spectrum* is

$$\sigma_{\text{ess}}(H) := \sigma(H) \setminus \sigma_{\text{disc}}(H).$$

Furthermore, a function $\psi : V(G) \to \mathbb{C}$ is called a *generalized eigenfunction of H corresponding to the eigenvalue λ* if $\psi \not\equiv 0$ and $H\psi(v) = \lambda\psi(v)$ for all $v \in V(G)$. With this at hand, $\sigma_\infty(H)$ denotes the set of all λ such that there exists a bounded generalized eigenfunction $\psi \in \ell^\infty(G)$ corresponding to the eigenvalue λ.

The *combinatorial graph distance* on G is defined by

$$\text{dist}(u, v) := \inf\{n \mid (v_0, v_1, \dots, v_n) \text{ is a path with } v_0 = u \text{ and } v_n = v\}.$$

For a rooted graph (G, v_0), the notation

$$|v| := \text{dist}(v, v_0)$$

is used for the distance of a vertex $v \in V(G)$ from the root v_0. Then $\mathcal{S}_r(v_0)$ denotes the *sphere* of radius $r \in \mathbb{N}$ about v_0 and $B_r(v_0)$ is the *ball* of radius $r \in \mathbb{N}$ about v_0, namely,

$$\mathcal{S}_r(v_0) := \{v \in G \mid \operatorname{dist}(v, v_0) = r\}, \quad B_r(v_0) := \{v \in G \mid \operatorname{dist}(v, v_0) \leq r\}.$$

Definition 2.1. A connected rooted graph (G, v_0) is of *sub-exponential growth rate* if for each $\gamma > 1$, there exists $C = C_{\gamma, v_0} > 0$ such that for every $r \in \mathbb{N}$,

$$\sharp B_r(v_0) < C\gamma^r.$$

Furthermore, a graph G is of *uniform sub-exponential growth rate* if the constant $C > 0$ can be chosen independently of the choice of the root. Specifically, for each $\gamma > 1$, there exists a constant $C = C_\gamma > 0$ such that

$$\sharp B_r(u) < C\gamma^r$$

holds for every $u \in G$ and $r \in \mathbb{N}$.

The concept of $\mathcal{R}$-limits of a Schrödinger operator H defined on a graph G was recently introduced in [8]. A precise mathematical definition is provided in Section 4.1.

Theorem 2.2. *Let (G, v_0) be an infinite and connected d-bounded graph of sub-exponential growth rate, and H be a Schrödinger operator on $\ell^2(G)$ of the form* (1.1). *Then*

$$\sigma_{\mathrm{ess}}(H) \subseteq \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H').$$

We point out that in the latter assertion it is not assumed that the graph is of uniform sub-exponential growth. The inclusion is preserved also for the adjacency operator $H := A_G$ on the d-regular tree $G := T_d$, although this graph has exponential growth rate. In this case, one can directly check that this inclusion is strict. Indeed, the only $\mathcal{R}$-limit is the same operator $H' = A_{T_d}$ on T_d. Then $\sigma_{\mathrm{ess}}(H) = \sigma(H) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$ and $[-d, d] \subset \sigma_\infty(H)$ holds (see e.g. [9, Theorem 1.1] and [16]). Thus, we derive

$$[-d, d] \setminus [-2\sqrt{d-1}, 2\sqrt{d-1}] \subset \sigma_\infty(H) \setminus \sigma_{\mathrm{ess}}(H)$$

where the set on the left hand side is non-empty. With this idea at hand, we also prove the following.

Proposition 2.3. *There exists an infinite and connected d-bounded graph G of sub-exponential growth rate so that the adjacency operator $H := A_G$ satisfies*

$$\Big(\bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H')\Big) \setminus \sigma_{\mathrm{ess}}(H) \neq \emptyset.$$

The proof of Proposition 2.3 is constructive and the corresponding graph is sub-exponentially growing but it does not admit uniform sub-exponential growth. If this is assumed then the following holds.

Theorem 2.4. *Let G be an infinite and connected, d-bounded graph of uniform sub-exponential growth rate and let H be a bounded Schrödinger operator on $\ell^2(G)$. Then,*

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{H' \in \mathcal{R}(H)} \sigma(H') = \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H').$$

As mentioned above, these equalities have been the subject of the previous works [8, 14], which are inspired by [27]. There the first equality is given on graphs of uniform polynomial growth rate [14, Theorem 2.1], and on regular trees [8, Theorem 4]. These results are complemented by [8, Theorem 3], by which the first equality is not satisfied in general. The proof of the latter statement includes an example for a graph on which the essential spectrum strictly includes the union over the spectra of the $\mathcal{R}$-limits. While there the growth rate of the graph is not sub-exponential, it can be adapted. Specifically, the construction is similar to the one in the proof of Proposition 2.3.

As remarked earlier, the equality

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{H' \in \mathcal{R}(H)} \sigma(H')$$

coincides with a recent result from [25]. The corresponding result there is obtained for metric spaces satisfying a certain set of assumptions. The assumption most relevant for us is known as Property A [48], and it is satisfied for graphs of uniform sub-exponential growth rate [47]. Note that the second equality

$$\sigma_{\mathrm{ess}}(H) = \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H')$$

in Theorem 2.4 is solely part of the current work in the context of graphs.

Following [14, Theorem 2.3], Property A is not satisfied for so called uniform graphs of exponential growth rate. While the terminology "property A" does not appear there, this property in fact follows, and is related to the earlier argument by [27] for proving Theorem 2.4 for $G = \mathbb{Z}^d$. Moreover, there are graphs of sub-exponential growth rate that do not satisfy property A. Such a graph can be constructed by adjusting the example in [8, Theorem 3].

3. Examples

The essential spectrum $\sigma_{\mathrm{ess}}(H)$ is computed here for some examples by using Theorem 2.4.

3.1. Variations of $\mathbb{Z}^n$.

The spectrum of an adjacency operator on a graph with bounded vertex degree by $2n$ admitting $\mathbb{Z}^n$ as an $\mathcal{R}$-limit is computed.

Proposition 3.1. *Let $1 < n \in \mathbb{N}$ and let G be a infinite graph of uniform sub-exponential growth such that $A_{\mathbb{Z}^n}$ appears as an $\mathcal{R}$-limit of the adjacency operator A_G. If the vertex degree of G is bounded by $2n$, then*

$$\sigma(A_G) = \sigma_{\mathrm{ess}}(A_G) = [-2n, 2n].$$

Proof. A short computation invoking the Cauchy–Schwarz inequality leads to

$$\|A_G f\|^2 = \sum_{v\in G} \Big| \sum_{u\sim v} f(u) \Big|^2 \le 2n \sum_{v\in G} \Big(\sum_{u\sim v} |f(u)|^2 \Big) \le (2n)^2 \sum_{u\in G} |f(u)|^2,$$

since in the last estimate each term is positive and appears at most $2n$ times in the total sum. Hence, the spectral radius $\rho(A_G)$ satisfies $\rho(A_G) \le 2n$ implying $\sigma(A_G) \subseteq [-2n, 2n]$.

For the converse inclusion, Theorem 2.4 together with [33, eq. (7.3)] assert $[-2n, 2n] = \sigma(A_{\mathbb{Z}^n}) \subseteq \sigma_{\mathrm{ess}}(A_G)$ since $\mathbb{Z}^n$ is an $\mathcal{R}$-limit of A_G and G is of uniform sub-exponential growth. □

For any $n \in \mathbb{N}$ we shall construct a graph which we denote by $Z_{n\times n}$ and is an example for a graph of this family of variations of $\mathbb{Z}_n$. The construction procedure is the following.

- Denote by B_L^n the subgraph of $\mathbb{Z}^n$ which is the restriction to the box of side length $2L+1$, centred at 0.
- For each point $x \in \mathbb{Z}^n$ we shall associate the graph $B_x^n \equiv B_{\|x\|_\infty}^n$.
- We connect each adjacent pair of boxes B_x^n and $B_{x+e_j}^n$ by a line. The connection is done between the center points of the corresponding boundary surfaces and includes a sequence of vertices and edges of length $\max(\|x\|_\infty, \|x+e_j\|_\infty)$.

For example (a portion of) the graph $Z_{2\times 2}$ is drawn in Figure 3.1. We conclude from the argument above that

$$\sigma(A_{Z_{n\times n}}) = \sigma_{\mathrm{ess}}(A_{Z_{n\times n}}) = [-2n, 2n].$$

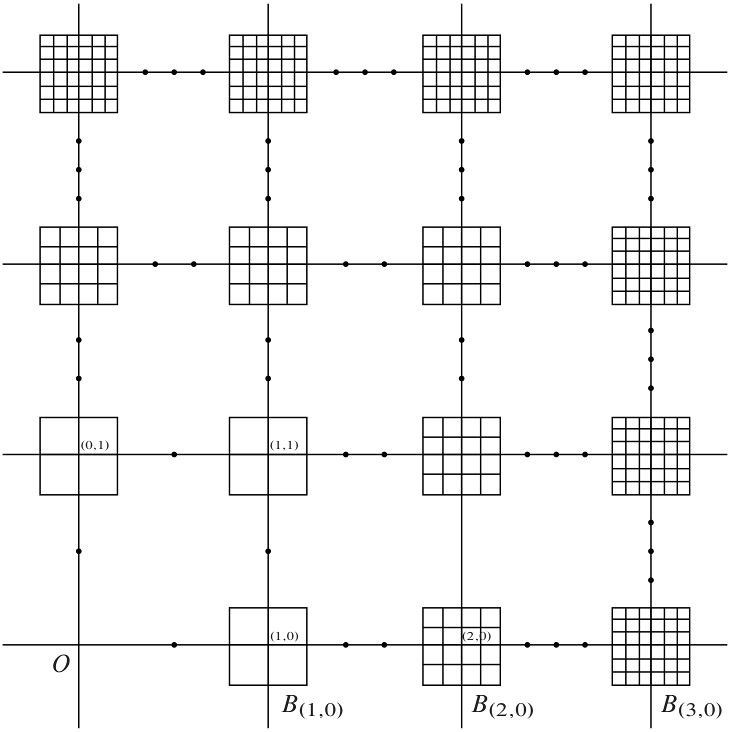

Figure 3.1. The graph $Z_{2\times 2}$.

3.2. Sparse trees with sparse cycles. In [7], so called sparse spherically homogeneous rooted trees were studied. It was shown there that under suitable assumptions the spectrum is purely singular continuous. These graphs are adjusted here by adding from time to time a circle in the graph while preserving the spherical symmetry. Invoking the current result, we compute the spectrum. We only provide a short discussion of these graphs and we refer the reader interested in more details to [14].

A rooted tree (T, v_0) is called *spherically homogeneous* if each vertex v is connected with $\kappa(|v| + 1)$ vertices with distance $|v| + 1$ from the root v_0. Let $\{L_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence with $L_n \in \mathbb{N}$ and $\{k_n\}_{n\in\mathbb{N}}$ be a bounded sequences with $k_n \in \mathbb{N}$ and $k_n > 1$. Following [7], a spherically homogeneous tree is called of type $\{L_n, k_n\}_{n\in\mathbb{N}}$ if $\kappa\colon \mathbb{N} \to \mathbb{N}$ is defined by

$$\kappa(j) := \begin{cases} k_j & \text{if } j \in \{L_n\}_{n\in\mathbb{N}}, \\ 1 & \text{otherwise.} \end{cases}$$

This graph is called *sparse* if $\lim_{n\to\infty}(L_{n+1} - L_n) = \infty$.

Suppose (T, v_0) is a sparse spherically homogeneous tree of type $\{L_n, k_n\}_{n\in\mathbb{N}}$. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of natural numbers satisfying

$$L_n \geq C_n \geq L_{n-1}, \quad \lim_{n\to\infty}(L_n - C_n) = \infty, \quad \text{and} \quad \lim_{n\to\infty}(C_n - L_{n-1}) = \infty.$$

With this at hand, the *sparse tree with sparse cycles* (G, v_0) of type $\{L_n, k_n, C_n\}_{n\in\mathbb{N}}$ is defined based on the spherically homogeneous tree (T, v_0) of type $\{L_n, k_n\}_{n\in\mathbb{N}}$ by adding edges for each $n \in \mathbb{N}$ between vertices in a sphere $\mathcal{S}_{C_n}(v_0)$ of (T, v_0) in the following way. Let $\mathcal{S}_{C_n}(v_0) = \{u_1, \ldots, u_m\}$ be some fixed ordering of the vertices in the sphere of radius C_n around v_0. Then, we add the edges (u_j, u_{j+1}) for $1 \leq j \leq m$ and the edge (u_m, u_1). Specifically, each vertex in the sphere of radius C_n in the graph (G, v_0) is adjacent to exactly two other vertices in the sphere and we create a circle, see e.g. a sketch of such a graph in Figure 3.2.

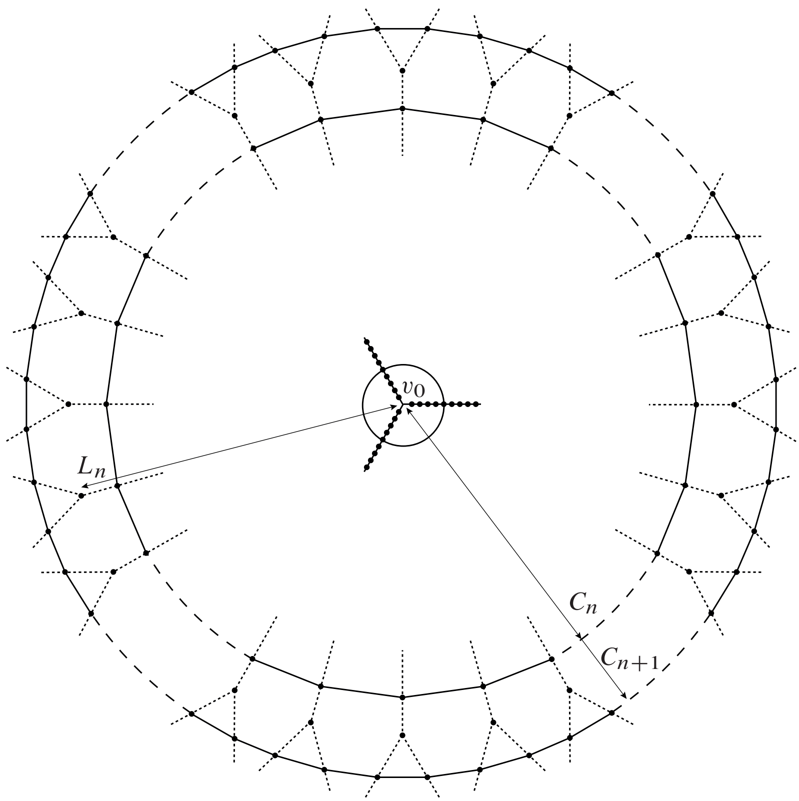

Figure 3.2. A sketch of a sparse tree with sparse cycles.

In order to apply Theorem 2.4, the sparse tree with sparse cycles need to be of uniform sub-exponential growth. If $\{L_n\}$ grows exponentially, then a direct computation shows that (G, v_0) is of uniform sub-exponential growth (for instance choose $L_n = 10^n$). Denote by $D \subseteq \mathbb{N}$ the set of all accumulation points of the sequence $\{k_n\}_{n\in\mathbb{N}}$, which by construction is finite. Then the possible $\mathcal{R}$-limits of A are the adjacency operators on the following graphs, see Figure 3.3:

- the line $\mathbb{Z}$;
- a two sided infinite comb graph, denoted by $\mathcal{C}$ and defined by

$$V(\mathcal{C}) = \{v = (k,\ell) \mid k,\ell \in \mathbb{Z}\},$$
$$E(\mathcal{C}) = \{((k,\ell),(k,\ell+1)) \mid k,\ell \in \mathbb{Z}\} \cup \{((k,0),(k+1,0)) \mid k \in \mathbb{Z}\}.$$

- the set of star graphs $\{S_{m+1}\}_{m\in D}$, where the star graph denoted by S_{m+1} is defined by $m+1$ copies of N glued together at 0.

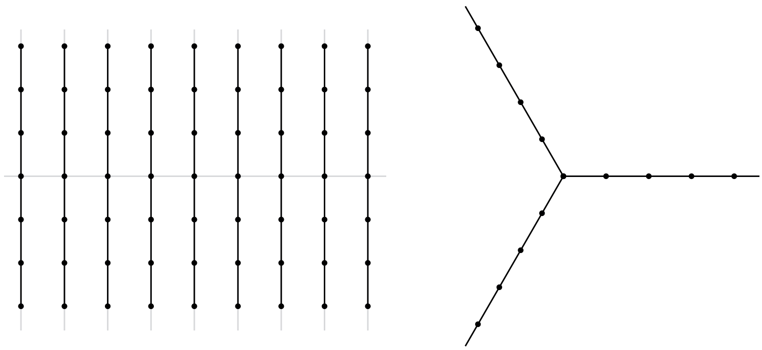

Figure 3.3. Some of the $\mathcal{R}$-limits of the sparse tree with sparse cycles (besides $\mathbb{Z}$): the infinite comb graph (left) and the star graph S_3 (right).

The spectrum of all of these graphs can be explicitly computed. The proofs are following standard ideas and can be found in [14].

Lemma 3.2. *The equality* $\sigma(A_{\mathcal{C}}) = [-2\sqrt{2}, 2\sqrt{2}]$ *holds.*

Proof. This follows by using the periodicity of the graph $A_{\mathcal{C}}$ (following e.g. [45, Chapter 5] and [40, Chapter XIII.16]) and the Aronszajn–Krein formula (see e.g. [44]). □

Lemma 3.3. *For* $m \in \mathbb{N}$ *with* $m > 1$ *we have*

$$\sigma(A_{S_m}) = [-2,2] \cup \left\{-\frac{m}{\sqrt{m-1}}, \frac{m}{\sqrt{m-1}}\right\}.$$

Proof. This is derived by standard computations invoking [7, Theorem 2.4] and coefficient stripping (see e.g. [45, Theorem 3.2.4]). □

Thus, Theorem 2.4 implies that the essential spectrum of the adjacency operator associated with the sparse tree with sparse cycles G of uniform sub-exponential growth is given by

$$\sigma_{\mathrm{ess}}(A_G) = [-2\sqrt{2}, 2\sqrt{2}] \cup \bigcup_{m \in D} \Big\{ -\frac{m+1}{\sqrt{m}}, \frac{m+1}{\sqrt{m}} \Big\}.$$

4. Characterizing $\sigma_{\mathrm{ess}}(H)$ using generalized eigenfunctions

The section is devoted to the proof of Theorem 2.2. The first part includes a more detailed review on $\mathcal{R}$-limits, and several related properties which we develop and use in this paper. The next three parts include the main tools which we use to show the existence of generalized eigenfunctions. The last part of this section is the actual proof of the theorem.

4.1. $\mathcal{R}$-limits. If $\eta\colon A \to B$ is a bijective map on two sets A and B, denote by $\mathcal{I}_\eta\colon \ell^2(A) \to \ell^2(B)$ the isomorphism defined via $\mathcal{I}_\eta(\delta_a) := \delta_{\eta(a)}$. Such maps will be mainly used for sets A and B that are balls in different graphs. Specifically, let (G, v_0) be an infinite, connected, rooted d-bounded graph. Since we assume that the graph G admits a uniform bound on the vertex degree, every ball $B_r(v)$ of radius $r \in \mathbb{N}$ about $v \in V(G)$ is finite. Throughout this work, $B_r(v)$ defines a subgraph of G by restricting the edge set only to those that connect to vertices in $B_r(v)$. For the sake of simplifying the notation, this induced subgraph is also denoted by $B_r(v)$.

Recall that a bijective map $\phi\colon V(G) \to V(G')$ between two graphs G and G' (finite or infinite) is called a *graph isomorphism* if the induced map

$$\phi_E\colon E(G) \to E(G'), \quad (u, v) \longmapsto (\phi(u), \phi(v))$$

is also bijective. Then two graphs G and G' are *isomorphic* ($G \sim G'$) if there exists a graph isomorphism between them. Clearly, $\deg(v) = \deg(\phi(v))$ holds for all $v \in V(G)$ where ϕ denotes the graph isomorphism. Let G, G' be two graphs. If there is an isomorphism between them, then G is a connected d-bounded graph if and only if G' is a connected d-bounded graph. We say that two balls $B_r(u)$ and $B_r(u')$ for $u \in V(G)$ and $u' \in V(G')$ are *isomorphic* ($B_r(u) \sim B_r(u')$) if the corresponding subgraph $B_r(u)$ is isomorphic to the subgraph $B_r(u')$.

Define the projection

$$P_{v_0,r}\colon \ell^2(G) \longrightarrow \ell^2(B_r(v_0)), \quad (P_{v_0,r}\psi)(v) := \chi_{B_r(v_0)}(v)\psi(v),$$

where $\chi_{B_r(v_0)}$ is the characteristic function of the ball $B_r(v_0)$. Note furthermore that $\ell^2(B_r(v_0))$ is naturally embedded into $\ell^2(G)$ by extending a function with zeros. In the following, we will not distinguish between the finite dimensional space $\ell^2(B_r(v_0))$ and its embedding in $\ell^2(G)$. For a linear bounded operator $H\colon \ell^2(G) \to \ell^2(G)$ on the rooted graph (G, v_0), define the operator

$$H_{v_0,r}\colon \ell^2(B_r(v_0)) \longrightarrow \ell^2(B_r(v_0)), \quad H_{v_0,r} := P_{v_0,r} H P_{v_0,r}.$$

It is worth pointing out that $H_{v_0,r}$ can be represented as a matrix acting on $\mathbb{C}^{\sharp B_r(v_0)}$. With this notion at hand, $\psi\colon V(G) \to \mathbb{C}$ is a generalized eigenfunction of H corresponding to the eigenvalue λ if and only if $P_{v_0,r} H\psi = \lambda P_{v_0,r}\psi$ for all $r \in \mathbb{N}$.

Recall that a sequence of vertices $\{v_n\}_{n\in\mathbb{N}}$ *goes to infinity* (or *converges to infinity*) if it leaves any finite subset of $V(G)$, or equivalently, if

$$\lim_{n\to\infty} \operatorname{dist}(v_0, v_n) = \infty.$$

Throughout this paper, we will usually assume, without loss of generality, that the convergence is monotone. Let (G, v_0) and (G', v_0') be two rooted graphs. A sequence of maps $f_r\colon B_r(v_0) \to B_r(v')$ for $r \in \mathbb{N}$ is called *coherent* (*for* v_0) if for $s > r$, the restriction of the map f_s to $B_r(v_0)$ equals f_r, namely

$$f_s(u) = f_r(u), \quad u \in B_r(v_0).$$

Definition 4.1. For $n \in \mathbb{N}$, let H_n be a Schrödinger operators on the connected rooted d-bounded graphs (G_n, v_n) and let $H'\colon \ell^2(G') \to \ell^2(G')$ be a linear bounded operator on the connected rooted graph (G', v_0'). Then the sequence $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ is called *convergent to* (H', G', v_0') if the following holds:

(C1) there are coherent maps $\{f_{n,r}\colon B_r(v_n) \to B_r(v_0')\}_{r\in\mathbb{N}}$ for each $n \in \mathbb{N}$, such that for every $r \in \mathbb{N}$, there exists an $N_r \in \mathbb{N}$ satisfying that $f_{n,r}\colon B_r(v_n) \to B_r(v_0')$ is a graph isomorphism for all $n \geq N_r$;

(C2) for each $r \in \mathbb{N}$,

$$\lim_{n\to\infty} \|\mathcal{I}_{f_{n,r}} P_{v_n,r} H_n P_{v_n,r} \mathcal{I}_{f_{n,r}}^{-1} - H'_{v_0',r}\| = 0.$$

Note that we require in the latter definition that the maps $f_{n,r}$ (in $n \in \mathbb{N}$) are eventually graph isomorphism between the balls $B_r(v_n)$ and $B_r(v_0')$. However, if $n < N_r$ these maps are not necessarily graph isomorphisms. Thus, for $n \geq N_r$, the map $\mathcal{I}_{f_{n,r}}: \ell^2(B_r(v_n)) \to \ell^2(B_r(v_0'))$ is an isomorphism and so $\mathcal{I}_{f_{n,r}}^{-1}$ is well-defined. If only (C1) holds, we call the sequence of connected, rooted d-bounded graphs $\{(G_n, v_n)\}_{n\in\mathbb{N}}$ *convergent* to the connected, rooted d-bounded graph (G', v_0').

Note, furthermore, that if $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ converges to (H', G', v_0'), then the limit point (H', G', v_0') is unique up to a graph isomorphisms. Specifically, if $(\widetilde{H}, \widetilde{G}, \tilde{v}_0)$ is another limit point of $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$, then there is a graph isomorphism $f: G' \to \widetilde{G}$ such that $f(v_0') = \tilde{v}_0$ and $\mathcal{I}_f H' \mathcal{I}_f^{-1} = \widetilde{H}$. This graph isomorphism f is defined by using the coherent maps, see (C1).

Additionally, note that (C1) in Definition 4.1 can be replaced with the requirement that there exist maps $f_{n,r}: B_r(v_n) \to B_r(v_0')$ that are eventually bijective. In this case, the maps $f_{n,r}$ are eventually graph isomorphism by (C2). It is worth pointing out that (C1) is only a combinatorial condition that guarantees that locally the graphs are eventually isomorphic. On the other hand, (C2) encodes further properties of the potential that in general cannot be kept by (C1). However, whenever the potential satisfies $W_n \equiv c_n$ for all $n \in \mathbb{N}$ with $c_n \in \mathbb{C}$ and $c_n \to c \in \mathbb{C}$, then (C1) implies (C2).

Definition 4.2. Let H be a Schrödinger operator on a graph G. Then a Schrödinger operator H' on a graph G' is called an *$\mathcal{R}$-limit of H* if there exist a vertex $v_0' \in V(G')$ and a sequence of vertices $v_n \in V(G)$ that monotonically converges to infinity such that $\{(H, G, v_n)\}_{n\in\mathbb{N}}$ converges to (H', G', v_0'). As mentioned before, the set of all $\mathcal{R}$-limits associated with H is denoted by $\mathcal{R}(H)$.

Whenever it is necessary to specify the coherent maps, we say that (H', G', v_0') is an $\mathcal{R}$-limit with respect to the coherent maps $\{f_{n,r}: B_r(v_n) \to B_r(v_0')\}_{r,n\in\mathbb{N}}$.

Notice that the above definition for $\mathcal{R}$-limits is equivalent to the definition given in [8, 14]. Nevertheless, the definition is presented here slightly different in order to relate it to the more general notion of convergence of a sequence of Schrödinger operators, introduced in Definition 4.1, which will be useful for us in this paper.

We start with some observation that will be helpful. They are inspired by previous considerations on $\mathbb{N}$, see [45].

Let G be a graph and $\psi: V(G) \to \mathbb{C}$ be a map. Then the *support of ψ* is defined by $\operatorname{supp}(\psi) := \{u \in V(G): \psi(u) \neq 0\}$. Denote by $\mathcal{C}_c(G)$ the set of all $\psi: V(G) \to \mathbb{C}$ such that $\operatorname{supp}(\psi)$ is finite. Clearly, $\mathcal{C}_c(G) \subseteq \ell^2(G)$ holds. Recall from equation (1.1) that a Schrödinger operator $H\colon \ell^2(G) \to \ell^2(G)$ on the graph G is given by

$$(H\psi)(v) := \sum_{u\sim v}(\psi(u) - \psi(v)) + W(v)\psi(v), \quad \psi \in \ell^2(G), v \in V(G),$$

where $W\colon V(G) \to \mathbb{R}$ satisfies $\|W\|_\infty := \sup_{v\in V(G)} |W(v)| < \infty$. The map W is called the potential. Although we mainly treat the case where H is self-adjoint (i.e. W is real-valued), the following statements of Lemma 4.3, Lemma 4.4, and Lemma 4.5 also hold for complex-valued potentials W.

Lemma 4.3. *For $n \in \mathbb{N}$, let H_n be a Schrödinger operators on a connected rooted d-bounded graph (G_n, v_n) with potential $W_n\colon V(G_n) \to \mathbb{C}$. Consider a linear bounded operator $H'\colon \ell^2(G') \to \ell^2(G')$ on a connected rooted graph (G', v_0'). If $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ converges to (H', G', v_0') and $\sup_{n\in\mathbb{N}} \|W_n\|_\infty \le C$ for some $C > 0$, then*

(a) *(G', v_0') is an infinite, connected rooted d-bounded graph,*

(b) *H' is a Schrödinger operator on (G', v_0') of the form (1.1) with potential W' and*

(c) *$\|H'\| \le 2(d + C)$ holds.*

In particular, every $\mathcal{R}$-limit H' of a Schrödinger operator H with potential W is a Schrödinger operator of the form (1.1) with bounded potential and $\|H'\| \le 2(d + C_H)$ for $C_H := \|W\|_\infty$.

Proof. Let $\{f_{n,r}\colon B_r(v_n) \to B_r(v_0')\}_{r\in\mathbb{N}}$ be the coherent maps (that are eventually isomorphic) such that $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ converges with respect to these maps to (H', G', v_0'). Claim (a) follows immediately from the definition (using the coherent maps $f_{n,r}$) since these properties are preserved under isomorphism and the fact that (G_n, v_n) is an infinite, connected, rooted d-bounded graph for each $n \in \mathbb{N}$.

Let $r \in \mathbb{N}$. By definition, there exists an $N_r \in \mathbb{N}$ such that $f_{n,r}\colon B_r(v_n) \to B_r(v_0')$ is an isomorphism for all $n \ge N_r$ and

$$\lim_{n\to\infty, n\ge N_r} \|\mathcal{I}_{f_{n,r}} P_{v_n,r} H_n P_{v_n,r} \mathcal{I}_{f_{n,r}}^{-1} - H'_{v_0',r}\| = 0.$$

Furthermore, $B_r(v_0')$ and $B_r(v_n)$ are finite sets of the same cardinality and so $\mathcal{I}_{f_{n,r}} P_{v_n,r} H_n P_{v_n,r} \mathcal{I}_{f_{n,r}}^{-1}$ and $H_{v_0',r}$ can be represented as matrices that converge in the matrix norm to each other. This is equivalent to the convergence of the coefficients. Since by (1.1), $(H_n\psi)(v)$ depends only on the values of ψ on the neighbours of v and v itself, H' is a Schrödinger operator of the form (1.1) proving (b).

In order to prove (c), let us first note that the operator norm estimate $\|H_n\| \leq 2d + C$. Let $\psi \in \mathcal{C}_c(G')$ be such that $\|\psi\| \leq 1$. Then there is an $r \in \mathbb{N}$ such that $\operatorname{supp}(\psi) \subseteq B_{r-1}(v_0')$. According to Definition 4.1, there exists an $n_0 \in \mathbb{N}$ such that

$$\|\mathcal{I}_{f_{n_0,r}} P_{v_{n_0},r} H_{n_0} P_{v_{n_0},r} \mathcal{I}^{-1}_{f_{n_0,r}} - H_{v_0',r}\| < C.$$

Since $\operatorname{supp}(\psi) \subseteq B_{r-1}(v_0')$ ($r-1$ is important here) and H' is a Schrödinger operator of the form (1.1), we have

$$H'\psi = H'_{v_0,r}\psi.$$

Thus, the previous considerations lead to

$$\begin{aligned}\|H'\psi\| &= \|H'_{v_0',r}\psi\| \\ &\leq \|H'_{v_0',r}\psi - \mathcal{I}_{f_{n_0,r}} P_{v_{n_0},r} H_{n_0} P_{v_{n_0},r} \mathcal{I}^{-1}_{f_{n_0,r}}\psi\| \\ &\quad + \|\mathcal{I}_{f_{n_0,r}} P_{v_{n_0},r} H_{n_0} P_{v_{n_0},r} \mathcal{I}^{-1}_{f_{n_0,r}}\psi\| \\ &< C + 2d + C\end{aligned}$$

as $\|\psi\| \leq 1$, $\mathcal{I}_{f_{n_0,r}}$ is an isomorphism and $\|P_{v_{n_0},r}\| \leq 1$.

The last statement in the lemma follows as every $\mathcal{R}$-limit (H', G', v_0') of H is a limit of a sequence $\{(H, G, v_n)\}_{n\in\mathbb{N}}$. Thus, $W_n := W$ holds implying $\sup_{n\in\mathbb{N}} \|W_n\|_\infty = \|W\|_\infty < \infty$. □

Next, we show that the operation of considering the $\mathcal{R}$-limits of H is a contraction in the sense that $\mathcal{R}(\mathcal{R}(H)) \subseteq \mathcal{R}(H)$.

Lemma 4.4. *Let (G, v_0) be a rooted d-bounded graph and H be a Schrödinger operator on $\ell^2(G)$ as defined in (1.1). If $\{(H'_m, G'_m, u'_m)\}_m$ is a sequence of $\mathcal{R}$-limits of H that converges in the sense of Definition 4.1 to $(\widetilde{H}, \widetilde{G}, \tilde{v}_0)$ then $(\widetilde{H}, \widetilde{G}, \tilde{v}_0)$ is an $\mathcal{R}$-limit of H. In particular, $\mathcal{R}(H') \subseteq \mathcal{R}(H)$ holds for every $\mathcal{R}$-limit H' of H, and thus also $\mathcal{R}(\mathcal{R}(H)) \subseteq \mathcal{R}(H)$.*

Proof. By assumption, we have the following:

(1) the sequence (H'_m, G'_m, u'_m) converges to $(\widetilde{H}, \widetilde{G}, \tilde{v}_0)$ along the sequence of coherent maps $\{f'_{m,r} \colon B_r(u'_m) \to B_r(\tilde{v}_0)\}_{m,r\in\mathbb{N}}$ that are eventually graph isomorphisms according to Definition 4.1;

(2) for $m \in \mathbb{N}$, there is a sequence $\{u_k^{(m)}\}_{k\in\mathbb{N}} \subseteq V(G)$ and a sequence of coherent maps $\{f_{k,r}^{(m)} \colon B_r(u_k^{(m)}) \to B_r(u'_m)\}_{k,r\in\mathbb{N}}$ such that $\{(H, G, u_k^{(m)})\}_{k\in\mathbb{N}}$ converges to (H'_m, G'_m, u'_m).

Let $\varepsilon > 0$ and $R \in \mathbb{N}$. Invoking (1) there is an $M_R \in \mathbb{N}$ such that

$$f'_{m,R}\colon B_R(u'_m) \longrightarrow B_R(\tilde{v}_0)$$

is a graph isomorphism for $m \geq M_R$ and

$$\|\mathcal{I}_{f'_{m,R}}(H'_m)_{u'_m,R}\mathcal{I}^{-1}_{f'_{m,R}} - \widetilde{H}_{\tilde{v}_0,R}\| < \frac{\varepsilon}{2}, \quad m \geq M_R.$$

Let $m \geq M_R$. Invoking (2) there is an $K(R,m) \in \mathbb{N}$ such that

$$f^{(m)}_{k,R}\colon B_R(u^{(m)}_k) \longrightarrow B_R(u'_m)$$

is a graph isomorphism for $k \geq K(R,m)$ and

$$\|\mathcal{I}_{f^{(m)}_{k,R}} H_{u^{(m)}_k,R}\mathcal{I}^{-1}_{f^{(m)}_{k,R}} - (H'_m)_{u'_m,R}\| < \frac{\varepsilon}{2}, \quad k \geq K(R,m).$$

For $m \geq M_R$, the map

$$g_{m,R} := f'_{n,R} \circ f^{(m)}_{K(R,m),R}\colon B_R(u^{(m)}_{K(R,m)}) \to B_R(\tilde{v}_0)$$

is a graph isomorphism and $\mathcal{I}_{f'_{m,R}}\mathcal{I}_{f^{(m)}_{K(R,m),R}} = \mathcal{I}_{g_{m,R}}$. Thus, we derive

$$\begin{aligned}
&\|\mathcal{I}_{g_{m,R}} H_{u^{(m)}_{K(R,m)},R}\mathcal{I}^{-1}_{g_{m,R}} - \mathcal{I}_{f'_{m,R}} H'_{u'_m,R}\mathcal{I}^{-1}_{f'_{m,R}}\| \\
&\quad = \|\mathcal{I}_{f^{(m)}_{K(R,m),R}} H_{u^{(m)}_{K(R,m)},R}\mathcal{I}^{-1}_{f^{(m)}_{K(R,m),R}} - H'_{u'_m,R}\| < \frac{\varepsilon}{2}.
\end{aligned}$$

With this at hand, the triangle inequality leads to

$$\begin{aligned}
&\|\mathcal{I}_{g_{m,R}} H_{u^{(m)}_{K(R,m)},R}\mathcal{I}^{-1}_{g_{m,R}} - \widetilde{H}_{\tilde{v}_0,R}\| \\
&\quad \leq \|\mathcal{I}_{g_{m,R}} H_{u^{(m)}_{K(R,m)},R}\mathcal{I}^{-1}_{g_{m,R}} - \mathcal{I}_{f'_{m,R}} H'_{u'_m,R}\mathcal{I}^{-1}_{f'_{m,R}}\| \\
&\qquad + \|\mathcal{I}_{f'_{m,R}} H'_{u'_m,R}\mathcal{I}^{-1}_{f'_{m,R}} - \widetilde{H}_{\tilde{v}_0,R}\| < \varepsilon
\end{aligned}$$

for all $m \geq M_R$. Since $\varepsilon > 0$ was arbitrary, $\widetilde{H}$ is an $\mathcal{R}$-limit of H. □

The following statement provides a (sequentially) compactness property of the set of triples (H, G, v) where the operators are uniformly bounded in the operator norm.

Lemma 4.5 (sequentially compactness). *Let $\{(G_n, v_n)\}_{n\in\mathbb{N}}$ be a sequence of connected, infinite, rooted d-bounded graphs and H_n be a Schrödinger operators of the form (1.1) on (G_n, v_n) such that* $\sup_{n\in\mathbb{N}} \|W_n\|_\infty < \infty$. *Then there exists a Schrödinger operator H' on a rooted d-bounded graph (G', v'_0) and a subsequence $\{(H_{n_k}, G_{n_k}, v_{n_k})\}_{k\in\mathbb{N}}$ that converges to (H', G', v'_0).*

Proof. By assumption, $\deg(v) \leq d$ holds for all $v \in V(G_n)$ and $n \in \mathbb{N}$. Thus, for each $r \in \mathbb{N}$, the set

$$\mathcal{B}_r := \{B_r(v) \colon v \in V(G_n), n \in \mathbb{N}\}/\sim$$

is finite, where $\sim$ is the equivalence relation induced by graph isomorphism. Then a Cantor diagonalization argument gives a convergent subsequence $\{(G_n, v_n)\}_{n\in\mathbb{N}}$ to a rooted d-bounded graph (G', v_0'). By a similar argument and by passing to another subsequence one gets the desired result that $\{(H_{n_k}, G_{n_k}, v_{n_k})\}_{k\in\mathbb{N}}$ converges to (H', G', v_0'). Since these arguments are standard, we only provide a sketch of the proof here.

Let $r = 1$. Since $\mathcal{B}_1$ is finite, there is a subsequence $\{N(1,k)\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $N(1,k) \to \infty$ such that $B_1(v_{N(1,k_1)}) \sim B_1(v_{N(1,k_2)})$ for all $k_1, k_2 \in \mathbb{N}$. Now let $r = 2$. By the same argument, there is a subsequence $\{N(2,k)\}_{k\in\mathbb{N}} \subseteq \{N(1,k)\}_{k\in\mathbb{N}}$ such that $B_R(v_{N(2,k_1)}) \sim B_R(v_{N(2,k_2)})$ for all $k_1, k_2 \in \mathbb{N}$ and $1 \leq R \leq 2$. By recursion we get for each $r \in \mathbb{N}$ a subsequence

$$\{N(r,k)\}_{k\in\mathbb{N}} \subseteq \{N(r-1,k)\}_{k\in\mathbb{N}} \subseteq \cdots \subseteq \{N(2,k)\}_{k\in\mathbb{N}} \subseteq \{N(1,k)\}_{k\in\mathbb{N}} \subseteq \mathbb{N} \tag{4.1}$$

such that

$$B_R(v_{N(r,k_1)}) \sim B_R(v_{N(r,k_2)}), \quad k_1, k_2 \in \mathbb{N}, 1 \leq R \leq r.$$

In particular, due to equation (4.1), $B_R(v_{N(r,k)})$ is isomorphic to $B_R(v_{N(R,1)})$ for every $k \in \mathbb{N}$ and $1 \leq R \leq r$.

In order to define the graph (G', v_0'), it suffices to define $B_R(v_0')$ for all $R \in \mathbb{N}$ modulo graph isomorphism. Define $B_R(v_0') := B_R(v_{N(R,1)})$. By construction $B_R(v_{N(R,1)}) \sim B_R(v_{N(r,k)})$ holds for all $k \in \mathbb{N}$ and $R \leq r$. Thus, the rooted graph (G', v_0') is well-defined (up to graph isomorphism). By construction, (G', v_0') is a connected and infinite (rooted) d-bounded graph.

We claim that the diagonal sequence $\{(G_{N(k,k)}, v_{N(k,k)})\}_{k\in\mathbb{N}}$ converges to (G', v_0'). This can be seen as follows. Define $f_{k,R} \colon B_R(v_{N(k,k)}) \to B_R(v_0')$ for $k \geq R$ to be the graph isomorphism between $B_R(v_{N(k,k)})$ and $B_R(v_0') = B_R(v_{N(R,1)})$ which exists by construction as $k \geq R$. If $k \leq R$, define $f_{k,R} \colon B_R(v_{N(k,k)}) \to B_R(v_0')$ by $f_{k,R}(u) := v_0'$ for all $u \in B_R(v_{N(k,k)})$. By construction, these maps are eventually graph isomorphisms (as $k \to \infty$), namely they satisfy the constraints given in Definition 4.1.

By the latter considerations, we have shown that there is a subsequence that converges to an infinite and connected (rooted) d-bounded graph (G', v_0'). In order to simplify the notation, suppose that $\{(G_n, v_n)\}_{n\in\mathbb{N}}$ converges to (G', v_0'). The operators H_n are uniformly bounded in $n \in \mathbb{N}$ since

$$\|H_n\| \leq 2d + \sup_{n\in\mathbb{N}} \|W_n\|_\infty .$$

Thus, for fixed $r \in \mathbb{N}$, there is a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that

$$\mathcal{I}_{f_{n_k,r}} P_{v_{n_k},r} H_{n_k} P_{v_{n_k},r} \mathcal{I}^{-1}_{f_{n_k,r}}$$

converges in norm (using that $\ell^2(B_r(v_{n_k}))$ is a finite dimensional vector space). By a similar argument, as for the graph sequence one can construct (with a Cantor diagonalization argument) an operator H' on (G', v_0') and a subsequence $\{n_k\}_{k\in\mathbb{N}}$ such that for each $r \in \mathbb{N}$

$$\lim_{k\to\infty, n_k \geq N_r} \|\mathcal{I}_{f_{n_k,r}} P_{v_{n_k},r} H_{n_k} P_{v_{n_k},r} \mathcal{I}^{-1}_{f_{n_k,r}} - H'_{v_0',r}\| = 0$$

where $N_r \in \mathbb{N}$ is chosen such that $f_{n_k,r}$ defines a graph isomorphism for $n_k \geq N_r$. According to Lemma 4.3, H' is a bounded Schrödinger operator of the form (1.1). □

4.2. Existence of bounded generalized eigenfunctions for $\mathcal{R}$-limits. This section is devoted to providing conditions such that a bounded generalized eigenfunction corresponding to the eigenvalue λ of an $\mathcal{R}$-limit exists. These are key ingredients for the proof of Theorem 2.2.

Proposition 4.6. *Let (H, G, v_0) and $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ be such that either*

(a) *each (H_n, G_n, v_n) is an $\mathcal{R}$-limit of (H, G, v_0), or*

(b) *$H_n = H$, $G_n = G$, and $\{v_n\}_{n\in\mathbb{N}}$ monotonically converges to infinity.*

Let $C > 0$ and for each $n \in \mathbb{N}$, let $\varphi^{(n)} : V(G_n) \to \mathbb{C}$, $\lambda_n \in \mathbb{C}$ and $R_n \in \mathbb{N}$ be such that $\lim_{n\to\infty} \lambda_n = \lambda'$, $\lim_{n\to\infty} R_n = \infty$,

$$P_{v_o,r} H_n \varphi^{(n)} = \lambda_n P_{v_0,r} \varphi^{(n)}, \quad \text{for all } r \leq R_n,\ n \in \mathbb{N},$$

and

$$\max_{u\in B_n(v_n)} |\varphi^{(n)}(u)| \leq C|\varphi^{(n)}(v_n)| \neq 0. \tag{4.2}$$

Then there exists a subsequence of $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ converging to an $\mathcal{R}$-limit (H', G', v_0') of H and a generalized eigenfunction $0 \neq \varphi' \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ'.

Proof. Due to Lemma 4.3, there is a constant $C > 0$ such that every $\mathcal{R}$-limit H' of H is a Schrödinger operator of the form (1.1) with potential W' satisfying $\|H'\| \leq 2(d + C)$. Hence, there is no loss of generality (by enlarging the constant C) that $\|W'\|_\infty \leq C$. Thus, $\sup_{n\in\mathbb{N}} \|W_n\|_\infty < \infty$ follows where W_n is the potential of H_n for both cases (a) or (b). Hence, Lemma 4.5 implies that $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ has a convergent subsequence.

Without loss of generality (by passing to a subsequence), assume that the sequence $\{(H_n, G_n, v_n)\}_{n\in\mathbb{N}}$ converges to (H', G', v_0') with respect to the coherent maps $\{f_{n,r}\}_{n,r\in\mathbb{N}}$ in the sense of Definition 4.1. In addition, there is no loss of generality (again, by passing to a subsequence) to assume that $f_{n,r}\colon B_r(v_n) \to B_r(v_0')$ is a graph isomorphism for all $r \leq n$. Define $\psi^{(n)}\colon V(G') \to \mathbb{C}$ by

$$\psi^{(n)}(u) := \frac{1}{\varphi^{(n)}(v_n)} (\mathcal{I}_{f_{n,n}} P_{v_n,n} \varphi^{(n)})(u)$$

Since $f_{n,n}^{-1}(v_0') = v_n$, we have $\psi^{(n)}(v_0') = 1$ and

$$\|\psi^{(n)}\| = \sup_{u\in G} |\psi^{(n)}(u)| = \sup_{u\in B_n(v_0')} |\psi^{(n)}(u)| \leq C$$

by the assumption (4.2). Thus, $\psi^{(n)} \in \ell^\infty(G')$.

By construction, the sequence $\psi^{(n)}(u)$ is uniformly bounded for every $u \in V(G')$. Hence, the Bolzano–Weierstrass theorem and a Cantor diagonalization argument yield that there is a subsequence $\{\psi^{(n(\ell))}\}_{\ell\in\mathbb{N}} \subseteq \mathbb{N}$ and a $\psi'\colon V(G') \to \mathbb{C}$ such that for all $r \in \mathbb{N}$,

$$\lim_{\ell\to\infty} \|P_{v_0',r}(\psi^{(n(\ell))} - \psi')\| = 0.$$

Furthermore, $|\psi'(u)| \leq C$ holds for all $u \in V(G')$, namely $\psi' \in \ell^\infty(G')$. In addition, $\psi'(v_0') = 1$ follows from $\psi^{(n)}(v_0') = 1$.

In case (a), each H_n is an $\mathcal{R}$-limit of H. Lemma 4.3 and Lemma 4.4 assert that H' is also an $\mathcal{R}$-limit of H. If (b) holds, H' is also an $\mathcal{R}$-limit of H as $\{v_n\}_{n\in\mathbb{N}}$ goes to infinity. Thus, it is left to show that $\psi' \in \ell^\infty(G')$ defines a generalized eigenfunction of H' corresponding to the eigenvalue λ'.

Let $\varepsilon > 0$ and fix $r \in \mathbb{N}$. Since H' is an $\mathcal{R}$-limit of H, it is of the form (1.1) (see Lemma 4.3). Thus, $P_{v_0',r} H'\psi' = P_{v_0',r} P_{v_0',r+1} H' P_{v_0',r+1}\psi'$ follows implying

$$\begin{aligned}\|P_{v_0',r}(H'\psi' - \lambda\psi')\| &\leq \|P_{v_0',r}\| \|H'_{v_0',r+1}\psi' - \lambda' P_{v_0',r+1}\psi'\| \\ &\leq \|H'_{v_0',r+1}\psi' - \lambda' P_{v_0',r+1}\psi'\|.\end{aligned}$$

In order to simplify the notation, set

$$L_{n,r} := \mathcal{I}_{f_{n,r}} P_{v_n,r} H_n P_{v_n,r} \mathcal{I}_{f_{n,r}}^{-1}$$

acting on $\ell^2(B_{r+1}(v_0'))$. Since $\lim_{\ell\to\infty}\lambda_{n(\ell)} = \lambda'$, there exists a $C_0 > 0$ such that $|\lambda_{n(\ell)}| \leq C_0$. Recall that for each $n \in \mathbb{N}$, there is an $R_n \in \mathbb{N}$ such that $P_{v_0,r}H_n\varphi^{(n)} = \lambda_n P_{v_0,r}\varphi^{(n)}$ for all $r \leq R_n$. Choose $\ell \in \mathbb{N}$ such that $n(\ell), R_{n(\ell)} > r$ and

$$\begin{aligned}\|L_{n(\ell),r+1} - H'_{v_0',r+1}\| &< \frac{\varepsilon}{3\|P_{v_0',r}\psi'\|},\\ \|P_{v_0',r+1}(\psi' - \psi^{(n(\ell))})\| &< \frac{\varepsilon}{6\max\{d+C, C_0\}},\\ |\lambda_{n(\ell)} - \lambda'| &< \frac{\varepsilon}{6\|P_{v_0',r}\psi'\|}.\end{aligned}$$

Note that $\|P_{v_0',r}\psi'\| \neq 0$ as $\psi'(v_0') = 1$ by construction. Furthermore,

$$\|P_{v_0',r}L_{n(\ell),r+1}\| \leq 2(d+C)$$

follows as H_n is an $\mathcal{R}$-limit and so $\|H_n\| \leq 2(d+C)$ holds by Lemma 4.3. Hence,

$$\begin{aligned}&\|P_{v_0',r}(H'\psi' - \lambda\psi')\|\\ &\quad\leq \|P_{v_0',r}H'_{v_0',r+1}\psi' - \lambda' P_{v_0',r+1}\psi'\|\\ &\quad\leq \|P_{v_0',r}H'_{v_0',r+1}\psi' - P_{v_0',r}L_{n(\ell),r+1}\psi'\|\\ &\qquad+ \|P_{v_0',r}L_{n(\ell),r+1}\psi' - P_{v_0',r}L_{n(\ell),r+1}\psi^{(n(\ell))}\|\\ &\qquad+ \|P_{v_0',r}L_{n(\ell),r+1}\psi^{(n(\ell))} - \lambda_{n(\ell)}P_{v_0',r}\psi'\|\\ &\qquad+ \|\lambda_{n(\ell)}P_{v_0',r}\psi' - \lambda' P_{v_0',r+1}\psi'\|\\ &\quad=: (1) + (2) + (3) + (4)\end{aligned}$$

follows by using the triangle inequality. We estimate each of the summands (1), (2), (3), and (4) separately. Specifically, the previous considerations and the choice of ℓ lead to

$$(1) \leq \|L_{n(\ell),r+1} - H'_{v_0',r+1}\|\|P_{v_0',r+1}\psi'\| \leq \frac{\varepsilon}{3}$$

and

$$(2) \leq \|P_{v_0',r}L_{n(\ell),r+1}\|\|P_{v_0',r+1}(\psi' - \psi^{(n(\ell))})\| \leq \frac{\varepsilon}{3}.$$

Since $R_{n(\ell)} \geq r$, we have $P_{v_0,r}H_n\varphi^{(n(\ell))} = \lambda_n P_{v_0,r}\varphi^{(n(\ell))}$. Hence, using the choice $n(\ell) > r$,

$$P_{v_0',r}L_{n(\ell),r+1}\psi^{(n(\ell))} = \lambda_{n(\ell)}P_{v_0',r}\psi^{(n(\ell))}$$

follows. Thus,

$$(3) = \|\lambda_{n(\ell)}P_{v_0',r}\psi^{(n(\ell))} - \lambda_{n(\ell)}P_{v_0',r}\psi'\| < \frac{\varepsilon}{6}.$$

Finally, using once more the choice of ℓ, we deduce

$$(4) \leq |\lambda_{n(\ell)} - \lambda'| \|P_{v_0',r}\psi'\| < \frac{\varepsilon}{6}.$$

Combining the latter estimates, we conclude

$$\|P_{v_0',r}(H'\psi' - \lambda\psi')\| < (1) + (2) + (3) + (4) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we derive that $\|P_{v_0',r}(H'\psi' - \lambda'\psi')\| = 0$ for any $r \in \mathbb{N}$, namely λ' is a generalized eigenvalue of H' with generalized eigenfunction $\psi' \in \ell^\infty(G')$. □

Corollary 4.7. *The union $\bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H')$ is closed.*

Proof. Assume $\{\lambda_n\}_{n\in\mathbb{N}} \subset \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H')$ and $\lim_{n\to\infty} \lambda_n = \lambda'$. For $n \in \mathbb{N}$, let H_n be an $\mathcal{R}$-limit on the d-bounded graph G_n and $0 \neq \varphi_n \in \ell^\infty(G_n)$ be such that $(H_n - \lambda_n)\varphi_n = 0$. Choose $u_n \in V(G_n)$ such that $\varphi_n(u_n) \geq \frac{\|\varphi_n\|_\infty}{2}$. Then the conditions of Proposition 4.6 are satisfied (for $C = 2$) and so we derive that $\lambda' \in \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H')$. □

Another consequence of Proposition 4.6 is the following statement. Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence of functions $\varphi_i : V(G) \to \mathbb{C}$. We will use the notation

$$q_{\ell,k}^{(i)} := \max_{\ell(k-1)\leq|u|<\ell k} |\varphi_i(u)|, \quad i \in \mathbb{N},$$

and, denote by $u_{\ell,k}^{(i)} \in V(G)$ a vertex satisfying $\ell(k-1) \leq |u_{\ell,k}^{(i)}| < \ell k$ and $q_{\ell,k}^{(i)} = |\varphi_i(u_{\ell,k}^{(i)})|$.

Lemma 4.8. *Let (H, G, v_0) be given where H is a Schrödinger operator on the infinite connected rooted d-bounded graph (G, v_0). Suppose we are given a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of generalized eigenfunctions of H corresponding to the eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfying*

(*a*) $\lim_{n\to\infty} \lambda_n = \lambda$;

(*b*) *there is an $s > 1$ such that for each $k \in \mathbb{N}$, there are subsequences $n_i \to \infty$ and $\ell_i \to \infty$ satisfying*

$$q_{\ell_i,k}^{(n_i)} = |\varphi_{n_i}(u_{\ell_i,k}^{(n_i)})| \geq \frac{1}{s} \max\{q_{\ell_i+1,k}^{(n_i)}, q_{\ell_i-1,k}^{(n_i)}\}.$$

Then there is an $\mathcal{R}$-limit (H', G', v_0') of H, and a generalized eigenfunction $0 \neq \varphi' \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ.

Proof. We will prove in the following that for each $k \in \mathbb{N}$, there is an $\mathcal{R}$-limit $(H^{(k)}, G^{(k)}, v_0^{(k)})$ of H together with a $\varphi^{(k)}\colon V(G^{(k)}) \to \mathbb{C}$ and $\lambda^{(k)} := \lambda$ satisfying all the assumptions of Proposition 4.6. This leads to the desired result.

Let $k \in \mathbb{N}$. For the sake of simplifying the notation and since k stays fixed until the last step of the proof, there is no loss of generality (by passing to a subsequence) in assuming that $n_i = i$ and $\ell_i \geq k$ for all $i \in \mathbb{N}$. Since $\ell_i \to \infty$, the vertices $u^{(i)}_{\ell_i,k}$ go to infinity if $i \to \infty$. Then there is no loss of generality in assuming that $\{(H, G, u^{(i)}_{\ell_i,k})\}_{i\in\mathbb{N}}$ converges to an $\mathcal{R}$-limit $(H^{(k)}, G^{(k)}, v_0^{(k)})$, otherwise we can pass to a convergent subsequence by Lemma 4.5. Since $\ell_i \geq k > k-1$, the requirement (b) yields

$$|\varphi_i(u)| \leq s|\varphi_i(u^{(i)}_{\ell_i,k})|$$

for all $u \in V(G)$ satisfying $(\ell_i - 1)(k-1) \leq |u| < (\ell_i + 1)k$. Let $u \in B_{k-1}(u^{(i)}_{\ell_i,k})$. Since $\ell_i(k-1) \leq |u^{(i)}_{\ell_i,k}| < \ell_i k$ holds by definition, we conclude

$$(\ell_i - 1)(k-1) \leq |u^{(i)}_{\ell_i,k}| - (k-1) \leq |u| \leq |u^{(i)}_{\ell_i,k}| + (k-1) < (\ell_i + 1)k.$$

Hence,

$$\max_{u \in B_{k-1}(u^{(i)}_{\ell_i,k})} |\varphi_i(u)| \leq s|\varphi_i(u^{(i)}_{\ell_i,k})|$$

follows. Define $\psi_i^{(k)}\colon V(G) \to \mathbb{C}$ by

$$\psi_i^{(k)}(u) := \begin{cases} \dfrac{\varphi_i(u)}{|\varphi_i(u^{(i)}_{\ell_i,k})|} & \text{if } u \in B_{k-1}(u^{(i)}_{\ell_i,k}), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\psi_i^{(k)}(u^{(i)}_{\ell_i,k}) = 1$ and $\|\psi_i^{(k)}\|_\infty \leq s$. Since $\{(H, G, u^{(i)}_{\ell_i,k})\}_{i\in\mathbb{N}}$ converges to $(H^{(k)}, G^{(k)}, v_0^{(k)})$ with respect to the coherent maps

$$\{f_{i,r}\colon B_r(u^{(i)}_{\ell_i,k}) \to B_r(v_0^{(k)})\}_{i,r\in\mathbb{N}},$$

there is an $i_0 \in \mathbb{N}$ such that $f_{i,k}$ is a graph isomorphism for all $i \geq i_0$. Define for $i \geq i_0$, $\varphi_i^{(k)}\colon V(G^{(k)}) \to \mathbb{C}$ by $\varphi_i^{(k)} := \mathcal{I}_{f_{i,k}} \psi_i^{(k)}$. We remind the reader that this formally just defines a function on $\ell^2(B_k(v_0^{(k)}))$ that we embed into $\ell^2(G^{(k)})$ by extending it by zero. Then, we deduce for $i \geq i_0$

$$\operatorname{supp}(\varphi_i^{(k)}) \subseteq B_{k-1}(v_0^{(k)}), \quad \varphi_i^{(k)}(v_0^{(k)}) = 1, \quad \text{and} \quad \|\varphi_i^{(k)}\|_\infty \leq s.$$

Since $B_{k-1}(v_0^{(k)})$ is finite, there is no loss of generality (by passing to another subsequence) that $\{\varphi_i^{(k)}\}_{i\in\mathbb{N}}$ converges pointwise (and so in ℓ^2-norm as all functions are supported on $B_{k-1}(v_0^{(k)})$) to a map $\varphi^{(k)}: V(G^{(k)}) \to \mathbb{C}$ such that $\operatorname{supp}(\varphi^{(k)}) \subseteq B_{k-1}(v_0^{(k)})$, $\varphi^{(k)}(v_0^{(k)}) = 1$ and $0 < \|\varphi^{(k)}\| \leq s$. Thus,

$$\sup_{v\in B_k(v_0^{(k)})} |\varphi^{(k)}(v)| \leq s = s|\varphi^{(k)}(v_0^{(k)})|.$$

Now, we are almost in the setting of Proposition 4.6. More precisely, we need to show that $\varphi^{(k)}$ are "approximate generalized eigenfunctions."

Let $r \leq k-2$. Then the convergence of $\{(H, G, u_{\ell_i,k}^{(i)})\}_{i\in\mathbb{N}}$ to $(H^{(k)}, G^{(k)}, v_0^{(k)})$ and the ℓ^2-convergence of $\{\varphi_i^{(k)}\}_{i\in\mathbb{N}}$ to $\varphi^{(k)}$ imply the following: for all $\varepsilon > 0$, there is an $i_1 \in \mathbb{N}$ such that for $i \geq i_1$,

$$\begin{aligned}
\|\mathcal{J}_{f_{i,r+1}} H_{u_{\ell_i,k}^{(i)},r+1} \mathcal{J}_{f_{i,r+1}} - H^{(k)}_{v_0^{(k)},r+1}\| &< \frac{\varepsilon}{2s\|\varphi^{(k)}\|_2},\\
\|\varphi_i^{(k)} - \varphi^{(k)}\|_2 &< \frac{\varepsilon}{8\max\{d + C_H, |\lambda_i|, 1\}},\\
|\lambda_i - \lambda| &< \frac{\varepsilon}{8\|\varphi^{(k)}\|_2}.
\end{aligned}$$

Note that $\varphi^{(k)}$ is supported on $B_{k-1}(v_0^{(k)})$ and so $\|\varphi^{(k)}\|_2$ is finite. Furthermore, $\lim_{i\to\infty} \lambda_i = \lambda$ holds implying $|\lambda_i|$ is uniformly bounded in $i \in \mathbb{N}$. According to Lemma 4.3, $H^{(k)}$ is a Schrödinger operator of the form (1.1) and $\|H^{(k)}\| \leq 2(d + C_H)$ for some constant $C_H > 0$ independent of k. Using (1.1), we derive

$$P_{v_0^{(k)},r} H^{(k)} \varphi^{(k)} = P_{v_0^{(k)},r} H^{(k)}_{v_0^{(k)},r+1} P_{v_0^{(k)},r+1} \varphi^{(k)}.$$

Let $v \in B_r(v_0^{(k)})$. Then

$$(\mathcal{J}_{f_{i,r+1}} P_{u_{\ell_i,k}^{(i)},r+1} \mathcal{J}_{f_{i,k}}^{-1} \zeta)(v) = \zeta(v), \quad \zeta \in \ell^2(G^{(k)}),$$

holds as the maps are coherent and $r+1 \leq k-1$. Denote by $\chi_{k,r}$ the characteristic function of $B_r(v_0^{(k)})$. Then, a short computation gives

$$\begin{aligned}
&G(P_{v_0^{(k)},r} \mathcal{J}_{f_{i,r+1}} H_{u_{\ell_i,k}^{(i)},r+1} \mathcal{J}_{f_{i,r+1}}^{-1} P_{v_0^{(k)},r+1} \varphi_i^{(k)})(v)\\
&\quad = \chi_{k,r}(v)(\mathcal{J}_{f_{i,r+1}} H_{u_{\ell_i,k}^{(i)},r+1} \psi_i^{(k)})(v)\\
&\quad = \chi_{k,r}(v)(H\psi_i^{(k)})(f_{i,r+1}^{-1}(v))\\
&\quad = \lambda_i (P_{v_0^{(k)},r} \mathcal{J}_{f_{i,r+1}} \psi_i^{(k)})(v)\\
&\quad = \lambda_i (P_{v_0^{(k)},r} \varphi_i^{(k)})(v)
\end{aligned}$$

invoking the definition of $\psi_i^{(k)}$ and the fact that φ_i is a generalized eigenfunction of H corresponding to the eigenvalue λ_i. With this at hand, the triangle inequality leads to

$$\| P_{v_0^{(k)},r} H^{(k)} \varphi^{(k)} - \lambda P_{v_0^{(k)},r} \varphi^{(k)} \| \le T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned}
T_1 &:= \| P_{v_0^{(k)},r} H^{(k)}_{v_0^{(k)},r+1} P_{v_0^{(k)},r+1} \varphi^{(k)} \\
&\qquad - P_{v_0^{(k)},r} \mathcal{I}_{f_{i,r+1}} H_{u^{(i)}_{l_i,k},r+1} \mathcal{I}^{-1}_{f_{i,r+1}} P_{v_0^{(k)},r+1} \varphi^{(k)} \|, \\
T_2 &:= \| P_{v_0^{(k)},r} \mathcal{I}_{f_{i,r+1}} H_{u^{(i)}_{l_i,k},r+1} \mathcal{I}^{-1}_{f_{i,r+1}} P_{v_0^{(k)},r+1} \varphi^{(k)} \\
&\qquad - P_{v_0^{(k)},r} \mathcal{I}_{f_{i,r+1}} H_{u^{(i)}_{l_i,k},r+1} \mathcal{I}^{-1}_{f_{i,r+1}} P_{v_0^{(k)},r+1} \varphi_i^{(k)} \|, \\
T_3 &:= \| P_{v_0^{(k)},r} \mathcal{I}_{f_{i,r+1}} H_{u^{(i)}_{l_i,k},r+1} \mathcal{I}^{-1}_{f_{i,r+1}} P_{v_0^{(k)},r+1} \varphi_i^{(k)} - \lambda_i P_{v_0^{(k)},r} \varphi^{(k)} \|, \\
T_4 &:= \| \lambda_i P_{v_0^{(k)},r} \varphi^{(k)} - \lambda P_{v_0^{(k)},r} \varphi^{(k)} \|.
\end{aligned}$$

Then the previous considerations yield

$$\begin{aligned}
T_1 &< \frac{\varepsilon}{2}, \\
T_2 &\le 2(d + C_H) \| \varphi^{(k)} - \varphi_i^{(k)} \| < \frac{\varepsilon}{4}, \\
T_3 &= |\lambda_i| \| P_{v_0^{(k)},r} \varphi_i^{(k)} - P_{v_0^{(k)},r} \varphi^{(k)} \| < \frac{\varepsilon}{8}, \\
T_4 &= |\lambda_i - \lambda| \| P_{v_0^{(k)},r} \varphi^{(k)} \| < \frac{\varepsilon}{8}.
\end{aligned}$$

Thus, $\| P_{v_0^{(k)},r} H^{(k)} \varphi^{(k)} - \lambda P_{v_0^{(k)},r} \varphi^{(k)} \| < \varepsilon$ follows implying

$$P_{v_0^{(k)},r} H^{(k)} \varphi^{(k)} = \lambda P_{v_0^{(k)},r} \varphi^{(k)}, \quad r \le k-2,$$

as $\varepsilon > 0$ was arbitrary.

Combining all the previous considerations, Proposition 4.6 applies for the sequence $\{(H^{(k)}, G^{(k)}, u^{(k)})\}_{k\in\mathbb{N}}$, the sequence $\{\varphi^{(k)}\}_{k\in\mathbb{N}}$, $R_k := k-2$, $\lambda_k := \lambda$ and $C := s$. Hence, there is an $\mathcal{R}$-limit (H', G', v_0') of H and a generalized eigenfunction $0 \ne \varphi' \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ finishing the proof. □

4.3. The existence and the behaviour at infinity of generalized eigenfunctions for $\mathcal{R}$-limits. The next two statements rely on Proposition 4.6. We show that under certain conditions the existence of a generalized eigenfunction φ of H results in the existence of a bounded generalized eigenfunction of some $\mathcal{R}$-limit of H or it gives constraints on the growth behaviour of φ at infinity. The first proposition treats bounded generalized eigenfunctions, while in the second proposition they are unbounded. Both proofs are based on elementary statements that are provided separately in a lemma to make the reading of the proofs more accessible.

Lemma 4.9. *Let $s > 1$ and $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative numbers such that $\lim_{n\to\infty} a_n = 0$. Then either there exists $C > 0$ such that*

$$a_n < C s^{-n}, \quad n \in \mathbb{N}. \tag{4.3}$$

or there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that for any $k \in \mathbb{N}$,

$$a_{n_k} > a_{n_k+1} \quad \text{and} \quad s a_{n_k} \geq a_{n_k-1}. \tag{4.4}$$

Proof. This follows by straightforward computations, see [14]. □

Proposition 4.10. *Let H be a Schrödinger operator on an infinite, connected rooted d-bounded graph (G, v_0) and $\varphi \in \ell^\infty(G)$ be a bounded generalized eigenfunction of H corresponding to the eigenvalue λ that does not vanish everywhere. Then one of the following assertions holds.*

(a) *There exists an $\mathcal{R}$-limit (H', G', v_0') of H, and a bounded generalized eigenfunction $0 \neq \varphi' \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ.*

(b) *There are constants $\gamma > 1$ and $C > 0$ such that*

$$|\varphi(u)| \leq C \cdot \gamma^{-|u|}, \quad u \in G.$$

Proof. Let $k \in \mathbb{N}$. Then there is a $u_{m,k} \in V(G)$ such that $m(k-1) \leq |u_{m,k}| < mk$ and

$$|\varphi(u_{m,k})| = \max_{m(k-1)\leq|u|<mk} |\varphi(u)| =: q_{m,k}.$$

Note that the latter is a maximum as $B_r(v_0)$ is finite for every $r \in \mathbb{N}$. We will treat the two cases

(A) $\{q_{m,k}\}_{m\in\mathbb{N}}$ does not tend to zero for some $k \in \mathbb{N}$ and

(B) $\{q_{m,k}\}_{m\in\mathbb{N}}$ does tend to zero for all $k \in \mathbb{N}$.

(A) Suppose that $k \in \mathbb{N}$ is chosen such that $\{q_{m,k}\}_{m\in\mathbb{N}}$ does not tend to zero. Let $\{q_{m_\ell,k}\}_{\ell\in\mathbb{N}}$ be a subsequence such that $\lim_{\ell\to\infty} q_{m_\ell,k} = q > 0$. Such a subsequence exists as $\{q_{m,k}\}_{m\in\mathbb{N}}$ is uniformly bounded since $\varphi \in \ell^\infty(G)$. Choose $\ell_0 \in \mathbb{N}$ such that $q_{m_\ell,k} > q/2$ for every $\ell > \ell_0$. Then,

$$\max_{u\in B_\ell(u_{m_\ell,k})} |\varphi(u)| \leq \|\varphi\|_\infty \leq \frac{2\|\varphi\|_\infty}{q} |\varphi(u_{m_\ell,k})| \neq 0.$$

Thus, all the requirements of Proposition 4.6 are satisfied for the sequence $\{(H, G, u_{m_\ell,k})\}_{\ell\in\mathbb{N}}$ with $\lambda_n = \lambda$, $\varphi^{(n)} = \varphi$ and $C := \frac{2\|\varphi\|_\infty}{q}$. Hence, statement (a) of Proposition 4.10 follows from Proposition 4.6.

(B) Suppose now that $\lim_{m\to\infty} q_{m,k} = 0$ for all $k \in \mathbb{N}$ and let $s > 1$. Due to Lemma 4.9, either

(B.1) there is a $k \in \mathbb{N}$ such that (4.3) holds for a suitable constant $C > 0$ and $a_m := q_{m,k}$ or

(B.2) for all $k \in \mathbb{N}$, there is a subsequence $\{a_{m_i}\}_{i\in\mathbb{N}}$ satisfying (4.4).

(B.1) There exists a $k \in \mathbb{N}$ and a constant $C > 0$ such that

$$q_{m,k} < C s^{-m}, \quad m \in \mathbb{N}.$$

If $u \in V(G)$, then there is an $m \in \mathbb{N}$ such that $m(k-1) \leq |u| < mk$ and so $-m < -\frac{|u|}{k}$. Consequently, the latter considerations yield

$$q_{m,k} < C s^{-\frac{|u|}{k}}.$$

Specifically, φ is exponentially decaying as claimed in (b) with $\gamma := s^{\frac{1}{k}} > 1$.

(B.2) For all $k \in \mathbb{N}$, there is a subsequence $\{q_{m_i,k}\}_{i\in\mathbb{N}}$ satisfying

$$q_{m_i,k} > q_{m_i+1,k} \quad \text{and} \quad s q_{m_i,k} \geq q_{m_i-1,k}. \tag{4.5}$$

Hence,

$$q_{m_i,k} \geq \frac{1}{s} \max\{q_{m_i-1,k}, q_{m_i+1,k}\}$$

follows and φ is a generalized eigenfunction of H corresponding to the eigenvalue λ. Thus, Lemma 4.8 applied to $\varphi_j := \varphi$ and $\lambda_j := \lambda$ implies (a). □

Next, unbounded generalized eigenfunctions are studied in Proposition 4.12. For this the following lemma will be useful.

Lemma 4.11. *Let $\{a_n\}_{n\in\mathbb{N}}$ be an unbounded sequence of non-negative numbers such that there is an $s > 1$ and a $C > 0$ satisfying*

$$a_n < Cs^n, \quad n \in \mathbb{N}.$$

Then, for each $\gamma > s$, there exists a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that for all $k \in \mathbb{N}$,

$$a_{n_k} > a_{n_k-1} \quad \textit{and} \quad \gamma a_{n_k} \geq a_{n_k+1}.$$

Proof. This follows by straightforward computations, see [14]. □

Proposition 4.12. *Let φ be an unbounded generalized eigenfunction of H corresponding to the eigenvalue λ. Then one of the following assertions holds.*

(a) *There exists an $\mathcal{R}$-limit (H', G', v_0') of H, and a bounded generalized eigenfunction $\varphi' \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ that does not vanish everywhere.*

(b) *There exists a constant $\gamma > 1$, such that for all $C > 0$, there is a $u = u(C) \in V(G)$ satisfying $|\varphi(u)| \geq C\gamma^{|u|}$.*

Proof. Suppose (b) is not satisfied. We will prove that then (a) holds. Let $k \in \mathbb{N}$. Define

$$q_{m,k} := \max_{m(k-1)\leq|u|<mk} |\varphi(u)|, \quad m \in \mathbb{N}.$$

Since (b) does not hold, for all $\gamma > 1$, there is a constant $C_{\gamma,k} > 0$ such that

$$|\varphi(u)| \leq C_{\gamma,k}\gamma^{|u|}, \quad u \in V(G).$$

Let $s > 1$ and set $\gamma := s^{\frac{1}{k}} > 1$. Thus, there is a constant $C_k = C(k,r) > 0$ such that

$$q_{m,k} < C_k\gamma^{mk} = C_k s^m.$$

Since φ is unbounded, $\lim_{m\to\infty} q_{m,k} = \infty$. Then Lemma 4.11 implies that there is a subsequence $\{q_{m_i,k}\}_{i\in\mathbb{N}}$ such that for every $\gamma' > s$,

$$q_{m_i,k} > q_{m_i-1,k} \quad \text{and} \quad \gamma' q_{m_i,k} \geq q_{m_i+1,k}.$$

Thus, Lemma 4.8 applied to $\varphi_j := \varphi$, $\lambda_j := \lambda$ and $s = \gamma'$ leads to (a). □

4.4. Generalized eigenfunctions and Shnol type theorems. As discussed in the introduction, Shnol type theorems connect a growth conditions for generalized eigenfunctions corresponding to the eigenvalue λ to the fact that λ belongs to the spectrum of the operator.

We say that a function $\varphi\colon V(G) \to \mathbb{C}$ on a graph G has *sub-exponential growth* (*with respect to the graph metric* dist) if for one (any) vertex $v_0 \in V(G)$, the map

$$V(G) \ni v \longmapsto e^{-\alpha \operatorname{dist}(v,v_0)}\varphi(v)$$

is an element of $\ell^2(G)$ for all $\alpha > 0$. With this at hand, the following Shnol-type theorem holds, proven in [24] in more general setting of bounded Jacobi operators on a graph. This theorem is used for the proof of Theorem 2.4.

Proposition 4.13 ([24, Theorem 4.8]). *Let H be a Schrödinger operator of the form (1.1) on an infinite, connected, rooted d-bounded graph (G, v_0). Suppose $\varphi\colon V(G) \to \mathbb{C}$ is a generalized eigenfunction of H corresponding to the eigenvalue λ. If φ is sub-exponentially bounded, then $\lambda \in \sigma(H)$.*

The proof of Theorem 2.2 depends on the existence of a generalized eigenfunction for each point in the spectrum admitting a suitable growth rate. The following statement provides a sufficient condition to get such generalized eigenfunction which can be found in [28, Theorem 3] in a more general setting.

Theorem 4.14 (reverse Shnol's Theorem, [28, Theorem 3]). *Let (G, v_0) be a connected, infinite, rooted graph and H be a Schrödinger operator on $\ell^2(G)$ of the form (1.1) with spectral measure μ. Suppose $\omega \in \ell^2(G)$ is real-valued and positive (i.e. $\omega(v) > 0$ for all $v \in G$). Then for μ-a.e. $\lambda \in \sigma(H)$, there exists a generalized eigenfunction φ of H corresponding to the eigenvalue λ satisfying $\varphi \cdot \omega \in \ell^2(G)$.*

Corollary 4.15. *Let (G, v_0) be a connected, infinite, rooted graph and H be a Schrödinger operator on $\ell^2(G)$ of the form (1.1). If $\omega \in \ell^2(G)$ is real-valued and positive, then*

$$\begin{aligned}&\overline{\{\lambda \in \sigma(H) \mid \text{there exists } \varphi\colon G \to \mathbb{C} \text{ such that } H\varphi = \lambda\varphi \text{ and } \varphi\omega \in \ell^2(G)\}}\\ &\quad = \sigma(H).\end{aligned}$$

As a consequence of Theorem 4.14 we derive

Corollary 4.16. *The set*

$$\Lambda_1 = \left\{ \lambda \in \sigma(H) \;\middle|\; \begin{array}{l} \text{there is a generalized eigenfunction } \varphi_\lambda \\ \text{corresponding to the eigenvalue } \lambda \\ \text{such that } |\varphi_\lambda(v)| \leq (|v|+1)\sqrt{\sharp \mathcal{S}_{|v|}(v_0)} \end{array} \right\}$$

is dense in $\sigma(H)$. In particular, if G has sub-exponential growth, then the set

$$\Lambda_2 = \left\{ \lambda \in \sigma(H) \;\middle|\; \begin{array}{l} \textit{there is a generalized eigenfunction } \varphi_\lambda \\ \textit{corresponding to the eigenvalue } \lambda \\ \textit{such that } \varphi_\lambda \textit{ is sub-exponentially growing} \end{array} \right\} \tag{4.6}$$

is dense in $\sigma(H)$.

Proof. The set Λ_1 is dense in $\sigma(H)$ by using Corollary 4.15 and $\omega_G \in \ell^2(G)$ where

$$\omega_G(v) := \frac{1}{(|v|+1)\sqrt{\sharp \mathcal{S}_{|v|}(v_0)}}, \quad v \in V(G).$$

The second part follows from the fact that if G is sub-exponentially growing, then for each $\gamma > 1$, there is a $C > 0$ such that

$$(|v|+1)\sqrt{\sharp \mathcal{S}_{|v|}(v_0)} \leq C \cdot \gamma^{|v|/2}(|v|+1).$$

Then the statement follows from the first part. □

4.5. Proof of Theorem 2.2. The strategy of the proof of Theorem 2.2 is as follows. Given $\lambda \in \sigma_{\mathrm{ess}}(H)$ we use the reverse Shnol's property (Theorem 4.14) to obtain a sequence of generalized eigenfunctions $\{\varphi^{(n)}\}_{n=1}^{\infty}$ of H corresponding to the eigenvalue λ_n such that $\lim_{n\to\infty} \lambda_n = \lambda$ and each φ_n is sub-exponentially growing. With this at hand, Proposition 4.6, Propositions 4.10 and Proposition 4.12 complete the proof of the theorem.

Recall that the discrete spectrum $\sigma_{\mathrm{disc}}(H)$ of an operator H is defined by the set of isolated eigenvalues of finite multiplicity. Furthermore, the essential spectrum $\sigma_{\mathrm{ess}}(H)$ is defined by $\sigma(H) \setminus \sigma_{\mathrm{disc}}(H)$. Thus, if λ is an element of the essential spectrum, then either λ is an eigenvalue of infinite multiplicity or in each neighbourhood of λ there are elements of the spectrum $\sigma(H)$ that are not equal to λ.

Proof of Theorem 2.2. Let $\lambda \in \sigma_{\mathrm{ess}}(H)$. Then one of the following cases holds:

(a) for every $\varepsilon > 0$ there is a $\lambda_\varepsilon \in \sigma(H)$ such that $0 < |\lambda - \lambda_\varepsilon| < \varepsilon$;

(b) there exists an infinite sequence $\varphi_n \in \ell^2(G)$ satisfying $H\varphi_n = \lambda\varphi_n$ and they are pairwise orthogonal.

We begin with case (a), due to Corollary 4.16 and since (G, v_0) has sub-exponential growth, the set Λ_2 (of (4.6)) is dense in $\sigma(H)$. Then by assumption (a) there is a sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ corresponding to generalized eigenfunctions $\varphi_n\colon V(G) \to \mathbb{C}$ such that $\lambda_n \neq \lambda_m$ for $m \neq n$, $\lim_{n\to\infty} \lambda_n = \lambda$ and φ_n

has sub-exponential growth. Then either infinitely many of the φ_n are unbounded or only finitely many of them are unbounded

If infinitely many of the φ_n are unbounded, there is no loss of generality in assuming that each φ_n is unbounded (otherwise pass to a subsequence). Then Proposition 4.12 asserts that $\lambda_n \in \sigma_\infty(H_n)$ for some $\mathcal{R}$-limit H_n of H or there is a $\gamma > 1$ such that for all $C > 0$, there exists a $u \in V(G)$ with $|\varphi(u)| \geq C\gamma^{|u|}$. However, the second assertion cannot hold as φ_n is sub-exponentially bounded. Hence, $\lambda_n \in \sigma_\infty(H_n)$ holds for some $H_n \in \mathcal{R}(H)$. Since $\lambda_n \to \lambda$, Corollary 4.7 yields $\lambda \in \bigcup_{H' \in \mathcal{R}(H)} \sigma_\infty(H')$ finishing the proof.

If only finitely many of the φ_n's are unbounded, there is no loss of generality in assuming that φ_n is bounded for each $n \in \mathbb{N}$ (otherwise pass to a subsequence). By Proposition 4.10 either infinitely many of the λ_n satisfy $\lambda_n \in \sigma_\infty(H_n)$ for some $H_n \in \mathcal{R}(H)$ or only finitely many. In the first case Corollary 4.7 finishes the proof as before. In the second case, Proposition 4.10 asserts that for n large enough

$$|\varphi_n(u)| \leq C_n \cdot \gamma_n^{-|u|}, \quad u \in G.$$

Thus, $\varphi_n \in \ell^2(G)$ and in particular, they are orthogonal as H is self-adjoint and $\lambda_n \neq \lambda_m$ for $m \neq n$. This is the same situation as in (b) (where the orthogonality of the eigenfunctions is assumed). We proceed proving the claim of the theorem in this last case which will also prove it for the case (b).

Set $\psi_n := \frac{\varphi_n}{\|\varphi_n\|_2}$ for each $n \in \mathbb{N}$. Then ψ_n is still an eigenfunction of H corresponding to the eigenvalue λ_n ($\lambda_n = \lambda$ in (b)) and the functions $\{\psi_n\}_{n\in\mathbb{N}}$ are pairwise orthogonal. Let $u_n \in V(G)$ be the vertex such that $\|\psi_n\|_\infty = |\psi_n(u_n)|$. We have two cases, either $\{u_n\}_{n\in\mathbb{N}}$ (monotonically, by passing to a subsequence) converges to infinity or $\{u_n\}_{n\in\mathbb{N}} \subseteq B_r(v)$ for some $r > 0$, $v \in V(G)$ and every $n \in \mathbb{N}$. In the first case, when $\{u_n\}_{n\in\mathbb{N}}$ converges to infinity, then

$$\max_{u \in B_n(u_n)} |\psi_n(u)| \leq \|\psi_n\|_\infty = |\psi_n(u_n)|$$

holds by construction. Thus, Proposition 4.6 yields that there exists an $\mathcal{R}$-limit (H', G', v_0') being a limit point of (H, G, u_n) and a generalized eigenfunction $0 \neq \psi \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ.

It is left to treat the case that $\{u_n\}_{n\in\mathbb{N}} \subseteq B_r(v)$ for some $r > 0$, $v \in V(G)$ and every $n \in \mathbb{N}$. Since $\{\psi_n\}_{n\in\mathbb{N}}$ are pairwise orthogonal, they converge weakly to zero, see e.g. [41, Theorem II.6]. Since $\|\psi_n\|_\infty = |\psi_n(u_n)|$ and $u_n \in B_r(v)$ for all $n \in \mathbb{N}$, the weak convergence to zero yields $\lim_{n\to\infty} \|\psi_n\|_\infty = 0$. For $k \in \mathbb{N}$ define

$$q_{\ell,k}^{(n)} := \max_{\ell(k-1) \leq |u| < \ell k} |\psi_n(u)|.$$

Since $\psi_n \in \ell^2(G)$, we derive $\lim_{\ell\to\infty} q_{\ell,k}^{(n)} = 0$. Define

$$L_{n,k} := \{\ell \in \mathbb{N}\colon q_{\ell,k}^{(n)} \geq \max_{i\geq\ell} q_{i,k}^{(n)}\}.$$

Since $\lim_{\ell\to\infty} q_{\ell,k}^{(n)} = 0$, we conclude that $L_{n,k}$ is infinite for all $n \in \mathbb{N}$. Next, we will prove the following lemma that together with Lemma 4.8 will conclude the proof.

Lemma 4.17. *Suppose we are in the setting as described before. For each $k \in \mathbb{N}$ there are two sequences $\{n_j\}_{j\in\mathbb{N}}$ (with $n_j \to \infty$) and $\{\ell_j\}_{j\in\mathbb{N}}$ (with $\ell_j \to \infty$) such that for each $j \in \mathbb{N}$, $\ell_j \in L_{n_j,k}$ and $q_{\ell_j-1,k}^{(n_j)} \leq 2q_{\ell_j,k}^{(n_j)}$.*

Before proving Lemma 4.17, let us explain why it finishes the proof. By construction, ψ_n is a generalized eigenfunction of H corresponding to the eigenvalue λ_n satisfying $0 \neq \psi_n \in \ell^\infty(G)$ and $\lim_{n\to\infty} \lambda_n = \lambda$. Lemma 4.17 implies that for each $k \in \mathbb{N}$, there are subsequences $n_j \to \infty$ and $\ell_j \to \infty$ satisfying $\ell_j \in L_{n_j,k}$ and $q_{\ell_j-1,k}^{(n_j)} \leq 2q_{\ell_j,k}^{(n_j)}$. The condition $\ell_j \in L_{n_j,k}$ leads to $q_{\ell_j,k}^{(n_j)} \geq q_{\ell_j+1,k}^{(n_j)}$. Thus, the constraint (b) of Lemma 4.8 is satisfied with $s = 2$. Hence, this lemma implies that there is an $\mathcal{R}$-limit (H', G', v_0') of H and a generalized eigenfunction $0 \neq \varphi' \in \ell^\infty(G')$ of H' corresponding to the eigenvalue λ proving Theorem 2.2. □

Proof of Lemma 4.17. Let us fix $k \in \mathbb{N}$ and introduce the following notation $q_\ell^{(n)} := q_{\ell,k}^{(n)}$ and $L_n := L_{n,k}$. Assume by contradiction that for all $\{n_j\}_j$ with $n_j \to \infty$, and $\{\ell_j\}_j$ with $\ell_j \to \infty$, there exists $j \in \mathbb{N}$ such that

$$\ell_j \notin L_{n_j} \quad \text{or} \quad q_{\ell_j-1}^{(n_j)} > 2q_{l_j}^{(n_j)}. \tag{4.7}$$

We first prove that this is equivalent to ask that there exist $n_0, \ell_0 \in \mathbb{N}$ such that

$$q_\ell^{(n)} < \max_{i\geq\ell} q_i^{(n)} \quad \text{or} \quad q_{\ell-1}^{(n)} > 2q_\ell^{(n)} \quad \text{for all } n \geq n_0, \ell \geq \ell_0. \tag{4.8}$$

That (4.8) implies (4.7) is straightforward. In order to show the implication (4.7) $\Longrightarrow$ (4.8), assume by contradiction that (4.7) holds but (4.8) does not hold. First note that $\ell \in L_n$ is equivalent to $q_\ell^{(n)} \geq \max_{i\geq\ell} q_i^{(n)}$. Let $A_{n,\ell}$ denotes the statement

$$(A_{n,\ell}) \qquad q_\ell^{(n)} \geq \max_{i\geq\ell} q_i^{(n)} \quad \text{and} \quad q_{\ell-1}^{(n)} \leq 2q_\ell^{(n)}.$$

Since (4.8) does not hold, we conclude

$$\text{for all } n_0, \ell_0 \in \mathbb{N} \text{ there exist } n \geq n_0, \ell \geq \ell_0 \text{ such that } A_{n,l} \text{ hold true.} \tag{4.9}$$

Then we can iteratively define a sequence (n_k, ℓ_k) with $n_k \to \infty$ and $\ell_k \to \infty$ such that A_{n_k,ℓ_k} is true for each $k \in \mathbb{N}$. More precisely, let (n_1, ℓ_1) be such that $1 \le \ell_1$, $1 \le n_1$ and A_{ℓ_1,n_1} is true, which is possible by (4.9). Suppose now we have (n_k, ℓ_k) for $1 \le k \le m$ such that A_{n_k,ℓ_k} is true. By (4.9), there is an $n > n_m$ and $\ell > \ell_m$ such that $A_{n,\ell}$ is true. Set $n_{m+1} := n$ and $\ell_{m+1} := \ell$. By construction $n_k \to \infty$ and $\ell_k \to \infty$ and A_{n_k,ℓ_k} is true for all $k \in \mathbb{N}$. This contradicts (4.7). Thus, we have proven that (4.7) and (4.8) are equivalent.

By our assumption and the previous considerations (4.8) holds. Let $\ell_0, n_0 \in \mathbb{N}$ be chosen according to (4.8). Furthermore, fix $n \ge n_0$, and $\ell \in L_n$ such that $\ell \ge \ell_0$ (exists since L_n is infinite). Then (4.8) leads to $q_{\ell-1}^{(n)} > 2q_\ell^{(n)}$. Thus,

$$q_{\ell-1}^{(n)} > 2q_\ell^{(n)} \ge \max_{i \ge \ell} q_i^{(n)}$$

follows implying that also $\ell - 1 \in L_n$. Since $\sharp L_n = \infty$, the latter considerations imply that $[\ell_0, \infty) \cap \mathbb{N} \subseteq L_n$. Thus, $q_{\ell-1}^{(n)} > 2q_\ell^{(n)}$ holds for all $\ell \ge \ell_0$ by (4.8). Altogether, we derive that

$$q_\ell^{(n)} < \frac{1}{2} q_{\ell-1}^{(n)} < \frac{1}{2^2} q_{\ell-2}^{(n)} < \cdots < \frac{1}{2^{\ell-\ell_0}} q_{\ell_0}^{(n)} \le \frac{1}{2^{\ell-l_0}} \|\psi_n\|_\infty$$

holds for all $n \ge n_0$ and $\ell \ge \ell_0$. Hence, for $v \in \mathcal{V}(G)$ with $|v| = r \ge \ell_0 k$, the estimates

$$|\psi_n(v)| \le q_{\lfloor r/k \rfloor}^{(n)} \le \frac{1}{2^{\lfloor r/k \rfloor - \ell_0}} \|\psi_n\|_\infty \le \|\psi_n\|_\infty 2^{\ell_0} e^{-\frac{ln(2)}{k} r}, \quad n \ge n_0,$$

are deduced where $\lfloor r/k \rfloor$ denotes the largest integer j satisfying $j \le \frac{r}{k}$. Recall that $\mathcal{S}_r(v_0)$ denotes the sphere of radius r and center v_0 in G. Since (G, v_0) has sub-exponential growth rate, there is a constant $C_k > 0$ such that $\sharp \mathcal{S}_r(v_0) < C_k \gamma_k^r$ for all $r \in \mathbb{N}$ where $\gamma_k := e^{\frac{ln(2)}{2k}} > 1$, see Definition 2.1. Then a short computation yields

$$\begin{aligned}\sum_{v \in V(G): |v| \ge \ell_0 k} |\psi_n(v)|^2 &\le \|\psi_n\|_\infty 2^{\ell_0} \sum_{r \ge \ell_0 k} \sharp \mathcal{S}_r(v_0) e^{-\frac{ln(2)}{k} r} \\ &\le \|\psi_n\|_\infty 2^{\ell_0} C_k \sum_{r \ge \ell_0 k} \gamma_k^{-r}.\end{aligned}$$

The latter sum is convergent by the root test as $\gamma_k^{-1} < 1$. Furthermore, the cardinality of all vertices $v \in V(G)$ with $|v| < \ell_0 k$ is finite as G is d-bounded graph. Hence,

$$\|\psi_n\|_2^2 = \sum_{v \in V(G)} |\psi_n(v)|^2 = \sum_{v \in V(G): |v| < \ell_0 k} |\psi_n(v)|^2 + \sum_{v \in V(G): |v| \ge \ell_0 k} |\psi_n(v)|^2 \le \widetilde{C}_k \|\psi_n\|_\infty$$

follows for a suitable constant $\widetilde{C}_k > 0$. Since $\lim_{n\to\infty} \|\psi_n\|_\infty = 0$, we derive $\lim_{n\to\infty} \|\psi_n\|_2^2 = 0$, a contradiction as $\|\psi_n\|_2 = 1$. Thus, Lemma 4.17 is proven. □

4.6. Proof of Proposition 2.3. Next we show that the inclusion of Theorem 2.2 can also be strict for graphs with sub-exponential growth rate. This is done by providing a specific example motivated by the considerations made in Section 2 about the d-regular tree. This construction is inspired by an example given in [14]. In order to do so a more general construction of so called chain graphs is introduced next.

Definition 4.18. Let $(G_k)_{k\in\mathbb{N}}$ be a sequence of finite graphs, $v_k^1, v_k^2 \in \mathcal{V}(G_k)$ for each $k \in \mathbb{N}$ and $\{k_\ell\}_{\ell\in\mathbb{N}} \subseteq \mathbb{N}$ be an increasing sequence. Then the corresponding *chain graph* is defined as follows:

- begin with the graph $\mathbb{N}$, namely $V = \mathbb{N}$ and $E = \{(n, n+1) \mid n \in \mathbb{N}\}$;
- for each $k \in \{k_\ell\}_{\ell=1}^\infty$, replace the vertex k with the graph G_k;
- the edges $(k-1, k)$, $(k, k+1)$ are replaced with the edges $(k-1, v_k^1)$ and $(v_k^2, k+1)$;
- in case that $k = k_\ell = k_{\ell-1} + 1$ the edge $(k-1, k)$ is replaced with the edge (v_{k-1}^2, v_k^1).

Proof of Proposition 2.3. Fix $d > 2$ and let T_d be the d-regular tree with root v_0, and denote by G_k the finite subgraph $B_k(v_0)$ in T_d. Let $\{k_\ell\}_{\ell=1}^\infty$ be the sequence defined by $k_\ell := d^{\ell+1}$. For each $\ell \in \mathbb{N}$, let v_ℓ^1, v_ℓ^2 be two vertices in G_ℓ of maximal distance from each other. Then let G^d be the corresponding chain graph, see Definition 4.18 and a sketch in Figure 4.1.

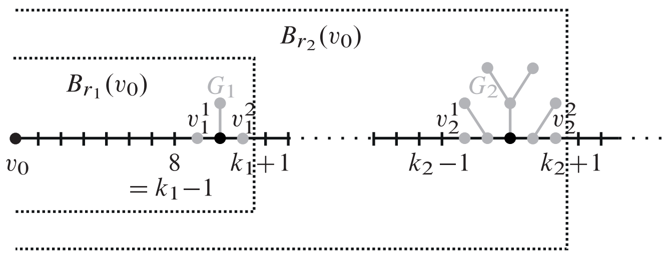

Figure 4.1. The graph G^d for $d = 3$.

Let A_{G^d} and A_{T_d} be the adjacency operator on G^d and T_d respectively, namely it is the Schrödinger operator of the form (1.1) with $W(v) := \deg(v)$. Next, we show that G^d with root $v_0 = 1$ is of sub-exponential growth.

A short computation gives that $\sharp G_k = 1+d\frac{(d-1)^k-1}{d-2} < \frac{d(d-1)^k}{d-2}$. Furthermore, $\delta_\ell := \operatorname{dist}(v_{\ell-1}^2, v_\ell^1) = d^{\ell+1} - d^\ell = (d-1)d^\ell$ holds implying $\sharp G_k < \delta_\ell$. Define $r_\ell := \operatorname{dist}(v_0, v_\ell^2)$. Then $\delta_\ell < r_\ell - r_{\ell-1}$ holds. Next, we will show $\sharp B_{r_\ell}(v_0) < 2r_\ell, \ell \in \mathbb{N}$, via induction. For the base case, we have

$$\sharp B_{r_1}(v_0) = k_1 - 1 + \sharp G_1 = d^2 + d < 2d^2 + 4 = 2(k_1 + 1) = 2r_1.$$

The induction step is deduced by the estimate

$$\sharp B_{r_\ell}(v_0) = \sharp B_{r_{\ell-1}}(v_0) + \delta_\ell + \sharp G_\ell < \sharp B_{r_{\ell-1}}(v_0) + 2\delta_\ell < 2r_\ell$$

invoking the induction hypothesis and the previous considerations. With this at hand, we derive by the definition of G^d that

$$\sharp B_{r_\ell+j}(v_0) = \sharp B_{r_\ell}(v_0) + j < 2(r_\ell + j), \quad 0 \le j \le k_{\ell+1} - k_\ell + 1.$$

On the other hand, decreasing the radius of $B_{r_\ell}(v_0)$ reduces the number of vertices in the ball by at least two in each step since $d > 2$. Thus,

$$\sharp B_{r_\ell - j}(v_0) < \sharp B_{r_\ell}(v_0) - \sum_{n=1}^{j} 2 < 2(r_\ell - j), \quad 1 \le j \le 2\ell - 1,$$

follows.

Putting all together, we derive $\sharp B_k(v_0) < 2k$ for each $k \in \mathbb{N}$. Furthermore, G^d is a sub-graph of T_d. Thus, [32, Corollary 4.5] together with [33, Section 7.c] lead to

$$\sigma_{\mathrm{ess}}(A_{G^d}) \subseteq \sigma(A_{G^d}) \subseteq [-2\sqrt{d-1}, 2\sqrt{d-1}] = \sigma(A_{T_d}).$$

On the other hand, one of the $\mathcal{R}$-limits of A_{G^d} is the adjacency operator on the d-regular tree, for which $[-d, d] \subseteq \sigma_\infty(A_{T_d})$, see [9, Theorem 1.1]. As a consequence, we conclude

$$\emptyset \neq [-d, d] \setminus [-2\sqrt{d-1}, 2\sqrt{d-1}] \subseteq \bigcup_{H' \in \mathcal{R}(A_{G^d})} \sigma_\infty(H') \setminus \sigma_{\mathrm{ess}}(A_{G^d})$$

since $d > 2$. □

5. Proof of Theorem 2.4

The proof follows from the results mentioned above, and the following theorem from [8].

Theorem 5.1 ([8, Theorem 2]). *Let G be an infinite, connected d-bounded graph and H a Schrödinger operator on G, then*

$$\bigcup_{H'\in\mathcal{R}(H)} \sigma(H') \subseteq \sigma_{\mathrm{ess}}(H).$$

With this at hand, we can prove Theorem 2.4.

Proof of Theorem 2.4. We already know from Theorem 5.1 and Theorem 2.2 that

$$\bigcup_{H'\in\mathcal{R}(H)} \sigma(H') \subseteq \sigma_{\mathrm{ess}}(H) \subseteq \bigcup_{H'\in\mathcal{R}(H)} \sigma_\infty(H').$$

Thus, it suffices to prove for each $H' \in \mathcal{R}(H)$, that

$$\sigma_\infty(H') \subseteq \sigma(H').$$

In order to do so, recall the notions introduced in Section 4.4. Since G admits a uniform sub-exponential growth also any $\mathcal{R}$-limit (H', G', v_0') of (H, G, v_0) is of uniform sub-exponential growth. Indeed, for each $\gamma > 1$ there exists a constant $C > 0$ (independent of the root) such that $\sharp\mathcal{S}_r(u) < C\gamma^r$ for all $u \in V(G)$ and $r \in \mathbb{N}$, see Definition 2.1. Let $\gamma > 1$ be arbitrary and $r \in \mathbb{N}$. Since (H', G', v_0') is an $\mathcal{R}$-limit of (H, G, v_0), there is a vertex $u \in V(G)$ such that the subgraphs $B_r(v_0')$ and $B_r(u)$ are isomorphic (see Definition 4.1). In particular, the spheres contained in these balls have the same cardinality, namely $\sharp\mathcal{S}_r(u) = \sharp\mathcal{S}_r(v_0')$. Since $\sharp\mathcal{S}_r(u) < C\gamma^r$ where the corresponding constant C is independent of $u \in V(G)$, we derive $\sharp\mathcal{S}_r(v_0') < C\gamma^r$. Thus, the map $V(G') \ni v' \mapsto e^{-\alpha\ \mathrm{dist}(v',v_0')}$ is an element $\ell^2(G')$ for any $\alpha > 0$.

Let $\lambda \in \sigma_\infty(H')$, then by definition there is a bounded generalized eigenfunction φ of H' corresponding to the eigenvalue λ. By the previous considerations

$$V(G') \ni v' \longmapsto e^{-\alpha\,\mathrm{dist}(v',v_0')}\varphi(v')$$

is an element of $\ell^2(G')$ as φ is uniformly bounded. Thus, φ is sub-exponentially bounded implying $\lambda \in \sigma(H')$ by Proposition 4.13. □

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Siegfried Beckus, Institut für Mathematik, Universität Potsdam, Campus Golm, Haus 9, Karl-Liebknecht-Straße 24–25, 14476 Potsdam, Germany

e-mail: beckus@uni-potsdam.de

Latif Eliaz, Mathematics Department, Technion – Israel Institute of Technology, Haifa, Israel

e-mail: latif@campus.technion.ac.il