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A theorem on the multiplicity of the singular spectrum of a general Anderson-type Hamiltonian

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Abstract. In this work, we study the multiplicity of the singular spectrum for operators of the form $A^\omega = A + \sum_n \omega_n C_n$ on a separable Hilbert space $\mathcal{H}$, where A is a self-adjoint operator and $\{C_n\}_n$ is a countable collection of non-negative finite-rank operators. When $\{\omega_n\}_n$ are independent real random variables with absolutely continuous distributions, we show that the multiplicity of the singular spectrum is almost surely bounded above by the maximum algebraic multiplicity of the eigenvalues of the operator $\sqrt{C_n}(A^\omega - z)^{-1}\sqrt{C_n}$ for all n and almost all (z, ω). The result is optimal in the sense that there are operators for which the bound is achieved. We also provide an effective bound on the multiplicity of the singular spectrum for some special cases.

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Contents

1. Introduction

Spectral theory of random operators is an important field of study, and within it, the Anderson tight binding model and the random Schrödinger operator have gained significant attention. Over the years considerable attention has been devoted to the nature of the spectrum of these operators. But to completely characterize the structure of the operator, information on the multiplicity is also important. Here we pay attention to the multiplicity of the singular spectrum for certain classes of random operators.

One of the well-studied classes of random operators is represented by the Anderson tight binding model. Many results about its spectrum are known, for example, the existence of pure point spectrum for the Anderson tight binding model over the integer lattice [1, 5, 12, 18]. Absolutely continuous spectrum is known to exist for the same model on the Bethe lattice [3, 11, 19] and anti-trees [31]. Other models where the pure point spectrum is known to exist include the random Schrödinger operator [4, 7, 13, 22], the multi-particle Anderson model [2, 6, 21] and magnetic Schrödinger operators [8, 35].

There are important results which also deal with the multiplicity of the singular spectrum. For the Anderson tight binding model, Simon [34], Klein and Molchanov [20] have shown the simplicity of the pure point spectrum. For Anderson-type models when the randomness acts as rank-one perturbations, Jakšić and Last [15, 17] showed that the singular spectrum is simple. For the random Schrödinger operator, in the regime of exponential decay of Green's function, Combes, Germinet, and Klein [9] and Dietlein and Elgart [10] showed that the spectrum is simple. Other work includes [32], where Sadel and Schulz-Baldes provided a multiplicity result for the absolutely continuous spectrum for random Dirac operators with time-reversal symmetry. However, no general results concerning the multiplicity of the spectrum are known. One of the difficulties hindering the derivation of multiplicity results for random Schrödinger operator or multi-particle Anderson model is that the randomness acts as perturbation over an infinite rank operator.

Randomness acting through perturbation by a finite-rank operator is an intermediate model between the Anderson tight binding model and the random Schrödinger operator. Examples of such a random operator are the Anderson dimer/polymer model, the Toeplitz/Hankel random matrix, and the random conductance model. Here we will deal with Anderson-type operators and provide a multiplicity result for the singular spectrum when the randomness acts through perturbation by a non-negative finite-rank operator. This work is similar to that by Jakšić and Last [15, 17] and is a generalization and extension of the work by Mallick [25]. The paper does not answer the question about the multiplicity of singular spectrum for the random Schrödinger operator, but it is a step towards it. The technique involved in the proof does not distinguish between point spectrum and singular continuous spectrum, so the stated results are true for the entire singular spectrum.

For a densely defined self-adjoint operator A with domain $\mathcal{D}(A)$ on a separable Hilbert space $\mathcal{H}$ and a countable collection of non-negative finite-rank operators $\{C_n\}_{n\in\mathcal{N}}$, define the random operator

$$A^\omega = A + \sum_{n\in\mathcal{N}} \omega_n C_n, \tag{1}$$

where $\{\omega_n\}_{n\in\mathcal{N}}$ are independent real random variables with absolutely continuous distributions. Let $(\Omega, \mathcal{B}, \mathbb{P})$ denote the probability space such that ω_n are random variables over Ω. We will assume that

$$A\colon \Omega \longrightarrow \mathcal{S}(\mathcal{H})$$

is an essentially self-adjoint operator-valued random variable.[1] This is a necessary assumption because otherwise there can be multiple self-adjoint extensions for the symmetric operator A^ω. The assumption itself is not too restrictive and a large class of operators satisfies this condition. For example, if A is bounded self-adjoint, $\{C_n\}_n$ are non-negative finite-rank operators satisfying $C_n C_m = C_m C_n = 0$ for any $n \neq m$, and the distributions of the random variables ω_n are supported in some fixed compact set $[-K, K]$, then the operator A^ω is almost surely bounded and self-adjoint. The Anderson polymer/dimer model falls into this category of operators.

For the main result we need to focus on the linear maps

$$G^\omega_{n,n}(z) := P_n (A^\omega - z)^{-1} P_n\colon P_n\mathcal{H} \longrightarrow P_n\mathcal{H}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$, where P_n is the projection onto the range of C_n. Using functional calculus, it is easy to see that the linear operator $G^\omega_{n,n}(z)$ can be viewed as a matrix on $P_n\mathcal{H}$ (after fixing a basis of $P_n\mathcal{H}$) that belongs to the set of matrix-valued Herglotz functions. Using the representation of matrix-valued Herglotz functions (see [14, Theorem 5.4]), we can extract all the properties of the spectral measure over the minimal closed A^ω-invariant subspace containing $P_n\mathcal{H}$.

Let $\mathrm{Mult}^\omega_n(z)$ denote the maximum multiplicity of the roots of the polynomial

$$\det(C_n G^\omega_{n,n}(z) - xI)$$

in the variable x, for $z \in \mathbb{C} \setminus \mathbb{R}$, where C_n and $G^\omega_{n,n}(z)$ are viewed as a linear operator on $P_n\mathcal{H}$, and so I denotes the identity operator on $P_n\mathcal{H}$. Since $C_n > 0$

[1] An "essentially self-adjoint operator" is an operator with a unique self-adjoint extension.

on $P_n\mathcal{H}$, we have

$$\det(C_n G^{\omega}_{n,n}(z) - xI) = \det(\sqrt{C_n}G^{\omega}_{n,n}(z)\sqrt{C_n} - xI),$$

because the similarity transformation preserves the determinant. This is the reason why the algebraic multiplicity of $\sqrt{C_n}(A^{\omega}-z)^{-1}\sqrt{C_n}$ can also be used instead of $C_n G^{\omega}_{n,n}(z)$. With these notations, we state our main result:

Theorem 1.1. *Let A be a densely defined self-adjoint operator with domain $\mathcal{D}(A)$ on a separable Hilbert space $\mathcal{H}$ and $\{C_n\}_{n\in\mathcal{N}}$ be a countable collection of non-negative finite-rank operators. Let P_n denote the projection onto the range of C_n and assume that $\sum_n P_n = I$. Let $\{\omega_n\}_{n\in\mathcal{N}}$ be a sequence of independent real random variables on the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with absolutely continuous distributions. Let A^{ω} given by (1) be a family of essentially self-adjoint operators. Then*

(1) *for any fixed $n \in \mathcal{N}$,*

$$\operatorname*{ess\,sup}_{z\in\mathbb{C}\setminus\mathbb{R}} \mathrm{Mult}^{\omega}_n(z)$$

is constant for almost all ω, and we denote its value by $\mathcal{M}_n$;

(2) *if $\sup_{n\in\mathcal{N}} \mathcal{M}_n < \infty$, then the multiplicity of the singular spectrum of the operator A^{ω} is bounded from above by $\sup_{n\in\mathcal{N}} \mathcal{M}_n$, for almost all ω.*

Remark 1.2. A few observations are in order.

(1) If range$(C_n) \subset \mathcal{D}(A)$ for all n, then the subspace

$$\mathcal{D} := \Big\{ \sum_{i=1}^{N} \phi_i : \phi_i \in \mathrm{range}(C_{n_i}), n_i \in \mathcal{N} \text{ for all } 1 \le i \le N,\ N \in \mathbb{N} \Big\},$$

is dense and is the domain of A^{ω}. If either A is bounded or $\sup_n |\omega_n|\,\|C_n\|$ is finite, then it is easy to show that A^{ω} is essentially self-adjoint.

(2) Note that although $\{C_n\}_n$ are finite-rank operators, a universal upper bound on their ranks does not necessarily exist. A simple example of such an operator is

$$H^{\omega} = \Delta + \sum_{n=0}^{\infty} \omega_n \chi_{\{x:\|x\|_\infty = n\}},$$

defined on the Hilbert space $\ell^2(\mathbb{Z}^d)$, where Δ is the discrete Laplacian and $\chi_{\{x:\|x\|_\infty=n\}}$ is the projection onto the subspace $\ell^2(\{x \in \mathbb{Z}^d : \|x\|_\infty = n\})$.

(3) We assume that $\sum_n P_n = I$ to ensure that the subspace $\sum_n \mathcal{H}^{\omega}_{P_n}$ is dense in $\mathcal{H}$. Here we set

$$\mathcal{H}^{\omega}_{P_n} = \overline{\langle f(A^{\omega})\phi \colon f \in C_c(\mathbb{R}), \phi \in P_n\mathcal{H}\rangle},$$

where $\overline{\langle S\rangle}$ denotes the closed linear span of the set S. Without this condition, infinite multiplicity could easily be achieved. For example, consider the Hilbert space $\bigoplus^2 \ell^2(\mathbb{Z})$, and define the operator

$$H^{\omega} = \Big(\Delta + \sum_{n\in\mathbb{Z}} \omega_n \chi_{\{nN,\ldots,(n+1)N-1\}}\Big) \oplus \Big(\sum_{n\in\mathbb{Z}} x_n \,|\delta_n\rangle\,\langle\delta_n|\Big),$$

where $\{x_n\}_{n\in\mathbb{Z}}$ is a fixed sequence and $\{\omega_n\}_{n\in\mathbb{Z}}$ are independent real random variables with absolutely continuous distributions. Notice that the first operator is an Anderson-like operator with simple point spectrum, but the second operator can have arbitrary multiplicity depending upon the sequence $\{x_n\}_n$.

Remark 1.3. To understand the conclusion of the theorem, consider the following examples.

(1) Consider the operator

$$H^{\omega} = \tilde{\Delta} + \sum_{n\in\mathbb{Z}} \omega_n P_n,$$

on the Hilbert space $\ell^2(\mathbb{Z}\times\{0,\ldots,N\})$, where

$$(\tilde{\Delta}u)(x,y) = u(x+1,y) + u(x-1,y) \quad \text{for all } (x,y) \in \mathbb{Z}\times\{0,\ldots,N\}$$

and the projections P_n are given by

$$(P_n u)(x,y) = \begin{cases} u(x,y) & \text{if } x = n,\\ 0 & \text{if } x \neq n.\end{cases}$$

First observe that the subspace

$$\mathcal{H}_k = \{u \in \ell^2(\mathbb{Z}\times\{0,\ldots,N\}) \colon u(x,y) = 0 \text{ for all } x \in \mathbb{Z}, y \neq k\}$$

is H^{ω}-invariant and $\{(H^{\omega},\mathcal{H}_k)\}_{k=0}^{N}$ are all unitarily equivalent. So any singular spectrum has multiplicity N. When the random variables $\{\omega_n\}_n$ are i.i.d., there are results [24, 23, 33] which show that $(H^{\omega},\mathcal{H}_0)$ has pure point spectrum (hence singular spectrum). It is easy to show that the matrix $G^{\omega}_{n,n}(z)$ is of the form $f(z)I$, where f is a Herglotz function and I is the identity on $\mathbb{C}^N$.

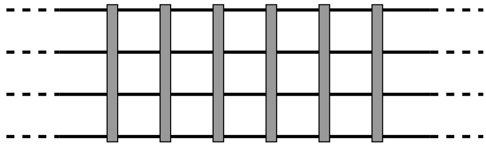

Figure 1. The operator described in the remark is visualized here for $N = 3$. The operator $\widetilde{\Delta}$ is the adjacency operator of the graph $\mathbb{Z} \times \{0, \ldots, 3\}$, where the edges are denoted by the dark lines. The shaded region denotes the support of the projections.

(2) Consider the operator

$$H^\omega = \widetilde{\Delta} + \sum_{(n,m)\in\mathbb{N}^2} \omega_{(n,m)} P_{(n,m)}$$

on the Hilbert space $\ell^2(\mathbb{N} \times \mathbb{N})$, where

$$(\widetilde{\Delta} u)(x, y) = \begin{cases} u(2, y) & \text{if } x = 1, \\ u(x+1, y) + u(x-1, y) & \text{if } x \neq 1, \end{cases} \quad \text{for all } (x, y) \in \mathbb{N} \times \mathbb{N},$$

and the projections $P_{(n,m)}$ are given by

$$P_{(n,m)} = \sum_{k=2^{n(m-1)}}^{2^{nm}-1} |\delta_{(n,k)}\rangle\langle\delta_{(n,k)}|$$

(where $|\phi\rangle\langle\phi|$ denotes the projection on the one-dimensional subspace generated by ϕ).

In this example $P_{(n,m)}(H^\omega - z)^{-1} P_{(n,m)}$ is diagonal (with respect to the Dirac basis $\{\delta_{(n,m)} : n, m \in \mathbb{N}\}$), and it is readily seen that

$$\sup_{(n,m)\in\mathbb{N}} \mathcal{M}_{(n,m)} = \infty.$$

Similarly to the previous example, the subspaces

$$\mathcal{H}_k = \{u \in \ell^2(\mathbb{N} \times \mathbb{N}) : u(x, y) = 0 \text{ for all } x \in \mathbb{N}, y \neq k\} \quad \text{for all } k \in \mathbb{N},$$

are invariant under the action of H^ω. The operators $\{(H^\omega, \mathcal{H}_k)\}_{k=2^m}^{2^{m+1}-1}$ are unitarily equivalent for any $m \in \mathbb{N}$. Consequently, the singular spectrum of H^ω has infinite multiplicity.

Thus, the conclusion of the theorem is optimal in the sense that there are random operators A^ω such that the multiplicity of their singular spectrum is $\sup_{n\in\mathcal{N}} \mathcal{M}_n$.

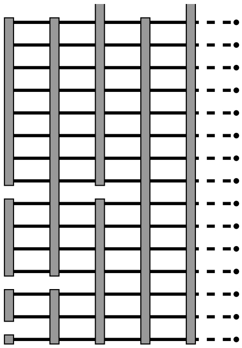

Figure 2. The operator described in the remark is visualized here. The operator $\widetilde{\Delta}$ is the adjacency operator of the graph $\mathbb{N}^2$ where the edges are denoted by the dark lines. The shaded region represents the support of the projections.

The main technique in the proof is the study of the behavior of the singular spectrum under a perturbation by a single non-negative operator. This is done via the resolvent identity, and so properties of matrix-valued Herglotz functions play an essential role. The steps of the proof will be further explained in Section 1.1. In general, this kind of result fails to hold without perturbation, and spectral averaging [7, Corollary 4.2] plays an important role. Since matrix-valued Herglotz functions are the primary tool, Poltoratskii's theorem [29] is used to obtain and characterize the singular measure.

It should be noted that our result (Theorem 1.1) extends the work of Jakšić and Last [15, 17], Naboko, Nichols, and Stolz [27], and Mallick [26] in the following way. In the case of Jakšić and Last [15, 17], since the rank of each operator P_n is one, the theorem above establishes the simplicity of the singular spectrum. Naboko, Nichols, and Stolz [27] showed the simplicity of the point spectrum for certain classes of Anderson-type operators on $\mathbb{Z}^d$, and Mallick [26] provided a bound on the multiplicity of the singular spectrum for a similar class of Anderson-type operators on $\mathbb{Z}^d$. In general, it is not possible to compute $G^{\omega}_{n,n}(z)$, and so other methods have to be devised to calculate $\mathcal{M}_n$. The following corollary is a possible way to bound $\mathcal{M}_n$ for certain classes of random operators.

Corollary 1.4. *Suppose that the operator A^ω defined by (1) satisfies the hypotheses of Theorem 1.1 on a separable Hilbert space $\mathcal{H}$. Let* $\operatorname{range}(C_n) \subset \mathcal{D}(A)$ *for all $n \in \mathcal{N}$, and let $M \in \mathbb{R}$ be such that $\sigma(A)$ and $\sigma(A^\omega)$ are subsets of (M, ∞) for almost all ω.*

(1) *If C_n is a finite-rank projection for all n, then the multiplicity of the singular spectrum of the operator A^ω is bounded from above by*

$$\max_{n\in\mathcal{N}} \max_{x\in\sigma(C_n A C_n)} \dim(\ker(C_n A C_n - xI)),$$

where $C_n A C_n$ is viewed as a linear operator on $P_n\mathcal{H}$.

(2) *If C_n is a non-negative finite rank operator for all n, then the multiplicity of the singular spectrum of the operator A^ω is bounded from above by*

$$\max_{n\in\mathcal{N}} \max_{x\in\sigma(C_n)} \dim(\ker(C_n - xI)),$$

where C_n is viewed as a linear operator on $P_n\mathcal{H}$.

Remark 1.5. It should be noted that the above bound is not optimal, but in many cases it can be computed easily. As an example, for the case of Remark 1.2 (1), all we have to do is count the multiplicity of the eigenvalues of the operator $\chi_{S_r}\Delta\chi_{S_r}$, where $S_r = \{x \in \mathbb{Z}^d : \|x\|_\infty = r\}$. For $d = 2$, this operator is same as the Laplacian on a set of $8n$ points arranged on a circle, so the multiplicity of the operator can be at most two. Another simple example is when C_n has simple spectrum; then the singular spectrum of A^ω is almost surely simple.

The corollary should be considered as a generalization of the technique developed in Naboko, Nichols, and Stolz [27]. There the authors used the simplicity of $P_n\Delta P_n$ to conclude the simplicity of the pure point spectrum for a certain type of Anderson operators on $\ell^2(\mathbb{Z}^d)$. Another similar work is [26], where the author bounded $\mathcal{M}_n$ by considering the first few terms of the Neumann series while keeping track of the perturbation.

Using an approach similar to [26], we can show that the singular spectrum of the Anderson-type operator on a Bethe lattice is simple. Let $\mathcal{B} = (V, E)$ denote the infinite tree with root e where each vertex has K neighbors. Set $K > 2$ so that the tree is not isomorphic to $\mathbb{Z}$. Consider random operators of the form

$$H^\omega = \Delta_{\mathcal{B}} + \sum_{x\in J} \omega_x \chi_{\widetilde{\Lambda}(x)}, \tag{2}$$

where $\Delta_{\mathcal{B}}$ is the adjacency operator of $\mathcal{B}$, and

$$\widetilde{\Lambda}(x) = \{y \in V : d(e,x) \le d(e,y) \text{ and } d(x,y) < l_x\},$$

for some $l: V \to \mathbb{N}$. One assumes that the indexing set $J \subset V$ is such that $\bigcup_{x\in J} \widetilde{\Lambda}(x) = V$ and

$$\widetilde{\Lambda}(x) \cap \widetilde{\Lambda}(y) = \emptyset \quad \text{for all } x \neq y \in J.$$

The random variables $\{\omega_x\}_{x\in J}$ are real-valued and independent, with absolutely continuous distributions. With these notation we have:

Theorem 1.6. *On a Bethe lattice $\mathcal{B}$ with $K > 2$, consider a family of random operators H^ω given by (2), where $\{\omega_x\}_{x\in J}$ are i.i.d. random variables with an absolutely continuous distribution with bounded support. Then the singular spectrum of the operator H^ω is almost surely simple.*

It can be seen that the spectrum of $\chi_{\widetilde{\Lambda}(x)}\Delta_{\mathcal{B}}\chi_{\widetilde{\Lambda}(x)}$ has non-trivial multiplicity (is exponential in terms of the diameter of $\widetilde{\Lambda}(x)$). So, the above result is not a consequence of the previous corollary.

1.1. Structure of the proof. The rest of the article is divided into four parts. In Section 2, we set up the notations and collect the results that will be used throughout. Section 3 deals with single perturbation results. Section 4 contains the proof of Theorem 1.1, which is divided into Lemma 4.1 and Lemma 4.2. Finally, in Section 5, we prove Corollary 1.4 and Theorem 1.6.

The proof of Theorem 1.1 is divided into three parts. First, we look at the operator $H_\lambda := H + \lambda C$, where H is a densely defined essentially self-adjoint operator and C is a non-negative finite-rank operator. Since all the results are obtained by resorting to properties of the Borel–Stieltjes transform, there is a set $S \subset \mathbb{R}$, independent of λ, of full Lebesgue measure, where all the analysis will be carried out. As a consequence of spectral averaging (see Lemma 2.1), it is enough to concentrate on S as long as we are working on the subspace

$$\mathcal{H}_C^\lambda = \overline{\langle f(H_\lambda)\phi \colon f \in C_c(\mathbb{R})\ \&\ \phi \in C\mathcal{H}\rangle}.$$

By spectral averaging, the spectrum of H_λ restricted to $\mathcal{H}_C^\lambda$ is contained in S for almost all λ. In Section 3, we establish a certain inclusion relation between singular subspaces. We show that for any finite-rank projection Q, the closed H_λ-invariant Hilbert subspace $\widetilde{\mathcal{H}}_Q^\lambda \subseteq \mathcal{H}_Q^\lambda$, such that the spectrum of H_λ restricted to $\widetilde{\mathcal{H}}_Q^\lambda$ is singular and is contained in S, is a subset of the singular subspace of $\mathcal{H}_C^\lambda$. This inclusion is shown in Lemma 3.1. This is the reason why the multiplicity of the singular subspace for $\mathcal{H}^\omega_{\sum_{i=1}^N P_{n_i}}$ does not depend on N. Lemma 4.2 uses this fact to get a bound on the multiplicity of the singular spectrum for $\mathcal{H}^\omega_{\sum_{i=1}^N P_{n_i}}$ for any finite collection of numbers $\{n_i\}_i$. Finally, a global bound on the multiplicity of the singular spectrum is obtained by observing that the set $\bigcup_{N\in\mathbb{N}} \mathcal{H}^\omega_{\sum_{i=1}^N P_{n_i}}$ is dense for any enumeration of $\mathcal{N}$.

Lemma 4.1 provides the first conclusion of the theorem and also establishes the relationship between $\mathcal{M}_n$ and the multiplicity of the singular spectrum for $\mathcal{H}^{\omega}_{P_n}$. The proof is mostly a consequence of properties of polynomial algebra where the coefficients of the polynomial under consideration are holomorphic function on $\mathbb{C} \setminus \mathbb{R}$. Part of the work is to establish a relation between the multiplicity of the singular spectrum and the multiplicity of the spectrum of the operators $\sqrt{C_n} G^{\omega}_{n,n}(z) \sqrt{C_n}$, which is achieved through the resolvent equation. After choosing a basis, we end up with matrix equations for functions that are holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Since we are only dealing with matrices, the multiplicity of the spectrum of $\sqrt{C_n} G^{\omega}_{n,n}(z) \sqrt{C_n}$ can be computed through its characteristic equation, and so we have polynomial equations where the coefficients are polynomials in the matrix elements. Most of the work is to show that it is independent of a single perturbation. The preceding argument also proves the independence on z. Indeed, the matrix elements are holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$, and so any non-zero polynomial can vanish only on a set of Lebesgue measure zero. Then by induction we show that $\mathrm{Mult}^{\omega}_n(z)$ is independent of any finite collection of random variables $\{\omega_{p_i}\}_i$. The Kolmogorov 0-1 law provides the stated result.

Finally, in Section 5, we prove Corollary 1.4 and Theorem 1.6. This is mostly done by writing the matrix $G^{\omega}_{n,n}(z)$ in a particular form. For the corollary, using the fact that range$(C_n) \subset \mathcal{D}(A)$, the matrix $C_n^{-\frac{1}{2}} A C_n^{-\frac{1}{2}}$ is well defined on $P_n \mathcal{H}$, and we have to estimate the number of eigenvalues of

$$C_n^{-\frac{1}{2}} A C_n^{-\frac{1}{2}} + \mu C^{-1}$$

that lie at a distance of at most $O(1/\mu)$ one another for $\mu \gg 1$. The corollary just deals with two extreme cases. For Theorem 1.6, most of the work is to show that for a tree (of finite depth), the adjacency operator perturbed at all the leaf nodes has simple spectrum. Then the particular structure of $G^{\omega}_{n,n}(z)$ yields the desired conclusion.

Even though $G^{\omega}_{n,m}(z)$ are defined over $\mathbb{C} \setminus \mathbb{R}$, part of the proof of Lemma 3.1 is carried out on $\mathbb{C}^+$ itself. The main problem that can arise upon restricting to $\mathbb{C}^+$ come from F. and R. Riesz's theorem [30], which states that *if the Borel–Stieltjes transform of a measure vanishes on $\mathbb{C}^+$, then the measure is equivalent to the Lebesgue measure* (see [17, Theorem 2.2] for a proof). This problem is avoided by using the fact that in case $G^{\omega}_{n,m}(z)$ vanishes $z \in \mathbb{C}^+$, one can repeat the proof by switching to $z \in \mathbb{C}^-$ and replacing $E + \iota\epsilon$ by $E - \iota\epsilon$ whenever necessary.

2. Preliminaries

In this section we introduce the notations and results used in the rest of the work. Mostly we will deal with the linear operators

$$G^{\omega}_{n,m}(z) := P_n(A^{\omega} - z)^{-1}P_m \colon P_m\mathcal{H} \longrightarrow P_n\mathcal{H} \quad \text{for all } n, m \in \mathcal{N},$$

which are well defined because of the assumption that A^{ω} is essentially self-adjoint. Here P_n denotes the orthogonal projection onto the range of C_n. We denote by

$$\mathcal{H}^{\omega}_{P_n} := \overline{\langle f(A^{\omega})\phi \colon f \in C_c(\mathbb{R}) \ \&\ \phi \in P_n\mathcal{H}\rangle},$$

the minimal closed A^{ω}-invariant subspace containing $P_n\mathcal{H}$. All the results are stated in a basis-independent form, but sometimes an explicit basis is fixed so that $G^{\omega}_{n,m}(z)$ can be viewed as a matrix-valued functions.

We mostly focus on a single perturbation, proceeding as follows. For $p \in \mathcal{N}$ we set $A^{\omega,\lambda}_p = A^{\omega} + \lambda C_p$ and define

$$G^{\omega,\lambda}_{p,n,m}(z) = P_n(A^{\omega,\lambda}_p - z)^{-1}P_m$$

as before. Using resolvent equation we have

$$G^{\omega,\lambda}_{p,p,p}(z) = G^{\omega}_{p,p}(z)(I + \lambda C_p G^{\omega}_{p,p}(z))^{-1}, \tag{3}$$

$$G^{\omega,\lambda}_{p,n,m}(z) = G^{\omega}_{n,m}(z) - \lambda G^{\omega}_{n,p}(z)(I + \lambda C_p G^{\omega}_{p,p}(z))^{-1}C_p G^{\omega}_{p,m}(z). \tag{4}$$

Another way to write (3) and (4) is

$$(I - \lambda C_p G^{\omega,\lambda}_{p,p,p}(z))(I + \lambda C_p G^{\omega}_{p,p}(z)) = I, \tag{5}$$

$$\begin{aligned} G^{\omega,\lambda}_{p,n,m}(z) &= G^{\omega}_{n,m}(z) - \lambda G^{\omega}_{n,p}(z)C_p G^{\omega}_{p,m}(z) \\ &\quad + \lambda^2 G^{\omega}_{n,p}(z)C_p G^{\omega,\lambda}_{p,p,p}(z)C_p G^{\omega}_{p,m}(z). \end{aligned} \tag{6}$$

Either of these equations will be used, depending on the situation. It should be noted that the identity operator in equations (3), (4), and (5) is the identity map on $P_p\mathcal{H}$. For a fixed basis in each of the subspaces $P_n\mathcal{H}$, using [15, Proposition 2.1] (which follows from a property of the Borel–Stieltjes transform) for every matrix element of $G^{\omega}_{n,m}(z)$, the limit

$$G^{\omega}_{n,m}(E \pm \iota 0) := \lim_{\epsilon \downarrow 0} G^{\omega}_{n,m}(E \pm \iota\epsilon)$$

exists for almost all E with respect to Lebesgue measure and for any $n, m \in \mathcal{N}$. Therefore, the linear operator $G^{\omega}_{n,m}(E \pm \iota 0)$ is well defined for almost all E and any $n, m \in \mathcal{N}$.

Using (5) we observe that for any $E \in \mathbb{R}$ such that $G^{\omega}_{p,p}(E \pm \iota 0)$ exists and for any function $f : (0, \infty) \to \mathbb{C}$ such that $\lim_{\epsilon \downarrow 0} f(\epsilon) = 0$, we have

$$\lim_{\epsilon \downarrow 0} f(\epsilon)(I - \lambda C_p G^{\omega,\lambda}_{p,p,p}(E \pm \iota\epsilon))(I + \lambda C_p G^{\omega}_{p,p}(E \pm \iota\epsilon)) = 0$$
$$\implies (\lim_{\epsilon \downarrow 0} f(\epsilon) C_p G^{\omega,\lambda}_{p,p,p}(E \pm \iota\epsilon) C_p)(C_p^{-1} + \lambda G^{\omega}_{p,p}(E \pm \iota 0)) = 0,$$

and similarly

$$(C_p^{-1} + \lambda G^{\omega}_{p,p}(E \pm \iota 0))(\lim_{\epsilon \downarrow 0} f(\epsilon) C_p G^{\omega,\lambda}_{p,p,p}(E \pm \iota\epsilon) C_p) = 0.$$

This implies that

$$\begin{aligned} \mathrm{range}((\lim_{\epsilon \downarrow 0} f(\epsilon) C_p G^{\omega,\lambda}_{p,p,p}(E \pm \iota\epsilon) C_p)) &\subseteq \ker(C_p^{-1} + \lambda G^{\omega}_{p,p}(E \pm \iota 0)) \\ &\subseteq \ker(\Im G^{\omega}_{p,p}(E \pm \iota 0)), \end{aligned} \tag{7}$$

which is used to determine the singular spectrum. One of the consequences of the fact that $\pm \Im G^{\omega}_{p,p}(E \pm \iota 0) \geq 0$ is the equality

$$G^{\omega}_{k,p}(E \pm \iota 0)\phi = G^{\omega}_{p,k}(E \pm \iota 0)^* \phi \quad \text{for all } \phi \in \ker(\pm \Im G^{\omega}_{p,p}(E \pm \iota 0)), \tag{8}$$

which plays an important role in the proof of Lemma 3.1.

Since most of the analysis is done using a single perturbation, one of the important results concerns the spectral averaging; we refer to [7, Corollary 4.2] for its proof. Here we will use the following version:

Lemma 2.1. *Let $E_\lambda(\cdot)$ be the spectral family for the operator $A_\lambda = A + \lambda C$, where A is a self-adjoint operator and C is a non-negative compact operator. For any set $M \subset \mathbb{R}$ of Lebesgue measure zero, $\sqrt{C} E_\lambda(M) \sqrt{C} = 0$ for almost all λ, with respect to Lebesgue measure.*

Since the set of points E where $\lim_{\epsilon \downarrow 0} G^{\omega}_{n,m}(E \pm \iota\epsilon)$ does not exist for some $n, m \in \mathbb{N}$, has Lebesgue measure zero, Lemma 2.1 guarantees that we can ignore this set in our analysis as long as we are only focusing on $A^{\omega,\lambda}_p$-invariant subspaces containing $P_p \mathcal{H}$. Another important result is

Lemma 2.2. *For a σ-finite positive measure space $(X, \mathcal{B}, m)$ and a collection of $\mathcal{B}$-measurable functions $a_i : X \to \mathbb{C}$ and $b_i : X \to \mathbb{C}$, define*

$$f(\lambda) = \frac{1 + \sum_{n=1}^{N} a_n(x)\lambda^n}{1 + \sum_{n=1}^{N} b_n(x)\lambda^n}.$$

Then the set

$$\Lambda_f = \{\lambda \in \mathbb{C} \colon m(x \in X \colon f(\lambda, x) = 0) > 0\}$$

is countable.

Its proof can be found in [25, Lemma 2.1]. This lemma ensures that the linear operator $G^{\omega,\lambda}_{p,p,p}(z)$ is well defined for almost all λ. This is the case because $G^{\omega,\lambda}_{p,p,p}(z)$ and $G^{\omega}_{p,p}(z)$ are related through the equation (3), and so the set $\{E \colon \det(I + \lambda C_p G^{\omega}_{p,p}(E \pm \iota 0)) = 0\}$ should have Lebesgue measure zero, otherwise the analysis will fail. This is also the set which contains the singular spectrum of the operator $A^{\omega,\lambda}_p$ restricted to $\mathcal{H}^{\omega}_{P_p}$ (it is easy to see that the space $\mathcal{H}^{\omega}_{P_p}$ is invariant under the action of $A^{\omega,\lambda}_p$).

The next result is Poltoratskii's theorem, which is the main tool allowing us to handle the singular part of the spectrum. Since we only deal with finite measures, we will denote the Borel–Stieltjes transform $F_\mu \colon \mathbb{C}^+ \to \mathbb{C}^+$ of a Borel measure μ by

$$F_\mu(z) = \int \frac{d\mu(x)}{x - z}.$$

For $f \in L^1(\mathbb{R}, d\mu)$, let $f\mu$ be the unique measure associated with the linear functional $C_c(\mathbb{R}) \ni g \mapsto \int g(x) f(x) d\mu(x)$. The version of the Poltoratskii's theorem we will use reads:

Lemma 2.3. *Let* μ *be a complex-valued Borel measure on* $\mathbb{R}$ *and let* $f \in L^1(\mathbb{R}, d\mu)$. *Then*

$$\lim_{\epsilon \downarrow 0} \frac{F_{f\mu}(E + \iota\epsilon)}{F_\mu(E + \iota\epsilon)} = f(E)$$

for a.e. E *with respect to the singular part of* μ.

The proof of this lemma can be found in [16]. With these results at hand, we can now prove our results.

3. Single perturbation results

This section is devoted to the case of a single perturbation. Lemma 3.1 will play an important role in proving the main result. For this section, we adopt a different notation, because it is not necessary to keep track of all the random variables $\{\omega_n\}_n$.

Let H be a densely defined self-adjoint operator on a separable Hilbert space $\mathcal{H}$ and C_1 be a non-negative finite-rank operator. Set $H_\lambda = H + \lambda C_1$ and let P_1 be the orthogonal projection onto the range of C_1. For any projection Q, let

$$\mathcal{H}_Q^\lambda := \overline{\langle f(H_\lambda)\psi \colon \psi \in Q\mathcal{H} \ \&\ f \in C_c(\mathbb{R})\rangle},$$

be the minimal closed H_λ-invariant subspace containing the range of Q. Let σ_1^λ denote the trace measure $\operatorname{tr}(P_1 E^{H_\lambda}(\cdot))$, where $E^{H_\lambda}(\cdot)$ is the spectral projection for the operator H_λ. The subscript "sing" will be used to denote the singular part of a measure whenever necessary. The main result of this section is the following:

Lemma 3.1. *Let Q be a finite-rank projection and let $\{e_i\}_i$ be an orthonormal basis of $Q\mathcal{H} + P_1\mathcal{H}$. Consider the set*

$$S = \{E \in \mathbb{R} \colon \langle e_i, (H - E \mp \iota 0)^{-1} e_j\rangle \text{ exists and finite}\},$$

and denote E_{sing}^λ the spectral measure on the singular part of the spectrum of the operator H_λ. Then

$$E_{\text{sing}}^\lambda(S)\mathcal{H}_Q^\lambda \subseteq E_{\text{sing}}^\lambda(S)\mathcal{H}_{P_1}^\lambda$$

for almost all λ with respect to the Lebesgue measure.

Remark 3.2. The spectral averaging result (Lemma 2.1) shows that $\sigma_1^\lambda(\mathbb{R} \setminus S) = 0$ for almost all λ with respect to the Lebesgue measure, so it is actually not necessary to write S on the right-hand side of the above inclusion. But $E_{\text{sing}}^\lambda(\mathbb{R} \setminus S)\mathcal{H}_Q^\lambda$ can be non-trivial.

Proof. In view of Lemma A.2, it is enough to show that

$$E_{\text{sing}}^\lambda(S)\mathcal{H}_{e_i}^\lambda \subseteq E_{\text{sing}}^\lambda(S)\mathcal{H}_{P_1}^\lambda,$$

where $\mathcal{H}_{e_i}^\lambda$ is the minimal closed H_λ-invariant subspace containing e_i. This is because applying Lemma A.2 for the operator $E_{\text{sing}}^\lambda(S)H_\lambda$ will then give the singular subspaces in the conclusion of the lemma.

Using the resolvent equation

$$(H_\lambda - z)^{-1} - (H - z)^{-1} = -\lambda(H_\lambda - z)^{-1} C_1 (H - z)^{-1}$$

and similarly

$$\begin{aligned}(H_\lambda - z)^{-1} &= (H - z)^{-1} - \lambda(H - z)^{-1} C_1 (H_\lambda - z)^{-1}\\ &= (H - z)^{-1} - \lambda(H - z)^{-1} C_1 (H - z)^{-1}\\ &\quad + \lambda^2 (H - z)^{-1} C_1 (H_\lambda - z)^{-1} C_1 (H - z)^{-1},\end{aligned}$$

we have

$$\begin{aligned}\langle e_i, (H_\lambda - z)^{-1} e_i\rangle &= \langle e_i, (H-z)^{-1} e_i\rangle - \lambda\langle e_i, (H-z)^{-1} C_1 (H-z)^{-1} e_i\rangle \\ &\quad + \lambda^2 \langle e_i, (H-z)^{-1} C_1 (H_\lambda - z)^{-1} C_1 (H-z)^{-1} e_i\rangle.\end{aligned} \tag{9}$$

Let $\{e_{1i}\}_{i=1}^{r_1}$, where $r_1 = \dim(P_1\mathcal{H})$, be an orthonormal basis of $P_1\mathcal{H}$ (so that the elements e_{1i} are linear combinations of $\{e_i\}_i$); hence $G^\lambda_{1,1}(z) = P_1(H_\lambda - z)^{-1} P_1$ is a matrix for this basis. Also, set

$$G_{i,1}(z) = \begin{pmatrix} \langle e_i, (H-z)^{-1} e_{11}\rangle \\ \langle e_i, (H-z)^{-1} e_{12}\rangle \\ \vdots \\ \langle e_i, (H-z)^{-1} e_{1r_1}\rangle \end{pmatrix}^t \quad \text{and} \quad G_{1,i}(z) = \begin{pmatrix} \langle e_{11}, (H-z)^{-1} e_i\rangle \\ \langle e_{12}, (H-z)^{-1} e_i\rangle \\ \vdots \\ \langle e_{1r_1}, (H-z)^{-1} e_i\rangle \end{pmatrix}.$$

Then equation (9) can be written as

$$\begin{aligned}\langle e_i, (H_\lambda - z)^{-1} e_i\rangle &= \langle e_i, (H-z)^{-1} e_i\rangle - \lambda G_{i,1}(z) C_1 G_{1,i}(z) \\ &\quad + \lambda^2 G_{i,1}(z) C_1 G^\lambda_{1,1}(z) C_1 G_{1,i}(z).\end{aligned}$$

Using the fact that the left-hand side is the Borel–Stieltjes transform of the measure $\langle e_i E^{H_\lambda}(\cdot) e_i\rangle$, the support of the singular part lies in the set of points $E \in \mathbb{R}$ where

$$\lim_{\epsilon\downarrow 0} \frac{1}{\langle e_i, (H_\lambda - E - \iota\epsilon)^{-1} e_i\rangle} = 0.$$

We don't need to consider the case when $\langle e_i, (H_\lambda - z)^{-1} e_i\rangle = 0$ for all $z \in \mathbb{C}^+$ because by F. and R. Riesz's theorem [30], the measure $\langle e_i, E^{H_\lambda}(\cdot) e_i\rangle$ is absolutely continuous. But by the definition of the set S, we have that $G_{i,1}(E \pm \iota 0)$, $G_{1,i}(E \pm \iota 0)$, and $\langle e_i, (H - E \mp \iota 0)^{-1} e_i\rangle$ exist for all $E \in S$. So the singular part of $\langle e_i, E^{H_\lambda}(\cdot) e_i\rangle$ can lie on the set $\mathbb{R} \setminus S$ or on the set of points $E \in S$ where $\lim_{\epsilon\downarrow 0}(\mathrm{tr}(G^\lambda_{1,1}(E + \iota\epsilon)))^{-1} = 0$.

For the points $E \in S$ where $\lim_{\epsilon\downarrow 0}(\mathrm{tr}(G^\lambda_{1,1}(E + \iota\epsilon)))^{-1} = 0$,

$$\begin{aligned}&\lim_{\epsilon\downarrow 0} \frac{\langle e_i, (H_\lambda - E - \iota\epsilon)^{-1} e_i\rangle}{\mathrm{tr}(G^\lambda_{1,1}(E + \iota\epsilon))} \\ &\quad = \lambda^2 G_{i,1}(E + \iota 0) C_1 \Big(\lim_{\epsilon\downarrow 0} \frac{G^\lambda_{1,1}(E + \iota\epsilon)}{\mathrm{tr}(G^\lambda_{1,1}(E + \iota\epsilon))}\Big) C_1 G_{1,i}(E + \iota 0).\end{aligned}$$

Using (8), we have

$$\begin{aligned}&\lim_{\epsilon\downarrow 0} \frac{\langle e_i, (H_\lambda - E - \iota\epsilon)^{-1} e_i\rangle}{\mathrm{tr}(G^\lambda_{1,1}(E + \iota\epsilon))} \\ &\quad = \lambda^2 [C_1 G_{1,i}(E + \iota 0)]^* \Big(\lim_{\epsilon\downarrow 0} \frac{G^\lambda_{1,1}(E + \iota\epsilon)}{\mathrm{tr}(G^\lambda_{1,1}(E + \iota\epsilon))}\Big) [C_1 G_{1,i}(E + \iota 0)].\end{aligned} \tag{10}$$

Since $G_{1,1}^{\lambda}(\cdot)$ is a matrix-valued Herglotz function for a positive operator-valued measure (it is the Borel transform of $P_1 E^{H_\lambda}(\cdot) P_1$), the Herglotz representation theorem for matrix-valued measures (see [14, Theorem 5.4]) provides a matrix-valued function $M_1^{\lambda} \in L^1(\mathbb{R}, \sigma_1^{\lambda}, M_{\mathrm{rank}(P_1)}(\mathbb{C}))$ such that

$$G_{1,1}^{\lambda}(z) = \int \frac{1}{x - z} M_1^{\lambda}(x) d\sigma_1^{\lambda}(x)$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. Using Poltoratskii's theorem (Lemma 2.3),

$$\lim_{\epsilon \downarrow 0} \frac{1}{\mathrm{tr}(G_{1,1}^{\lambda}(E + \iota\epsilon))} G_{1,1}^{\lambda}(E + \iota\epsilon) = M_1^{\lambda}(E)$$

for almost all E with respect to $\sigma_{1,\mathrm{sing}}^{\lambda}$. Since the measure $P_1 E^{H_\lambda}(\cdot) P_1$ is non-negative, the matrix-valued function $M_1^{\lambda}(E) \geq 0$ for almost all E with respect to $\sigma_{1,\mathrm{sing}}^{\lambda}$.

Let $U_1^{\lambda}(E)$ be the unitary matrix which diagonalizes $M_1^{\lambda}(E)$, i.e.,

$$U_1^{\lambda}(E) M_1^{\lambda}(E) U_1^{\lambda}(E)^* = \mathrm{diag}(f_j^{\lambda}; 1 \leq j \leq r_1),$$

where some of the f_j^{λ} can be zero. Using the Hahn–Hellinger theorem (see [28, Theorem 1.34]), U_1^{λ} can be chosen to be a Borel measurable unitary matrix-valued function. Since we are only interested on the singular part, we set $U_1^{\lambda}(E) = 0$ for E not in the support of $\sigma_{1,\mathrm{sing}}^{\lambda}$ and define $\psi_j^{\lambda} = U_1^{\lambda}(H_\lambda)^* e_{1j}$. Now observe that

$$\begin{aligned}
&\langle \psi_k^{\lambda}, (H_\lambda - z)^{-1} \psi_l^{\lambda} \rangle \\
&\quad = \int \frac{1}{x - z} \langle \psi_k^{\lambda}, E^{H_\lambda}(dx) \psi_l^{\lambda} \rangle \\
&\quad = \int \frac{1}{x - z} \langle U_1^{\lambda}(x)^* e_{1k}, E^{H_\lambda}(dx) U_1^{\lambda}(x)^* e_{1l} \rangle \\
&\quad = \int \frac{1}{x - z} \sum_{p,q} \langle U_1^{\lambda}(x)^* e_{1k}, e_{1p} \rangle \langle e_{1q}, U_1^{\lambda}(x)^* e_{1l} \rangle \langle e_{1p}, E^{H_\lambda}(dx) e_{1q} \rangle \\
&\quad = \sum_{p,q} \int \frac{1}{x - z} \langle U_1^{\lambda}(x)^* e_{1k}, e_{1p} \rangle \langle e_{1q}, U_1^{\lambda}(x)^* e_{1l} \rangle \langle e_{1p}, E^{H_\lambda}(dx) e_{1q} \rangle,
\end{aligned}$$

and so using Poltoratskii's theorem (Lemma 2.3) we get

$$\begin{aligned}
&\lim_{\epsilon \downarrow 0} \frac{\langle \psi_k^{\lambda}, (H_\lambda - E - \iota\epsilon)^{-1} \psi_l^{\lambda} \rangle}{\mathrm{tr}(G_{1,1}^{\lambda}(E + \iota\epsilon))} \\
&\quad = \sum_{p,q} \langle U_1^{\lambda}(E)^* e_{1k}, e_{1p} \rangle \langle e_{1q}, U_1^{\lambda}(E)^* e_{1l} \rangle \Big(\lim_{\epsilon \downarrow 0} \frac{\langle e_{1p}, (H_\lambda - E - \iota\epsilon)^{-1} e_{1q} \rangle}{\mathrm{tr}(G_{1,1}^{\lambda}(E + \iota\epsilon))} \Big) \\
&\quad = \langle e_{1k}, U_1^{\lambda}(E) M_1^{\lambda}(E) U_1^{\lambda}(E)^* e_{1l} \rangle = f_k^{\lambda}(E) \delta_{k,l}
\end{aligned}$$

for almost all E with respect to $\sigma^{\lambda}_{1,\mathrm{sing}}$. By construction of ψ^{λ}_j, the spectral measure $\langle \psi^{\lambda}_j, E^{H_\lambda}(\cdot)\psi^{\lambda}_j\rangle$ is purely singular with respect to the Lebesgue measure, so above computation implies $\langle \psi^{\lambda}_k, (H_\lambda - z)^{-1}\psi^{\lambda}_l\rangle = 0$ for all z for $k \neq l$, which implies that the measure $\langle \psi^{\lambda}_k, E^{H_\lambda}(\cdot)\psi^{\lambda}_l\rangle$ is zero, and in particular we have $\mathcal{H}^{\lambda}_{\psi^{\lambda}_k} \perp \mathcal{H}^{\lambda}_{\psi^{\lambda}_l}$ for $k \neq l$.

Next, using the resolvent equation, we obtain

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{\langle \psi^{\lambda}_k, (H_\lambda - E - \iota\epsilon)^{-1} e_i\rangle}{\mathrm{tr}(G^{\lambda}_{1,1}(E+\iota\epsilon))} \\
&\quad = \lim_{\epsilon\downarrow 0} -\lambda \frac{\langle \psi^{\lambda}_k, (H_\lambda - E - \iota\epsilon)^{-1} C_1 (H - E - \iota\epsilon)^{-1} e_i\rangle}{\mathrm{tr}(G^{\lambda}_{1,1}(E+\iota\epsilon))} \\
&\quad = -\lambda f^{\lambda}_k(E)\langle e_{1k}, U^{\lambda}_1(E) C_1 G_{1,i}(E+\iota 0)\rangle,
\end{aligned} \tag{11}$$

for a.e. E with respect to $\sigma^{\lambda}_{1,\mathrm{sing}}$. Using Lemma A.1 and (11) in equation (10), we conclude that

$$\lim_{\epsilon\downarrow 0} \frac{\langle e_i, (H_\lambda - E - \iota\epsilon)^{-1} e_i\rangle}{\mathrm{tr}(G^{\lambda}_{1,1}(E+\iota\epsilon))} = \sum_j |(Q^{\lambda}_{\psi^{\lambda}_j} e_i)(E)|^2 f^{\lambda}_j(E)$$

for almost all E with respect to $\sigma^{\lambda}_{1,\mathrm{sing}}$, where $Q^{\lambda}_{\psi^{\lambda}_j} e_i$ is the projection of e_i on the Hilbert subspace $\mathcal{H}^{\mu}_{\psi^{\lambda}_j}$. So for $g \in C_c(\mathbb{R})$, we can write

$$\langle e_i, E^{\lambda}_{\mathrm{sing}}(S) g(H_\lambda) e_i\rangle = \sum_j \int g(E) |(Q^{\lambda}_{\psi^{\lambda}_j} e_i)(E)|^2 f^{\lambda}_j(E)\, d\sigma^{\lambda}_{1,\mathrm{sing}}(E),$$

which implies that the projection of $E^{\lambda}_{\mathrm{sing}}(S)e_i$ onto $\mathcal{H}^{\lambda}_{P_1}$ is an isometry, hence

$$E^{\lambda}_{\mathrm{sing}}(S)\mathcal{H}^{\lambda}_{e_i} \subseteq E^{\lambda}_{\mathrm{sing}}(S)\mathcal{H}^{\lambda}_{P_1}.$$

The lemma follows by an application of Lemma A.2. □

4. Proof of Theorem 1.1

The proof of the main result is divided into Lemma 4.1 and Lemma 4.2. It should be noted that the conclusion of Lemma 4.1 is similar to the conclusion reached by combining [26, Lemma 2.2 and Lemma 2.1]. This section deals with the operator A^{ω} itself and so the notations introduced in Section 2 will be used. Following the notations from the previous section, set $\mathcal{H}^{\omega}_P$ to be the minimal closed A^{ω}-invariant subspace containing the range of the projection P.

Lemma 4.1. *For any $n \in \mathbb{N}$,*

$$\mathcal{M}_n^\omega := \operatorname*{ess\,sup}_{z\in\mathbb{C}\setminus\mathbb{R}} \mathrm{Mult}_n^\omega(z)$$

is almost surely constant; denote it by $\mathcal{M}_n$. The multiplicity of the singular spectrum for $\mathcal{H}_{P_n}^\omega$ is bounded above by $\mathcal{M}_n$.

Proof. First we prove that $\mathcal{M}_n^\omega$ is independent of ω. This is done using the Kolmogorov 0-1 law. So the first step is to show that $\mathcal{M}_n^\omega$ is independent of any finite collection of random variables $\{\omega_{p_i}\}_i$.

Following the notations from Section 2, set $A_p^{\omega,\lambda} = A^\omega + \lambda C_p$ for $p \in \mathbb{N}\setminus\{n\}$, we have the equation (4)

$$G_{p,n,n}^{\omega,\lambda}(z) = G_{n,n}^\omega(z) - \lambda G_{n,p}^\omega(z)(I + \lambda C_p G_{p,p}^\omega(z))^{-1} C_p G_{p,n}^\omega(z).$$

Looking at $G_{i,j}^\omega(z)$ as a matrix, observe that

$$\begin{aligned}\tilde{g}_{\lambda,z}^\omega(x) &= \det(C_n G_{p,n,n}^{\omega,\lambda}(z) - xI)\\ &= \det(C_n G_{n,n}^\omega(z) - \lambda C_n G_{n,p}^\omega(z)(I + \lambda C_p G_{p,p}^\omega(z))^{-1} C_p G_{p,n}^\omega(z) - xI)\\ &= \frac{p_l^\omega(z,\lambda)x^l + p_{l-1}^\omega(z,\lambda)x^{l-1} + \cdots + p_0^\omega(z,\lambda)}{\det(C_p^{-1} + \lambda G_{n,n}^\omega(z))},\end{aligned}$$

where $l = \operatorname{rank}(P_n)$. Here $\{p_i^\omega(z,\lambda)\}_{i=0}^l$ are polynomials in the elements of the matrices $\{G_{i,j}^\omega(z)\}_{i,j\in\{n,p\}}$ and λ. We are not interested in the denominator, so let us set

$$g_{\lambda,z}^\omega(x) = p_l^\omega(z,\lambda)x^l + p_{l-1}^\omega(z,\lambda)x^{l-1} + \cdots + p_0^\omega(z,\lambda).$$

The maximum algebraic multiplicity of $G_{p,n,n}^{\omega,\lambda}(z)$ is at most k if the function

$$\mathcal{F}_{\lambda,z}^{\omega,k}(x) = \gcd\left(g_{\lambda,z}^\omega(x), \frac{dg_{\lambda,z}^\omega}{dx}(x), \ldots, \frac{d^k g_{\lambda,z}^\omega}{dx^k}(x)\right)$$

is constant with respect to x. Using the fact that

$$\gcd(f_1(x),\ldots,f_m(x)) = \gcd(f_1(x),\ldots,f_{m-2}(x),\gcd(f_{m-1}(x),f_m(x)))$$

and Euclid's algorithm for polynomials, we get

$$\mathcal{F}_{\lambda,z}^{\omega,k}(x) = q_{k,0}^\omega(\lambda,z) + q_{k,1}^\omega(\lambda,z)x + \cdots + q_{k,s}^\omega(\lambda,z)x^s,$$

where $\{q_{k,i}^\omega(\lambda,z)\}_{i=0}^s$ are rational polynomials of $\{p_i^\omega(z,\lambda)\}_i$. We need to examine the numerators of $q_{k,i}^\omega$, which we denote by $\tilde{q}_{k,i}^\omega$. Since $\{\tilde{q}_{k,i}^\omega\}$ are polynomials in the matrix elements $\{G_{i,j}^\omega(z)\}_{i,j\in\{n,p\}}$ and λ, we can write

$$\tilde{q}_{k,i}^\omega(\lambda,z) = \sum_j a_{k,i,j}^\omega(z)\lambda^j,$$

where $\{a^{\omega}_{k,i,j}\}_{k,i,j}$ are holomorphic functions on $\mathbb{C} \setminus \mathbb{R}$. Hence, for each i the functions $\{\tilde{q}^{\omega}_{k,i}\}$ are well defined for $(\lambda, z) \in \mathbb{R} \times (\mathbb{C} \setminus \mathbb{R})$.

Now suppose $\mathcal{M}^{\omega}_n = k$. Then $q^{\omega}_{k,0}(0,\cdot) \neq 0$ and $q^{\omega}_i(0,\cdot) = 0$ identically, which implies $a^{\omega}_{k,i,0}(\cdot) = 0$ for $i \neq 0$. On other hand, the assumption $\mathcal{M}^{\omega}_n = k$ implies that $\mathcal{F}^{\omega,k-1}_{0,z}(x)$ is a non-constant polynomial (with respect to x), hence $q^{\omega}_{k-1,i_1}(0,\cdot) \neq 0$ for some $i_1 > 0$. Hence, there exists indices i_1, j_1 such that $a^{\omega}_{k-1,i_1,j_1}(\cdot) \neq 0$, which implies that $\mathcal{F}^{\omega,k-1}_{\lambda,z}(x)$ is a non-constant polynomial (with respect to x) for almost all λ. This implies $G^{\omega,\lambda}_{p,n,n}(z)$ have multiplicity strictly greater than $k-1$. Setting $\tilde{\omega}^p$ to be such that $\tilde{\omega}^p_k = \omega_k$ for $k \neq p$ and $\tilde{\omega}^p_p = \omega_p + \lambda$, gives $\mathcal{M}^{\omega}_n \leq \mathcal{M}^{\tilde{\omega}^p}_n$ for almost all λ. Since $\mathcal{M}^{\tilde{\omega}^p}_n$ can be at most $\operatorname{rank}(P_n)$, this implies $\mathcal{M}^{\tilde{\omega}^p}_n$ is independent of λ.

For the case $p = n$, we can follow above argument, but a simpler reasoning is available. Indeed, observe that

$$\begin{aligned}
\tilde{g}^{\omega}_{\lambda,z}(x) &= \det(C_n G^{\omega,\lambda}_{n,n,n}(z) - xI) \\
&= \det(C_n G^{\omega}_{n,n}(z)(I + \lambda C_n G^{\omega}_{n,n}(z))^{-1} - xI) \\
&= \frac{\det((1 - x\lambda) C_n G^{\omega}_{n,n}(z) - xI)}{\det(I + \lambda C_n G^{\omega}_{n,n}(z))},
\end{aligned}$$

which implies that the roots of $\tilde{g}^{\omega}_{\lambda,z}(x)$ are given by $\frac{x_i^{\omega,z}}{1+\lambda x_i^{\omega,z}}$, where $x_i^{\omega,z}$ are the roots of $\tilde{g}^{\omega}_{0,z}(x)$. The multiplicities of the roots are also preserved in this process. We conclude that $\mathcal{M}^{\tilde{\omega}^n}_n$ is independent of λ.

Now repeating the proof inductively for a collection of sites $\{p_i\}_{i=1}^N$ proves the independence of $\mathcal{M}^{\omega}_n$ from the random variables $\{\omega_{p_i}\}_{i=1}^N$. Hence, using the Kolmogorov 0-1 law, $\mathcal{M}^{\omega}_n$ is independent of ω.

Assume that $\mathcal{M}_n = k$, which implies that the maximum multiplicity for the matrix $G^{\omega}_{n,n}(z)$ is k for almost every z. Using above argument for the polynomial

$$g^{\omega}_z(x) = \det(C_n G^{\omega}_{n,n}(z) - xI) = (-x)^l + (-x)^{l-1} p^{\omega}_{l-1}(z) + \cdots + p^{\omega}_0(z),$$

we see that the function

$$\gcd\left(g^{\omega}_z(x), \frac{dg^{\omega}_z}{dx}(x), \ldots, \frac{d^k g^{\omega}_z}{dx^k}(x)\right)$$

is a rational polynomial in the matrix elements of $G^{\omega}_{n,n}(z)$, and so the numerator is holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Since it is non-zero for a set of positive Lebesgue measure, it is non-zero for almost all $z \in \mathbb{C} \setminus \mathbb{R}$, which implies

$$k = \operatorname*{ess\,sup}_{E \in \mathbb{R}} \{\text{maximum multiplicity of roots of } \det(C_n G^{\omega}_{n,n}(E \pm \iota 0) - xI)\}. \tag{12}$$

Now we address the second assertion of the lemma, i.e., that multiplicity of the singular spectrum on $\mathcal{H}^{\omega}_{P_n}$ is bounded by $\mathcal{M}_n$. Consider the set

$$S = \{E \in \mathbb{R}\colon \text{maximum multiplicity of the roots of } \det(C_n G^{\omega}_{n,n}(E \pm \iota 0) - xI) \text{ is } k\}, \tag{13}$$

which as shown above has full Lebesgue measure.

Using the spectral theorem (see [25, Theorem A.3]) for the operator $A^{\omega,\lambda}_n = A^{\omega} + \lambda C_n$ gives

$$(\mathcal{H}^{\omega,\lambda,n}_{P_n}, A^{\omega,\lambda}_n) \cong (L^2(\mathbb{R}, P_n E^{A^{\omega,\lambda}_n}(\cdot) P_n, P_n\mathcal{H}), M_{\mathrm{Id}}).$$

Here $E^{A^{\omega,\lambda}_n}$ is the spectral measure for $A^{\omega,\lambda}_n$ and $\mathcal{H}^{\omega,\lambda,n}_{Q}$ is the minimal closed $A^{\omega,\lambda}_n$-invariant space containing the subspace $Q\mathcal{H}$ for a projection Q. Since the measure $P_n E^{A^{\omega,\lambda}_n}(\cdot) P_n$ is absolutely continuous with respect to the trace measure $\sigma^{\omega,\lambda}_n(\cdot) = \mathrm{tr}(P_n E^{A^{\omega,\lambda}_n}(\cdot) P_n)$, after a choice of basis, there exists a non-negative matrix-valued function $M^{\omega,\lambda}_n \in L^1(\mathbb{R}, \sigma^{\omega,\lambda}_n, M_{\mathrm{rank}(P_n)}(\mathbb{C}))$ such that

$$P_n E^{A^{\omega,\lambda}_n}(dx) P_n = M^{\omega,\lambda}_n(x) \sigma^{\omega,\lambda}_n(dx),$$

and applying Poltoratskii's theorem (Lemma 2.3) we see that

$$\lim_{\epsilon \downarrow 0} \frac{1}{\mathrm{tr}(G^{\omega,\lambda}_{n,n,n}(E + \iota\epsilon))} G^{\omega,\lambda}_{n,n,n}(E + \iota\epsilon) = M^{\omega,\lambda}_n(E)$$

for almost all E with respect to $\sigma^{\omega,\lambda}_{n,\mathrm{sing}}$. Here we are assuming that $\sigma^{\omega,\lambda}_n$ has a non-trivial singular component, so $G^{\omega,\lambda}_{n,n,n}(z) \neq 0$ for almost all $z \in \mathbb{C}^+$. In much the same ways as in (5), we also have

$$(I + \lambda C_n G^{\omega}_{n,n}(z))(I - \lambda C_n G^{\omega,\lambda}_{n,n,n}(z)) = I,$$

which implies (using steps involved in the derivation of (7)) that

$$(I + \lambda C_n G^{\omega}_{n,n}(E + \iota 0))\Big[C_n \lim_{\epsilon \downarrow 0} \frac{1}{\mathrm{tr}(G^{\omega,\lambda}_{n,n,n}(E + \iota\epsilon))} G^{\omega,\lambda}_{n,n,n}(E + \iota\epsilon)\Big] = 0,$$

for E such that $\lim_{\epsilon\downarrow 0} \frac{1}{\mathrm{tr}(G^{\omega,\lambda}_{n,n,n}(E+\iota\epsilon))} = 0$. Consequently,

$$(I + \lambda C_n G^{\omega}_{n,n}(E + \iota 0)) C_n M^{\omega,\lambda}_n(E) = 0$$

for almost all E with respect to $\sigma^{\omega,\lambda}_{n,\mathrm{sing}}$. Using the fact that $\sigma^{\omega,\lambda}_n(\mathbb{R} \setminus S) = 0$ for almost all λ and the above equation, which implies that the rank of $M^{\omega,\lambda}_n(E)$ is bounded above by the dimension of the kernel $(I + \lambda C_n G^{\omega}_{n,n}(E + \iota 0))$, which in turn is bounded above by k over the set S (as it follows from (13)), we conclude that the multiplicity of the singular spectrum for $A_n^{\omega,\lambda}$ is bounded above by k over $\mathcal{H}^{\omega,\lambda,n}_{P_n}$.

This completes the proof, as the conclusion is true for almost all (ω, λ). □

Note that the above lemma establishes a bound for the multiplicity of the singular spectrum for the subspace $\mathcal{H}^{\omega}_{P_n}$, and not on the entire Hilbert space. Lemma 3.1 is used to obtain the final result, which is as follows:

Lemma 4.2. *Under the hypotheses of Theorem 1.1, assume that $\mathcal{M}_n \leq K$ for all $n \in \mathbb{N}$. Then the multiplicity of the singular spectrum of the operator A^{ω} is bounded above by K almost surely.*

Proof. The proof carried out in two steps. First we show that for any finite collection $\{p_i\}_{i=1}^N \subset \mathbb{N}$, the multiplicity of the singular spectrum of the operator A^{ω} restricted to the subspace $\mathcal{H}^{\omega}_{\sum_{i=1}^N P_{p_i}}$ is bounded by K.

Then the proof is completed using the denseness of $\bigcup_{N=1}^{\infty} \mathcal{H}^{\omega}_{\sum_{i=1}^N P_{p_i}}$.

The first part uses induction, so let $\{p_i\}_{i\in\mathbb{N}}$ be an enumeration of the set $\mathbb{N}$. The induction statement $\mathcal{S}_N$ is: *the multiplicity of the singular spectrum of A^{ω} restricted to the subspace $\mathcal{H}^{\omega}_{\sum_{i=1}^N P_{p_i}}$ is at most K.*

For $N = 1$, the conclusion follows from Lemma 4.1, i.e., the multiplicity of the singular spectrum over $\mathcal{H}^{\omega}_{P_{p_1}}$ is at most K.

Now assume $\mathcal{S}_N$ is true, i.e., the multiplicity of the singular spectrum on $\mathcal{H}^{\omega}_{\sum_{i=1}^N P_{p_i}}$ is bounded by K. Before going on to show that $\mathcal{S}_{N+1}$ holds, note that

$$\mathcal{H}^{\omega}_{\sum_{i=1}^{N+1} P_{p_i}} = \mathcal{H}^{\omega}_{\sum_{i=1}^{N} P_{p_i}} + \mathcal{H}^{\omega}_{P_{p_{N+1}}},$$

It is obvious that the right-hand side is a subset of the left-hand side; for the opposite inclusion observe that the right-hand side is dense and closed in the left-hand side.

Now consider the operator $A^{\omega,\lambda}_{p_{N+1}} = A^{\omega} + \lambda C_{p_{N+1}}$. By Lemma 4.1, the multiplicity of singular spectrum of $A^{\omega,\lambda}_{p_{N+1}}$ on $\mathcal{H}^{\omega,\lambda,p_{N+1}}_{P_{p_{N+1}}}$ is bounded by K. By $\mathcal{S}_N$, the multiplicity of singular spectrum for

$$(\mathcal{H}^{\omega,\lambda,p_{N+1}}_{\sum_{i=1}^N P_{p_i}}, A^{\omega,\lambda}_{p_{N+1}})$$

is at most K. Next, by Lemma 3.1, there exists a set S^{ω} of full Lebesgue measure such that

$$E^{A^{\omega,\lambda}_{p_{N+1}}}_{\text{sing}}(S^{\omega})\mathcal{H}^{\omega,\lambda,p_{N+1}}_{\sum_{i=1}^N P_{p_i}} \subseteq E^{A^{\omega,\lambda}_{p_{N+1}}}_{\text{sing}}(S^{\omega})\mathcal{H}^{\omega,\lambda,p_{N+1}}_{P_{p_{N+1}}}.$$

Spectral averaging implies that

$$E_{\mathrm{sing}}^{A_{p_{N+1}}^{\omega,\lambda}}(\mathbb{R}\setminus S^{\omega})\mathcal{H}_{P_{p_{N+1}}}^{\omega,\lambda,p_{N+1}}=\{0\}$$

for almost all λ (with respect to the Lebesgue measure). Now the decomposition

$$\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega,\lambda,p_{N+1}}=E^{A_{p_{N+1}}^{\omega,\lambda}}(S^{\omega})\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega,\lambda,p_{N+1}}\oplus E^{A_{p_{N+1}}^{\omega,\lambda}}(\mathbb{R}\setminus S^{\omega})\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega,\lambda,p_{N+1}}$$

gives

$$\begin{aligned}E_{\mathrm{sing}}^{A_{p_{N+1}}^{\omega,\lambda}}\mathcal{H}_{\sum_{i=1}^{N+1}P_{p_i}}^{\omega,\lambda,p_{N+1}}&=E_{\mathrm{sing}}^{A_{p_{N+1}}^{\omega,\lambda}}\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega,\lambda,p_{N+1}}+E_{\mathrm{sing}}^{A_{p_{N+1}}^{\omega,\lambda}}\mathcal{H}_{P_{p_{N+1}}}^{\omega,\lambda,p_{N+1}}\\&=E_{\mathrm{sing}}^{A_{p_{N+1}}^{\omega,\lambda}}(\mathbb{R}\setminus S^{\omega})\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega,\lambda,p_{N+1}}\oplus E_{\mathrm{sing}}^{A_{p_{N+1}}^{\omega,\lambda}}(S^{\omega})\mathcal{H}_{P_{p_{N+1}}}^{\omega,\lambda,p_{N+1}},\end{aligned}$$

where both subspaces have multiplicity at most K. The supports of the singular spectrum of $A_{p_{N+1}}^{\omega,\lambda}$ restricted to the two subspaces are disjoint, and so assertion $\mathcal{S}_{N+1}$ holds. This completes the first part of the proof.

With the induction completed, note that

$$\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega}\subseteq\mathcal{H}_{\sum_{i=1}^{N+1}P_{p_i}}^{\omega}\quad\text{for all } N\in\mathbb{N},$$

which implies that $\widetilde{\mathcal{H}}^{\omega}:=\bigcup_{n\in\mathbb{N}}\mathcal{H}_{\sum_{i=1}^{N}P_{p_i}}^{\omega}$ is a linear subspace of $\mathcal{H}$, and it is dense because $\sum_{p\in\mathcal{N}}P_p=I$. Clearly, $\widetilde{\mathcal{H}}^{\omega}$ is invariant under the action of the operator A^{ω}. For any finite collection $\{\phi_i\}_{i=1}^{N}\in\widetilde{\mathcal{H}}^{\omega}$, there exists $M\in\mathbb{N}$ such that $\phi_i\in\mathcal{H}_{\sum_{j=1}^{M}P_{p_j}}^{\omega}$ for all i. Therefore, the multiplicity of the singular spectrum for $\widetilde{\mathcal{H}}^{\omega}$ is bounded by K. Finally, since $\widetilde{\mathcal{H}}^{\omega}$ is dense in $\mathcal{H}$, we conclude that the multiplicity of the singular spectrum is bounded by K. □

5. An application

To prove Corollary 1.4 and Theorem 1.6, we need some results about the multiplicity of the matrix $\sqrt{C_n}G_{n,n}^{\omega}(z)\sqrt{C_n}$. These are obtained by using the resolvent equation for a special decomposition of A^{ω}.

Fix $n\in\mathcal{N}$. Then using the fact that range$(C_n)\subset\mathcal{D}(A)$, the operators P_nAP_n, $(I-P_n)AP_n$ and $P_nA(I-P_n)$ are well defined, and since they are finite-rank operators, they are bounded. Hence, using the resolvent equation connecting the operators A^{ω} and

$$\widetilde{A}^{\omega}=P_nAP_n+(I-P_n)A(I-P_n)+\sum_{m\in\mathcal{N}}\omega_mC_m,$$

we see that

$$G^{\omega}_{n,n}(z) = [P_n A P_n + \omega_n C_n - z P_n - P_n A(I - P_n)(\widetilde{A}^{\omega} - z)^{-1}(I - P_n) A P_n]^{-1}, \tag{14}$$

where the right-hand side is viewed as a linear operator on $P_n \mathcal{H}$.

Thus, the maximum algebraic multiplicity of the eigenvalues of the matrix $\sqrt{C_n} G^{\omega}_{n,n}(z) \sqrt{C_n}$ is the same as the maximum algebraic multiplicity of the eigenvalues of the matrix

$$C_n^{-\frac{1}{2}} A C_n^{-\frac{1}{2}} - z C_n^{-1} - C_n^{-\frac{1}{2}} A(I - P_n)(\widetilde{A}^{\omega} - z)^{-1}(I - P_n) A C_n^{-\frac{1}{2}}. \tag{15}$$

Notice that (15) is independent of ω_n. The basic difference between the proof of Corollary 1.4 and that of Theorem 1.6 is in how the term

$$C_n^{-\frac{1}{2}} A(I - P_n)(\widetilde{A}^{\omega} - z)^{-1}(I - P_n) A C_n^{-\frac{1}{2}}$$

is handled. Since the norm of this operator is $O((\Im z)^{-1})$, it is clear that we can ignore this term by choosing $\Im z$ large enough, but it is this term that provides the simplicity of the spectrum in Theorem 1.6.

We will be using the following lemma:

Lemma 5.1. *Suppose the operators A^{ω} and A satisfy the hypotheses of Corollary 1.4. Let $I \subset (-\infty, M)$ be a bounded interval such that the maximum algebraic multiplicity of the eigenvalues of $\sqrt{C_n} G^{\omega}_{n,n}(E) \sqrt{C_n}$ is bounded by K for all $E \in I$. Then for almost all z the maximum algebraic multiplicity of the eigenvalues of $\sqrt{C_n} G^{\omega}_{n,n}(z) \sqrt{C_n}$ is bounded by K.*

Remark 5.2. The main advantage of this lemma is that instead of looking for a bound in $\mathbb{C} \setminus (M, \infty)$, we can work with $z \in \mathbb{R} \setminus (\sigma(A^{\omega}) \cup \sigma(A))$ and so the operator $P_n(A^{\omega} - E)^{-1} P_n = \lim_{\epsilon \downarrow 0} P_n(A^{\omega} - E - \iota\epsilon)^{-1} P_n$ is self-adjoint, hence the algebraic and geometric multiplicities coincides.

The proof follows the same steps as the proof of Lemma 4.1 and is omitted. Now we are ready to prove our other two results.

5.1. Proof of Corollary 1.4.

Using Lemma 5.1 and the fact that the algebraic multiplicity of $\sqrt{C_n} G^{\omega}_{n,n}(E) \sqrt{C_n}$ is same as the algebraic multiplicity of

$$C_n^{-\frac{1}{2}} A C_n^{-\frac{1}{2}} - E C_n^{-1} - C_n^{-\frac{1}{2}} A(I - P_n)(\widetilde{A}^{\omega} - E)^{-1}(I - P_n) A C_n^{-\frac{1}{2}}, \tag{16}$$

it suffices to boud the multiplicity of this last matrix for $E \ll M$.

First we handle the case when C_n are projections. The maximum algebraic multiplicity of (16) is same as that of

$$P_n A P_n - P_n A(I - P_n)(\widetilde{A}^{\omega} - E)^{-1}(I - P_n)A P_n; \tag{17}$$

the term EC_n^{-1} can be ignored because it is the identity operator, and so it does not affect the multiplicity. Let

$$\delta = \min_{\substack{x,y\in\sigma(P_n A P_n)\\ x\neq y}} |x - y|.$$

Then for $E < -M - \frac{3}{\delta}\,\|P_n A(I - P_n)\|^2$ we have

$$\|P_n A(I - P_n)(\widetilde{A}^{\omega} - E)^{-1}(I - P_n)A P_n\| < \frac{\delta}{3}.$$

Hence, viewing $P_n A(I - P_n)(\widetilde{A}^{\omega} - E)^{-1}(I - P_n)A P_n$ as a perturbation, we see that any eigenvalue of (17) is in the $\frac{\delta}{3}$-neighborhood of the set of eigenvalues of $P_n A P_n$. So the multiplicity of any eigenvalue of the operator (17) cannot exceed the multiplicity of the eigenvalues of the operator $P_n A P_n$. This completes the proof for the case of a projection.

For general C_n, the maximum algebraic multiplicity of (16) is same as the maximum algebraic multiplicity for

$$-C_n^{-1} + \frac{1}{E}(C_n^{-\frac{1}{2}} A C_n^{-\frac{1}{2}} - C_n^{-\frac{1}{2}} A(I - P_n)(\widetilde{A}^{\omega} - E)^{-1}(I - P_n)A C_n^{-\frac{1}{2}}), \tag{18}$$

so setting

$$\delta = \min_{\substack{x,y\in\sigma(C_n^{-1})\\ x\neq y}} |x - y|$$

and choosing

$$E < -2M - \frac{3}{\delta}(\|C_n^{-\frac{1}{2}} A C_n^{-\frac{1}{2}}\| + \|C_n^{-\frac{1}{2}} A(I - P_n)\|^2),$$

we see that the eigenvalues of (18) are in the $\frac{\delta}{3}$-neighborhood of the set of eigenvalues of C_n^{-1}. So, following the argument for the projection case, we conclude that the multiplicity of any eigenvalue of (16) is bounded above by the multiplicity of the eigenvalues of C_n^{-1}.

5.2. Proof of Theorem 1.6. Since $P_n \Delta_{\mathcal{B}} P_n$ has a non-trivial multiplicity, the previous argument does not give us the desired result. So we have to concentrate on (17), which in this case is

$$P_n \Delta_{\mathcal{B}} P_n - P_n \Delta_{\mathcal{B}}(I - P_n)(\widetilde{H}^{\omega} - E)^{-1}(I - P_n)\Delta_{\mathcal{B}} P_n, \tag{19}$$

where

$$\widetilde{H}^{\omega} = P_n \Delta_{\mathcal{B}} P_n + (I - P_n)\Delta_{\mathcal{B}}(I - P_n) + \sum_{x \in J} \omega_x P_x.$$

Here we denote $P_x = \chi_{\widetilde{\Lambda}(x)}$. For simplicity of notation, we set

$$\partial\widetilde{\Lambda}(x) = \{(p,q) \in \widetilde{\Lambda}(x) \times \widetilde{\Lambda}(x)^c : d(p,q) = 1\},$$

i.e., we pair all the leaf nodes of the tree $\widetilde{\Lambda}(x)$ with their neighbors outside the tree.

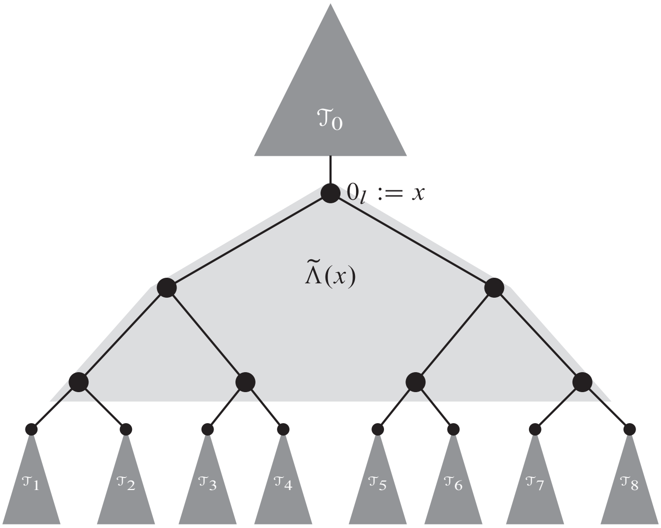

Figure 3. A representation of the rooted tree with three neighbors. Observe that removing the sub-tree $\widetilde{\Lambda}(x)$ divides the graphs into nine connected components.

Using the Dirac bra and ket notation, we observe that

$$\begin{aligned} &P_n \Delta_{\mathcal{B}}(I - P_n)(\widetilde{H}^{\omega} - E)^{-1}(I - P_n)\Delta_{\mathcal{B}} P_n \\ &= \sum_{(p,q)\in\partial\widetilde{\Lambda}(x)} |\delta_p\rangle\langle\delta_p| \, \langle\delta_q, (\widetilde{H}^{\omega} - E)^{-1}\delta_q\rangle. \end{aligned}$$

This holds because

$$\langle\delta_q, (I - P_n)\Delta_{\mathcal{B}} P_n \delta_p\rangle = \begin{cases} 1 & \text{if } (p,q) \in \partial\widetilde{\Lambda}(n), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\langle\delta_{q_1}, (\widetilde{H}^{\omega})^k \delta_{q_2}\rangle = 0 \quad \text{for all } k \in \mathbb{N},$$

for $(p_1, q_1), (p_2, q_2) \in \partial\widetilde{\Lambda}(n)$ and $q_1 \neq q_2$. This is also the reason why the random variables

$$\{\langle \delta_q, (\widetilde{H}^\omega - E)^{-1}\delta_q\rangle\}_{(p,q)\in\partial\widetilde{\Lambda}(x)}$$

are independent of each other. The random variable $\langle \delta_q, (\widetilde{H}^\omega - E)^{-1}\delta_q\rangle$ is real for $E \in \mathbb{R}$, and has an absolutely continuous distribution, which follows from the continuos fraction expression

$$\langle \delta_q, (\widetilde{H}^\omega - E)^{-1}\delta_q\rangle = \cfrac{1}{\omega_q - E - \sum_{x_1\in N_q} \cfrac{1}{\omega_{x_1} - E - \sum_{x_2\in N_{x_1}} \cfrac{1}{\ddots - \sum_{x_l\in N_{x_{l-1}}} a^\omega_{x_l}(E)}}},$$

where $\{a^\omega_{x_l}(E)\}$ are independent of ω_q, and the distribution of ω_q is absolutely continuous with respect to the Lebesgue measure. Now Theorem 1.6 follows from Theorem 5.3.

But first a few notations are needed. Let $\mathcal{T}_L$ denote a rooted tree with root 0_L and such that every vertex has $K+1$ neighbors, except for the root 0_L (which has K neighbors) and the vertices in the boundary

$$\partial\mathcal{T}_L := \{x \in \mathcal{T}_L : d(0_L, x) = L\},$$

which have one neighbor each.

Theorem 5.3. *Let $\Delta_{\mathcal{T}_L}$ denote the adjacency matrix of the tree $\mathcal{T}_L$ and set*

$$B_\tau = \sum_{x\in\partial\mathcal{T}_L} t_x \, |\delta_x\rangle \, \langle\delta_x|$$

for $\tau = \{t_x\}_{x\in\partial\mathcal{T}_L} \in \mathbb{R}^{\partial\mathcal{T}_L}$. Then, for almost all τ with respect to the Lebesgue measure, the spectrum of $H_\tau = \Delta_{\mathcal{T}_L} + B_\tau$ is simple.

Proof. The proof is done by induction on L. Let $H_{\tau,l}$ denote the operator

$$H_{\tau,l} = \Delta_{\mathcal{T}_l} + \sum_{x\in\partial\mathcal{T}_l} \tau_x \, |\delta_x\rangle \, \langle\delta_x| \, ,$$

where $\Delta_{\mathcal{T}_l}$ is the adjacency operator of the rooted tree $\mathcal{T}_l$ with root 0_l.

The induction is done over the statement: *for almost all τ, $H_{\tau,l}$ has simple spectrum with the property that all the eigenfunctions are non-zero at the root, and $\sigma(H_{\tau,l}) \cap \sigma(H_{\omega,l}) = \emptyset$ for almost all ω.*

For $l = 0$, the statement is trivial because $H_{\tau,0}$ is the operator of multiplication by the random variable τ_{0_l} on $\mathbb{C}$.

Suppose that the induction statement holds for $l = N - 1$. Observe that

$$H_{\tau,N} = \sum_{x:d(0_N,x)=1} (|\delta_{0_N}\rangle\langle\delta_x| + |\delta_x\rangle\langle\delta_{0_N}|) + \sum_{x:d(0_N,x)=1} H_{\tau,x},$$

where $H_{\tau,x} := \chi_{\mathcal{T}_x} H_{\tau,l} \chi_{\mathcal{T}_x}$ for the sub-tree

$$\mathcal{T}_x := \{y \in \mathcal{T}_l : d(0_N, y) = d(0_N, x) + d(x, y)\}.$$

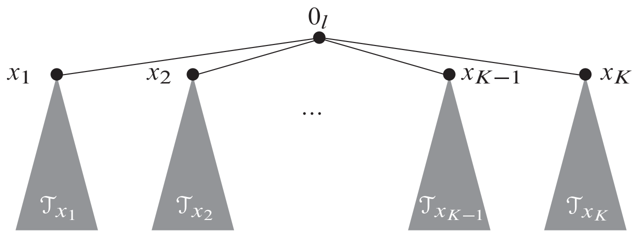

Figure 4. The tree $\mathcal{T}_l$ can be viewed as a union of K disjoint trees $\{\mathcal{T}_{x_i}\}_i$ which are connected through their roots $\{x_1, \dots, x_K\}$ to a separate node 0_l.

First notice that $H_{\tau,x}$ is unitarily equivalent to $H_{\tilde{\tau},N-1}$, where $\tilde{\tau}$ is restriction of τ to the $\partial\mathcal{T}_x$. Next note that the $\{\tau_y\}_y$ that appear in H_{τ,x_i} are disjoint for any two subtrees $\mathcal{T}_{x_1}$ and $\mathcal{T}_{x_2}$ with $x_1 \neq x_2$. Hence, by the induction hypothesis, $\sigma(H_{\tau,x}) \cap \sigma(H_{\tau,y}) = \emptyset$ for $x \neq y$ and the spectrum of the operator $H_{\tau,x}$ is simple, with the property that the eigenfunctions corresponding to the eigenvalues are non-zero at the root, for each x.

Since we are working on tree graphs, we have

$$\begin{aligned}\langle\delta_{0_N}, (H_{\tau,N} - z)^{-1}\delta_{0_N}\rangle &= \frac{1}{-z - \sum_{x:d(0_N,x)=1} \langle\delta_x, (H_{\tau,x} - z)^{-1}\delta_x\rangle} \\ &= \frac{1}{-z - \sum_{x:d(0_N,x)=1} \sum_{E\in\sigma(H_{\tau,x})} \frac{|\langle\psi_{\tau,x,E},\delta_x\rangle|^2}{E-z}}, \end{aligned} \tag{20}$$

where $\psi_{\tau,x,E}$ is the eigenfunction of $H_{\tau,x}$ corresponding to the eigenvalue E. By the induction hypothesis we have $\langle\psi_{\tau,x,E}, \delta_x\rangle \neq 0$ for each $E \in \sigma(H_{\tau,x})$ and x a neighbor of 0_N. Next, using the fact that $\sigma(H_{\tau,x}) \cap \sigma(H_{\tau,y}) = \emptyset$ for $x \neq y$, we get that

$$z + \sum_{x:d(0_N,x)=1} \sum_{E\in\sigma(H_{\tau,x})} \frac{|\langle\psi_{\tau,x,E}, \delta_x\rangle|^2}{E - z}$$

has $\sum_{x:d(0_N,x)=1} \#\sigma(H_{\tau,x})$ poles, and consequently equation (20) has

$$1 + \sum_{x:d(0_N,x)=1} \#\sigma(H_{\tau,x})$$

roots, that is, $|\mathcal{T}_N|$ roots. But using functional calculus we also have

$$\langle \delta_{0_N}, (H_{\tau,N} - z)^{-1} \delta_{0_N} \rangle = \sum_{E \in \sigma(H_{\tau,N})} \frac{|\langle \psi_{\tau,N,E}, \delta_{0_N} \rangle|^2}{E - z},$$

where $\psi_{\tau,N,E}$ is the eigenfunction of the matrix $H_{\tau,N}$ corresponding to the eigenvalue E. So each pole $\langle \delta_{0_N}, (H_{\tau,N} - z)^{-1} \delta_{0_N} \rangle$ corresponds to an eigenvalue, and the previous computation shows that there are $|\mathcal{T}_N|$ many poles, which establishes the simplicity of the spectrum of the operator $H_{\tau,N}$. Finally, the eigenfunction $\psi_{\tau,N,E}$ is non-zero at the root 0_N, because if $\langle \psi_{\tau,N,E}, \delta_{0_N} \rangle = 0$, then the pole corresponding to E will not be present in the above expression.

Finally, we have to prove that $\sigma(H_{\tau,l}) \cap \sigma(H_{\omega,l}) = \emptyset$ for almost all τ, ω. But first we need the following claim:

Claim. *For any solution $\psi \in \mathbb{C}^{\mathcal{T}_l} \setminus \{0\}$ of the equation $H_{\tau,l}\psi = E\psi$ for $E \in \mathbb{R}$, there exists $x \in \partial\mathcal{T}_l$ such that $\psi_x \neq 0$.*

Proof of the claim. If for some $E \in \mathbb{R}$ there exists $\psi \in \mathbb{C}^{\mathcal{T}_l}$ such that $H_{\tau,l}\psi = E\psi$ and

$$\psi_x = 0 \quad \text{for all } x \in \partial\mathcal{T}_l,$$

then, for any $x \in \partial\mathcal{T}_l$,

$$\begin{aligned}(H_{\tau,l}\psi)_x = E\psi_x = 0 &\implies \psi_{Px} + t_x\psi_x = 0 \\ &\implies \psi_{Px} = 0,\end{aligned}$$

where Px is the unique neighbor of x satisfying $d(0_l, x) = d(0_l, Px) + 1$. So we get that $\psi_x = 0$ for all $x \in \mathcal{T}_l$ such that $d(0, x) = l - 1$. Repeating this argument for x satisfying $d(0, x) = l - 1$ shows that $\psi_x = 0$ for all x such that $d(0_l, x) = l - 2$. Repeating the last step recursively gives $\psi \equiv 0$, a contradiction. This completes the proof of the claim. △

Now, to prove that $\sigma(H_{\tau,l}) \cap \sigma(H_{\omega,l}) = \emptyset$, for almost all τ, ω, denote $\tau = \{\tau_x\}_{x\in\partial\mathcal{T}_l}$, $\omega = \{\omega_x\}_{x\in\partial\mathcal{T}_l}$, and let $\{E_i^\tau\}_i$ and $\{\psi_i^\tau\}$ denote the eigenvalues and the corresponding eigenfunctions for $H_{\tau,l}$, and similarly for $H_{\omega,l}$. Using the Feynman–Hellmann theorem for rank-one perturbations, we have

$$\frac{dE_i^\tau}{d\tau_x} = |\langle \psi_i^\tau, \delta_x \rangle|^2 \quad \text{for all } x \in \partial\mathcal{T}_l \text{ and all } i,$$

and similarly

$$\frac{dE_i^\omega}{d\omega_x} = |\langle \psi_i^\omega, \delta_x \rangle|^2 \quad \text{for all } x \in \partial\mathcal{T}_l \text{ and all } i.$$

For each i, by the previous claim, there exists $x_i^\tau \in \partial\mathcal{T}_l$ such that $\langle \psi_i^\tau, \delta_{x_i^\tau} \rangle \neq 0$, and similarly for ω. Now using the implicit function theorem over the function $g(\omega, \tau) = E_i^\tau - E_j^\omega = 0$, we conclude that the dimension of the manifold

$$\{(\tau, \omega) \in \mathbb{R}^{\partial\mathcal{T}_l} \times \mathbb{R}^{\partial\mathcal{T}_l} : E_i^\tau = E_j^\omega\}$$

is smaller than $2|\partial\mathcal{T}_l|$. In particular, the Lebesgue measure of the set

$$\{(\tau, \omega) \in \mathbb{R}^{\partial\mathcal{T}_l} \times \mathbb{R}^{\partial\mathcal{T}_l} : E_i^\tau = E_j^\omega\}$$

is equal to zero, which completes the proof of the induction step. □

A. Appendix

Lemma A.1. *Let $\mathcal{H}$ be a separable Hilbert space, and let H be a self-adjoint operator on $\mathcal{H}$, and for $\phi, \psi \in \mathcal{H}$ set $\sigma_\phi(\cdot) = \langle \phi, E_H(\cdot)\phi \rangle$ and $\sigma_{\phi,\psi}(\cdot) = \langle \phi, E_H(\cdot)\psi \rangle$. Let f be the Radon–Nikodym derivative of $\sigma_{\phi,\psi}$ with respect to σ_ϕ. Then $f(H)\phi$ is the projection of ψ on the minimal closed H-invariant subspace containing ϕ.*

Proof. Let $\mathcal{H}_\phi$ denote the minimal closed H-invariant subspace containing ϕ. Then, the pair $(\mathcal{H}_\phi, H)$ is unitarily equivalent to $(L^2(\mathbb{R}, \sigma_\phi), M_{\mathrm{Id}})$, where M_{Id} is multiplication by the identity map on $\mathbb{R}$. Consider the linear functional

$$g \longmapsto \langle g(H)\phi, \psi - f(H)\phi \rangle$$

for $g \in L^2(\mathbb{R}, \sigma_\phi)$. Observe that

$$\begin{aligned}\langle g(H)\phi, \psi - f(H)\phi \rangle &= \langle g(H)\phi, \psi \rangle - \langle g(H), f(H)\phi \rangle \\ &= \int g(x) d\sigma_{\phi,\psi}(x) - \int g(x) f(x) d\sigma_\phi(x) = 0.\end{aligned}$$

Since $g(H)\phi$ are dense in $\mathcal{H}_\phi$ for $\phi \in L^2(\mathbb{R}, \sigma_\phi)$, we have

$$\psi - f(H)\phi \perp \mathcal{H}_\phi,$$

hence $f(H)\phi$ is the projection of ψ on to $\mathcal{H}_\phi$. □

Lemma A.2. *Let $\mathcal{H}$ be separable Hilbert space, H be a self-adjoint operator $\mathcal{H}$, and Q be a finite-rank projection. Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis for the subspace $Q\mathcal{H}$ and denote*

$$\mathcal{H}_i = \overline{\langle f(H)e_i \colon f \in C_c(\mathbb{R})\rangle}$$

and

$$\mathcal{H}_Q = \overline{\langle f(H)\phi \colon f \in C_c(\mathbb{R})\ \&\ \phi \in Q\mathcal{H}\rangle}.$$

Then

$$\mathcal{H}_Q = \sum_i \mathcal{H}_i,$$

where $\sum_i \mathcal{H}_i$ denotes the closed linear span of $\mathcal{H}_i$.

Proof. Since $\mathcal{H}_i \subseteq \mathcal{H}_Q$ for any i, we always have

$$\sum_i \mathcal{H}_i \subseteq \mathcal{H}_Q.$$

For the other way round, note that we only have to show $f(H)\phi \in \sum_i \mathcal{H}_i$ whenever $\phi \in Q\mathcal{H}$. Since $\{e_i\}_i$ is a basis, we have

$$\phi = \sum_i a_i e_i.$$

Using it, define

$$\psi_N = \sum_{i=1}^{N} a_i f(H)e_i,$$

which satisfies $\psi_N \in \sum_i \mathcal{H}_i$ for any $N \in \mathbb{N}$. Now the conclusion of the lemma holds, since $\sum_i \mathcal{H}_i$ is closed. □

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