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Semiclassical Gevrey operators and magnetic translations

Michael Hitrik, Richard Lascar, Johannes Sjöstrand, and Maher Zerzeri

Abstract. We study semiclassical Gevrey pseudodifferential operators acting on the Bargmann space of entire functions with quadratic exponential weights. Using some ideas from time frequency analysis, we show that such operators are uniformly bounded on a natural scale of exponentially weighted spaces of holomorphic functions, provided that the Gevrey index is greater than or equal to 2.

In memory of Misha Shubin

1. Introduction and statement of results

The purpose of this paper is to study continuity properties of semiclassical Gevrey pseudodifferential operators acting on exponentially weighted spaces of entire holomorphic functions on $\mathbf{C}^n$, providing an alternative approach to some of the results established in the recent work [11]. Let us proceed to describe the assumptions and state the main results.

Let $\Omega \subset \mathbf{R}^m$ be open and let $s \geq 1$. The Gevrey class $\mathcal{G}^s(\Omega)$ consists of all functions $u \in C^\infty(\Omega)$ such that for any $K \subset \Omega$ compact there exist $A > 0$, $C > 0$ such that for all $\alpha \in \mathbf{N}^m$ we have

$$|\partial^\alpha u(x)| \leq AC^{|\alpha|}(\alpha!)^s, \quad x \in K. \tag{1.1}$$

The class $\mathcal{G}^1(\Omega)$ is the space of real analytic functions on Ω, while, for $s > 1$, we have $\mathcal{G}_0^s(\Omega) := \mathcal{G}^s(\Omega) \cap C_0^\infty(\Omega) \neq \{0\}$, see [15, Theorem 1.3.5]. In this work we shall specifically be concerned with the subspace $\mathcal{G}_b^s(\mathbf{R}^m) \subset \mathcal{G}^s(\mathbf{R}^m)$ of functions $u \in C^\infty(\mathbf{R}^m)$ satisfying the Gevrey condition (1.1) uniformly on all of $\mathbf{R}^m$, for some $s > 1$: we have $u \in \mathcal{G}_b^s(\mathbf{R}^m)$ precisely when there exist $A > 0$, $C > 0$ such that for all $\alpha \in \mathbf{N}^m$,

$$|\partial^\alpha u(x)| \leq AC^{|\alpha|}(\alpha!)^s, \quad x \in \mathbf{R}^m. \tag{1.2}$$

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Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbf{C}^n$ and let us introduce the real linear subspace

$$\Lambda_{\Phi_0} = \left\{\left(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\right): x \in \mathbf{C}^n\right\} \subset \mathbf{C}^{2n} = \mathbf{C}^n_x \times \mathbf{C}^n_\xi. \tag{1.3}$$

Identifying Λ_{Φ_0} linearly with $\mathbf{C}^n_x$, via the projection map $\pi_x: \Lambda_{\Phi_0} \ni (x, \xi) \mapsto x \in \mathbf{C}^n_x$, we may define the Gevrey spaces $\mathscr{G}^s(\Lambda_{\Phi_0})$, $\mathscr{G}^s_0(\Lambda_{\Phi_0})$, $\mathscr{G}^s_b(\Lambda_{\Phi_0})$.

Given $a \in \mathscr{G}^s_b(\Lambda_{\Phi_0})$, for some $s > 1$, and $u \in \mathrm{Hol}(\mathbf{C}^n)$ such that for all $N \geq 0$ we have $u(x) = \mathcal{O}_{h,N}(1)\langle x\rangle^{-N} e^{\Phi_0(x)/h}$, let us introduce the semiclassical Weyl quantization of a acting on u,

$$\mathrm{Op}^w_h(a)u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\left(\frac{x+y}{2}, \theta\right) u(y)\, dy \wedge d\theta. \tag{1.4}$$

Here $0 < h \leq 1$ is the semiclassical parameter and $\Gamma(x) \subset \mathbf{C}^{2n}_{y,\theta}$ is the natural integration contour given by

$$\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right). \tag{1.5}$$

The operator $\mathrm{Op}^w_h(a)$ extends to a uniformly bounded map

$$\mathrm{Op}^w_h(a) = \mathcal{O}(1): H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n), \tag{1.6}$$

see [13, 27]. Here $H_{\Phi_0}(\mathbf{C}^n)$ is the Bargmann space defined by

$$H_{\Phi_0}(\mathbf{C}^n) = \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_0/h} L(dx)), \tag{1.7}$$

with $L(dx)$ being the Lebesgue measure on $\mathbf{C}^n$. Now the mapping property (1.6) follows merely from the fact that $\nabla^k a \in L^\infty(\Lambda_{\Phi_0})$ for all $k \in \mathbf{N}$, and the Gevrey smoothness of a allows us to consider other weights as well. The effect of modifying the exponential weight has been considered in [11], and the following result has been established there, see [11, Theorems 3.3 and 3.4].

Theorem 1.1. *Let $a \in \mathscr{G}^s_b(\Lambda_{\Phi_0})$, $s > 1$, and let $\Phi_1 \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$ be such that*

$$\|\nabla^k(\Phi_1 - \Phi_0)\|_{L^\infty(\mathbf{C}^n)} \leq \frac{1}{C} h^{1-\frac{1}{s}}, \quad k = 0, 1, 2, \tag{1.8}$$

where $C > 0$ is large enough. Then, the operator $\mathrm{Op}^w_h(a)$ extends to a uniformly bounded map

$$\mathrm{Op}^w_h(a) = \mathcal{O}(1): H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_1}(\mathbf{C}^n). \tag{1.9}$$

Here, similarly to (1.7), we have set

$$H_{\Phi_1}(\mathbf{C}^n) = \mathrm{Hol}(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, e^{-2\Phi_1/h} L(dx)).$$

The proof of Theorem 1.1 in [11] proceeds by a contour deformation argument in (1.4), introducing a Gevrey almost holomorphic extension of $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$ and making use of Stokes's theorem. A noteworthy aspect of the proof developed in [11] is that performing a deformation to a contour of the form

$$\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\Big(\frac{x+y}{2}\Big) + \frac{i}{C}\overline{(x-y)}, \quad C > 0, \tag{1.10}$$

natural in the analytic theory [13,22,27], does not lead to some exponentially accurate remainder estimates of the form

$$\mathcal{R} = \mathcal{O}(1)\exp\Big(-\frac{1}{\mathcal{O}(1)}h^{-\frac{1}{s}}\Big)\colon H_{\Phi_0}(\mathbf{C}^n) \to L^2(\mathbf{C}^n, e^{-2\Phi_0/h}L(dx)), \tag{1.11}$$

natural in the Gevrey theory. To overcome this issue, the argument in [11] proceeds by deforming to a suitable "mixed" contour, using (1.10) in the region

$$|x-y| \leq \frac{1}{\mathcal{O}(1)}h^{1-\frac{1}{s}}$$

only. An additional deformation to a "mixed" contour adapted to the weight Φ_1 leads then to the uniform boundedness in (1.9) in the range $s \in (1,2]$ only, and some further work, involving another change of contour, is required to recover the mapping property (1.9) in the full range $s > 1$. See [11, Theorem 3.4]. Let us also remark that, as explained in [11], when obtaining a uniformly bounded realization of the operator in (1.9) for $s > 2$, via a contour deformation, one has to accept a remainder which is larger than the one in (1.11).

Our purpose here is to give a direct proof of Theorem 1.1 in the "complementary" region $s \geq 2$, avoiding the use of contour deformations entirely. Specifically, the following is the main result of this work.

Theorem 1.2. *Let $a \in \mathcal{G}_b^s(\Lambda_{\Phi_0})$, for some $s \geq 2$, and let $\Phi_1 \in C^{1,1}(\mathbf{C}^n;\mathbf{R})$ be such that*

$$\|\nabla^k(\Phi_1 - \Phi_0)\|_{L^\infty(\mathbf{C}^n)} \leq \frac{1}{C}h^{1-\frac{1}{s}}, \quad k = 0,1, \tag{1.12}$$

where $C > 0$ is large enough. Then, the operator $\mathrm{Op}_h^w(a)$ extends to a uniformly bounded map

$$\mathrm{Op}_h^w(a) = \mathcal{O}(1)\colon H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_1}(\mathbf{C}^n). \tag{1.13}$$

When establishing Theorem 1.2, rather than performing a contour deformation in (1.4), we shall proceed by following some basic ideas of the time frequency analysis [7, 9, 28], decomposing the symbol a into a superposition of coherent states of the form $\Lambda_{\Phi_0} \ni X \mapsto e^{2i\sigma(X,Y)/h}\chi_0((X-T)/h^{1/2})$, for $Y, T \in \Lambda_{\Phi_0}$. Here σ is the complex symplectic form on $\mathbf{C}^{2n}_{x,\xi}$ and χ_0 is a fixed function in $\mathcal{G}_b^s(\Lambda_{\Phi_0})$, which we can choose essentially as a real Gaussian. Passing to the Weyl quantizations leads to the representation of the operator $\mathrm{Op}_h^w(a)$ as a direct integral of certain rank-one projections expressed in terms of the "magnetic translations" $e^{i\sigma((x,hD_x),Y)/h}$, for $Y \in \Lambda_{\Phi_0}$, unitary on $H_{\Phi_0}(\mathbf{C}^n)$, whose operator norm on $H_{\Phi_1}(\mathbf{C}^n)$ can be controlled. Theorem 1.2 follows from these observations, by an application of Schur's lemma, when combined with the Wiener type characterization of the Gevrey space $\mathcal{G}_b^s(\Lambda_{\Phi_0})$. We refer to [25, 26] for the original works on the Wiener algebras of pseudodifferential operators. See also [8, 28], and the references given there. It is perhaps of note that the time frequency approach to the proof of Theorem 1.2 appears to work for the Gevrey indices $s \geq 2$ only, which is precisely the range where the contour deformation method of [11] encounters some difficulties. The restriction $s \geq 2$ appears at the very end of the proof of Theorem 1.2 for some seemingly technical reason, and the question whether the methods developed in the present work could be extended to obtain the uniform boundedness results in the entire range $s > 1$ seems interesting to us, demanding further attention. We hope to be able to address it in some future work.

Let us point out that the present work, as well as [11], are largely motivated by the relevance of results such as Theorem 1.2 in the study of microlocal Gevrey regularity questions for solutions of Gevrey pseudodifferential equations, say. To motivate this a bit further, let us recall the characterization of the semiclassical wave front set in the Gevrey framework, see [4]. Let $u(h)$ be a tempered family in $L^2(\mathbf{R}^n)$, so that $\|u(h)\|_{L^2(\mathbf{R}^n)} \leq \mathcal{O}(h^{-K})$, for some $K \geq 0$, and let $\mathcal{T}\colon L^2(\mathbf{R}^n) \to H_{\Phi_0}(\mathbf{C}^n)$ be a generalized Bargmann transformation with the associated canonical transformation $\kappa_{\mathcal{T}}$, see [13, 27], and also (4.28) below. Given $(y_0, \eta_0) \in T^*\mathbf{R}^n = \mathbf{R}^{2n}$, we have $(y_0, \eta_0) \notin \mathrm{WF}_{s,h}(u(h))$, for some $s > 1$, precisely when there exists an open neighborhood $V \subset \mathbf{C}^n$ of $x_0 = \pi_x(\kappa_{\mathcal{T}}(y_0, \eta_0)) \in \mathbf{C}^n$ and $C > 0$ such that

$$|\mathcal{T}u(x;h)| \leq \mathcal{O}(1)e^{\frac{\Phi_0(x)}{h}} \exp\Big(-\frac{1}{C}h^{-\frac{1}{s}}\Big), \quad x \in V,\ 0 < h \leq 1.$$

When studying microlocal properties of solutions to a pseudodifferential equation of the form $\mathrm{Op}_h^w(\tilde{a})u(h) = 0$, say, where $\tilde{a} \in \mathcal{G}_b^s(\mathbf{R}^{2n})$, rather than looking at high order derivatives directly, it is therefore natural to work on the Bargmann transform side, considering deformations Φ_1 of the quadratic weight function Φ_0, letting the conjugated operator $\mathrm{Op}_h^w(a) = \mathcal{T} \circ \mathrm{Op}_h^w(\tilde{a}) \circ \mathcal{T}^{-1}$ act on the new exponentially weighted spaces $H_{\Phi_1}(\mathbf{C}^n)$ of holomorphic functions. The result of Theorem 1.2 may therefore be viewed as the first step in the implementation of this general program in the Gevrey framework. Let us also emphasize that in the analytic case, this approach has proven to be quite fruitful, see for instance [22, 23, 27].

We would also like to mention that an interesting class of problems, where operators such as those considered in Theorem 1.2 occur naturally, comes from the study of scattering poles for semiclassical Schrödinger operators with Gevrey potentials, see [19, 20].

The plan of the paper is as follows. In Section 2, we study mapping properties of magnetic translations on the weighted spaces $H_{\Phi_0}(\mathbf{C}^n)$, $H_{\Phi_1}(\mathbf{C}^n)$. In Section 3, as a preparation for the proof of Theorem 1.2, we consider compactly supported Gevrey symbols $a \in \mathscr{G}_0^s(\Lambda_{\Phi_0})$. In this case, the mapping property (1.13) can be established in the full range $s > 1$, by decomposing the operator $\mathrm{Op}_h^w(a)$ into a superposition of magnetic translations directly. Section 4 is devoted to the proof of Theorem 1.2 in the general case. As alluded to above, an essential role in the proof is played by a decomposition of the operator $\mathrm{Op}_h^w(a)$ into a direct integral of rank-one projections, and we would like to emphasize that it can be viewed as the Bargmann space analogue of the corresponding decomposition established in [9] in the real setting. In Appendix A we recall the composition formulas for the Weyl h-pseudodifferential calculus in the complex domain, obtained by the method of magnetic translations.

We dedicate this paper to the memory of Misha Shubin and would like to acknowledge his pioneering contributions to the global theory of pseudodifferential operators [21, 32], of which this work is a more recent descendant.

2. Magnetic translations on weighted spaces

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbf{C}^n$ and let $\Lambda_{\Phi_0} \subset \mathbf{C}^{2n}$ be defined as in (1.3). The real $2n$-dimensional linear subspace Λ_{Φ_0} is I-Lagrangian and R-symplectic, in the sense that the restriction of the complex symplectic (2,0)-form

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j \tag{2.1}$$

on $\mathbf{C}^{2n} = \mathbf{C}_x^n \times \mathbf{C}_\xi^n$ to Λ_{Φ_0} is real and non-degenerate. In particular, Λ_{Φ_0} is maximally totally real.

Let $\ell(x, \xi)$ be a complex linear form on $\mathbf{C}^{2n}$ so that

$$\ell(x, \xi) = \ell'_x \cdot x + \ell'_\xi \cdot \xi = \sigma((x, \xi), H_\ell), \quad H_\ell = \ell'_\xi \cdot \partial_x - \ell'_x \cdot \partial_\xi. \tag{2.2}$$

Let us notice that the restriction $\ell|_{\Lambda_{\Phi_0}}$ is real precisely when the Hamilton vector field $H_\ell \in T\Lambda_{\Phi_0}$. Here we identify the holomorphic (constant) vector field H_ℓ with the corresponding real vector field $H_\ell^\rho = H_\ell + \overline{H_\ell}$. We have therefore

$$-\ell'_x = \frac{2}{i}\big((\Phi_0)''_{xx}\ell'_\xi + (\Phi_0)''_{x\bar{x}}\overline{\ell'_\xi}\big) = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\ell'_\xi), \tag{2.3}$$

and we may write

$$\ell(x, \xi) = \sigma((x, \xi), H_\ell) = -\frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x^*) \cdot x + x^* \cdot \xi, \quad (x, \xi) \in \mathbf{C}^{2n}, \tag{2.4}$$

for some unique $\mathbf{C}^n \ni x^* = \ell'_\xi$.

To a complex linear form ℓ on $\mathbf{C}^{2n}$, such that $\ell|_{\Lambda_{\Phi_0}}$ is real, we associate the operator

$$\mathrm{Op}_h^w(e^{-i\ell/h}) = e^{-i\ell(x,hD_x)/h}, \tag{2.5}$$

and recall from [27, Proposition 1.4] that

$$e^{-i\ell(x,hD_x)/h} = e^{-\frac{i}{2h}\ell'_x\cdot x} \circ \tau_{\ell'_\xi} \circ e^{-\frac{i}{2h}\ell'_x\cdot x}. \tag{2.6}$$

Here τ_z is the operator of translation by $z \in \mathbf{C}^n$, $(\tau_z u)(x) = u(x-z)$. Throughout this paper, operators of the form (2.6) will be referred to as *magnetic translations,* in view of the fact that such operators appear naturally in the study of Schrödinger operators with magnetic fields, see [5, 24].

Let us recall from [27] that the first order differential operator $\ell(x, hD_x)$ is self-adjoint on the Bargmann space $H_{\Phi_0}(\mathbf{C}^n)$, when equipped with the maximal domain $\{u \in H_{\Phi_0}(\mathbf{C}^n); \ell(x,hD_x)u \in H_{\Phi_0}(\mathbf{C}^n)\}$. In particular, the operator in (2.5) is unitary when acting on $H_{\Phi_0}(\mathbf{C}^n)$. For future reference, it will be convenient for us to start by doing the exercise mentioned in the proof of [27, Proposition 1.4], verifying the unitarity directly, using the expression (2.6). See also [28, Section 3].

Given $u \in H_{\Phi_0}(\mathbf{C}^n)$, we shall show that

$$|(e^{-\Phi_0/h}e^{-i\ell(x,hD_x)/h}u)(x)| = |(e^{-\Phi_0/h}u)(x-\ell'_\xi)|, \quad x \in \mathbf{C}^n. \tag{2.7}$$

When doing so, let us write using (2.6),

$$e^{-i\ell(x,hD_x)/h}u(x) = e^{\frac{i}{2h}\ell'_x\cdot\ell'_\xi}e^{-\frac{i}{h}\ell'_x\cdot x}u(x-\ell'_\xi). \tag{2.8}$$

A simpler expression is obtained, by exploiting the natural symmetry in the Weyl quantization, if we express $e^{-i\ell(x,hD_x)/h}u$ at the point $x+\ell'_\xi/2$ in terms of u at the point $x-\ell'_\xi/2$, which gives that

$$(e^{-i\ell(x,hD_x)/h}u)\Big(x+\frac{\ell'_\xi}{2}\Big) = e^{-i\ell'_x\cdot x/h}u\Big(x-\frac{\ell'_\xi}{2}\Big). \tag{2.9}$$

It follows that

$$\begin{aligned}(e^{-\frac{1}{h}\Phi_0}&e^{-i\ell(x,hD_x)/h}u)\Big(x+\frac{\ell'_\xi}{2}\Big)\\ &= e^{-\frac{1}{h}(\Phi_0(x+\frac{1}{2}\ell'_\xi)-\Phi_0(x-\frac{1}{2}\ell'_\xi))}e^{-i\ell'_x\cdot x/h}(e^{-\frac{1}{h}\Phi_0}u)\Big(x-\frac{\ell'_\xi}{2}\Big).\end{aligned} \tag{2.10}$$

We next observe that

$$\Phi_0\Big(x+\frac{1}{2}\ell'_\xi\Big) - \Phi_0\Big(x-\frac{1}{2}\ell'_\xi\Big) = 2\,\mathrm{Re}(\partial_x\Phi_0(x)\cdot\ell'_\xi), \tag{2.11}$$

in view of the following standard consequence of Taylor's formula:

$$\Phi_0(x) - \Phi_0(y) = 2\,\mathrm{Re}\Big(\partial_x \Phi_0\Big(\frac{x+y}{2}\Big) \cdot (x-y)\Big), \tag{2.12}$$

valid for the real valued quadratic form Φ_0. Moreover, using (2.3), we see that the right-hand side of (2.11) takes the form

$$\begin{aligned} 2\,\mathrm{Re}(\partial_x \Phi_0(x) \cdot \ell'_\xi) &= 2\,\mathrm{Re}(\partial_x \Phi_0(\ell'_\xi) \cdot x) \\ &= -\mathrm{Re}\Big(i\Big(-\frac{2}{i}\partial_x \Phi_0(\ell'_\xi) \cdot x\Big)\Big) = -\mathrm{Re}(i\ell'_x \cdot x). \end{aligned} \tag{2.13}$$

Here we have also used the general relation

$$\mathrm{Re}(\partial_x \Phi_0(x) \cdot y) = \mathrm{Re}(\partial_x \Phi_0(y) \cdot x), \quad x, y \in \mathbf{C}^n. \tag{2.14}$$

Combining (2.11) and (2.13), we obtain that

$$\Phi_0\Big(x + \frac{1}{2}\ell'_\xi\Big) - \Phi_0\Big(x - \frac{1}{2}\ell'_\xi\Big) + \mathrm{Re}(i\ell'_x \cdot x) = 0, \tag{2.15}$$

and the absolute value of the prefactor in the right-hand side of (2.10) is therefore equal to 1. We get from (2.10) and (2.15),

$$\Big|(e^{-\frac{1}{h}\Phi_0} e^{-i\ell(x,hD_x)/h} u)\Big(x + \frac{\ell'_\xi}{2}\Big)\Big| = \Big|(e^{-\frac{1}{h}\Phi_0}) u\Big(x - \frac{\ell'_\xi}{2}\Big)\Big|,$$

and (2.7) follows, by replacing x by $x - \ell'_\xi/2$.

It follows from (2.7) that the operator

$$e^{-i\ell(x,hD_x)/h} \colon H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n) \tag{2.16}$$

is an isometry, and is therefore unitary, since it is a bijection, with the inverse given by $e^{i\ell(x,hD_x)/h} \colon H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n)$.

Remark. Magnetic translations play a role also in the theory of Toeplitz operators, where they appear under the name *Weyl operators,* see [1].

We shall now consider weights other than Φ_0. To this end, let $\Phi_1 \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$, the space of C^1-functions on $\mathbf{C}^n$ with a globally Lipschitz gradient. In particular, we know, thanks to Rademacher's theorem, that

$$\nabla^2 \Phi_1 \in L^\infty(\mathbf{C}^n). \tag{2.17}$$

Let us also assume that $\|\nabla^k(\Phi_1 - \Phi_0)\|_{L^\infty(\mathbf{C}^n)}$ are small enough, for $k = 0, 1$. We would like to consider magnetic translations acting on the weighted space

$$H_{\Phi_1}(\mathbf{C}^n) = L^2(\mathbf{C}^n, e^{-2\Phi_1/h} L(dx)) \cap \mathrm{Hol}(\mathbf{C}^n). \tag{2.18}$$

To this end, viewing the operator in (2.6) as a Fourier integral operator associated to the complex canonical transformation $\exp(H_\ell)$, we shall first determine the image of the Lipschitz manifold

$$\Lambda_{\Phi_1} = \left\{\left(x, \frac{2}{i}\frac{\partial \Phi_1}{\partial x}(x)\right) : x \in \mathbf{C}^n\right\} \subset \mathbf{C}^{2n}, \tag{2.19}$$

under the map $\exp(H_\ell)\colon \mathbf{C}^{2n} \ni \rho \mapsto \rho + H_\ell \in \mathbf{C}^{2n}$. Here we may notice that the manifold Λ_{Φ_1} is I-Lagrangian, in the sense that its almost everywhere defined tangent plane is Lagrangian with respect to $\operatorname{Im}\sigma$.

We have the following result, where we write $\Phi_1(x) = \Phi_0(x) + f(x)$.

Proposition 2.1. *Let $\ell(x,\xi)$ be a complex linear form on $\mathbf{C}^{2n}$ such that $\ell|_{\Lambda_{\Phi_0}}$ is real, and let us represent ℓ in the form* (2.4)*. Then,*

$$\exp(H_\ell)(\Lambda_{\Phi_1}) = \Lambda_{\Phi_2},$$

where $\Phi_2 \in C^{1,1}(\mathbf{C}^n)$ is given by

$$\Phi_2(x) = \Phi_1(x) + f(x - x^*) - f(x), \quad x \in \mathbf{C}^n. \tag{2.20}$$

Proof. Following the proof of [3, Lemma 2.2], let us consider the real Hamilton–Jacobi equation

$$\frac{\partial \Psi}{\partial t}(x,t) - \operatorname{Im}\ell\left(x, \frac{2}{i}\frac{\partial \Psi}{\partial x}(x,t)\right) = 0, \quad \Psi(x,0) = \Phi_1(x), \tag{2.21}$$

for $x \in \mathbf{C}^n$, $t \in \mathbf{R}$, $t \geq 0$. Associated to the function $\Psi(x,t) \in C^{1,1}(\mathbf{C}^n \times \mathbf{R}; \mathbf{R})$ in (2.21) is the Lipschitz manifold

$$L_\Psi = \left\{\left(t, \frac{\partial \Psi}{\partial t}(x,t), x, \frac{2}{i}\frac{\partial \Psi}{\partial x}(x,t)\right)\right\} \subset \mathbf{R}^2_{t,\tau} \times \mathbf{C}^{2n}_{x,\xi}, \tag{2.22}$$

which is Lagrangian with respect to the real symplectic form

$$d\tau \wedge dt - \operatorname{Im}\sigma. \tag{2.23}$$

The function $\tau - \operatorname{Im}\ell$ vanishes along L_Ψ, in view of (2.21), and therefore its Hamilton vector field, computed with respect to the real symplectic form (2.23), is tangent to L_Ψ, almost everywhere. Using the general relation

$$H^\rho_\ell = H^{-\operatorname{Im}\sigma}_{-\operatorname{Im}\ell}, \tag{2.24}$$

valid for any $\ell(x,\xi)$ holomorphic, where $H^\rho_\ell = H_\ell + \overline{H_\ell}$ is the real vector field naturally associated to the holomorphic vector field H_ℓ, see [18,22], we conclude that the vector field

$$\partial_t + H^{-\operatorname{Im}\sigma}_{-\operatorname{Im}\ell} = \partial_t + H^\rho_\ell$$

is tangent to L_Ψ. Identifying H_ℓ^ρ and H_ℓ, we get therefore

$$\Lambda_{\Psi(\cdot,t)} = \exp(tH_\ell)(\Lambda_{\Phi_1}), \quad t \in \mathbf{R}.$$

Referring to [18, Section 1] for the detailed proof of (2.24), let us just indicate that it is based on the usual pointwise relations

$$\langle \sigma, t \wedge H_\ell \rangle = \langle d\ell, t \rangle, \quad \langle \sigma, t \wedge \overline{H_\ell} \rangle = 0,$$

valid for $\ell(x,\xi)$ holomorphic and for all $t \in T(\mathbf{C}^{2n}) \otimes \mathbf{C}$, as well as the relation

$$\langle \operatorname{Im} \sigma, t \wedge H_{\operatorname{Im} \ell}^{\operatorname{Im} \sigma} \rangle = \langle d \operatorname{Im} \ell, t \rangle,$$

which holds for all real tangent vectors $t \in T(\mathbf{C}^{2n})$.

We claim now that the unique $C^{1,1}$-solution of (2.21) is given by

$$\Psi(x,t) = \Phi_1(x) + f(x - tx^*) - f(x) = \Phi_0(x) + f(x - tx^*). \tag{2.25}$$

Indeed, using (2.25), (2.4), and the fact that $\ell|_{\Lambda_{\Phi_0}}$ is real, we see that

$$\begin{aligned} \operatorname{Im} \ell\Big(x, \frac{2}{i}\frac{\partial \Psi}{\partial x}(x,t)\Big) &= \operatorname{Im}\Big(x^* \cdot \frac{2}{i}\frac{\partial f}{\partial x}(x - tx^*)\Big) \\ &= -2 \operatorname{Re}\Big(x^* \cdot \frac{\partial f}{\partial x}(x - tx^*)\Big), \end{aligned} \tag{2.26}$$

which agrees with

$$\frac{\partial \Psi}{\partial t}(x,t) = \partial_t(f(x - tx^*)) = -(x^* \cdot f_x'(x - tx^*) + \overline{x^*} \cdot f_{\bar{x}}'(x - tx^*)).$$

This shows (2.25) and completes the proof. ∎

Proposition 2.1 can be viewed as an indication that we have a uniformly bounded Fourier integral operator,

$$e^{-i\ell(x,hD_x)/h} = \mathcal{O}(1) \colon H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_2}(\mathbf{C}^n). \tag{2.27}$$

Here the weighted space of holomorphic functions $H_{\Phi_2}(\mathbf{C}^n)$ is defined analogously to (2.18). In order to give a direct verification of the mapping property (2.27), we observe that (2.7) gives, for $u \in H_{\Phi_1}(\mathbf{C}^n)$,

$$|(e^{-\Phi_2/h} e^{-i\ell(x,hD_x)/h} u)(x)| = |(e^{-\Phi_1/h} u)(x - x^*)|, \quad x \in \mathbf{C}^n. \tag{2.28}$$

Here Φ_2 is given by (2.20). It follows from (2.28) that the operator in (2.27) is an isometry, and therefore unitary.

The discussion in this section may be summarized in the following result.

Theorem 2.2. *Let $\ell(x,\xi)$ be a complex linear form on $\mathbf{C}^{2n}$ such that $\ell|_{\Lambda_{\Phi_0}}$ is real, and let us represent ℓ in the form (2.4). We have the unitary operators*

$$e^{-i\ell(x,hD_x)/h}\colon H_{\Phi_0}(\mathbf{C}^n) \to H_{\Phi_0}(\mathbf{C}^n), \tag{2.29}$$

and

$$e^{-i\ell(x,hD_x)/h}\colon H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_2}(\mathbf{C}^n). \tag{2.30}$$

Here in (2.30), the weight function

$$\Phi_1 = \Phi_0 + f \in C^{1,1}(\mathbf{C}^n;\mathbf{R})$$

is such that $\|\nabla^k(\Phi_1 - \Phi_0)\|_{L^\infty(\mathbf{C}^n)}$ are small enough, for $k = 0, 1$, and $\Phi_2(x) = \Phi_0(x) + f(x - x^*)$.

Remark. The purpose of this remark is to outline an alternative approach to the Hamilton–Jacobi equation (2.21), leading directly the unitarity of the operator in equation (2.30). To this end, let $u \in H_{\Phi_1}(\mathbf{C}^n)$, and let us differentiate formally the scalar product

$$(e^{-it\ell(x,hD_x)/h}u, e^{-it\ell(x,hD_x)/h}u)_{H_{\Psi_t}} = (u(t),u(t))_{H_{\Psi_t}} \tag{2.31}$$

with respect to $t \in \mathbf{R}$. Here $\Psi_t \in C^{1,1}$ is to be chosen so that the time derivative of (2.31) vanishes. We refer to [10, 12, 14, 29] for other instances of this approach, which is particularly straightforward in the present linear setting. We get

$$\begin{aligned} h\partial_t(u(t),u(t))_{H_{\Psi_t}} &= h\partial_t \int u(t,x)\overline{u(t,x)}e^{-2\Psi_t(x)/h}L(dx) \\ &= -i(\ell(x,hD_x)u(t),u(t))_{H_{\Psi_t}} + i(u(t),\ell(x,hD_x)u(t))_{H_{\Psi_t}} \\ &\quad - \int 2\partial_t\Psi_t(x)|u(t,x)|^2 e^{-2\Psi_t(x)/h}L(dx). \end{aligned} \tag{2.32}$$

Using that $hD_x(\overline{u(t,x)}) = 0$, we obtain that

$$\begin{aligned} (\ell(x,hD_x)u(t),u(t))_{H_{\Psi_t}} &= \int \ell(x,hD_x)u(t,x)\overline{u(t,x)}e^{-2\Psi_t(x)/h}L(dx) \\ &= \int |u(t,x)|^2 \ell^{\mathrm{t}}(x,hD_x)(e^{-2\Psi_t(x)/h})L(dx). \end{aligned} \tag{2.33}$$

Here

$$\ell^{\mathrm{t}}(x,hD_x) = \ell(x,-hD_x)$$

is the transpose of the first order differential operator $\ell(x,hD_x)$, and therefore we get the quantization-multiplication formula in its exact version, see also [13],

$$(\ell(x,hD_x)u(t),u(t))_{H_{\Psi_t}} = \int \ell\Big(x, \frac{2}{i}\frac{\partial\Psi_t}{\partial x}(x)\Big)|u(t,x)|^2 e^{-2\Psi_t(x)/h}L(dx). \tag{2.34}$$

Combining (2.32) and (2.34), we get

$$\begin{aligned} &h\partial_t (u(t), u(t))_{H_{\Psi_t}} \\ &\quad = -2 \int \Big(\partial_t \Psi_t(x) - \operatorname{Im} \ell\Big(x, \frac{2}{i}\frac{\partial \Psi_t}{\partial x}(x)\Big)\Big) |u(t,x)|^2 e^{-2\Psi_t(x)/h} L(dx). \end{aligned} \tag{2.35}$$

The unitarity of the map

$$e^{-it\ell(x,hD_x)/h} \colon H_{\Phi_1}(\mathbf{C}^n) \to H_{\Psi_t}(\mathbf{C}^n), \quad t \in \mathbf{R},$$

is therefore implied by the vanishing of the first factor under the integral sign in the right-hand side of (2.35), which agrees with the Hamilton–Jacobi equation (2.21).

3. Fourier inversion and compactly supported Gevrey symbols

The purpose of this section is to prove Theorem 1.2 in the special case when $a \in \mathcal{G}_0^s(\Lambda_{\Phi_0})$ is compactly supported. Here we shall let $s > 1$ be arbitrary. When doing so, let us start by recalling the invariant form of the Fourier inversion formula on a real symplectic vector space (W, σ) of dimension $2n$, see [30]. Let

$$\mathcal{F}u(X) = \hat{u}(X) = \frac{1}{\pi^n} \int e^{2i\sigma(X,Y)} u(Y)\, dY, \quad u \in \mathcal{S}(W), \tag{3.1}$$

be the (twisted) Fourier transformation on W. Here dY is the symplectic volume form on W. Then, $\mathcal{F}^2 = I$ on $\mathcal{S}(W)$ and the Fourier transformation extends to a unitary selfadjoint involution of $L^2(W)$. After a change of variables, we get the following semiclassical version of the Fourier inversion formula:

$$a(X) = \frac{1}{\pi^n h^{2n}} \int e^{2i\sigma(X,Y)/h} \hat{a}\Big(\frac{Y}{h}\Big)\, dY, \quad a \in \mathcal{S}(W). \tag{3.2}$$

Specializing (3.2) to the case when $W = \Lambda_{\Phi_0}$, with the symplectic form given by $\sigma|_{\Lambda_{\Phi_0}}$, where σ is the complex symplectic $(2,0)$-form on $\mathbf{C}^{2n}$ defined in (2.1), $a \in \mathcal{G}_0^s(\Lambda_{\Phi_0})$, and passing to the Weyl quantizations, we get

$$a^w(x, hD_x) := \operatorname{Op}_h^w(a) = \frac{1}{\pi^n h^{2n}} \int_{\Lambda_{\Phi_0}} \hat{a}\Big(\frac{Y}{h}\Big) e^{2i\sigma((x,hD_x),Y)/h}\, dY. \tag{3.3}$$

Here for $Y \in \Lambda_{\Phi_0}$, the complex linear form

$$\ell_Y(x,\xi) = \sigma((x,\xi),Y), \quad (x,\xi) \in \mathbf{C}^{2n} \tag{3.4}$$

is real along Λ_{Φ_0} and Theorem 2.2 provides us therefore with some precise mapping properties for the magnetic translations $e^{2i\sigma((x,hD_x),Y)/h}$, for $Y \in \Lambda_{\Phi_0}$.

It is now easy to finish the proof of Theorem 1.2 in the special case of compactly supported Gevrey symbols. Let $\Phi_1 = \Phi_0 + f \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$ be such that

$$\|\nabla^k f\|_{L^\infty(\mathbf{C}^n)} \le \frac{1}{C} h^{1-\frac{1}{s}}, \quad k = 0, 1, \tag{3.5}$$

for some $C > 0$. For each $Y = (y, \eta) \in \Lambda_{\Phi_0}$, in view of Theorem 2.2, we have the unitary operators

$$e^{2i\sigma((x,hD_x),Y)/h}\colon H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_2}(\mathbf{C}^n), \tag{3.6}$$

where $\Phi_2(x) = \Phi_0(x) + f(x + 2y)$. It follows that

$$\begin{aligned} &\|e^{2i\sigma((x,hD_x),Y)/h}\|_{\mathcal{L}(H_{\Phi_1}(\mathbf{C}^n), H_{\Phi_1}(\mathbf{C}^n))} \\ &\quad \le \exp\Big(\frac{\|\Phi_2 - \Phi_1\|_{L^\infty(\mathbf{C}^n)}}{h}\Big) = \exp\Big(\frac{\|f(\cdot + 2y) - f(\cdot)\|_{L^\infty(\mathbf{C}^n)}}{h}\Big). \end{aligned} \tag{3.7}$$

Here

$$\|f(\cdot + 2y) - f(\cdot)\|_{L^\infty(\mathbf{C}^n)} \le \min(2\|f\|_{L^\infty(\mathbf{C}^n)}, 2\|\nabla f\|_{L^\infty(\mathbf{C}^n)}|y|), \tag{3.8}$$

and combining (3.5), (3.7), and (3.8), we get with the same constant $C > 0$ as in (3.5),

$$\begin{aligned} &\|e^{2i\sigma((x,hD_x),Y)/h}\|_{\mathcal{L}(H_{\Phi_1}(\mathbf{C}^n), H_{\Phi_1}(\mathbf{C}^n))} \\ &\quad \le \exp\Big(\frac{2}{C} h^{-1/s} \min(1, |y|)\Big), \quad Y = (y, \eta) \in \Lambda_{\Phi_0}. \end{aligned} \tag{3.9}$$

We claim that the integral in the right-hand side of (3.3) converges in the space $\mathcal{L}(H_{\Phi_1}(\mathbf{C}^n), H_{\Phi_1}(\mathbf{C}^n))$, provided that $C > 0$ in (3.5) is large enough. To this end, let us recall from [11, Section 2] the following decay estimate for the (twisted) semi-classical Fourier transform of $a \in \mathcal{G}_0^s(\Lambda_{\Phi_0})$, see also [15, Lemma 12.7.4]:

$$\Big|\hat{a}\Big(\frac{Y}{h}\Big)\Big| \le C_0 \exp\Big(-\frac{1}{C_0}\Big(\frac{|y|}{h}\Big)^{1/s}\Big), \quad Y = (y, \eta) \in \Lambda_{\Phi_0},\ C_0 > 0. \tag{3.10}$$

We get, using (3.3), (3.9), and (3.10), for some $\widetilde{C} > 0$,

$$\begin{aligned} &\|\operatorname{Op}_h^w(a)\|_{\mathcal{L}(H_{\Phi_1}(\mathbf{C}^n), H_{\Phi_1}(\mathbf{C}^n))} \\ &\quad \le \frac{1}{\pi^n h^{2n}} \int_{\Lambda_{\Phi_0}} \Big|\hat{a}\Big(\frac{Y}{h}\Big)\Big| \|e^{2i\sigma((x,hD_x),Y)/h}\|_{\mathcal{L}(H_{\Phi_1}(\mathbf{C}^n), H_{\Phi_1}(\mathbf{C}^n))}\, dY \\ &\quad \le \frac{\widetilde{C}}{h^{2n}} \int_{\mathbf{C}^n} \exp\Big(\frac{1}{h^{1/s}}\Big(-\frac{1}{C_0}|y|^{1/s} + \frac{2}{C}\min(1, |y|)\Big)\Big) L(dy). \end{aligned} \tag{3.11}$$

Here $L(dy)$ is the Lebesgue measure on $\mathbf{C}^n$. When estimating the integral in the right-hand side of (3.11), we write, using that $\min(1,|y|) \le |y|^{1/s}$ for $y \in \mathbf{C}^n$, and taking $C > 0$ large enough,

$$\begin{aligned}
&\frac{1}{h^{2n}} \int_{\mathbf{C}^n} \exp\Big(\frac{1}{h^{1/s}}\Big(-\frac{1}{C_0}|y|^{1/s} + \frac{2}{C}\min(1,|y|)\Big)\Big) L(dy) \\
&\le \frac{1}{h^{2n}} \int_{\mathbf{C}^n} \exp\Big(\frac{1}{h^{1/s}}\Big(-\frac{1}{C_0}|y|^{1/s} + \frac{2}{C}|y|^{1/s}\Big)\Big) L(dy) \\
&= \frac{1}{h^{2n}} \int_{\mathbf{C}^n} \exp\Big(-\frac{|y|^{1/s}}{h^{1/s}}\Big(\frac{1}{C_0} - \frac{2}{C}\Big)\Big) L(dy) \\
&= \int_{\mathbf{C}^n} \exp\Big(-|z|^{1/s}\Big(\frac{1}{C_0} - \frac{2}{C}\Big)\Big) L(dz) = \mathcal{O}(1). \qquad (3.12)
\end{aligned}$$

Combining (3.11) and (3.12), we get

$$\|\mathrm{Op}_h^w(a)\|_{\mathcal{L}(H_{\Phi_1}(\mathbf{C}^n), H_{\Phi_1}(\mathbf{C}^n))} \le \mathcal{O}(1). \qquad (3.13)$$

This completes the proof of Theorem 1.2, in the full range $s > 1$, in the case when $a \in \mathcal{G}_0^s(\Lambda_{\Phi_0})$.

4. Wiener conditions and rank-one decompositions

In the beginning of this section, rather than working on Λ_{Φ_0}, for simplicity we shall work on $\mathbf{R}^m \simeq T^*\mathbf{R}^n$, where $m = 2n$. Following [25, 26], we shall first establish a Wiener type characterization of the Gevrey space $\mathcal{G}_b^s(\mathbf{R}^m)$, for $s > 1$. See also [31]. When doing so, let $e_1, \dots, e_m$ be a basis of $\mathbf{R}^m$ and let

$$\Gamma = \bigoplus_{j=1}^{m} \mathbf{Z} e_j \qquad (4.1)$$

be the corresponding integer lattice. Let $0 \le \chi_0 \in C_0^\infty(\mathbf{R}^m)$ be such that

$$\sum_{j \in \Gamma} \chi_j = 1, \quad \chi_j(x) = \chi_0(x - j). \qquad (4.2)$$

It was remarked in [25] that we have $a \in S^0_{0,0}(\mathbf{R}^m)$, the space of C^∞ functions on $\mathbf{R}^m$ bounded together with all of their derivatives, precisely when $a \in \mathcal{S}'(\mathbf{R}^m)$ is such that

$$\sup_{j \in \Gamma} |\mathcal{F}(\chi_j a)(\xi)| \le \mathcal{O}_N(1)\langle \xi \rangle^{-N}, \quad N = 1, 2, \dots, \ \xi \in \mathbf{R}^m. \qquad (4.3)$$

Here $\mathscr{F}$ stands for the standard Fourier transformation,

$$\mathscr{F}u(\xi) = \hat{u}(\xi) = \int e^{-ix\cdot\xi}u(x)\,dx.$$

When establishing an analogous characterization of the space $\mathscr{G}_b^s(\mathbf{R}^m) \subset S_{0,0}^0(\mathbf{R}^m)$, it will be natural to assume that the compactly supported function χ_0 in (4.2) satisfies $\chi_0 \in \mathscr{G}_0^s(\mathbf{R}^m)$.

Proposition 4.1. *Let $0 \le \chi_0 \in \mathscr{G}_0^s(\mathbf{R}^m)$ be such that (4.2) holds. We have $a \in \mathscr{G}_b^s(\mathbf{R}^m)$, for some $s > 1$, if and only if $a \in \mathcal{S}'(\mathbf{R}^m)$ has the property that*

$$\sup_{j\in\Gamma}|\mathscr{F}(\chi_j a)(\xi)| \le \mathcal{O}(1)\exp\Big(-\frac{1}{C}|\xi|^{1/s}\Big), \quad \xi \in \mathbf{R}^m, \tag{4.4}$$

for some $C > 0$.

Proof. Let us first verify the necessity of (4.4). When doing so, we remark that given $\chi_0 \in \mathscr{G}_0^s(\mathbf{R}^m)$, $a \in \mathscr{G}_b^s(\mathbf{R}^m)$, we have, in view of the Leibniz formula,

$$|\partial^\alpha(\chi_j(x)a(x))| \le C^{|\alpha|+1}(\alpha!)^s, \quad x \in \mathbf{R}^m,\ \alpha \in \mathbf{N}^m, \tag{4.5}$$

for some $C > 0$, uniformly in $j \in \Gamma$. Following an argument in [11, Section 2] and writing

$$\mathscr{F}\big((1-\Delta)^{\frac{N}{2}}(\chi_j a)\big)(\xi) = \langle\xi\rangle^N \mathscr{F}(\chi_j a)(\xi), \tag{4.6}$$

for some even integer N large to be chosen, we get in view of (4.5) and (4.6),

$$|\mathscr{F}(\chi_j a)(\xi)| \le \langle\xi\rangle^{-N}\|(1-\Delta)^{N/2}(\chi_j a)\|_{L^1(\mathbf{R}^m)} \le C^{N+1}\langle\xi\rangle^{-N}(N!)^s, \tag{4.7}$$

uniformly in $j \in \Gamma$. Choosing $N \sim (|\xi|/C)^{1/s}$, as explained in [11], we get (4.4).

As for the sufficiency of (4.4), let $a \in \mathcal{S}'(\mathbf{R}^m)$ be such that (4.4) holds. As remarked above, we then have $a \in S_{0,0}^0(\mathbf{R}^m)$, and we only need to control the growth of the derivatives of a. When doing so, let us set $U_j(\xi) = (1/(2\pi)^m)\mathscr{F}(\chi_j a)(\xi)$. Let $0 \le \tilde{\chi}_0 \in \mathscr{G}_0^s(\mathbf{R}^m)$ be such that $\tilde{\chi}_0 = 1$ near supp χ_0 and let us put $\tilde{\chi}_j(x) = \tilde{\chi}_0(x-j)$, $j \in \Gamma$. Using the Fourier inversion formula and (4.2), we may write

$$a(x) = \sum_{j\in\Gamma}\tilde{\chi}_j(x)\int e^{ix\cdot\xi}U_j(\xi)\,d\xi, \tag{4.8}$$

and therefore, when $\alpha \in \mathbf{N}^m$, we have

$$D^\alpha a(x) = \sum_{j\in\Gamma}\sum_{\beta\le\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta}\tilde{\chi}_0)(x-j)\int e^{ix\cdot\xi}\xi^\beta U_j(\xi)\,d\xi. \tag{4.9}$$

Using (4.4), we get, passing to polar coordinates and making a change of variables,

$$\begin{aligned}\left|\int e^{ix\cdot\xi}\xi^\beta U_j(\xi)\,d\xi\right| &\leq \mathcal{O}(1)\int |\xi|^{|\beta|}\exp\Big(-\frac{1}{C}|\xi|^{1/s}\Big)\,d\xi \\ &\leq C_1^{1+|\beta|}\Gamma(s(|\beta|+m)),\end{aligned} \tag{4.10}$$

for some $C_1>0$, uniformly in j. Here $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}\,dt$, $x>0$, is the Γ-function. Using Stirling's formula

$$\Gamma(\lambda)\sim\sqrt{\frac{2\pi}{\lambda}}\Big(\frac{\lambda}{e}\Big)^\lambda,\quad \lambda\to\infty, \tag{4.11}$$

and the general inequality $|\beta|!\leq m^{|\beta|}\beta!$, valid for $\beta\in\mathbf{N}^m$, we see therefore that

$$\left|\int e^{ix\cdot\xi}\xi^\beta U_j(\xi)\,d\xi\right|\leq C^{1+|\beta|}(\beta!)^s, \tag{4.12}$$

for a (new) constant $C>0$, uniformly in j. Combining (4.9) and (4.12), we obtain that

$$|\partial^\alpha a(x)|\leq C^{1+|\alpha|}(\alpha!)^s,\quad x\in\mathbf{R}^m. \tag{4.13}$$

The proof is complete. ■

Remark. The proof of the sufficiency of the condition (4.4) can alternatively be carried out by staying on the Fourier transform side. Indeed, let $\chi\in\mathcal{G}_0^s(\mathbf{R}^m)$, and let us write, using (4.2),

$$\chi a=\sum_{j;\,\mathrm{supp}\,\chi\cap\,\mathrm{supp}\,\chi_j\neq\emptyset}\chi\chi_j a, \tag{4.14}$$

$$\mathcal{F}(\chi a)=\frac{1}{(2\pi)^m}\sum_{j;\,\mathrm{supp}\,\chi\cap\,\mathrm{supp}\,\chi_j\neq\emptyset}\mathcal{F}\chi*\mathcal{F}(\chi_j a), \tag{4.15}$$

where $*$ is the convolution star. Using that

$$|\mathcal{F}\chi(\xi)|\leq\mathcal{O}(1)\exp\Big(-\frac{1}{C}|\xi|^{1/s}\Big),\quad \xi\in\mathbf{R}^m, \tag{4.16}$$

and (4.4), we obtain that

$$\begin{aligned}|(\mathcal{F}\chi*\mathcal{F}(\chi_j a))(\xi)| &\leq\int|\mathcal{F}(\chi)(\xi-\eta)||\mathcal{F}(\chi_j a)(\eta)|\,d\eta \\ &\leq\mathcal{O}(1)\int\exp\Big(-\frac{1}{C}\big(|\xi-\eta|^{1/s}+|\eta|^{1/s}\big)\Big)\,d\eta \\ &\leq\mathcal{O}(1)\exp\Big(-\frac{1}{\mathcal{O}(1)}|\xi|^{1/s}\Big).\end{aligned} \tag{4.17}$$

Here the last inequality in (4.17) follows by estimating separately the contributions coming from the regions of integration $|\xi - \eta| \geq |\xi|/2$ and $|\xi - \eta| \leq |\xi|/2$. Using equations (4.15) and (4.17), we get

$$|\mathscr{F}\chi(\xi)| \leq \mathcal{O}(1)\exp\Big(-\frac{1}{\mathcal{O}(1)}|\xi|^{1/s}\Big), \quad \xi \in \mathbf{R}^m. \tag{4.18}$$

In this estimate we can replace χ by any translate of χ, and by Fourier's inversion formula we can conclude therefore that $a \in \mathscr{G}_b^s(\mathbf{R}^m)$.

Remark. Let $\chi_0 \in \mathscr{G}_0^s(\mathbf{R}^m)$ be real valued such that $\|\chi_0\|_{L^2} = 1$, and let us set $\chi_t(x) = \chi_0(x - t)$, $t \in \mathbf{R}^m$. We then have the following natural analog of Proposition 4.1: $a \in \mathscr{G}_b^s(\mathbf{R}^m)$, for some $s > 1$, precisely when we have

$$\sup_{t\in\mathbf{R}^m} |\mathscr{F}(\chi_t a)(\xi)| \leq \mathcal{O}(1)\exp\Big(-\frac{1}{C}|\xi|^{1/s}\Big), \quad \xi \in \mathbf{R}^m, \tag{4.19}$$

for some $C > 0$. Indeed, this follows by arguing as in the proof of Proposition 4.1, replacing (4.8) by the following consequence of the Fourier inversion formula:

$$a(x) = \frac{1}{(2\pi)^m} \iint e^{ix\cdot\xi} \chi_t(x)\mathscr{F}(\chi_t a)(\xi)\, d\xi\, dt. \tag{4.20}$$

Here we may also notice that the same characterization of the class $\mathscr{G}_b^s(\mathbf{R}^m)$ remains valid when $\chi_0(x) = 2^{m/4}\pi^{-m/4}e^{-|x|^2}$ is the L^2-normalized real Gaussian. Indeed, the derivatives of χ_0 obey the pointwise bounds

$$|\partial^\alpha \chi_0(x)| \leq C^{1+|\alpha|}(\alpha!)^{1/2}e^{-|x|^2/C}, \quad x \in \mathbf{R}^m,\ \alpha \in \mathbf{N}^m, \tag{4.21}$$

for some $C \geq 1$, which is sufficient for the arguments to go through. Let us also remark that such derivative bounds are well known [16], and can also be obtained directly by means of the Cauchy inequalities.

Let us proceed to make some additional remarks in the real setting, in preparation for the discussion on the FBI-Bargmann transform side. Let us set

$$e_0(x) = Ch^{-n/4}e^{-x^2/2h}, \quad x \in \mathbf{R}^n, \tag{4.22}$$

where $C > 0$ is chosen so that $\|e_0\|_{L^2} = 1$. The distribution kernel of the orthogonal projection $L^2(\mathbf{R}^n) \ni u \mapsto (u, e_0)_{L^2}e_0$ onto $\mathbf{C}e_0$ is given by $K(x, y) = e_0(x)\overline{e_0(y)}$, and the semiclassical Weyl symbol of the orthogonal projection has the form

$$\int e^{-iy\cdot\xi/h} K\Big(x + \frac{y}{2}, x - \frac{y}{2}\Big)\, dy. \tag{4.23}$$

A simple computation shows that the integral in (4.23) is given by $\varphi_0((x,\xi)/h^{1/2})$, where

$$\varphi_0(x,\xi) = (4\pi)^{n/2} C^2 e^{-(x^2+\xi^2)}. \tag{4.24}$$

See also [26, Section 3].

We shall now pass to work on Λ_{Φ_0}, and to this end let $\phi(x,y)$ be a holomorphic quadratic form on $\mathbf{C}^n_x \times \mathbf{C}^n_y$ such that

$$\operatorname{Im} \phi''_{yy} > 0, \quad \det \phi''_{xy} \neq 0, \tag{4.25}$$

and with the property that the associated complex linear canonical transformation

$$\kappa_\phi \colon \mathbf{C}^{2n} \ni (y, -\phi'_y(x,y)) \mapsto (x, \phi'_x(x,y)) \in \mathbf{C}^{2n} \tag{4.26}$$

satisfies

$$\kappa_\phi(\mathbf{R}^m) = \Lambda_{\Phi_0}. \tag{4.27}$$

Associated to the quadratic form ϕ is the generalized Bargmann transformation

$$\mathcal{T} u(x,h) = C h^{-3n/4} \int e^{i\phi(x,y)/h} u(y)\, dy, \tag{4.28}$$

where $C > 0$ is chosen suitably so that the map $\mathcal{T}$ is unitary,

$$\mathcal{T} \colon L^2(\mathbf{R}^n) \to H_{\Phi_0}(\mathbf{C}^n), \tag{4.29}$$

see [13, 17, 27].

Let $a \in \mathcal{G}^s_b(\Lambda_{\Phi_0})$ be uniformly Gevrey, for some $s > 1$, and let us set

$$\chi_0 = \varphi_0 \circ \kappa_\phi^{-1} \in \mathcal{S}(\Lambda_{\Phi_0}),$$

where φ_0 is given in (4.24). Put also $\chi_T(X) = \chi_0(X - T)$, $T \in \Lambda_{\Phi_0}$. The discussion above, see (4.19), shows that

$$|\mathcal{F}(\chi_T a)(Y)| \le \mathcal{O}(1) \exp\Big(-\frac{1}{C_0}|Y|^{1/s}\Big), \quad Y \in \Lambda_{\Phi_0}, \tag{4.30}$$

for some $C_0 > 0$, uniformly in $T \in \Lambda_{\Phi_0}$. Here $\mathcal{F}$ is the symplectic Fourier transformation on Λ_{Φ_0}, introduced in (3.1). Letting $\mathcal{F}_h$ be the semiclassical symplectic Fourier transformation on Λ_{Φ_0}, given in (A.1), we can write, in view of the Fourier inversion formula,

$$\chi_T(X) a(X) = \frac{1}{(\pi h)^n} \int_{\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h} \mathcal{F}_h(\chi_T a)(Y)\, dY. \tag{4.31}$$

Multiplying (4.31) by $\chi_0((X-T)/h^{1/2}) = \varphi_0(\kappa_\phi^{-1}(X-T)/h^{1/2})$ and integrating with respect to $T \in \Lambda_{\Phi_0}$, we get

$$M(h)a(X) = \frac{1}{(\pi h)^n} \iint\limits_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h}\chi_0\Big(\frac{X-T}{h^{1/2}}\Big)\mathcal{F}_h(\chi_T a)(Y)\, dY\, dT, \tag{4.32}$$

where

$$M(h) = \int\limits_{\Lambda_{\Phi_0}} \chi_0(T)\chi_0\Big(\frac{T}{h^{1/2}}\Big)\, dT \asymp h^n. \tag{4.33}$$

We would like to pass to the Weyl quantizations in (4.32), and to this end we shall first take a closer look at the Weyl quantization of the Schwartz function

$$\Lambda_{\Phi_0} \ni X \mapsto b(X) = b_{Y,T}(X) = e^{2i\sigma(X,Y)/h}\chi_0\Big(\frac{X-T}{h^{1/2}}\Big), \quad Y, T \in \Lambda_{\Phi_0}.$$

Let us start with some preliminary observations and computations, closely related to [28, Section 3]. Let ℓ be a complex linear form on $\mathbf{C}^{2n}$ such that $\ell|_{\Lambda_{\Phi_0}}$ is real, so that $H_\ell \in \Lambda_{\Phi_0}$, and let $a \in \mathcal{S}(\Lambda_{\Phi_0})$. A straightforward computation using (A.15) shows that

$$(e^{i\ell/h} \# a)(X) = e^{i\ell(X)/h} a\Big(X + \frac{H_\ell}{2}\Big), \tag{4.34}$$

and similarly we find

$$(a \# e^{i\ell/h})(X) = e^{i\ell(X)/h} a\Big(X - \frac{H_\ell}{2}\Big). \tag{4.35}$$

It follows from (4.34), (4.35) that

$$(e^{i\ell/h} \# a \# e^{i\ell/h})(X) = e^{2i\ell(X)/h} a(X), \tag{4.36}$$

$$(e^{i\ell/h} \# a \# e^{-i\ell/h})(X) = a(X + H_\ell). \tag{4.37}$$

Following equation (3.4), let us write $\ell_Y(X) = \sigma(X,Y)$, for $Y \in \Lambda_{\Phi_0}$, and notice that $H_{\ell_T} = T$. Using (4.36) and (4.37), we get

$$\begin{aligned} b(X) &= e^{2i\sigma(X,Y)/h}\chi_0\Big(\frac{X-T}{h^{1/2}}\Big) \\ &= \Big(e^{i\ell_Y/h} \# e^{-i\ell_T/h} \# \chi_0\Big(\frac{\cdot}{h^{1/2}}\Big) \# e^{i\ell_T/h} \# e^{i\ell_Y/h}\Big)(X), \end{aligned} \tag{4.38}$$

and therefore

$$\begin{aligned} b^w(x, hD_x) = e^{i\ell_Y(x,hD_x)/h} &\circ e^{-i\ell_T(x,hD_x)/h} \circ \chi_0^w\Big(\frac{(x,hD_x)}{h^{1/2}}\Big) \\ &\circ e^{i\ell_T(x,hD_x)/h} \circ e^{i\ell_Y(x,hD_x)/h}. \end{aligned} \tag{4.39}$$

Using (A.4), we infer that

$$\begin{aligned} e^{i\ell_Y(x,hD_x)/h} \circ e^{-i\ell_T(x,hD_x)/h} &= e^{i\ell_{Y-T}(x,hD_x)/h} e^{i\sigma(Y,-T)/2h} \\ &= e^{i\ell_{Y-T}(x,hD_x)/h} e^{i\sigma(T,Y)/2h}, \end{aligned} \tag{4.40}$$

$$e^{i\ell_T(x,hD_x)/h} \circ e^{i\ell_Y(x,hD_x)/h} = e^{i\ell_{T+Y}(x,hD_x)/h} e^{i\sigma(T,Y)/2h}, \tag{4.41}$$

and, combining (4.39), (4.40), and (4.41), we get

$$b^w(x,hD_x) = e^{i\sigma(T,Y)/h} e^{i\sigma((x,hD_x),Y-T)/h} \circ \chi_0^w\Big(\frac{(x,hD_x)}{h^{1/2}}\Big) \circ e^{i\sigma((x,hD_x),Y+T)/h}. \tag{4.42}$$

An application of the exact Egorov theorem [13, 27] gives that

$$\chi_0^w\Big(\frac{(x,hD_x)}{h^{1/2}}\Big) = \mathcal{T} \circ \varphi_0^w\Big(\frac{(x,hD_x)}{h^{1/2}}\Big) \circ \mathcal{T}^{-1}, \tag{4.43}$$

and we conclude therefore that the operator in (4.43) is a rank-one orthogonal projection on $H_{\Phi_0}(\mathbf{C}^n)$, given by

$$\chi_0^w\Big(\frac{(x,hD_x)}{h^{1/2}}\Big)u = (u, v_0)v_0, \quad u \in H_{\Phi_0}(\mathbf{C}^n). \tag{4.44}$$

Here $v_0 = \mathcal{T} e_0$ and $(\cdot,\cdot)$ stands for the scalar product in $H_{\Phi_0}(\mathbf{C}^n)$. Let us mention explicitly that we owe the idea of using Gaussians to pass to rank-one projections to [9].

For future reference, let us also notice that an application of the exact (quadratic) stationary phase together with (4.22), (4.28) allows us to conclude that

$$v_0(x) = (\mathcal{T} e_0)(x,h) = C h^{-n/2} e^{ig(x)/h}, \quad C \neq 0, \tag{4.45}$$

where g is a holomorphic quadratic form on $\mathbf{C}^n$. The strict positivity of the complex Lagrangian plane $\eta = iy$, $y \in \mathbf{C}^n$, associated to the state e_0 in (4.22) implies that

$$\Phi_0(x) + \operatorname{Im} g(x) \asymp |x|^2, \quad x \in \mathbf{C}^n, \tag{4.46}$$

see [2, Theorem 2.1].

Using (4.42) and (4.44), we get, using the unitarity of magnetic translations on $H_{\Phi_0}(\mathbf{C}^n)$,

$$\begin{aligned} &b^w(x,hD_x)u \\ &\quad = e^{i\sigma(T,Y)/h} e^{i\sigma((x,hD_x),Y-T)/h} \circ \chi_0^w\Big(\frac{(x,hD_x)}{h^{1/2}}\Big) \circ e^{i\sigma((x,hD_x),Y+T)/h} u \\ &\quad = e^{i\sigma(T,Y)/h} (u, e^{-i\sigma((x,hD_x),Y+T)/h} v_0) e^{i\sigma((x,hD_x),Y-T)/h} v_0. \end{aligned} \tag{4.47}$$

Passing to the Weyl quantizations in (4.32), we obtain therefore, in view of (4.47),

$$M(h)a^w(x,hD_x)u = \frac{1}{(\pi h)^n} \iint_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{\frac{i\sigma(T,Y)}{h}} \mathcal{F}_h(\chi_T a)(Y)(u, e^{-\frac{i\sigma((x,hD_x),Y+T)}{h}} v_0) \times e^{\frac{i\sigma((x,hD_x),Y-T)}{h}} v_0 \, dY \, dT. \tag{4.48}$$

Making the change of variables $Y' = Y - T$, $T' = -Y - T$, and using that $2\sigma(T,Y) = \sigma(Y',T')$, we obtain, after dropping the primes,

$$M(h)a^w(x,hD_x)u = \frac{C}{h^n} \iint_{(\Lambda_{\Phi_0})^2} e^{\frac{i\sigma(Y,T)}{2h}} \mathcal{F}_h(\chi_{-\frac{Y+T}{2}} a)\Big(\frac{Y-T}{2}\Big) \times (u, e^{\frac{i\sigma((x,hD_x),T)}{h}} v_0) e^{\frac{i\sigma((x,hD_x),Y)}{h}} v_0 \, dY \, dT. \tag{4.49}$$

Here we have incorporated the non-vanishing constant Jacobian into the (new) constant $C \neq 0$. We have therefore represented the operator $a^w(x,hD_x)$ as a superposition of rank-one kernels. The decomposition (4.49) can be regarded as the Bargmann transform side analogue of the corresponding decomposition in the real setting, established in [9].

Let us next record the following observation, closely related to the computations in Section 2.

Lemma 4.2. *Let $\ell(x,\xi)$ be a complex linear form on $\mathbf{C}^{2n}$ such that the restriction $\ell|_{\Lambda_{\Phi_0}}$ is real, and let us represent ℓ in the form (2.4),*

$$\ell(x,\xi) = -\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x^*)\cdot x + x^*\cdot\xi,$$

for some unique $x^* \in \mathbf{C}^n$. There exists $C_{x^*,h} \in \mathbf{C}$ with $|C_{x^*,h}| = 1$, such that

$$e^{i\ell(x,hD_x)/h}u(x) = C_{x^*,h} e^{-2i\,\mathrm{Im}(\Phi''_{0,xx}x^*\cdot x)/h} e^{i\sigma(X,X^*)/2h} e^{(\Phi_0(x)-\Phi_0(x+x^*))/h} u(x+x^*). \tag{4.50}$$

Here $u \in H_{\Phi_0}(\mathbf{C}^n)$, and $X \in \Lambda_{\Phi_0}$, $X^* = H_\ell \in \Lambda_{\Phi_0}$ are the points in Λ_{Φ_0} above x, $x^* \in \mathbf{C}^n$, respectively.

Proof. This result follows by a direct computation, using (2.3), (2.7), (2.8), as well as the following general expression for the complex symplectic (2,0)-form σ on $\mathbf{C}^{2n}$, restricted to Λ_{Φ_0},

$$\sigma(X,X^*) = -4\,\mathrm{Im}(\Phi''_{0,\bar{x}x}x\cdot\overline{x^*}), \quad X, X^* \in \Lambda_{\Phi_0}. \qquad \blacksquare$$

Remark. Let $u, v \in H_{\Phi_0}(\mathbf{C}^n)$. It follows from Lemma 4.2 that the scalar product $(e^{i\ell(x,hD_x)/h}u, v)_{H_{\Phi_0}}$, viewed as a function of $x^* \in \mathbf{C}^n$, or equivalently as a function of $X^* \in \Lambda_{\Phi_0}$, can be regarded as the twisted convolution of the functions $e^{-\Phi_0/h}u$, $e^{-\Phi_0/h}v \in L^2(\mathbf{C}^n)$ in the sense of (A.7), after these have been modified by some unimodular factors. It follows, in particular, that

$$\mathbf{C}^n \ni x^* \mapsto (e^{i\ell(x,hD_x)/h}u, v)_{H_{\Phi_0}} \in L^2(\mathbf{C}^n),$$

see [30]. Ignoring the unimodular factors in (4.50), we get the more elementary pointwise estimate,

$$|(e^{i\ell(x,hD_x)/h}u, v)_{H_{\Phi_0}}(x^*)| \leq \int_{\mathbf{C}^n} e^{-\Phi_0(x+x^*)/h}|u(x+x^*)|e^{-\Phi_0(x)/h}|v(x)|L(dx), \tag{4.51}$$

which will be sufficient in what follows.

We now come to complete the proof of Theorem 1.2. When doing so, let us write, using (4.49), (4.50), and (4.51),

$$\begin{aligned} M(h)&|a^w(x,hD_x)u(x)|e^{-\Phi_0(x)/h} \\ &\leq \frac{\mathcal{O}(1)}{h^{2n}} \iiint_{(\mathbf{C}^n)^3} \mathcal{U}\Big(\frac{y-t}{h}\Big) e^{-\Phi_0(z+t)/h}|v_0(z+t)|e^{-\Phi_0(x+y)/h}|v_0(x+y)| \\ &\qquad\qquad \times |u(z)|e^{-\Phi_0(z)/h}L(dy)L(dt)L(dz), \end{aligned} \tag{4.52}$$

where, in view of (4.30),

$$\mathcal{U}(y) = \exp\Big(-\frac{1}{C_0}|y|^{1/s}\Big), \quad y \in \mathbf{C}^n. \tag{4.53}$$

Letting $\Phi_1 = \Phi_0 + f \in C^{1,1}(\mathbf{C}^n; \mathbf{R})$ be such that (3.5) holds, we obtain next, making use of (4.33), (4.45), (4.46), and (4.52),

$$|a^w(x,hD_x)u(x)|e^{-\Phi_1(x)/h} \leq \int_{\mathbf{C}^n} K(x,z)|u(z)|e^{-\Phi_1(z)/h}L(dz), \tag{4.54}$$

where

$$K(x,z) \leq \frac{\mathcal{O}(1)}{h^{4n}} \iint_{\mathbf{C}^n\times\mathbf{C}^n} \mathcal{U}\Big(\frac{y-t}{h}\Big) e^{-|z+t|^2/Ch} e^{-|x+y|^2/Ch} e^{(f(z)-f(x))/h} L(dt)L(dy). \tag{4.55}$$

We would like to show that the kernel $K(x,z)$ is dominated pointwise by an L^1 convolution kernel, in order to be able to apply Schur's lemma to (4.54). To this end let us consider the t-integration in (4.55) first, estimating the integral

$$\frac{1}{h^{2n}}\int_{\mathbf{C}^n}\mathcal{U}\Big(\frac{y-t}{h}\Big)e^{-|z+t|^2/Ch}L(dt) = \frac{1}{h^{2n}}\int_{\mathbf{C}^n}\mathcal{U}\Big(\frac{t}{h}\Big)e^{-|z+y-t|^2/Ch}L(dt) = I_1+I_2. \tag{4.56}$$

Here

$$I_1 = \frac{1}{h^{2n}}\int_{|z+y-t|\geq|z+y|/2}\mathcal{U}\Big(\frac{t}{h}\Big)e^{-|z+y-t|^2/Ch}L(dt) \leq \|\mathcal{U}\|_{L^1}e^{-|z+y|^2/4Ch}, \tag{4.57}$$

and, in view of (4.53), we have

$$I_2 = \frac{1}{h^{2n}}\int_{|z+y-t|\leq|z+y|/2}\mathcal{U}\Big(\frac{t}{h}\Big)e^{-|z+y-t|^2/Ch}L(dt) \leq \frac{1}{h^{2n}}\int_{|z+y-t|\leq|z+y|/2}\mathcal{U}\Big(\frac{t}{h}\Big)L(dt) \leq \mathcal{O}(1)\exp\Big(-\frac{1}{C_0}\Big(\frac{|z+y|}{h}\Big)^{1/s}\Big). \tag{4.58}$$

Here we have also used that $|z+y|\leq 2|t|$ in the region of integration in (4.58).

Combining (4.55), (4.56), (4.57), and (4.58), we see that

$$K(x,z)\leq K_1(x,z)+K_2(x,z), \tag{4.59}$$

where

$$K_1(x,z)\leq e^{(f(z)-f(x))/h}\frac{\mathcal{O}(1)}{h^{2n}}\int_{\mathbf{C}^n}e^{-|z+y|^2/Ch}e^{-|x+y|^2/Ch}L(dy) = e^{(f(z)-f(x))/h}\frac{\mathcal{O}(1)}{h^{2n}}\int_{\mathbf{C}^n}e^{-|y|^2/Ch}e^{-|z-x+y|^2/Ch}L(dy), \tag{4.60}$$

and

$$K_2(x,z)\leq e^{(f(z)-f(x))/h}\frac{\mathcal{O}(1)}{h^{2n}}\int_{\mathbf{C}^n}\exp\Big(-\frac{1}{C_0}\Big(\frac{|z+y|}{h}\Big)^{1/s}\Big)e^{-|x+y|^2/Ch}L(dy) = e^{(f(z)-f(x))/h}\frac{\mathcal{O}(1)}{h^{2n}}\int_{\mathbf{C}^n}\exp\Big(-\frac{1}{C_0}\Big(\frac{|z-x+y|}{h}\Big)^{1/s}\Big)e^{-|y|^2/Ch}L(dy). \tag{4.61}$$

When estimating the contribution $K_1(x,z)$ in (4.60), we notice that considering separately the regions of integration $|z-x+y| \leq |z-x|/2$ and $|z-x+y| \geq |z-x|/2$, and using that

$$\frac{1}{h^n}\int_{\mathbf{C}^n} e^{-|y|^2/Ch} L(dy) = \mathcal{O}(1),$$

we get

$$\frac{\mathcal{O}(1)}{h^n}\int_{\mathbf{C}^n} e^{-|y|^2/Ch} e^{-|z-x+y|^2/Ch} L(dy) \leq \mathcal{O}(1) e^{-|x-z|^2/Ch}, \tag{4.62}$$

and therefore,

$$K_1(x,z) \leq \frac{\mathcal{O}(1)}{h^n} e^{(f(z)-f(x))/h} e^{-|x-z|^2/Ch}. \tag{4.63}$$

Alternatively, the estimate (4.62) can be obtained by an application of the exact stationary phase to the integral in the left-hand side of (4.62). Arguing similarly, we find that

$$\begin{aligned} &\frac{\mathcal{O}(1)}{h^n}\int_{\mathbf{C}^n} \exp\Big(-\frac{1}{C_0}\Big(\frac{|z-x+y|}{h}\Big)^{1/s}\Big) e^{-|y|^2/Ch} L(dy) \\ &\quad \leq \mathcal{O}(1)\exp\Big(-\frac{1}{C_0}\Big(\frac{|z-x|}{h}\Big)^{1/s}\Big) + \mathcal{O}(1) e^{-|x-z|^2/Ch}. \end{aligned} \tag{4.64}$$

Combining (4.59), (4.63), (4.61), and (4.64), we get

$$\begin{aligned} K(x,z) \leq{}& \frac{\mathcal{O}(1)}{h^n} e^{(f(z)-f(x))/h} e^{-|x-z|^2/Ch} \\ &+ \frac{\mathcal{O}(1)}{h^n} e^{(f(z)-f(x))/h} \exp\Big(-\frac{1}{C_0}\Big(\frac{|z-x|}{h}\Big)^{1/s}\Big). \end{aligned} \tag{4.65}$$

To handle the second term in the right-hand side of (4.65), we write, following (3.8) and using (3.5),

$$f(z)-f(x) \leq \frac{1}{\mathcal{O}(1)} h^{1-\frac{1}{s}} \min(1,|z-x|) \leq \frac{1}{\mathcal{O}(1)} h^{1-\frac{1}{s}} |z-x|^{1/s}. \tag{4.66}$$

The Schur norm of the second term in the right-hand side of (4.65) is therefore $\mathcal{O}(1)$, provided that the implicit constant in (4.66) is large enough, and we only need to estimate the Schur norm of the first term in the right-hand side of (4.65). Using (4.66), we see that it suffices to control the L^1-norm

$$\frac{1}{h^n}\int_{\mathbf{C}^n} e^{-|x|^2/Ch} \exp\Big(\frac{h^{1-\frac{1}{s}}|x|}{\mathcal{O}(1)h}\Big) L(dx) = I_1 + I_2, \tag{4.67}$$

where

$$I_1 = \frac{1}{h^n} \int\limits_{|x| \geq \widetilde{C} h^{1-\frac{1}{s}}} e^{-|x|^2/Ch} \exp\Big(\frac{h^{1-\frac{1}{s}}|x|}{\mathcal{O}(1)h}\Big) L(dx), \tag{4.68}$$

and

$$I_2 = \frac{1}{h^n} \int\limits_{|x| \leq \widetilde{C} h^{1-\frac{1}{s}}} e^{-|x|^2/Ch} \exp\Big(\frac{h^{1-\frac{1}{s}}|x|}{\mathcal{O}(1)h}\Big) L(dx). \tag{4.69}$$

Taking the constant $\widetilde{C} > 0$ sufficiently large, we get

$$I_1 \leq \frac{1}{h^n} \int \exp\Big(-\frac{|x|^2}{\mathcal{O}(1)h}\Big) L(dx) = \mathcal{O}(1). \tag{4.70}$$

Furthermore,

$$I_2 \leq \Big(\frac{1}{h^n} \int\limits_{|x| \leq \widetilde{C} h^{1-\frac{1}{s}}} e^{-|x|^2/Ch} L(dx)\Big) \exp\Big(\frac{\widetilde{C} h^{2-\frac{2}{s}}}{\mathcal{O}(1)h}\Big) \leq \mathcal{O}(1)(\mathcal{O}(1)h^{1-\frac{2}{s}}). \tag{4.71}$$

Recalling that $s \geq 2$, we conclude, in view of (4.65), (4.66), (4.67), (4.70), and (4.71), that the Schur norm of the kernel $K(x,z)$ is $\mathcal{O}(1)$. Applying Schur's lemma to (4.54), we get therefore,

$$\mathrm{Op}_h^w(a) = \mathcal{O}(1) \colon H_{\Phi_1}(\mathbf{C}^n) \to H_{\Phi_1}(\mathbf{C}^n).$$

The proof of Theorem 1.2 is complete.

Remark. The purpose of this remark is to verify that the decomposition (4.49) can also be used to give a direct proof of the L^2-boundedness result for the Wiener algebra of pseudodifferential operators, established in [25, 26]. Indeed, the fact that decompositions such as (4.49) are useful to this end is well known in the real setting [9], and the observation here is that working on the FBI–Bargmann transform side seems to make the computations and estimates particularly natural. See also [26, Section 5].

Let us therefore replace (4.30) by the weaker assumption,

$$|\mathcal{F}(\chi_T a)(Y)| \leq U(Y), \quad Y \in \Lambda_{\Phi_0}, \tag{4.72}$$

uniformly in $T \in \Lambda_{\Phi_0}$. Here $U \in L^1(\Lambda_{\Phi_0})$. Setting

$$F(T) = (u, e^{\frac{i\sigma((x,hD_x),T)}{h}} v_0)_{H_{\Phi_0}}, \quad T \in \Lambda_{\Phi_0} \simeq \mathbf{C}^n, \tag{4.73}$$

we get, in view of (4.51) and the Young inequality,

$$\|F\|_{L^2(\Lambda_{\Phi_0})} \leq \mathcal{O}(1)\|u\|_{H_{\Phi_0}} \|e^{-\Phi_0/h} v_0\|_{L^1} \leq \mathcal{O}(h^{n/2})\|u\|_{H_{\Phi_0}}. \tag{4.74}$$

Here we have also used (4.45), (4.46). An application of Schur's lemma together with (4.72) and (4.74) allows us next to conclude that the function

$$G(Y) := \frac{C}{h^n} \int_{\Lambda_{\Phi_0}} e^{\frac{i\sigma(Y,T)}{2h}} \mathcal{F}_h(\chi_{-\frac{(Y+T)}{2}} a)\Big(\frac{Y-T}{2}\Big) F(T)\, dT \in L^2(\Lambda_{\Phi_0}) \tag{4.75}$$

and we have

$$\|G\|_{L^2(\Lambda_{\Phi_0})} \leq \mathcal{O}(1)\|F\|_{L^2(\Lambda_{\Phi_0})} \leq \mathcal{O}(h^{n/2})\|u\|_{H_{\Phi_0}}. \tag{4.76}$$

Using (4.49), (4.73), and (4.75), we can write

$$M(h)a^w(x, hD_x)u(x) = \int_{\Lambda_{\Phi_0}} G(Y) e^{\frac{i\sigma((x,hD_x),Y)}{h}} v_0(x)\, dY, \tag{4.77}$$

and an application of Lemma 4.2 gives the pointwise estimate,

$$M(h)|a^w(x, hD_x)u(x)|e^{-\Phi_0(x)/h} \leq \int_{\Lambda_{\Phi_0}} |G(Y)|e^{-\Phi_0(x+y)/h}|v_0(x+y)|\, dY. \tag{4.78}$$

Here we have written $Y = (y, \eta) \in \Lambda_{\Phi_0}$. Applying the Young inequality once more we get, using also (4.76),

$$\begin{aligned} M(h)\|a^w u\|_{H_{\Phi_0}} &\leq \|G\|_{L^2}\|e^{-\Phi_0/h} v_0\|_{L^1} \leq \mathcal{O}(h^{n/2})\|G\|_{L^2} \\ &\leq \mathcal{O}(h^n)\|u\|_{H_{\Phi_0}}. \end{aligned} \tag{4.79}$$

Recalling finally (4.33), we conclude that

$$\|a^w u\|_{H_{\Phi_0}} \leq \mathcal{O}(1)\|u\|_{H_{\Phi_0}}, \quad u \in H_{\Phi_0}(\mathbf{C}^n), \tag{4.80}$$

provided that (4.72) holds. We have therefore recovered the L^2-boundedness result of [25, 26] in the H_{Φ_0}-setting.

A. Weyl composition of symbols

Let (W, σ) be a real symplectic vector space of dimension $2n$ and let

$$\mathcal{F}_h u(X) = \frac{1}{h^n}\mathcal{F}u\Big(\frac{X}{h}\Big) = \frac{1}{(\pi h)^n}\int e^{2i\sigma(X,Y)/h} u(Y)\, dY, \quad u \in \mathcal{S}(W), \quad 0 < h \leq 1, \tag{A.1}$$

be the semiclassical (twisted) Fourier transformation on W. Here the map $\mathcal{F}$ is given in (3.1). We have $\mathcal{F}_h^2 = I$ on $\mathcal{S}'(W)$.

We shall carry out a familiar computation composing two semiclassical Weyl quantizations, see [6, Chapter 7] for such computations on the real side. Let $a, b \in \mathcal{S}'(\Lambda_{\Phi_0})$ be such that $\mathcal{F}a$, $\mathcal{F}b \in L^1(\Lambda_{\Phi_0})$ and let us write, following (3.3),

$$a^w(x, hD_x) = \frac{1}{(\pi h)^n} \int\limits_{\Lambda_{\Phi_0}} e^{2i\sigma((x,hD_x),Y)/h} \mathcal{F}_h a(Y)\, dY, \tag{A.2}$$

$$b^w(x, hD_x) = \frac{1}{(\pi h)^n} \int\limits_{\Lambda_{\Phi_0}} e^{2i\sigma((x,hD_x),Y)/h} \mathcal{F}_h b(Y)\, dY. \tag{A.3}$$

Using the composition law for magnetic translations

$$e^{2i\sigma((x,hD_x),Y)/h} e^{2i\sigma((x,hD_x),Z)/h} = e^{2i\sigma((x,hD_x),Y+Z)/h} e^{2i\sigma(Y,Z)/h}, \tag{A.4}$$

see [6, (7.11)] for the corresponding result in the real domain, and making the change of variables $(Y, Z) \mapsto (Y + Z, Z)$, we get that the composition

$$a^w(x, hD_x) \circ b^w(x, hD_x)$$

of the operators in (A.2), (A.3), is given by

$$\begin{aligned} &\frac{1}{(\pi h)^{2n}} \iint\limits_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma((x,hD_x),Y+Z)/h} e^{2i\sigma(Y,Z)/h} \mathcal{F}_h a(Y) \mathcal{F}_h b(Z)\, dY\, dZ \\ &\quad = \frac{1}{(\pi h)^n} \int\limits_{\Lambda_{\Phi_0}} e^{2i\sigma((x,hD_x),Y)/h} \mathcal{F}_h c(Y)\, dY = c^w(x, hD_x). \end{aligned} \tag{A.5}$$

Here

$$\mathcal{F}_h c(X) = \frac{1}{(\pi h)^n} \int\limits_{\Lambda_{\Phi_0}} e^{2i\sigma(X,Z)/h} \mathcal{F}_h a(X - Z) \mathcal{F}_h b(Z)\, dZ \in L^1(\Lambda_{\Phi_0}). \tag{A.6}$$

We shall now compute the semiclassical Fourier transform of the expression in the right-hand side of (A.6), leading to an integral representation formula for the symbol $c = a \# b \in L^\infty(\Lambda_{\Phi_0}) \cap C(\Lambda_{\Phi_0})$. To this end, following [30], it will be convenient to introduce the (non-commutative) twisted convolution product on Λ_{Φ_0},

$$(u *_\sigma v)(X) = \int\limits_{\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h} u(X - Y) v(Y)\, dY, \tag{A.7}$$

where $u, v \in L^1(\Lambda_{\Phi_0})$, so that

$$\mathcal{F}_h(a \# b) = \frac{1}{(\pi h)^n} \mathcal{F}_h a *_\sigma \mathcal{F}_h b. \tag{A.8}$$

We have the following result, due to [30], whose proof we give for the convenience of the reader only.

Proposition A.1. *If* $u, v \in L^1(\Lambda_{\Phi_0})$*, then*

$$\mathcal{F}_h(u *_\sigma v) = (\mathcal{F}_h u) *_\sigma v. \tag{A.9}$$

Proof. Using (A.1), (A.7), let us write

$$\begin{aligned}\mathcal{F}_h(u *_\sigma v)(X) &= \frac{1}{(\pi h)^n} \int_{\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h}(u *_\sigma v)(Y)\, dY \\ &= \frac{1}{(\pi h)^n} \iint_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h} e^{2i\sigma(Y,Z)/h} u(Y-Z)v(Z)\, dY\, dZ.\end{aligned} \tag{A.10}$$

On the other hand, we compute

$$\begin{aligned}((\mathcal{F}_h u) *_\sigma v)(X) &= \int_{\Lambda_{\Phi_0}} e^{2i\sigma(X,Z)/h}(\mathcal{F}_h u)(X-Z)v(Z)\, dZ \\ &= \frac{1}{(\pi h)^n} \iint_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X,Z)/h} e^{2i\sigma(X-Z,Y)/h} u(Y)v(Z)\, dY\, dZ.\end{aligned} \tag{A.11}$$

Making the change of variables $(Y, Z) \mapsto (Y - Z, Z)$, we can rewrite (A.11) as follows:

$$\begin{aligned}&((\mathcal{F}_h u) *_\sigma v)(X) \\ &\quad= \frac{1}{(\pi h)^n} \iint_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X,Z)/h} e^{2i\sigma(X-Z,Y-Z)/h} u(Y-Z)v(Z)\, dY\, dZ.\end{aligned} \tag{A.12}$$

Here

$$\begin{aligned}\sigma(X,Z) + \sigma(X-Z, Y-Z) &= \sigma(X,Z) + \sigma(X,Y) - \sigma(X,Z) - \sigma(Z,Y) \\ &= \sigma(X,Y) + \sigma(Y,Z),\end{aligned}$$

and therefore the expressions (A.10) and (A.12) agree. ■

Combining Proposition A.1 and (A.8) with the fact that $\mathcal{F}_h^2 = I$, we get

$$c = a \# b = \frac{1}{(\pi h)^n} a *_\sigma \mathcal{F}_h b, \tag{A.13}$$

and therefore, assuming for simplicity that $a, b \in \mathcal{S}(\Lambda_{\Phi_0})$,

$$\begin{aligned} c(X) &= \frac{1}{(\pi h)^n} \int\limits_{\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h} a(X-Y) \mathcal{F}_h b(Y)\, dY \\ &= \frac{1}{(\pi h)^{2n}} \iint\limits_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X,Y)/h} e^{2i\sigma(Y,Z)/h} a(X-Y) b(Z)\, dY\, dZ \\ &= \frac{1}{(\pi h)^{2n}} \iint\limits_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{2i\sigma(X-Z,Y)/h} a(X-Y) b(Z)\, dY\, dZ. \end{aligned} \tag{A.14}$$

We obtain finally, after a change of variables,

$$c(X) = (a \# b)(X) = \frac{1}{(\pi h)^{2n}} \iint\limits_{\Lambda_{\Phi_0}\times\Lambda_{\Phi_0}} e^{-2i\sigma(Y,Z)/h} a(X+Y) b(X+Z)\, dY\, dZ. \tag{A.15}$$

Remark. The integral representation formula (A.15) can also be inferred from the corresponding expression for the Weyl symbol of the composition $a^w(x, hD_x) \circ b^w(x, hD_x)$ in the real domain [33, Chapter 4], thanks to the metaplectic invariance of the Weyl calculus [13, 27].

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Michael Hitrik
Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA; hitrik@math.ucla.edu

Richard Lascar
JAD – UMR 7351, Université Côte d'Azur, Parc Valrose 06108, Nice Cedex 02, France; richard.lascar@univ-cotedazur.fr

Johannes Sjöstrand
IMB, Université de Bourgogne 9, Av. A. Savary, BP 47870, 21078 Dijon; and UMR 5584, CNRS, France; johannes.sjostrand@u-bourgogne.fr

Maher Zerzeri
LAGA – UMR7539 CNRS, Université Sorbonne Paris-Nord, 99 avenue J.-B. Clément, 93430 Villetaneuse, France; zerzeri@math.univ-paris13.fr