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Erratum to "Asymptotic shape optimization for Riesz means of the Dirichlet Laplacian over convex domains"

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We correct two errors in [6]. All the main results of the paper remain unchanged, but two of the proofs need to be altered. Firstly, the proof of Theorem 2.4 in [6] contains a gap. Indeed, the proof of the theorem is based on the bound [3, eq. 4.3]. However, the proof of this equation in [3] contains an error. We shall therefore give a alternative proof of [6, Theorem 2.4] which avoids using [3, eq. 4.3]. Secondly, there is a mistake in the proof of Corollary 5.3 which we also fix. The statements of two results are:

Theorem 1 ([6, Theorem 2.4]). *Let $\Omega \in \mathcal{K}^n$. For $\gamma \geq 1$ there exists a constant $c(\gamma, n) > 0$ such that*

- *if $\Lambda \leq \frac{\pi^2}{4r(\Omega)^2}$, then*

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma = 0;$$

- *if $\Lambda > \frac{\pi^2}{4r(\Omega)^2}$, then*

$$\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma \leq L_{\gamma,n}^{cl} |\Omega| \Lambda^{\gamma+n/2} - c(\gamma, n) L_{\gamma,n-1}^{cl} |\partial\Omega| \Lambda^{\gamma+(n-1)/2}.$$

Corollary 2 ([6, Corollary 5.3]). *Let $\Omega \in \mathcal{K}^n$. There exists a constant $c(n) > 0$ such that, for all $m \geq 1$,*

$$\frac{1}{m} \sum_{k=1}^{m} \lambda_k(\Omega) \geq A_n \Big(\frac{m}{|\Omega|}\Big)^{2/n} + c(n) B_n \frac{|\partial\Omega|}{|\Omega|} \Big(\frac{m}{|\Omega|}\Big)^{1/n}.$$

As in [6] the constants $L^{\mathrm{cl}}_{\gamma,n}$, A_n, and B_n are defined by

$$L^{\mathrm{cl}}_{\gamma,n} = \frac{\Gamma(\gamma+1)}{(4\pi)^{n/2}\Gamma\big(\gamma+1+\frac{n}{2}\big)},$$

$$A_n = \frac{4\pi n\Gamma\big(\frac{n}{2}+1\big)^{2/n}}{n+2},$$

$$B_n = \frac{2\pi\Gamma\big(\frac{n}{2}+1\big)^{1+1/n}}{(n+1)\Gamma\big(\frac{n+1}{2}\big)}.$$

1. Proof of Theorem 1

The first part of the theorem is a direct consequence of the bound $\lambda_1(\Omega) \geq \frac{\pi^2}{4r(\Omega)^2}$ proved in [4, 7]. For the second part, Theorem 1.2 in [2] implies that there exist universal constants $c_1, c_2 > 0$ such that

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_- \leq L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2} - c_2 L^{\mathrm{cl}}_{1,n-1}|\partial\Omega|\Lambda^{1+(n-1)/2}, \tag{1}$$

for all $\Lambda \geq \frac{c_1}{r(\Omega)^2}$. In order to extend (1) to all $\Lambda \geq \frac{\pi^2}{4r(\Omega)^2}$ we recall Theorem 3.5 in [5] which for $\Omega \in \mathcal{K}^n$ implies that

$$\mathrm{Tr}(-\Delta_\Omega - \Lambda)_- \leq L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2} - \frac{L^{\mathrm{cl}}_{1,n}}{72} r(\Omega)^{-3/2}|\Omega|\Lambda^{n/2+1/4}, \tag{2}$$

for all $\Lambda \geq \frac{\pi^2}{4r(\Omega)^2}$. Note that this is almost what we aim to prove but the negative correction term on the right-hand side differs from that in (1) both in terms of the dependence on Ω and in the power of Λ. However, provided the quantity $r(\Omega)^2\Lambda$ remains bounded the two correction terms can be related as follows.

We claim that there is a universal constant $c_3 > 0$ such that

$$c_3 L^{\mathrm{cl}}_{1,n-1}|\partial\Omega|\Lambda^{1+(n-1)/2} \leq \frac{L^{\mathrm{cl}}_{1,n}}{72} r(\Omega)^{-3/2}|\Omega|\Lambda^{n/2+1/4}$$

for $\frac{\pi^2}{4r(\Omega)^2} \leq \Lambda \leq \frac{c_1}{r(\Omega)^2}$, which allows us to extend (1) to all $\Lambda \geq \frac{\pi^2}{4r(\Omega)^2}$ (possibly with a smaller constant c_2). The existence of such a constant follows from

$$\inf_{\Omega\in\mathcal{K}^n} \inf_{\frac{\pi^2}{4r(\Omega)^2}\leq\Lambda\leq\frac{c_1}{r(\Omega)^2}} \frac{|\Omega|}{|\partial\Omega| r(\Omega)^{3/2}}\Lambda^{-1/4} = \inf_{\Omega\in\mathcal{K}^n} \frac{|\Omega| c_1^{-1/4}}{|\partial\Omega| r(\Omega)} \geq \frac{1}{n c_1^{1/4}}, \tag{3}$$

where we in the second step used [6, eq. (13)],

$$\frac{|\Omega|}{|\partial\Omega|} \leq r(\Omega) \leq n\frac{|\Omega|}{|\partial\Omega|}. \tag{4}$$

This completes the proof of Theorem 1 for $\gamma = 1$.

To prove the case $\gamma > 1$ we apply the Aizenman–Lieb identity [1] (see also [6, Section 6]). For all $\Lambda \geq 0$ and $b \in [0,1]$ the case $\gamma = 1$ implies that

$$\begin{aligned}
\operatorname{Tr}(-\Delta_\Omega - \Lambda)_- &\leq \Big(L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2} - cL^{\mathrm{cl}}_{1,n-1}|\partial\Omega|\Lambda^{1+(n-1)/2}\Big)_+ \\
&\leq L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2}\Big(1 - \frac{bcL^{\mathrm{cl}}_{1,n-1}}{L^{\mathrm{cl}}_{1,n}}\frac{|\partial\Omega|}{|\Omega|}\Lambda^{-1/2} \\
&\qquad\qquad + \frac{b^2c^2(L^{\mathrm{cl}}_{1,n-1})^2}{4(L^{\mathrm{cl}}_{1,n})^2}\frac{|\partial\Omega|^2}{|\Omega|^2}\Lambda^{-1}\Big)_+ \\
&= L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2}\Big(1 - \frac{bcL^{\mathrm{cl}}_{1,n-1}}{2L^{\mathrm{cl}}_{1,n}}\frac{|\partial\Omega|}{|\Omega|}\Lambda^{-1/2}\Big)^2 \\
&= L^{\mathrm{cl}}_{1,n}|\Omega|\Lambda^{1+n/2} - c_1bL^{\mathrm{cl}}_{1,n-1}|\partial\Omega|\Lambda^{1+(n-1)/2} \\
&\quad + c_2b^2L^{\mathrm{cl}}_{1,n-2}\frac{|\partial\Omega|^2}{|\Omega|}\Lambda^{1+(n-2)/2},
\end{aligned}$$

for some positive constants c_1, c_2. Here the positive part in the first inequality is necessary as we do not distinguish between the two cases of Theorem 1. An application of the Aizenman–Lieb identity yields

$$\begin{aligned}
\operatorname{Tr}(-\Delta_\Omega - \Lambda)_-^\gamma &\leq L^{\mathrm{cl}}_{\gamma,n}|\Omega|\Lambda^{\gamma+n/2} \\
&\quad - c_1bL^{\mathrm{cl}}_{\gamma,n-1}|\partial\Omega|\Lambda^{\gamma+(n-1)/2} + c_2b^2L^{\mathrm{cl}}_{\gamma,n-2}\frac{|\partial\Omega|^2}{|\Omega|}\Lambda^{\gamma+(n-2)/2},
\end{aligned} \tag{5}$$

for all $\Lambda \geq 0$ and $b \in [0,1]$. From (4) we have $\frac{|\partial\Omega|}{|\Omega|}\Lambda^{-1/2} \leq \frac{2n}{\pi}$ for all $\Lambda \geq \frac{\pi^2}{4r(\Omega)^2}$ and thus the claimed bound follows by choosing b sufficiently small. □

2. Proof of Corollary 2

The proof is as before based on the fact that

$$\sup_{\Lambda\geq 0}\Big(m\Lambda - \sum_{k:\lambda_k(\Omega)<\Lambda}(\Lambda - \lambda_k(\Omega))\Big) = \sum_{k=1}^{m}\lambda_k(\Omega). \tag{6}$$

To find a bound for the right-hand side of (6) we plug in a suitable choice of $\Lambda \geq 0$ in the left-hand side and use Theorem 1 to bound the Riesz mean.

The natural choice to make is

$$\Lambda = \Lambda_0 := \frac{4^{1/n}}{(n+2)^{2/n}(L^{\mathrm{cl}}_{1,n})^{2/n}}\Big(\frac{m}{|\Omega|}\Big)^{2/n},$$

however we need to distinguish whether this quantity is less or greater than $\frac{\pi^2}{4r(\Omega)^2}$. This case distinction is precisely what was missed in the erroneous proof given in [6]. If $\Lambda_0 \geq \frac{\pi^2}{4r(\Omega)^2}$ Theorem 1 implies that

$$\begin{aligned}\sum_{k=1}^{m} \lambda_k(\Omega) &\geq m\Lambda_0 - L_{1,n}^{\mathrm{cl}}|\Omega|\Lambda_0^{1+n/2} + cL_{1,n-1}^{\mathrm{cl}}|\partial\Omega|\Lambda_0^{(n+1)/2} \\ &= mA_n\Big(\frac{m}{|\Omega|}\Big)^{2/n} + mc'B_n\frac{|\partial\Omega|}{|\Omega|}\Big(\frac{m}{|\Omega|}\Big)^{1/n}.\end{aligned}$$

If $\Lambda_0 \leq \frac{\pi^2}{4r(\Omega)^2}$ choose $\Lambda = \frac{\pi^2}{4r(\Omega)^2}$. The results of [4, 7] (or Theorem 1) implies that $\frac{1}{m}\sum_{k=1}^{m} \lambda_k(\Omega) \geq \frac{\pi^2}{4r(\Omega)^2}$. What remains is to show that if $\Lambda_0 \leq \frac{\pi^2}{4r(\Omega)^2}$, then

$$A_n\Big(\frac{m}{|\Omega|}\Big)^{2/n} + cB_n\frac{|\partial\Omega|}{|\Omega|}\Big(\frac{m}{|\Omega|}\Big)^{1/n} \leq \frac{\pi^2}{4r(\Omega)^2}, \tag{7}$$

provided $c = c(n) > 0$ is small enough. Let

$$a = \frac{4^{1/n}}{(n+2)^{2/n}(L_{1,n}^{\mathrm{cl}})^{2/n}}$$

so that $A_n = a - L_{a,n}^{\mathrm{cl}}a^{1+n/2}$. Therefore, by (4) and $\Lambda_0 = a\big(\frac{m}{|\Omega|}\big)^{2/n} \leq \frac{\pi^2}{4r(\Omega)^2}$,

$$A_n\Big(\frac{m}{|\Omega|}\Big)^{2/n} + cB_n\frac{|\partial\Omega|}{|\Omega|}\Big(\frac{m}{|\Omega|}\Big)^{1/n} \leq \Big[(1 - L_{1,n}^{\mathrm{cl}}a^{n/2}) + cB_n\frac{2n}{\pi a^{1/2}}\Big]\frac{\pi^2}{4r(\Omega)^2}, \tag{8}$$

from which (7) follows for c sufficiently small.

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