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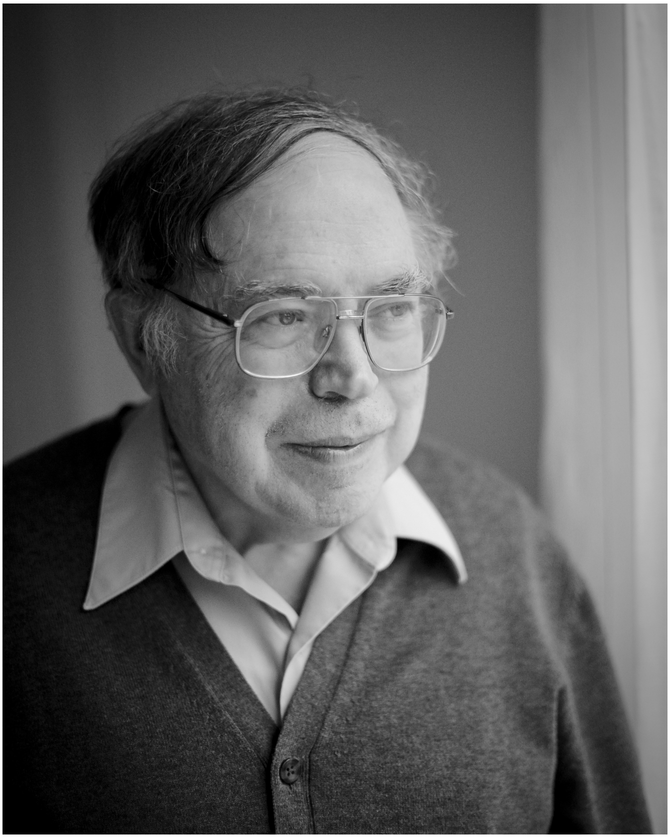
Mikhail Shubin (1944–2020)

Maxim Braverman and Leonid Friedlander



This volume is dedicated to the memory of Mikhail (Misha) Shubin who passed away on May 13, 2020. The articles that are presented in the volume are written by people who knew Misha well; some of them were Misha's students or collaborators. The diversity of topics reflects the diversity of Misha's interests. One of the authors of this preface (LF) knew Misha from 1974 when he took his pseudodifferential operators class as an undergraduate student. The second one (MB) was Misha's colleague and collaborator.

Misha was born on December 19, 1944, in Kujbyshev (now Samara), an old Russian city that lies along the Volga river. In his early years, music was his main passion, and he was seriously considering choosing it as a profession. When in high school, Misha developed an interest to mathematics, he successfully participated in Mathematical Olympiads, and mathematics became his final choice. He was admitted to the Moscow State University in 1961; after the graduation he went to the Graduate School. Marko Vishik was his Ph.D. advisor.

In his Ph.D. dissertation that Misha defended in 1969, he derived formulas for computing index of matrix-valued Wiener–Hopf operators. Soon after that, Misha turned to micro-local analysis that remained an important part of almost all his future works. In [18] he developed the calculus of operators in an Euclidean space with the Weyl symbols $p(y) = p(x, \xi)$ that satisfy the condition

$$|D^\alpha p(y)| \le C_\alpha (1 + |y|)^{m-\rho|\alpha|}$$

where $0 < \rho \le 1$. This class of operators is now called *Shubin class.*

The first paper by Misha that directly dealt with spectral theory was his joint paper with Tulovsky [20] where they studied operators in an n-dimensional Euclidean space from the Shubin class that satisfy additional assumptions

$$|D^\alpha p(y)| \le C_\alpha p(y)|y|^{-\rho|\alpha|}$$

and

$$p(y) \ge a|y|^{m_0}$$

where a and m_0 are positive numbers. Notice that these operators need not be elliptic. They proved that such operators have discrete spectrum. If the symbol $p(y)$ is either a polynomial for large values of $|y|$ or it satisfies the estimate

$$p(y)^{1-\alpha} \le C|\langle y, \nabla p(y)\rangle|$$

for large values of $|y|$ (α is an appropriate, sufficiently small number), then the eigenvalues of the corresponding operator obey a Weyl-type asymptotics. Namely, let $V(\lambda)$ be the volume of the region $\{y\colon p(y) < \lambda\}$, divided by $(2\pi)^n$. Then the counting function $N(\lambda)$ of the eigenvalues satisfies the asymptotics

$$N(\lambda) = V(\lambda)((1 + O(\lambda^{-\beta}))$$

as $\lambda \to \infty$. Here a positive number β is related to ρ, m, and α. The method of the proof was remarkable, and it became quite popular later. Tulovsky and Shubin introduced an approximate spectral projection. Let $\chi(t, \lambda, \epsilon)$ be a smooth function that equals 1 when $t \le \lambda$, that equals 0 when $t > \lambda + \epsilon$, and such that

$$|(d/dt)^k \chi(t, \lambda, \epsilon)| \le C_k \epsilon^{-k}.$$

Let $E_\epsilon(\lambda)$ be the pseudodifferential operator with the Weyl symbol $e(y, \lambda, \epsilon) = \chi(p(y), \lambda, \epsilon)$. The proof boils down to estimating the trace norm of the difference $E(\lambda) - E_\epsilon(\lambda)$ and to finding the asymptotics in λ of the trace of $E_\epsilon(\lambda)$. Here $E(\lambda)$ is the spectral projections of the operator.

In the mid-seventies, Misha was mostly working on operators with almost-periodic coefficients (see [19] for the detailed exposition). He introduced a class of pseudodifferential operators in the Euclidean space with symbols $a(x, \xi)$ that belong to the Hörmander class S^m, and that are almost-periodic in x for every ξ. For elliptic, self-adjoint differential operators $A(x, D)$ with almost-periodic coefficients, he proved the existence of the density of states. Let V_k be a system of expanding domains in the Euclidean space that satisfies the admissibility condition. Roughly speaking, the admissibility condition says that the ratio of the area of the boundary of V_k to the volume of V_k goes to 0 as $k \to \infty$ (the precise definition is a bit more involved). Let $N_k(\lambda)$ be the counting function of the eigenvalues of the operatorA_k, that is $A(x, D)$ in V_k, with appropriate boundary conditions (the Dirichlet condition will fit the bill). The density of states $N(\lambda)$ of the operator $A(x, D)$ in the whole space is the limit as $k \to \infty$ of $N_k(\lambda)$ divided by the volume of V_k. Misha also proved the existence of the Fermi energy $E^F(\rho)$, that is the limit

$$\frac{1}{p(k)}(\lambda_1^{(k)} + \cdots + \lambda_{p(k)}^{(k)})$$

where $k \to \infty$ and $p(k)/|\Omega_k| \to \rho$. Here $\lambda_j^{(k)}$ are eigenvalues of the operator A_k. Misha introduced a II_∞ factor $\mathcal{A}_B$, and he used it to define the trace of an almost-periodic pseudodifferential operator of sufficiently small negative order; he used this notion of a trace to define the ζ-function of an elliptic operator, and he proved the theorem about analytic continuation of the ζ-function to a meromorphic function in the whole complex plane. The technique developed by Shubin was used by his student Kiselyov to prove the Weyl asymptotics for the density of states of an elliptic differential operator with almost-periodic coefficients. Finally, Misha used the II_∞ factor $\mathcal{A}_B$ to define the index of a matrix-valued elliptic differential operator with almost-periodic coefficients (usual dimensions are replaced by the $\mathcal{A}_B$-dimensions), and he derived the explicit formula for the index:

$$-(2\pi i)^{-n} \frac{(n-1)!}{(2n-1)!} \lim_{T \to \infty} \frac{1}{T^n} \int\limits_{Q_T \times S^{n-1}} \operatorname{tr}(a_m^{-1} d a_m)^{2n-1}.$$

Here n is the dimension of the space, a_m is the principal symbol of the operator, and $Q_T = \{x \colon -T/2 \le x_j \le T/2\}$.

In the mid-eighties, together with Sergey Novikov [16,17], Misha studied geometric invariants of a compact manifold M constructed from the spectrum of operators acting on its Galois covering $\widetilde{M}$ with covering group Γ. If A is a Γ-invariant operator, $k_A(x,y)$ is its Schwartz kernel, and F is the fundamental domain for the Γ-action on $\widetilde{M}$, then the Γ-*trace* of A is defined as

$$\mathrm{Tr}_\Gamma(A) := \int_F \mathrm{tr}\, k_A(x,x)\, dx.$$

A Γ-*dimension* of a Γ-invariant subspace of the space of L^2-section of some Γ-equivariant bundle over M is defined as the Γ-trace of the orthogonal projection on this subspace.

Atiyah [1] showed that the kernel of a Γ-invariant elliptic operator on $\widetilde{M}$ has finite Γ-dimension and proved the index theorem for the Γ-index of such operators. He also defined the L^2-Betti numbers $\beta_k^{(2)}(M)$ as the Γ-dimension of the kernel of the Laplace operator Δ_k, acting on k-differential forms on $\widetilde{M}$.

In a joint paper with Sergey Novikov [16], Misha showed that the L^2-Betti numbers of Atiyah satisfy the usual Morse inequalities and gave examples where the obtained inequalities give stronger bounds on the numbers of critical points of a Morse function than the classical Morse inequality for the usual Betti numbers.

Next, Novikov and Shubin [16, 17] studied the behavior of the spectrum of Δ_k near zero. They showed that the dilation class of the Γ-spectral function $N_\Gamma^k(\lambda)$ is a differential invariant of M, i.e., is independent of the Riemannian metric. Later, in a join paper with M. Gromov [7], Misha showed that, in fact, this is a homotopy invariant of M.

To make the dilation class of the spectral function more concrete, Novikov and Shubin introduced the Γ-*theta function*

$$\theta_k(t) := \mathrm{Tr}_\Gamma(e^{-t\Delta_k} - \beta_k^{(2)}(M)).$$

In many examples, it behaves as a power function as $t \to \infty$. This led Novikov and Shubin to the following numeric invariants of M, which are now known as *Novikov–Shubin invariants,*

$$\alpha_k(M) := \sup\left\{\beta\colon \theta_k \text{ is } O(t^\beta) \text{ as } t \to \infty\right\} \in [0,\infty],$$
$$\bar{\alpha}_k(M) := \inf\left\{\beta\colon \theta_k \text{ is } O(t^\beta) \text{ as } t \to \infty\right\} \in [0,\infty].$$

Again, these numbers are homotopy invariants of M by [7]. They were a subject of extensive studies by many mathematician. In particular, Farber [4] and Lück [11] gave a homological interpretation of the Novikov–Shubin invariants.

In two joint papers with Gromov and Henkin [5, 6], Misha studied the spectrum of the holomorphic Laplace operator on a Galois covering of a pseudo-convex domain M. They proved the following version of the Oka–Grauert theorem: if the closure of M is compact, then the cohomology spaces $H^{p,q}(\widetilde{M})$ have infinite Γ-dimension for all $q > 0$.

Misha was very interested in the spectral theory of Shrödinger operators with non-positive potentials. His student I. Oleinik [14, 15] extended the classical criterium of self-adjointness of scalar Schrödinger operators with unbounded below potential from $\mathbb{R}^n$ to complete Riemannian manifolds. Several authors, inspired by these papers and by communication with Misha, worked on the subject (see, for example, [2] or [10]), but the strongest and most general results were obtained in Misha's joint paper with Braverman and Miltatovich [3], where the previous results were generalized in three different directions. First, the authors considered more general operators acting on sections of arbitrary vector bundles. For a first-order elliptic operator D, they considered a Schrödinger-type operator

$$H_V := D^*D + V,$$

where D^* is the formal adjoint of D. Here D is not necessarily a Dirac-type operator and, hence, the leading symbol of H_V is not necessarily scalar. Second, they allowed singular potentials $V \in L^p_{\mathrm{loc}}$, where $p > \dim M/2$ for $\dim M \geq 4$ and $p = 2$ for $n \leq 3$. This includes the Coulomb potential and many other interesting examples. Finally, the authors did not assume that the manifold is complete in the original metric. Instead, they introduced a new metric, defined by the leading symbol of H_V and by the potential V. The main result of [3] is that if this new metric is complete, then the operator H_V is essentially self-adjoint. This gives a very precise geometric meaning to the classical principle that the essential self-adjointness of a Schrödinger operator (the *quantum completeness* of the system) is equivalent to the classical completeness.

Another related question, which Misha addressed in a series of papers [8, 9, 12], was the condition for discreteness of the spectrum of a Schrödinger operator on $\mathbb{R}^n$. It is well known that if the potential V tends to $+\infty$ at infinity, then the spectrum of the Schrödinger operator $H_V = -\Delta + V$ is discrete. A more subtle question is: when the spectrum is discrete without $V \to +\infty$? Let Q_d denote an open cube with the edge length d and with the edges parallel to coordinate axes. It was shown by A. Molchanov [13] that the spectrum of H_V is discrete if

$$\inf_F \int\limits_{Q_d \setminus F} V(x)\,dx \to +\infty \quad \text{as } Q_d \to \infty,$$

for every $d > 0$, where the infimum is taken over all *negligible* subsets. A subset is negligible if its Wiener capacity satisfies the inequality

$$\operatorname{cap} F \leq \gamma \operatorname{cap} Q_d \tag{1}$$

and γ is a sufficiently small constant. In 1953, I. Gel'fand raised a question of finding the best possible value of γ. This question was answered in the seminal paper [12] which Misha wrote with V. Maz'ya. They showed that any constant $\gamma \in (0, 1)$ can be used. They also gave a stronger result by replacing condition (1) with

$$\operatorname{cap} F \le \gamma(d) \operatorname{cap} Q_d, \tag{2}$$

and giving a complete description of the class of functions $\gamma(d)$ for which (2) implies discreteness of the spectrum of H_V.

Let us mention some of other results of Misha, not necessarily in chronological order. There are 137 publications of Misha listed in MatSciNet; that list is probably incomplete.

Gromov and Shubin obtained a far-reaching generalization of the classical Riemann–Roch theorem: formulas for dimensions of solutions of elliptic differential equations with given zeroes and singularities. Later, Misha proved the L^2 version of this theorem.

Mazya and Shubin proved a beautiful theorem that generalizes a theorem by W. K. Heiman saying that the first Dirichlet eigenvalue in a simply connected domain in $\mathbb{R}^2$ is bounded from both above and below by constants divided by the square of the inradius. If one modifies the definition of the inradius, then the theorem holds for all domains in all dimensions.

In collaboration with Th. Kappeler, P. Perry, and P. Topalov, Misha studied solutions of some integrable systems that are unbounded or of low regularity.

Misha worked in other areas of mathematics: holomorphic projection-valued functions, discrete problems, non-standard analysis.

Misha's impact is not limited to his research publications. His book *Pseudodifferential operators and spectral theory* that was first published in 1978 is still one of the major introductory texts in microlocal analysis. The book *The Schrödinger operator* that Misha wrote in collaboration with F. A. Berezin treats many topics in a unique way. This book was finished by Misha after the tragic death of Berezin. Misha wrote a textbook in PDE, and he authored and co-authored numerous expository articles.

Misha was a remarkable lecturer. His style was a bit dry, and he paid attention to all small details. He was actually rather enthusiastic a person, but he was able to keep his emotions in check. Most of his life he taught in the Moscow State University, and from 1992 in the Northeastern University in Boston, where he was a Professor and later the Distinguished Professor. In the seventies and in the eighties, he was a regular lecturer in the Voronezh Winter Mathematical Schools. One year he would give a course in Fedosov's approach to index theory, another year in non-standard analysis – and so on. Starting from the time when Misha was a graduate student, he worked a lot with high school students, mostly in summer math. camps. About 20 students defended their Ph.D. dissertations under his directions, and he influenced many mathematicians who were not his students.

Misha was famous for taking detailed notes of every talk that he attended. From 1964, when he was still an undergraduate student, he attended the Gel'fand seminar that was one of the main focal points of the mathematical life in the USSR. Now his Gel'fand seminar notes that cover 25 years, with almost no interruption, are available on the internet.

Misha loved books, both mathematical and on other subjects. His flat was covered with bookshelves and his office was filled with piles of books from the library. Though this book collection sometimes looked chaotic to an outsider, Misha knew precisely where each of these thousands of books was located and what was significant about it. He had a sort of a personal relationship with his books. When he shared one of them with a friend or a colleague, he always explained why that was a good book, why he bought it, and why it was important to him.

Though mathematics was Misha's main passion, he had many other interests: music, literature, art, linguistics. He was a kind person, but he was also a person with a strong moral compass. That was not always beneficial for his career in the Soviet Union. He was eager to help, and he was actually helping people who had difficulties because of political reasons. Misha was a remarkable mathematician and a wonderful human being. We are grateful to fate for having known him.

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Maxim Braverman
College of Science, Northeastern University, 567 LA (Lake Hall), Boston, MA 02115, USA;
m.braverman@northeastern.edu

Leonid Friedlander
Department of Mathematics, The University of Arizona, 617 N. Santa Rita Ave.,
P.O. Box 210089, Tucson, AZ 85721-0089, USA; friedlan@math.arizona.edu