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On the Benjamin–Ono equation on $\mathbb{T}$ and its periodic and quasiperiodic solutions

Patrick Gérard, Thomas Kappeler, and Petar Topalov

Abstract. In this paper, we survey our recent results on the Benjamin–Ono equation on the torus. As an application of the methods developed we construct large families of periodic or quasiperiodic solutions, which are not C^∞-smooth.

Dedicated to the memory of our friend and colleague Misha Shubin

1. Introduction

In this paper, we consider the Benjamin–Ono (BO) equation on $\mathbb{T}$,

$$\partial_t u = H\partial_x^2 u - \partial_x(u^2), \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R}, \tag{1}$$

where $u \equiv u(t,x)$ is real valued and H denotes the Hilbert transform, defined as the Fourier multiplier

$$f = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx} \mapsto \sum_{n\neq 0} -i \operatorname{sign}(n)\hat{f}(n)e^{inx}.$$

We refer to the recent survey [25] for a discussion of the origin of this equation as a model for long, unidirectional internal gravity waves in a two layer fluid and for a comprehensive bibliography.

Our study of (1) focuses on the following topics: wellposedness in Sobolev spaces, traveling waves and their orbital stability, long time behaviour of solutions (i.e., on properties of their orbits such as boundedness, orbital stability, recurrence), aspects of integrability, and the construction of periodic and quasiperiodic solutions.

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Sharp wellposedness in Sobolev spaces. The wellposedness of (1) in Sobolev spaces $H^s_r \equiv H^s(\mathbb{T}, \mathbb{R})$ has been extensively studied in the last 40 years – see e.g. references in [20]. To state our results we need to recall the following result of [1] (cf. also [24]). It says that for any initial data u in H^s_r with $s > 3/2$, there exists a unique solution v in $C(\mathbb{R}, H^s_r)$ of (1) with $v|_{t=0} = u$ such that the solution map $\mathcal{S}(t)u := v(t)$,

$$\mathcal{S}: \mathbb{R} \times H^s_r \to H^s_r, \quad (t, u) \mapsto \mathcal{S}(t, u) \equiv \mathcal{S}(t)u,$$

is continuous.

Theorem 1 ([12]). (i) *For any $s > -1/2$, the map $\mathcal{S}$ extends as a continuous map, $\mathcal{S}: \mathbb{R} \times H^s_r \to H^s_r$.*

(ii) *No such extension exists for $s = -1/2$. In more detail, there exists a sequence $(u^{(k)})_{k \geq 1}$ in $C^\infty(\mathbb{T}, \mathbb{R})$ with the property that $u^{(k)} \to 0$ in $H_r^{-1/2}$, but the function $t \mapsto \langle \mathcal{S}(t)u^{(k)} \mid e^{ix} \rangle$ does not converge to 0 on any time interval $I \subseteq \mathbb{R}$ with $|I| > 0$.*

Note that $\int_0^{2\pi} u dx$ is a prime integral of (1). In particular, for any $a \in \mathbb{R}$ and $s > -1/2$, the solution map $\mathcal{S}(t)$ leaves the subspace

$$H^s_{r,a} := \left\{ u \in H^s_r \;\middle|\; \int_0^{2\pi} u \, dx = a \right\}$$

invariant. We denote by $\mathcal{S}_a(t)$ the restriction of $\mathcal{S}(t)$ to H^s_a.

Addendum to Theorem 1. *In [13, 14] we prove that for any $t \in \mathbb{R}$, $a \in \mathbb{R}$, and $-1/2 < s < 0$, $\mathcal{S}_a(t): H^s_{r,a} \to H^s_{r,a}$ is nowhere locally uniformly continuous and that for any $s \geq 0$, $\mathcal{S}_a(t): H^s_{r,a} \to H^s_{r,a}$ is real analytic.*

Remark 1. Theorem 1 (i) improves on a result by Molinet [18] (cf. also [20]), saying that for any $s \geq 0$, $\mathcal{S}: \mathbb{R} \times H^s_r \to H^s_r$ extends as a continuous map and Theorem 1 (ii) improves on a result by Angulo Pava and Hakkaev [3], saying that no such extension exists for $s < -1/2$. The Addendum to Theorem 1 improves on a result by Molinet [18, 19], saying that for any $a \in \mathbb{R}$ and $s \geq 0$, $\mathcal{S}_a(t): H^s_{r,a} \to H^s_{r,a}$ is analytic near the zero solution.

We refer to [20] for a comprehensive bibliography on the wellposedness of (1). We note that our method of proof is different from the methods used in the papers cited above (cf. Theorem 4 below).

Traveling waves. Recall that a solution $t \mapsto \mathcal{S}(t)U$ of (1) with $U \in H^s_r$, $s > -1/2$, is said a *traveling wave with profile U and velocity $c \in \mathbb{R}$* if $\mathcal{S}(t)U = U(\cdot - ct)$. Amick and Toland [2] listed all C^∞-smooth profiles of traveling waves of (1),

$$U_{r,N,\alpha,a}(x) = N U_r(Nx + \alpha) + a, \quad N \in \mathbb{N}, \alpha \in \mathbb{T}, a \in \mathbb{R},$$

where $U_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$, $0 < r < 1$, are the traveling wave profiles with corresponding velocity $c_r := \frac{1+r^2}{1-r^2}$, found by Benjamin [5].

Theorem 2 ([12]). *For any $s > -1/2$ the following holds:*

(i) *any traveling wave in H^s_r has a profile of the form $U_{r,N,\alpha,a}$ and hence in particular is C^∞-smooth;*

(ii) *any traveling wave is orbitally stable in H^s_r (see also Remark 5 below).*

Remark 2. Theorem 2 (ii) improves on a result by Angulo Pava and Natali [4], saying that the traveling waves with profile of the form $U_{r,N,\alpha,a}$ are orbitally stable in $H^{1/2}_r$.

Long time behaviour of solutions. For solutions of the Benjamin–Ono equation on the line, the question of main interest concerning their long time behaviour is to know whether they admit an asymptotic description as $t \to \infty$. Since $\mathbb{T}$ is compact, such a description typically does not exist for solutions of (1). In such a case, one is interested to know properties of the orbits of solutions such as boundedness, orbital stability, or recurrence.

Theorem 3 ([12]). *For any $u \in H^s_r$ with $s > -1/2$ the following holds:*

(i) *the orbit $\{\mathcal{S}(t)u \mid t \in \mathbb{R}\}$ is relatively compact in H^s_r;*

(ii) *the solution $\mathbb{R} \to H^s_r, t \mapsto \mathcal{S}(t)u$, is almost periodic;*

(iii) $\sup_{t\in\mathbb{R}} \|\mathcal{S}(t)u\|_s \le M$ *where $M > 0$ can be chosen uniformly on bounded subsets of H^s_r.*

Addendum to Theorem 3. The solutions of Theorem 1 of (1) are orbitally stable in the sense explained in Remark 5 below.

Remark 3. We point out that a solution $\mathbb{R} \to H^s_r, t \mapsto \mathcal{S}(t)u$ of (1) being almost periodic implies that it is Poincaré recurrent. In particular, Theorem 3 (ii) improves on results by Deng, Tzvetkov, and Visciglia [8] and Deng [7]. In these papers (cf. also references in [7, 8]) invariant measures are constructed on Sobolev spaces of various order of regularity, which then are used to show that for a.e. initial data, the corresponding solutions are Poincaré recurrent.

Theorem 3 (iii) improves on results of similar type, which can be derived from the BO hierarchy, obtained in [6, 22]. The BO hierarchy consists of a sequence $\mathcal{H}_j(u)$, $j \ge 0$, of prime integrals of (1). The boundedness of $(\mathcal{H}_j)_{0\le j\le n}$ can be shown to be equivalent to the boundedness of the $H^{n/2}$-norm. Furthermore, Talbut [26] proved such estimates for the H^s-norms, $-1/2 < s < 0$, for smooth solutions of (1).

Nonlinear Fourier transform. Our proofs of Theorems 1–3 rely on the integrability of the BO equation. In fact, we show that this equation is integrable in the strongest possible sense. To state this result, we first need to introduce some more notation. As already mentioned above, $\int_0^{2\pi} u dx$ is a prime integral of (1). Furthermore, for any solution $u(t,x)$ of (1) and any $a \in \mathbb{R}$, $u_a(t,x) = a + u(t, x - 2at)$ is again a solution. We therefore restrict ourselves to consider equation (1) on the Sobolev spaces $H^s_{r,0}$. By $h^\sigma_+ \equiv h^\sigma(\mathbb{N},\mathbb{C})$, $\sigma \in \mathbb{R}$, we denote the weighted ℓ^2-sequence spaces defined by $h^\sigma_+ := \{z = (z_n)_{n\geq 1} \mid z_n \in \mathbb{C}; \|z\|_\sigma < \infty\}$, where

$$\|z\|_\sigma := \Big(\sum_{n=1}^{\infty} n^{2\sigma}|z_n|^2\Big)^{1/2}.$$

Theorem 4 ([10, 12]). *There exists a map*

$$\Phi\colon \bigsqcup_{s>-1/2} H^s_{r,0} \to \bigsqcup_{s>-1/2} h^{s+1/2}_+, \quad u \mapsto \zeta(u) := (\zeta_n(u))_{n\geq 1}$$

so that the following properties hold for any $s > -\frac{1}{2}$:

(NF1) $\Phi\colon H^s_{r,0} \to h^{s+1/2}_+$ *is a homeomorphism and Φ and its inverse map bounded subsets to bounded ones;*

(NF2) *for any $u \in H^s_{r,0}$, and any $n \geq 1$, $\zeta_n(\mathcal{S}(t)u) = e^{i\omega_n t}\zeta_n(u)$ where*

$$\omega_n \equiv \omega_n(u) := n^2 - 2\sum_{k=1}^{n} k|\zeta_k(u)|^2 - 2n\sum_{k>n}|\zeta_k(u)|^2. \tag{2}$$

It follows that for any $n \geq 1$, $|\zeta_n(\mathcal{S}(t)u)|^2$ is independent of t.

(NF3) *The map Φ does not continuously extend to a map $H^{-1/2} \to h^0_+$.*

Addendum to Theorem 4. In [13, 14], we prove that for any $s > -1/2$, the maps $\Phi\colon H^s_{r,0} \to h^{s+1/2}_+$ and $\Phi^{-1}\colon h^{s+1/2}_+ \to H^s_{r,0}$ are real analytic.

Remark 4. (i) The differential $d_0\Phi$ of Φ at 0 is given by the weighted Fourier transform, $\mathcal{F}[v] = -(\frac{1}{\sqrt{n}}\hat{v}(n))_{n\geq 1}$. Furthermore, the linearization of (1) at the zero solution is given by $\partial_t v = H\partial_x^2 v$. The solutions of the latter equation in $H^s_{r,0}$ are given by $\sum_{n\neq 0} e^{i\,\mathrm{sign}(n)n^2 t}\hat{v}(n)e^{inx}$. For this reason, we refer to Φ as a nonlinear Fourier transform.

(ii) It is well known that (1) is Hamiltonian,

$$\partial_t u = \partial_x \nabla\mathcal{H}, \quad \mathcal{H}(u) := \frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{2}(|\partial_x|^{1/2}u)^2 - \frac{1}{3}u^3\Big)\,dx,$$

where ∂_x is the Poisson structure, which corresponds to the Poisson bracket, defined for functionals F, G on $H^s_{r,0}$ with sufficiently regular L^2-gradients,

$$\{F, G\}(u) = \frac{1}{2\pi} \int_0^{2\pi} (\partial_x \nabla F) \nabla G \, dx.$$

We prove that for any $n, m \geq 1$,

$$\{\zeta_n, \zeta_m\} = 0, \quad \{\zeta_n, \bar{\zeta}_m\} = -i\delta_{nm},$$

implying that $\{|\zeta_n|^2, |\zeta_m|^2\} = 0$. In addition, we show that $\mathcal{H} \circ \Phi^{-1}$ is a function of $|\zeta_n|^2$, $n \geq 1$, alone. Hence, Φ is *canonical*, $|\zeta_n|^2$, $n \geq 1$, are actions, and the phases of ζ_n, $n \geq 1$, angles. In this way, the quantities ζ_n, $n \geq 1$, are globally defined Birkhoff coordinates of (1) on $H^s_{r,0}$ for any $s > -1/2$.

(iii) For any $n \geq 1$, ω_n is referred to as the nth BO frequency. By (2), it is an *affine* function of the actions.

Remark 5. By Theorem 4, one infers that for $\xi \in h^{s+1/2}_+$, $s > -1/2$,

$$\mathrm{Iso}(\xi) := \{u \in H^s_{r,0} \mid |\zeta_n(u)|^2 = |\xi_n|^2 \text{ for all } n \geq 1\},$$

is an invariant torus for (1). Any such torus is *Lyapunov stable* in the sense that for any initial data $u \in H^s_{r,0}$ near $\mathrm{Iso}(\xi)$, the solution $\mathcal{S}(t)u$ stays close to $\mathrm{Iso}(\xi)$ for all $t \in \mathbb{R}$.

In the remaining part of this introduction, we briefly comment on applications of Theorem 4 and on elements of its proof. We keep our exposition as short as possible and refer to our papers for more details.

The Benjamin–Ono equation admits finite-dimensional integrable subsystems. To define them, we need to introduce some more notation. We say that $u \in \bigcup_{s>-1/2} H^s_{r,0}$ is a *finite gap potential* if there exists $N \in \mathbb{N}$ so that $\zeta_n(u) = 0$ for any $n > N$. We denote by $\mathcal{U}_N$ the set of all such potentials in $\bigcup_{s>-1/2} H^s_{r,0}$ with $\zeta_N \neq 0$. Furthermore, we say that $u \in \bigcup_{s>-1/2} H^s_{r,0}$ is a *one gap potential* if there exists $N \geq 1$ so that $\zeta_n(u) \neq 0$ if and only if $n = N$. In particular, such a potential is in $\mathcal{U}_N$. Theorem 4 implies that, for any $N \geq 1$, $\mathcal{U}_N$ is contained in $\bigcap_{s>-1/2} H^s_{r,0}$. An element $u \in \mathcal{U}_N$ is of the form

$$u(x) = -2\mathrm{Re}\Big(e^{ix} \frac{Q'_N(e^{ix})}{Q_N(e^{ix})}\Big), \quad Q_N(z) = \prod_{j=1}^{N} (1 - q_j z),$$

where $0 < |q_j| < 1$ for any $1 \leq j \leq N$ (see [10]). The time evolution of potentials in $\mathcal{U}_N$ can be explicitly described, using the frequencies, defined in (NF2) of Theorem 4.

These solutions coincide with the ones constructed by Satsuma and Ishimori [23] and further studied by Dobrokhotov and Krichever [9]. We refer to these solutions as *finite gap solutions* or *(periodic in x) multi-solitons.* They are quasiperiodic in time. The one gap solutions coincide with the traveling waves of Theorem 2 and are periodic in time.

In Section 3 we address the questions whether there are periodic and quasiperiodic solutions in time of (1) which are *not* (multi-)solitons. Both questions are answered affirmatively – see Theorem 5, Proposition 3, and Theorem 6. The proof of these results is based on the action to frequency map, studied in Section 2. To the best of our knowledge, results of this type are not known for integrable PDEs such as the Korteweg–de Vries (KdV) equation or the nonlinear Schrödinger (NLS) equation. We expect, but have not verified, that such results also hold for many of these PDEs although the action to frequency map might be significantly more complicated and hence the results more difficult to prove. In this connection, we only mention that the Hessian of the KdV and the NLS Hamiltonian are known to be strictly convex in a neighborhood of the zero solution (cf. [16, 21] for details).

A key ingredient of the proof of Theorem 4 is the Lax pair formulation of (1), $\partial_t L_u = B_u L_u - L_u B_u$, where

$$L_u = -i\partial_x - T_u, \quad B_u := i(T_{|\partial_x|u} - T_u^2), \tag{3}$$

and T_u denotes the Toeplitz operator, defined for potentials u in $H^s_{r,0}$, $s > -1/2$. Here, the pseudo-differential operators L_u and B_u act on the Hardy space

$$H_+ := \{u \in H^0 \mid \hat{u}(n) = 0 \text{ for all } n < 0\}$$

with L_u being self-adjoint (cf. [12, Corollary 2]). The Lax pair formulation implies that the spectrum of L_u is preserved by (1). For any $u \in H^s_{r,0}$, the latter is discrete, bounded from below and consists of a sequence of simple real eigenvalues, which we list in increasing order, $\lambda_0 < \lambda_1 < \cdots$ (cf. [10] ($s = 0$) and [12] ($-1/2 < s < 0$) as well as [11]). They satisfy

$$\gamma_n := \lambda_n - \lambda_{n-1} - 1 \geq 0 \quad \text{for all } n \geq 1, \tag{4}$$

where γ_n, referred to as the nth gap of the spectrum of L_u (cf. [10, Appendix C]), turns out to be the action variable $|\zeta_n|^2$, mentioned in Remark 4. The spectrum of L_u is encoded by the generating function,

$$\mathcal{H}_\lambda(u) := \langle (L_u + \lambda)^{-1} 1 \mid 1 \rangle. \tag{5}$$

This function is at the heart of the construction of the map Φ. It admits an expansion at $\lambda = \infty$, whose coefficients constitute the BO-hierarchy, mentioned in Remark 3.

In Appendix A we show that $\mathcal{H}_\lambda(u)$ can be viewed as the relative determinant of $L_u + \lambda + 1$ with respect to $L_u + \lambda$.

2. Action to frequency map

The aim of this section is to study the action to frequency map, mentioned in Section 1. We restrict ourselves to potentials $u \in H^0_{r,0}$.

Recall that for any $n \geq 1$, the actions $|\zeta_n(u)|^2$ of (1), associated to a potential $u \in L^2_{r,0} \equiv H^0_{r,0}$, coincide with the gap lengths $\gamma_n \equiv \gamma_n(u)$, $n \geq 1$, defined in (4). By Theorem 4, the actions $\gamma(u) := (\gamma_n(u))_{n\geq 1}$ fill out the positive quadrant $\ell^{1,1}_{\geq 0}$ of the ℓ^1-sequence space $\ell^{1,1} \equiv \ell^{1,1}(\mathbb{N}, \mathbb{R})$,

$$\ell^{1,1}_{\geq 0} := \Big\{ (x_n)_{n\geq 1} \in \ell^1(\mathbb{N}, \mathbb{R}) \;\Big|\; x_n \geq 0 \text{ for all } n \geq 1, \sum_{n\geq 1} n x_n < \infty \Big\}.$$

The frequencies $\omega_n \equiv \omega_n(\gamma)$, $n \geq 1$, (cf. (2)), when viewed as functions on the space of actions $\ell^{1,1}_{\geq 0}$, can then be conveniently written as

$$\omega_n(\gamma) = n^2 - 2\breve{\omega}_n(\gamma), \quad \breve{\omega}_n(\gamma) := \sum_{k\geq 1} k\gamma_k - \sum_{k>n} (k-n)\gamma_k. \tag{6}$$

Note that $\breve{\omega}_n \equiv \breve{\omega}_n(\gamma)$, $n \geq 1$, satisfy

$$\lim_{n\to\infty} \breve{\omega}_n = \sum_{k\geq 1} k\gamma_k, \quad \breve{\omega}_n - \breve{\omega}_{n-1} = \sum_{k\geq n} \gamma_k \geq 0 \quad \text{for all } n \geq 1, \tag{7}$$

(with $\breve{\omega}_0(\gamma) := 0$) and

$$(\breve{\omega}_n - \breve{\omega}_{n-1}) - (\breve{\omega}_{n+1} - \breve{\omega}_n) = \gamma_n \geq 0 \quad \text{for all } n \geq 1. \tag{8}$$

To describe the range of the map $\gamma \mapsto (\breve{\omega}_n(\gamma))_{n\geq 1}$, we introduce the Banach space c, defined as the $\mathbb{R}$-vector space of real valued, convergent sequences $\mathrm{y} := (y_n)_{n\geq 1}$, endowed with the sup-norm $\|\mathrm{y}\| := \sup_{n\geq 1} |y_n|$. For the sequel, it is convenient to set for any $\mathrm{y} = (y_n)_{n\geq 1}$ in c,

$$y_0 := 0, \quad y_\infty := \lim_{n\to\infty} y_n.$$

Denote by $\mathrm{c}_\uparrow$ the subset of c of sequences $\mathrm{y} = (y_n)_{n\geq 1}$ satisfying

$$0 \leq y_n \leq y_{n+1}, \quad (y_n - y_{n-1}) - (y_{n+1} - y_n) \geq 0 \quad \text{for all } n \geq 1.$$

By (7)–(8), for any $\gamma \in \ell^{1,1}_{\geq 0}$, the sequence $(\breve{\omega}_n(\gamma))_{n\geq 1}$ is in $\mathrm{c}_\uparrow$. The *normalized action to frequency map* is defined as

$$\breve{\Omega} \colon \ell^{1,1}_{\geq 0} \to \mathrm{c}_\uparrow, \quad \gamma \mapsto (\breve{\omega}_n(\gamma))_{n\geq 1}.$$

Proposition 1. *The map* $\breve{\Omega}\colon \ell^{1,1}_{\geq 0} \to c_\uparrow$ *is a homeomorphism.*

Remark 6. To the best of our knowledge, comparable results for integrable PDEs such as the KdV equation or the NLS equation are not known. For partial results in this direction for the KdV equation, we refer to [16] and references therein.

Proof. By (8), $\breve{\Omega}$ is one-to-one. To see that $\breve{\Omega}$ is onto, consider $y = (y_n)_{n\geq 1}$ in $c_\uparrow$. Let

$$\gamma_n := (y_n - y_{n-1}) - (y_{n+1} - y_n) \quad \text{for all } n \geq 1. \tag{9}$$

Then, by the definition of $c_\uparrow$, $\gamma_n \geq 0$ for any $n \geq 1$. By telescoping, one has for any $n \geq 1$,

$$y_n - y_{n-1} = \lim_{N\to\infty} \sum_{k=n}^{N} \gamma_k = \sum_{k\geq n} \gamma_k.$$

and for any $n \geq 0$,

$$y_\infty - y_n = \sum_{k\geq n} (y_{k+1} - y_k) = \sum_{k\geq n} \sum_{j>k} \gamma_j = \sum_{j>n} \gamma_j (j - n),$$

or $y_n = y_\infty - \sum_{j>n} \gamma_j (j-n)$. In particular, for $n = 0$, $y_\infty = \sum_{k\geq 1} k\gamma_k$. Hence, $\gamma := (\gamma_n)_{n\geq 1} \in \ell^{1,1}_{\geq 0}$ and $y_n = \sum_{k\geq 1} k\gamma_k - \sum_{k>n}(k-n)\gamma_k$ for any $n \geq 1$. We thus have proved that $\breve{\Omega}(\gamma) = y$.

Finally, note that $\breve{\Omega}$ is the restriction of a linear map $\ell^{1,1} \to c$, whose norm is bounded by 2. Indeed, since for any $(x_n)_{n\geq 1} \in \ell^{1,1}$ and any $n \geq 1$,

$$\sum_{k\geq 1} kx_k - \sum_{k>n}(k-n)x_k = \sum_{k=1}^{n} kx_k + n\sum_{k>n} x_k,$$

one has

$$\Big|\sum_{k\geq 1} kx_k - \sum_{k>n}(k-n)x_k\Big| \leq \sum_{k=1}^{n} k|x_k| + n\sum_{k>n} |x_k| \leq 2\sum_{k\geq 1} k|x_k|. \tag{10}$$

This implies that $\breve{\Omega} : \ell^{1,1}_{\geq 0} \to c_\uparrow$ is continuous. Going through the proof of the ontoness of $\breve{\Omega}\colon \ell^{1,1}_{\geq 0} \to c_\uparrow$, one sees that $\breve{\Omega}^{-1}\colon c_\uparrow \to \ell^{1,1}_{\geq 0}$ is continuous as well. Indeed, assume that $(y^{(k)})_{k\geq 1}$ is a sequence in $c_\uparrow$, converging to $y = (y_n)_{n\geq 1} \in c_\uparrow$. By (9), it follows that for any $n \geq 1$, the nth component $\gamma_n^{(k)}$ of $\gamma^{(k)} := \breve{\Omega}^{-1}(y^{(k)})$ converges to the nth component γ_n of $\gamma := \breve{\Omega}^{-1}(y)$ and $\sum_{n\geq 1} n\gamma_n^{(k)} = y_\infty^{(k)}$ converges to $\sum_{n\geq 1} n\gamma_n = y_\infty$. One then infers that for any $N \geq 1$, $\sum_{n\geq N} n\gamma_n^{(k)}$ converges to $\sum_{n\geq N} n\gamma_n$. Given any $\varepsilon > 0$, choose $N \geq 1$ so that $\sum_{n\geq N} n\gamma_n < \varepsilon/4$ and $k_\varepsilon \geq 1$ so that for any $k \geq k_\varepsilon$,

$|\sum_{n\geq N} n\gamma_n^{(k)} - \sum_{n\geq N} n\gamma_n| < \varepsilon/4$. It then follows that $|\sum_{n\geq N} n\gamma_n^{(k)}| < \varepsilon/2$ for any $k \geq k_\varepsilon$. From

$$\sum_{n\geq 1} n|\gamma_n^{(k)} - \gamma_n| \leq \sum_{n\geq N} n\gamma_n^{(k)} + \sum_{n\geq N} n\gamma_n + \sum_{n<N} n|\gamma_n^{(k)} - \gamma_n|,$$

one then concludes that $\gamma^{(k)}$ converges to γ in $\ell^{1,1}$. ∎

Remark 7. A result similar to the one of Proposition 1 can be derived for the restriction $\breve{\Omega}_J$ of $\breve{\Omega}$ to the subset $\ell^{1,1}_{\geq 0,J}$ of $\ell^{1,1}_{\geq 0}$,

$$\breve{\Omega}_J : \ell^{1,1}_{\geq 0,J} \to \mathrm{c}_{\uparrow,J}, \quad (\gamma_{n_p})_p \mapsto (\breve{\omega}_{n_p})_p.$$

Here $J := \{n_1 < n_2 < \cdots < n_*\}$ is a subset of $\mathbb{N}$ with either $n_* = \infty$ or $n_* = N+1$ for some integer $N \geq 0$. The sets $\ell^{1,1}_{\geq 0,J}$ and $\mathrm{c}_{\uparrow,J}$ are defined in these two cases as follows.

(i) *Case J infinite.* In this case, the subset J is of the form $J := \{n_p \mid p \geq 1\}$, $\ell^{1,1}_{\geq 0,J}$ is the subset

$$\ell^{1,1}_{\geq 0,J} := \{(\gamma_n)_{n\geq 1} \in \ell^{1,1}_{\geq 0} \mid \gamma_n > 0 \text{ for all } n \in J, \gamma_n = 0, n \notin J\},$$

and $\mathrm{c}_{\uparrow,J}$ the set of strictly increasing sequences $\mathrm{y}_J := (y_{n_p})_{p\geq 1}$ of positive numbers, satisfying

$$y_\infty := \lim_{p\to\infty} y_{n_p} < \infty, \quad \frac{y_{n_p} - y_{n_{p-1}}}{n_p - n_{p-1}} - \frac{y_{n_{p+1}} - y_{n_p}}{n_{p+1} - n_p} > 0 \quad \text{for all } p \geq 1,$$

where we set $n_0 = 0$ and $y_0 = 0$. Note that for any $(\gamma_n)_{n\geq 1}$ in $\ell^{1,1}_{\geq 0,J}$,

$$\breve{\omega}_{n_p} - \breve{\omega}_{n_{p-1}} = (n_p - n_{p-1}) \sum_{q\geq p} \gamma_{n_q} \quad \text{for all } p \geq 1,$$

where we recall that $\breve{\omega}_0 = 0$. For any $p \geq 1$, one then has

$$\frac{\breve{\omega}_{n_p} - \breve{\omega}_{n_{p-1}}}{n_p - n_{p-1}} - \frac{\breve{\omega}_{n_{p+1}} - \breve{\omega}_{n_p}}{n_{p+1} - n_p} = \gamma_{n_p}.$$

(ii) *Case J finite.* In this case, the subset J is of the form $J := \{n_p \mid 1 \leq p \leq N\}$, $\ell^{1,1}_{\geq 0,J}$ denotes the subset

$$\ell^{1,1}_{\geq 0,J} := \{(\gamma_n)_{n\geq 1} \in \ell^{1,1}_{\geq 0} \mid \gamma_n > 0 \text{ for all } n \in J;\ \gamma_n = 0 \text{ for all } n \notin J\},$$

and $\mathrm{c}_{\uparrow,J}$ the set of strictly increasing finite sequences $\mathrm{y}_J := (y_{n_p})_{1\leq p\leq N}$ of real numbers, satisfying

$$\frac{y_{n_p} - y_{n_{p-1}}}{n_p - n_{p-1}} - \frac{y_{n_{p+1}} - y_{n_p}}{n_{p+1} - n_p} \geq 0 \quad \text{for all } 1 \leq p \leq N,$$

where we set $n_0 = 0$, $y_0 = 0$ and $n_{N+1} = n_N + 1$, $y_{n_{N+1}} = y_{n_N}$. Note that for any $(\gamma_n)_{n\geq 1}$ in $\ell^{1,1}_{\geq 0,J}$,

$$\breve{\omega}_{n_p} - \breve{\omega}_{n_{p-1}} = (n_p - n_{p-1}) \sum_{q\geq p} \gamma_{n_q} \quad \text{for all } 1 \leq p \leq N.$$

In this case, $\breve{\omega}_n = \breve{\omega}_{n_N}$ for any $n \geq n_N$ and for any $1 \leq p \leq N$,

$$\frac{\breve{\omega}_{n_p} - \breve{\omega}_{n_{p-1}}}{n_p - n_{p-1}} - \frac{\breve{\omega}_{n_{p+1}} - \breve{\omega}_{n_p}}{n_{p+1} - n_p} = \gamma_{n_p}.$$

It is convenient to extend $\breve{\Omega}$ to a linear map $\Omega\colon \ell^{1,1} \to \mathrm{c}$. This extension is given by

$$\Omega[\mathrm{x}] = \Big(\sum_{k=1}^{n} k x_k + n \sum_{k>n} x_k\Big)_{n\geq 1} \quad \text{for all } \mathrm{x} = (x_n)_{n\geq 1} \in \ell^{1,1}.$$

Then Ω is a bounded by (10). Denote by Q the quadratic form, induced by Ω. For any $\mathrm{x} \in \ell^{1,1}$, $Q(\mathrm{x})$ is given by

$$Q(\mathrm{x}) = \langle \mathrm{x} \mid \Omega(\mathrm{x})\rangle = \sum_{n\geq 1} x_n \sum_{k=1}^{n} k x_k + \sum_{n\geq 1} n x_n \sum_{k>n} x_k.$$

Since $\sum_{n\geq 1} x_n \sum_{k=1}^{n} k x_k = \sum_{k\geq 1} k x_k \sum_{n\geq k} x_n$, $Q(\mathrm{x})$ can be written as

$$Q(\mathrm{x}) = \sum_{n\geq 1} n x_n^2 + 2 \sum_{n\geq 1} n x_n \sum_{k>n} x_k. \tag{11}$$

As a quadratic form, Q extends to $\ell^{1,1/2} \equiv \ell^{1,1/2}(\mathbb{N}, \mathbb{R})$ and

$$\begin{aligned} |Q(\mathrm{x})| &\leq \sum_{n\geq 1} \big(\sqrt{n}|x_n|\big)^2 + 2 \sum_{n\geq 1} \sqrt{n}|x_n| \sum_{k>n} \sqrt{k}|x_k| \\ &\leq 2 \sum_{n\geq 1} \sqrt{n}|x_n| \sum_{k\geq n} \sqrt{k}|x_k| \leq 2\|\mathrm{x}\|^2_{\ell^{1,1/2}}. \end{aligned}$$

(We mention that the quadrant $\ell^{1,1/2}_{\geq 0}$, filled out by the actions, corresponds to the phase space of potentials $H^{-1/4}_{r,0}$, for which the map Φ is well defined by Theorem 4.) By (11) one has

$$Q(\mathrm{x}) = \sum_{n\geq 1} n\Big(x_n^2 + 2x_n \sum_{k>n} x_k\Big).$$

Hence, by completing squares one obtains

$$Q(\mathrm{x}) = \sum_{n\geq 1} n\Big(x_n + \sum_{k>n} x_k\Big)^2 - \sum_{n\geq 1} n\Big(\sum_{k>n} x_k\Big)^2$$

or $Q(\mathrm{x}) = \sum_{n\geq 1} n(s_n^2 - s_{n+1}^2)$ where $s_n := \sum_{k\geq n} x_k$. It implies

$$Q(\mathrm{x}) = s_1^2 + \sum_{n\geq 2} s_n^2 = \sum_{n\geq 1} s_n^2.$$

In particular, one sees that Q is a positive semidefinite quadratic form on $\ell^{1,1/2}$ and that $-Q$, when restricted to $\ell^{1,1}_{\geq 0}$, coincides with the quadratic part of the Hamiltonian $\mathcal{H}$ of the BO equation, when expressed in the action variables $\gamma = (\gamma_n)_{n\geq 1}$ (cf. [10, Proposition 8.1]),

$$\mathcal{H} = \sum_{n\geq 1} n^2\gamma_n - \sum_{n\geq 1}\Big(\sum_{k\geq n}\gamma_n\Big)^2. \tag{12}$$

Since $-2Q$ is the Hessian of $\mathcal{H}$, the latter can thus be viewed as a concave function. In summary, Q has the following properties.

Proposition 2. *The quadratic form Q is well defined on $\ell^{1,1/2}$. It is positive semidefinite and satisfies*

$$|Q(\mathrm{x})| \leq 2\|\mathrm{x}\|^2_{\ell^{1,1/2}} \quad \textit{for all } \mathrm{x}\in\ell^{1,1/2}.$$

Furthermore,

$$\inf\{Q(\mathrm{x}) \mid \|\mathrm{x}\|_{\ell^{1,1/2}} = 1\} = 0, \tag{13}$$

hence Q is not *positive definite.*

Proof. It remains to prove (13). For an arbitrary integer $N \geq 2$, consider the sequence $\mathrm{x}_N := (x_n^{(N)})_{n\geq 1} \in \ell^{1,1/2}$ with $x_n^{(N)} = 0$ for any $n > N$ and

$$x_n^{(N)} = \frac{1}{a_N}(-1)^{n+1} \quad \text{for all } 1\leq n\leq N, \quad a_N := \sum_{n=1}^{N}\sqrt{n}.$$

Then $\|\mathrm{x}_N\|_{\ell^{1,1/2}} = 1$ and $(s_n^{(N)})^2 \leq 1/a_N^2$, implying that $Q(\mathrm{x}_N) \leq \frac{N}{a_N^2}$. Since

$$a_N \geq \int_0^N \sqrt{x}dx = \frac{2}{3}N^{3/2}, \quad a_N^2 \geq \frac{4}{9}N^3,$$

it follows that $Q(\mathrm{x}_N) \leq \frac{9}{4N^2}$. As $N \geq 2$ is arbitrary, (13) holds. ∎

3. Applications

As an illustration of our analysis of the action to frequency map we apply our results to construct families of periodic solutions of the BO equation, which are *not* traveling waves, and families of quasiperiodic solutions, which are *not* finite gap solutions.

Periodic solutions. Our first result addresses the question, whether there are periodic in time solutions of (1), which are *not finite gap solutions.*

Theorem 5. (i) *For $T > 0$ with T/π rational, any T-periodic solution in $L^2_{r,0}$ of* (1) *is a finite gap solution.*

(ii) *For any positive irrational number b, there exists a strictly increasing sequence $(n_p)_{p\geq 1}$ in $\mathbb{N}$ and a sequence of actions $\gamma = (\gamma_n)_{n\geq 1}$ in $\ell^{1,1}_{\geq 0,J}$, $J := \{n_p \colon p \in \mathbb{N}\}$, satisfying*

$$\sum_{p\geq 1} n_p^3 \gamma_{n_p} = \infty, \quad \omega_{n_p}(\gamma) \in b\mathbb{Z}, \quad \textit{for all } p \geq 1,$$

where $\omega_n(\gamma)$ is given by (6)*. As a consequence, any potential u_0 in the torus (cf.* [10, Section 3]*)*

$$\mathrm{Iso}_\gamma := \{u \in L^2_{r,0} \mid \gamma_n(u) = \gamma_n \textit{ for all } n \geq 1\} \tag{14}$$

is not *in $H^1_{r,0}$ and the solution $u(t)$ of the BO equation with $u(0) = u_0$ (cf. Theorem* 1*) is periodic in time with period $T = 2\pi/b$. Therefore,* Iso_γ *is entirely filled up with T-periodic solutions.*

Proof. (i) Let $u(t)$ be the solution of (1) with initial condition $u(0) = u_0 \in L^2_{r,0}$. By Theorem 1, $\Phi(u(t)) = (\zeta_n(t))_{n\geq 1} \in h^{1/2}_+$ is given by

$$\zeta_n(t) = \zeta_n(0)\mathrm{e}^{it\omega_n}, \quad |\zeta_n(0)|^2 = \gamma_n \equiv \gamma_n(u_0) \quad \text{for all } n \geq 1,$$

Hence, u is T-periodic if and only if, for every $n \geq 1$ with $\zeta_n(0) \neq 0$,

$$\omega_n \in \omega\mathbb{Z}, \quad \omega := \frac{2\pi}{T}.$$

By assumption, ω is rational. Choose $p, q \in \mathbb{Z}$ with $q \geq 1$ so that $\omega = p/q$. Since u_0 is in $L^2_{r,0}$, $\gamma = (\gamma_n)_{n\geq 1} \in \ell^{1,1}_{\geq 0}$ and formula (6) for the frequencies hold. It then follows that for any $n \geq 1$ with $\zeta_n(0) \neq 0$,

$$-2\sum_{k\geq 1} k\gamma_k + 2\sum_{k>n}(k-n)\gamma_k \in \frac{1}{q}Z.$$

Assume that there are *infinitely* many integers n with $\zeta_n(0) \neq 0$. Since one has $\sum_{k>n}(k-n)\gamma_k \to 0$ as $n \to \infty$, one concludes that $-2\sum_{k\geq 1} k\gamma_k \in \frac{1}{q}\mathbb{Z}$. Consequently, for infinitely many integers n, one has $2\sum_{k>n}(j-n)|\zeta_k(0)|^2 \in \frac{1}{q}\mathbb{Z}$, which contradicts that $\sum_{k>n}(k-n)\gamma_k$ converges to 0. Hence, there are only finitely many integers n with $\zeta_n(0) \neq 0$, which implies that $u(t)$ is a finite gap solution.

(ii) Our task is to find a strictly increasing sequence $(n_p)_{p\geq 1}$ of $\mathbb{N}$ and a sequence $\gamma = (\gamma_n)_{n\geq 1}$ in $\ell^{1,1}_{\geq 0,J}$, $J := \{n_p \mid p \in \mathbb{N}\}$, (cf. Remark 7 (i)), so that there exists a sequence $(m_p)_{p\geq 1}$ in $\mathbb{Z}$ with the property that

$$n_p^2 - 2\check{\omega}_{n_p} = m_p b \quad \text{for all } p \geq 1, \tag{15}$$

and

$$\breve{\omega}_{n_p} = y_\infty - \sum_{q>p}(n_q - n_p)\gamma_{n_q}, \quad y_\infty = \sum_{p\geq 1} n_p\gamma_{n_p}. \tag{16}$$

To find such sequences, we use that by a result due to Weyl [27], the set

$$D_b := \{n^2 + kb \mid n \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. Given an arbitrary positive real number y_∞, choose a sequence $(\varepsilon_p)_{p\geq 1}$ of the form

$$\varepsilon_p := \varepsilon_0 4^{-p} \quad \text{for all } p \geq 1, \tag{17}$$

where $\varepsilon_0 > 0$ is chosen so that

$$\varepsilon_0 < 4y_\infty. \tag{18}$$

For any $p \geq 1$, we then choose integers $n_p \geq 0$ and k_p so that

$$\rho_p := 2y_\infty - n_p^2 + k_p b \in [\varepsilon_p, 2\varepsilon_p]. \tag{19}$$

By the definition of the sequence $(\varepsilon_p)_{p\geq 1}$, $(\rho_p)_{p\geq 1}$ is a strictly decreasing sequence of positive numbers, converging to 0 as $n \to \infty$,

$$0 < \rho_{p+1} \leq 2\varepsilon_{p+1} = \varepsilon_p/2 < \varepsilon_p \leq \rho_p.$$

By induction on p, it is possible to choose $n_p \geq 1$ for any $p \geq 1$, so that $n_1 \geq 1$ and $n_{p+1} \geq 2n_p$. (Indeed, for every integer N, the set

$$\{n^2 + kb \mid 0 \leq n \leq N,\ k \in \mathbb{Z}\}$$

is discrete, so what is left over after removing it from D_b is still dense in $\mathbb{R}$.) Thinking of ρ_p as $2y_\infty - 2\breve{\omega}_{n_p}$ and hence of $2y_\infty - \rho_p$ as $2\breve{\omega}_{n_p}$, we define for any $p \geq 2$ (cf. Remark 7 (i)),

$$\gamma_{n_p} := a_p - a_{p+1}, \quad a_p := \frac{\rho_{p-1} - \rho_p}{2(n_p - n_{p-1})} > 0. \tag{20}$$

Using that $n_p \geq n_p - n_{p-1}$ and $n_{p+1} - n_p \geq n_p$, one sees that

$$\frac{n_p}{n_p - n_{p-1}}(\rho_{p-1} - \rho_p) - \frac{n_p}{n_{p+1} - n_p}(\rho_p - \rho_{p+1}) \geq \rho_{p-1} - 2\rho_p + \rho_{p+1}. \tag{21}$$

Since by (19), $\varepsilon_p \leq \rho_p \leq 2\varepsilon_p$ and by (17), $\varepsilon_{p-1} - 4\varepsilon_p = 0$, one gets from (20) and (21),

$$2n_p\gamma_{n_p} \geq \varepsilon_{p-1} - 4\varepsilon_p + \varepsilon_{p+1} = \varepsilon_{p+1}.$$

Consequently, $\gamma_{n_p} > 0$ and, for some $c > 0$,

$$n_p^3 2\gamma_{n_p} \geq (2^{p-1}n_1)^2 2n_p\gamma_{n_p} \geq 4^{p-1}n_1^2\varepsilon_{p+1} = 4^{p-1}n_1^2\frac{\varepsilon_0}{4^{p+1}} \geq c > 0,$$

implying that $\sum n_p^3 \gamma_{n_p} = \infty$. On the other hand, by (20),

$$2n_p \gamma_{n_p} \le \frac{n_p}{n_p - n_{p-1}} (\rho_{p-1} - \rho_p) \le \rho_{p-1} - \rho_p;$$

hence, by telescoping and by the bound (18) of ε_0,

$$\sum_{p \ge 2} n_p \gamma_{n_p} \le \rho_1/2 \le \varepsilon_1 = \varepsilon_0/4 < y_\infty.$$

Now, define $\gamma_{n_1} > 0$ so that the second identity in (16) holds,

$$\gamma_{n_1} := \frac{1}{n_1} \Big(y_\infty - \sum_{p \ge 2} n_p \gamma_{n_p} \Big) > 0.$$

It remains to check the identities in (15). Using the definition (20) of γ_{n_p}, $p \ge 2$, one verifies that for any $p \ge 1$, $\rho_p = 2 \sum_{q > p} (n_q - n_p) \gamma_{n_q}$ and thus, by the definition of ρ_p,

$$\check{\omega}_p = n_p^2 - 2y_\infty + 2 \sum_{q > p} (n_q - n_p) \gamma_{n_q} = k_p b \quad \text{for all } p \ge 1.$$

This completes the proof of item (ii). ■

Remark 8. (i) It is possible to choose the sequence $(\varepsilon_p)_{p \ge 1}$, constructed in the proof of Theorem 5 (ii), so that $u \notin H^s_{r,0}$ for some $0 < s < 1$.

(ii) The sequence $(n_p)_{p \ge 1}$, constructed in the proof of Theorem 5 (ii), needs to be sparse in the following sense: if $(n_p)_{p \ge 1}$ is a strictly increasing sequence in $\mathbb{N}$ and $\gamma = (\gamma_k)_{k \ge 1}$ a sequence of actions in $\ell^{1,1}_{\ge 0, J}$, $J := \{n_p \mid p \ge 1\}$, with the property that there exists an infinite set of integers n in J so that $n - 1$ and $n + 1$ are also contained in J, then the frequencies cannot satisfy (15). Indeed, the frequencies satisfy on $\ell^{1,1}_{\ge 0}$ the identities (with $\omega_0 = 0$)

$$\omega_{k+1} - 2\omega_k + \omega_{k-1} = 2 + 2\gamma_k \quad \text{for all } k \ge 1.$$

Hence, if $\omega_{n_p} \in b\mathbb{Z}$ for any $p \ge 1$, it then would follow that $2 + 2\gamma_n(u) \in b\mathbb{Z}$ for infinitely many n in J, implying that $2 \in b\mathbb{Z}$. This however contradicts the assumption of b being irrational.

The following result says that there are many finite gap solutions of the BO equation which are periodic in time, but not traveling waves.

Proposition 3. *For any rational number of the form $1/a$, $a \in \mathbb{N}$, any $N \in \mathbb{N}$, and any strictly increasing sequences $(n_p)_{1 \le p \le N}$, $(k_p)_{1 \le p \le N}$ in $\mathbb{N}$ with*

$$\frac{k_p - k_{p-1}}{n_p - n_{p-1}} - \frac{k_{p+1} - k_p}{n_{p+1} - n_p} > 0 \quad \textit{for all } 1 \le p \le N,$$

(where we set $n_0 := 0$, $n_{N+1} := n_N + 1$, $k_0 := 0$, and $k_{N+1} := k_N$), the following holds: the sequence of actions, $\gamma = (\gamma_n)_{n\geq 1} \in \ell^{1,1}_{\geq 0,J}$, defined by

$$\gamma_{n_p} := \frac{1}{a}\Big(\frac{k_p - k_{p-1}}{n_p - n_{p-1}} - \frac{k_{p+1} - k_p}{n_{p+1} - n_p}\Big) \quad \textit{for all } 1 \leq p \leq N,$$

and $J := \{n_p \mid 1 \leq p \leq N\}$ (cf. Remark 7(ii)), has frequencies $\omega_{n_p} \equiv \omega_{n_p}(\gamma)$, $1 \leq p \leq N$, given by

$$\omega_{n_p} = n_p^2 - 2\check{\omega}_{n_p} \in \frac{1}{a}\mathbb{Z}, \quad \check{\omega}_{n_p} = \frac{1}{a}k_p \quad \textit{for all } 1 \leq p \leq N.$$

As a consequence, any potential u_0 in the torus Iso_γ (cf. (14)) is a finite gap potential, the solution $u(t)$ of (1) with $u(0) = u_0$ periodic in time with period $T = 2\pi a$, and hence Iso_γ entirely filled with T-periodic solutions.

Remark 9. Since by Theorem 2, the traveling waves of the BO equation coincide with the one gap solutions, it follows from Proposition 3 that there is a plenitude of periodic in time solutions of (1) which are finite gap solutions, but not traveling waves.

Proof. The claimed results follow from Remark 7 (ii). ■

Quasiperiodic solutions. The aim of this paragraph to construct quasiperiodic solutions of (1), which are *not* finite gap solutions. We begin with describing the ω-quasiperiodic in time solutions of (1) in terms of the map Φ of Theorem 4 where ω is a frequency vector in $\mathbb{R}^d$, $d \geq 2$, with $\mathbb{Q}$-linearly independent components.

Definition 1. Let E be a Banach space and $\omega \in \mathbb{R}^d$, $d \geq 2$ with $\mathbb{Q}$-linearly independent components. A function $u \in C(\mathbb{R}, E)$ is said *ω-quasiperiodic* if there exists a function $U \in C(\mathbb{T}^d, E)$, so that $u(t) = U(t\omega)$ for any $t \in \mathbb{R}$. Here, by notational convenience, the vector $t\omega$ denotes also the class of vectors $t\omega + (2\pi\mathbb{Z})^d$. The function U is referred to as the *profile* of u.

Proposition 4. *Let $U : \mathbb{T}^d \to H^s_{r,0}$ with $s > -1/2$ and let ω be a vector in $\mathbb{R}^d$, $d \geq 2$, with $\mathbb{Q}$-linearly independent components. Then U is the profile of a ω-quasiperiodic solution of (1) in $H^s_{r,0}$ if and only if $\Phi(U(\varphi))$ is of the form*

$$\Phi(U(\varphi)) = (\zeta_n e^{ik^{(n)}\cdot\varphi})_{n\geq 1} \quad \textit{for all } \varphi \in \mathbb{T}^d, \tag{22}$$

where $(\zeta_n)_{n\geq 1} \in h^{s+1/2}_+$ and $(k^{(n)})_{n\geq 1}$ is a sequence in $\mathbb{Z}^d$ with the property that for any $n \geq 1$,

$$\zeta_n = 0 \quad \textit{or} \quad k^{(n)} \cdot \omega = n^2 - 2\sum_{k=1}^{n} k|\zeta_k|^2 - 2n\sum_{k>n} |\zeta_k|^2. \tag{23}$$

Remark 10. For any ω-quasiperiodic solution of (1) with action variables

$$\gamma = (\gamma_n)_{n\ge 1} \in \ell^{1,1+2s}_{\ge 0},$$

the invariant torus

$$\mathrm{Iso}_\gamma := \{u \in H^s_{r,0} \mid \gamma_n(u) = \gamma_n \text{ for all } n \ge 1\}$$

is filled with ω-quasiperiodic solutions of (1). The corresponding profiles are given by (22) with $(\zeta_n)_{n\ge 1}$ being an arbitrary element in the set $\Phi(\mathrm{Iso}_\gamma)$.

Proof. Let U be the profile of an ω-quasiperiodic solution $u(t)$ in $H^s_{r,0}(\mathbb{T})$ of (1). It is to show that (22)–(23) hold. Let

$$(\xi_n(\varphi))_{n\ge 1} := \Phi(U(\varphi)) \in h^{s+1/2}_+ \quad \text{for all } \varphi \in \mathbb{T}^d.$$

Since by Definition 1, U is in $C(\mathbb{T}^d, H^s_{r,0})$, the map $\mathbb{T}^d \to h^{s+1/2}_+, \varphi \mapsto (\xi_n(\varphi))_{n\ge 1}$ is continuous and by (NF2) in Theorem 1, for any $n \ge 1$,

$$\zeta_n \mathrm{e}^{it\omega_n} = \xi_n(t\omega) \quad \text{for all } t \in \mathbb{R}, \tag{24}$$

where $(\zeta_n)_{n\ge 1} := (\xi_n(0))_{n\ge 1} \in h^{s+1/2}_+$. Since by assumption, the components of ω are linearly independent in $\mathbb{Q}$, the Fourier coefficients $\hat{\xi}_n(k)$, $k \in \mathbb{Z}^d$, of ξ_n can be computed as

$$\hat{\xi}_n(k) = (2\pi)^{-d} \int_{\mathbb{T}^d} \xi_n(\varphi)\mathrm{e}^{-ik\cdot\varphi}\, d\varphi = \lim_{T\to\infty} T^{-1} \int_0^T \xi_n(t\omega)\mathrm{e}^{-itk\cdot\omega}\, dt.$$

Furthermore, by formula (24) for $\xi_n(t\omega)$, one has

$$\lim_{T\to\infty} T^{-1} \int_0^T \xi_n(t\omega)\mathrm{e}^{-itk\cdot\omega}\, dt = \zeta_n \lim_{T\to\infty} T^{-1} \int_0^T \mathrm{e}^{it(\omega_n - k\cdot\omega)}\, dt.$$

Note that the right-hand side of the latter identity vanishes if $\omega_n \ne k\cdot\omega$, and equals ζ_n if $\omega_n = k\cdot\omega$. Consequently, for any given $n \ge 1$, the following dichotomy holds: in the case where there is no $k \in \mathbb{Z}^d$, satisfying $\omega_n = k\cdot\omega$, it follows that $\hat{\xi}_n(k) = 0$ for any $k \in \mathbb{Z}^d$. Hence, the continuous function ξ_n vanishes, implying that $\zeta_n = \xi_n(0) = 0$. Otherwise, since the components of ω are linearly independent over $\mathbb{Q}$, there exists exactly one $k^{(n)} \in \mathbb{Z}^d$ such that $\omega_n = k^{(n)}\cdot\omega$ and $\xi_n(\varphi)$ equals $\zeta_n \mathrm{e}^{ik^{(n)}\cdot\varphi}$. We thus have proved that (22)–(23) hold.

Conversely, if $\Phi\circ U$ is given by the expression (22), U is a continuous map $\mathbb{T}^d \to H^s_{r,0}$ since Φ^{-1} is continuous. Furthermore, by (23), $\Phi(U(t\omega)) = (\zeta_n \mathrm{e}^{it\omega_n})_{n\ge 1}$ so that $t \mapsto U(t\omega)$ is a ω-quasiperiodic solution of (1) with profile U. ∎

The following result illustrates how Proposition 4 can be used to construct ω-quasiperiodic solutions of (1), which are not C^∞-smooth, hence in particular not finite gap solutions.

Theorem 6. *Let b be an irrational real number and $\omega := (1, b) \in \mathbb{R}^2$. For any $s > -1/2$, there exists an ω-quasiperiodic solution of (1) in $H^s_{r,0} \setminus \bigcup_{\sigma>s} H^\sigma_{r,0}$.*

Proof. Let $s > -1/2$ be given. In view of Proposition 4, it suffices to find a sequence $(\zeta_n)_{n\geq 1}$ in $h^{s+1/2}_+ \setminus \bigcup_{\sigma>s} h^{\sigma+1/2}_+$ with $\zeta_n = |\zeta_n| > 0$ and a sequence $(k^{(n)})_{n\geq 1}$ in $\mathbb{Z}^2$ so that

$$k^{(n)} \cdot \omega = n^2 - 2\sum_{k=1}^{n} k\zeta_k^2 - 2n\sum_{k>n}\zeta_k^2 \quad \text{for all } n \geq 1.$$

The latter identities imply that for any $n \geq 1$,

$$(k^{(n+1)} - k^{(n)}) \cdot \omega = 2n + 1 - 2\sum_{j=n+1}^{\infty}\zeta_j^2,$$
$$(k^{(n+1)} - 2k^{(n)} + k^{(n-1)}) \cdot \omega = 2 + 2\zeta_n^2,$$

with $k^{(0)} := 0 \in \mathbb{Z}^2$. It is convenient to reformulate our problem. Let

$$\ell^{(n)} := k^{(n+1)} - 2k^{(n)} + k^{(n-1)} \quad \text{for all } n \geq 1.$$

Since

$$k^{(n+1)} = \ell^{(n)} + 2k^{(n)} - k^{(n-1)}, \quad n \geq 1,$$

our problem can be described equivalently as follows: find a sequence $(\gamma_n)_{n\geq 1}$ in $\mathbb{R}_{>0}$, belonging to $\ell^{1,1+2s}_+ \setminus \bigcup_{\sigma>s} \ell^{1,1+2\sigma}_+$, a sequence $(\ell^{(n)})_{n\geq 1}$ in $\mathbb{Z}^2$, and $k^{(1)} \in \mathbb{Z}^2$ so that $k^{(0)} = 0$ and

$$\ell^{(n)} \cdot \omega = 2 + 2\gamma_n \quad \text{for all } n \geq 1, \quad k^{(1)} \cdot \omega = 1 - 2\sum_{j=1}^{\infty}\gamma_j, \tag{25}$$

where, for any $n \geq 1$, γ_n is related to ζ_n by $\zeta_n = \sqrt{\gamma_n}$.

Using the density of the additive subgroup $\omega \cdot \mathbb{Z}^2 = \mathbb{Z} + b\mathbb{Z}$ in $\mathbb{R}$, it is straightforward to construct sequences $(\gamma_n)_{n\geq 1}$ and $(\ell^{(n)})_{n\geq 1}$, which satisfy the first set of identities in (25). But it is more involved to construct such sequences satisfying at the same time the second identity in (25), which can be rephrased as

$$1 - 2\sum_{j=1}^{\infty}\gamma_j \in \omega \cdot \mathbb{Z}^2. \tag{26}$$

Accordingly, we proceed in two steps.

Step 1. Let $(\varepsilon_n)_{n\geq 1}$ be a sequence in $\mathbb{R}_{>0}$ that belongs to $\ell^{1,1+2s}_{\geq 0} \setminus \bigcup_{\sigma>s} \ell^{1,1+2\sigma}_{\geq 0}$ and satisfies

$$4\sum_{n=1}^{\infty} \varepsilon_n < 1. \tag{27}$$

By the density of $\omega \cdot \mathbb{Z}^2$ in $\mathbb{R}$, there exist $m^{(n)} \in \mathbb{Z}^2$, $n \geq 1$, so that

$$2 + \varepsilon_n \leq m^{(n)} \cdot \omega \leq 2 + 2\varepsilon_n.$$

For any $n \geq 1$, let $\bar{\gamma}_n$ be the number in $[\varepsilon_n/2, \varepsilon_n]$, defined by

$$m^{(n)} \cdot \omega = 2 + 2\bar{\gamma}_n.$$

By (27), it then follows that

$$x := 1 - 2\sum_{j=1}^{\infty} \bar{\gamma}_j \in \Big(\frac{1}{2}, 1\Big).$$

Step 2. We correct $m^{(n)}$ and $\bar{\gamma}_n$ so that (26) is satisfied with $k^{(1)} = 0$. To this end, we inductively construct a sequence $(\delta_n)_{n\geq 1}$ in $\mathbb{R}_{>0}$, which belongs to $\omega \cdot \mathbb{Z}^2$ and satisfies

$$x = \sum_{j=1}^{\infty} \delta_j, \quad 0 < \delta_n < 2^{1-n}, \quad \text{for all } n \geq 1. \tag{28}$$

We begin with δ_1. Since $\frac{1}{2} < x < 1$, there exists $\delta_1 \in \omega \cdot \mathbb{Z}^2$ so that $x - \frac{1}{2} < \delta_1 < x - \frac{1}{4}$. It follows that $0 < \delta_1 < 1$ and that $y_1 := x - \delta_1$ satisfies

$$\frac{1}{4} < y_1 < \frac{1}{2}, \quad x = \delta_1 + y_1.$$

Since $\frac{1}{4} < y_1 < \frac{1}{2}$, there exists $\delta_2 \in \omega \cdot \mathbb{Z}^2$ so that $y_1 - \frac{1}{4} < \delta_2 < y_1 - \frac{1}{8}$. One concludes that $0 < \delta_2 < 2^{-1}$ and that $y_2 := y_1 - \delta_2$ satisfies

$$\frac{1}{8} < y_2 < \frac{1}{4}, \quad x = \delta_1 + \delta_2 + y_2.$$

Continuing inductively in this way, we construct sequences $(y_n)_{n\geq 1}$, $(\delta_n)_{n\geq 1}$ in $\mathbb{R}_{>0}$ with $(\delta_n)_{n\geq 1}$ belonging to $\omega \cdot \mathbb{Z}^2$, so that for any $n \geq 1$,

$$0 < \delta_n < \frac{1}{2^{n-1}}, \quad \frac{1}{2^{n+1}} < y_n < \frac{1}{2^n}, \quad x = \sum_{j=1}^{n} \delta_j + y_n.$$

Hence, we obtain $x = \sum_{j=1}^{\infty} \delta_j$. By construction, δ_n is of the form $\delta_n = p^{(n)} \cdot \omega$ with $p^{(n)} \in \mathbb{Z}^2$ and hence we define

$$\gamma_n := \bar{\gamma}_n + \frac{\delta_n}{2} > 0, \quad \ell^{(n)} := m^{(n)} + p^{(n)} \in \mathbb{Z}^2, \quad \text{for all } n \geq 1.$$

Since $(\bar{\gamma}_n)_{n\geq 1}$ is in $\ell^{1,1+2s}_{\geq 0} \setminus \bigcup_{\sigma>s} \ell^{1,1+2\sigma}_{\geq 0}$, and since δ_n satisfies $0 < \delta_n < 2^{1-n}$ for any $n \geq 1$, $(\gamma_n)_{n\geq 1}$ is also in $\ell^{1,1+2s}_{\geq 0} \setminus \bigcup_{\sigma>s} \ell^{1,1+2\sigma}_{\geq 0}$. Furthermore,

$$2 + 2\gamma_n = 2 + 2\bar{\gamma}_n + \delta_n = m^{(n)} \cdot \omega + p^{(n)} \cdot \omega = \ell^{(n)} \cdot \omega$$

and $k^{(1)} = 0$, as $1 - 2\sum_{j=1}^{\infty} \gamma_j = 1 - 2\sum_{j=1}^{\infty} \bar{\gamma}_j - \sum_{j=1}^{\infty} \delta_j = 0 = 0 \cdot \omega$. ■

Remark 11. In contrast to the periodic in time solutions of (1) of Theorem 5 (cf. also Remark 8 (ii)), the action variables of the ω-quasiperiodic solutions constructed in the proof of Theorem 6 are all strictly positive.

A. Generating function

The aim of this appendix is to show that the generating function $\mathcal{H}_\lambda(u)$, defined in (5) (cf. [10, 12]), is the relative determinant of $L_u + \lambda + 1$ by $L_u + \lambda$, where L_u is the Lax operator of (1) (cf. (3)).

First, let us introduce the notion of a relative determinant in a setup, sufficient for our purposes. To motivate the definition of such a determinant, consider a positive Hermitian $N \times N$ matrix A with complex valued coefficients. We list its eigenvalues in increasing order and with their multiplicities, $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ and consider the one-parameter family of matrices $A + \lambda$. It is then straightforward to verify that for any $\lambda > -\mu_1$, the following, well-known formula holds,

$$\frac{d}{d\lambda} \log(\det(A + \lambda)) = \operatorname{trace}(A + \lambda)^{-1}.$$

This formula motivates the following definition of a relative determinant. Let A and A_0 be two self-adjoint operators, acting on the Hardy space H_+ and assume that both are semibounded (from below) and have compact resolvents. Hence, their spectrum is discrete, real, and bounded from below. The spectrum of A and the one of A_0 then consist of increasing sequences $(\mu_n)_{n\geq 0}$ and $(\nu_n)_{n\geq 0}$, respectively, of real eigenvalues, converging to ∞. Let $\lambda_* := \max\{-\mu_0, -\nu_0\}$. Then for any $\lambda > \lambda_*$, $A + \lambda$ and $A_0 + \lambda$ are invertible. We say that $A + \lambda$ *admits a determinant relative to* $A_0 + \lambda$ (for $\lambda > \lambda_*$) if for any $\lambda > \lambda_*$, $(A + \lambda)^{-1} - (A_0 + \lambda)^{-1}$ is of trace class and if there exists a C^1-function,

$$(\lambda_*, \infty) \to \mathbb{R}_{>0}, \quad \lambda \mapsto \det_{A_0+\lambda}(A + \lambda),$$

satisfying the normalization condition $\det_{A_0+\lambda}(A + \lambda) \to 0$ as $\lambda \to \infty$, and for any $\lambda_* < \lambda < \infty$, the variational formula

$$\frac{d}{d\lambda} \log\big(\det_{A_0+\lambda}(A + \lambda)\big) = \operatorname{trace}\big((A + \lambda)^{-1} - (A_0 + \lambda)^{-1}\big).$$

To state our result on $\mathcal{H}_\lambda(u)$, recall that for any $u \in H^s_{r,0}$ with $s > -1/2$, the spectrum of the Lax operator L_u is given by a sequence of simple real eigenvalues,

$$\lambda_0 < \lambda_1 < \cdots,$$

and (cf. [10, 12])

$$\lambda_n = n - \sum_{k>n} \gamma_k, \quad \gamma_k := \lambda_k - \lambda_{k-1} - 1 \geq 0, \quad \text{for all } k \geq 1. \tag{29}$$

The sequence $(\gamma_k)_{k\geq 1}$ is an element in the weighted ℓ^1-space $\ell^{1,1+2s}$.

Proposition 5. *For any $u \in H^s_{r,0}$ with $s > -1/2$ the following holds:*

(i) *for any $\lambda > -\lambda_0$, $(L_u + 1 + \lambda)^{-1} - (L_u + \lambda)^{-1}$ is of trace class;*

(ii) *$L_u + 1 + \lambda$ admits a determinant relative to $L_u + \lambda$ and*

$$\mathcal{H}_\lambda(u) = \det_{L_u+\lambda}(L_u + 1 + \lambda) \quad \textit{for all } \lambda > -\lambda_0.$$

Proof. We use arguments developed in the proof of [10, Lemma 3.2].

(i) By functional calculus, $(L_u + 1 + \lambda)^{-1} - (L_u + \lambda)^{-1}$ is a self-adjoint operator on H_+ with eigenvalues

$$\frac{1}{\lambda_n + 1 + \lambda} - \frac{1}{\lambda_n + \lambda} = -\frac{1}{(\lambda_n + 1 + \lambda)(\lambda_n + \lambda)}.$$

One then concludes from (29) and the decay properties of the γ_n that

$$(L_u + 1 + \lambda)^{-1} - (L_u + \lambda)^{-1}$$

is of trace class for any $\lambda > -\lambda_0$ and that its trace is given by

$$-\sum_{n\geq 0} (\lambda_n + 1 + \lambda)^{-1} (\lambda_n + \lambda)^{-1}.$$

(ii) Denote by S the shift operator on H_+ and by S^* its adjoint,

$$S\colon H_+ \to H_+, \quad f \mapsto e^{ix} f,$$
$$S^*\colon H_+ \to H_+, \quad f \mapsto e^{-ix}(f - \langle f | 1 \rangle 1).$$

By a straightforward computation, one has (cf. [10, (3.3)])

$$S^*(L_u + \lambda)S = L_u + \lambda + 1 \tag{30}$$

and (cf. [10, Lemma 3.1])

$$(S^*(L_u + \lambda)S)^{-1} = S^*(L_u + \lambda)^{-1} S - \frac{\langle \cdot \mid S^* w_\lambda \rangle}{\langle w_\lambda \mid 1 \rangle} S^* w_\lambda, \tag{31}$$

where $w_\lambda := (L_u + \lambda)^{-1}1$. Combining the identities (30)–(31), one gets

$$(L_u + 1 + \lambda)^{-1} - (L_u + \lambda)^{-1} = \mathrm{I}_\lambda - \mathrm{II}_\lambda \,, \tag{32}$$

where

$$\mathrm{I}_\lambda := S^*(L_u + \lambda)^{-1} S - (L_u + \lambda)^{-1}, \quad \mathrm{II}_\lambda := \frac{\langle \cdot \mid S^* w_\lambda \rangle}{\langle w_\lambda \mid 1 \rangle} S^* w_\lambda .$$

Note that II_λ is an operator of rank 1 and hence in particular of trace class. Its trace can be computed as follows. Denote by $(f_n)_{n\geq 0}$ the orthonormal basis of eigenfunctions of L_u, introduced in [10] ($s = 0$) and [12] ($-1/2 < s < 0$). Computing the trace of II_λ with respect to this basis one obtains from Parseval's identity,

$$\mathrm{trace}(\mathrm{II}_\lambda) = \sum_{n\geq 0} \frac{\langle f_n \mid S^* w_\lambda \rangle}{\langle w_\lambda \mid 1 \rangle} \langle S^* w_\lambda \mid f_n \rangle = \frac{\|S^* w_\lambda\|^2}{\langle w_\lambda \mid 1 \rangle}.$$

Using that $S^* w_\lambda = e^{-ix} w_\lambda - \langle w_\lambda \mid 1 \rangle e^{-ix}$ and that

$$\langle w_\lambda | 1 \rangle = \langle (L_u + \lambda)^{-1} 1 \mid 1 \rangle \geq 0,$$

since $(L_u + \lambda)^{-1}$ is a positive operator, one infers

$$\mathrm{trace}(II_\lambda) = \frac{\|S^* w_\lambda\|^2}{\langle w_\lambda \mid 1 \rangle} = \frac{\|w_\lambda\|^2}{\langle w_\lambda \mid 1 \rangle} - \frac{|\langle w_\lambda \mid 1 \rangle|^2}{\langle w_\lambda \mid 1 \rangle} = \frac{\|w_\lambda\|^2}{\langle w_\lambda \mid 1 \rangle} - \langle w_\lambda \mid 1 \rangle. \tag{33}$$

Since, by (32), the operator I_λ is the sum of two operators of trace class, it is itself of trace class. Computing its trace with respect to the orthonormal basis $(e^{inx})_{n\geq 0}$, one obtains

$$\begin{aligned} \mathrm{trace}(I_\lambda) &= \sum_{n\geq 0} \langle (L_u + \lambda)^{-1} S e^{inx} \mid S e^{inx} \rangle - \langle (L_u + \lambda)^{-1} e^{inx} \mid e^{inx} \rangle \\ &= -\langle (L_u + \lambda)^{-1} 1 \mid 1 \rangle = -\langle w_\lambda \mid 1 \rangle. \end{aligned} \tag{34}$$

Combining (32), (33), and (34) we arrive at

$$\mathrm{trace}\big((L_u + 1 + \lambda)^{-1} - (L_u + \lambda)^{-1}\big) = -\frac{\|w_\lambda\|^2}{\langle w_\lambda \mid 1 \rangle}.$$

Since

$$\langle w_\lambda \mid 1 \rangle = \sum_{n\geq 0} \frac{|\langle 1 \mid f_n \rangle|^2}{\lambda_n + \lambda}, \quad \|w_\lambda\|^2 = \sum_{n\geq 0} \frac{|\langle 1 \mid f_n \rangle|^2}{(\lambda_n + \lambda)^2}$$

one has,[1]

$$-\frac{\|w_\lambda\|^2}{\langle w_\lambda \mid 1 \rangle} = \frac{d}{d\lambda} \log\Big(\sum_{n\geq 0} \frac{|\langle 1 \mid f_n \rangle|^2}{\lambda_n + \lambda}\Big)$$

[1]Note that by (5) for a given $u \in H^s_{r,0}$, $\mathcal{H}_\lambda(u)$ is real analytic for $\lambda > -\lambda_0$.

and hence

$$\frac{d}{d\lambda}\log\Big(\sum_{n\geq 0}\frac{|\langle 1 \mid f_n\rangle|^2}{\lambda_n+\lambda}\Big) = \operatorname{trace}\big((L_u+1+\lambda)^{-1}-(L_u+\lambda)^{-1}\big). \qquad (35)$$

Since when expanding $\mathcal{H}_\lambda(u) = \langle (L_u + \lambda\,\mathrm{Id})^{-1}1 \mid 1\rangle$ with respect to the orthonormal basis $(f_n)_{n\geq 0}$, one obtains

$$\mathcal{H}_\lambda(u) = \sum_{n\geq 0}\frac{|\langle 1 \mid f_n\rangle|^2}{\lambda_n+\lambda}$$

and since $\lim_{\lambda\to\infty}\mathcal{H}_\lambda(u) = 0$ we proved that $\mathcal{H}_\lambda(u)$ is the determinant of $L_u+1+\lambda$ relative to $L_u+\lambda$. ■

In the remaining part of this appendix we study in more detail how for any given $u \in H^s_{r,0}$, $s > -1/2$, $\mathcal{H}_\lambda(u)$ is related to the spectrum of L_u. First, note that it follows from (35) that

$$\frac{d}{d\lambda}\log\mathcal{H}_\lambda = \frac{d}{d\lambda}\Big(\log\Big(\frac{1}{\lambda_0+\lambda}\Big) + \sum_{n\geq 1}\log\Big(\frac{\lambda_{n-1}+1+\lambda}{\lambda_n+\lambda}\Big)\Big),$$

implying that (cf. [10, Proposition 3.1] and [12])

$$\mathcal{H}_\lambda(u) = \frac{1}{\lambda_0+\lambda}\prod_{n\geq 1}\frac{\lambda_{n-1}+1+\lambda}{\lambda_n+\lambda}. \qquad (36)$$

Therefore, $\mathcal{H}_\lambda(u)$ is determined by the periodic spectrum of L_u. Since the latter is invariant by the flow of (1), $\mathcal{H}_\lambda(u)$ is a one-parameter family of prime integrals of this equation. The question arises if, conversely, $\mathcal{H}_\lambda(u)$ determines the spectrum of L_u. To this end, we take a closer look at the product representation (36) of $\mathcal{H}_\lambda(u)$. Setting $\nu_n := \lambda_{n-1}+1$ for any $n \geq 1$, one has

$$\lambda_0 < \nu_1 \leq \lambda_1 < \nu_2 \leq \lambda_2 < \cdots.$$

Furthermore, for any $n \geq 1$, $\nu_n = \lambda_n$ if and only if $\gamma_n = 0$. Hence,

$$\mathcal{H}_\lambda(u) = \frac{1}{\lambda_0+\lambda}\prod_{n\in J_u}\frac{\nu_n+\lambda}{\lambda_n+\lambda}, \qquad J_u := \{n \geq 1 \mid \gamma_n > 0\}.$$

Note that $-\nu_n$, $n \in J_u$, are the zeros of $\mathcal{H}_\lambda(u)$ and $-\lambda_n$, $n \in J_u \cup \{0\}$, its poles. All the poles and zeros of $\mathcal{H}_\lambda(u)$ are simple. The question raised above can now be rephrased as follows: does $\mathcal{H}_\lambda(u)$ besides $\nu_n(u)$, $n \in J_u$, and $\lambda_n(u)$, $n \in J_u \cup \{0\}$, also determine the eigenvalues $\lambda_n(u)$ with $\gamma_n(u) = 0$? The following result says that this is indeed the case.

Proposition 6. *For any $u \in H^s_{r,0}$, $s > -1/2$, the generating function $\mathcal{H}_\lambda(u)$ determines the entire spectrum of L_u.*

Proof. Since $-\lambda_0$ is a pole of $\mathcal{H}_\lambda(u)$, it is determined by the generating function. If $-\nu_1$ is a zero of $\mathcal{H}_\lambda(u)$, then $\nu_1 = \lambda_0 + 1 < \lambda_1$ and hence $-\lambda_1$ is a pole of $\mathcal{H}_\lambda(u)$. If $-(\lambda_0 + 1)$ is not a zero of $\mathcal{H}_\lambda(u)$, then $\lambda_0 + 1$ is the periodic eigenvalue λ_1 of L_u and hence also determined by $\mathcal{H}_\lambda(u)$. Arguing inductively, the claimed result follows. ■

Remark 12. With the help of a conformal map, Hochstadt proved a result corresponding to the one of Proposition 6 for Hill's operator, which is a Lax operator for the Korteweg–de Vries equation (cf. [15, 17]). Note that in contrast, the proof of Proposition 6 is elementary.

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References

[1] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut, Nonlocal models for nonlinear, dispersive waves. *Phys. D* **40** (1989), no. 3, 360–392 Zbl 0699.35227 MR 1044731

[2] C. J. Amick and J. F. Toland, Uniqueness and related analytic properties for the Benjamin–Ono equation – a nonlinear Neumann problem in the plane. *Acta Math.* **167** (1991), no. 1–2, 107–126 Zbl 0755.35108 MR 1111746

[3] J. Angulo Pava and S. Hakkaev, Ill-posedness for periodic nonlinear dispersive equations. *Electron. J. Differential Equations* **2010**, article no. 119 Zbl 1402.35237 MR 2685029

[4] J. Angulo Pava and F. M. A. Natali, Positivity properties of the Fourier transform and the stability of periodic travelling-wave solutions. *SIAM J. Math. Anal.* **40** (2008), no. 3, 1123–1151 Zbl 1162.76012 MR 2452883

[5] T. Benjamin, Internal waves of permanent form in fluids of great depth. *J. Fluid Mech.* **29** (1967), 559–592 Zbl 0147.46502

[6] T. L. Bock and M. D. Kruskal, A two-parameter Miura transformation of the Benjamin–Ono equation. *Phys. Lett. A* **74** (1979), no. 3-4, 173–176 MR 591320

[7] Y. Deng, Invariance of the Gibbs measure for the Benjamin–Ono equation. *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 5, 1107–1198 Zbl 1379.37135 MR 3346690

[8] Y. Deng, N. Tzvetkov, and N. Visciglia, Invariant measures and long time behaviour for the Benjamin–Ono equation III. *Comm. Math. Phys.* **339** (2015), no. 3, 815–857 Zbl 1379.37136 MR 3385985

[9] S. Y. Dobrokhotov and I. M. Krichever, Multiphase solutions of the Benjamin–Ono equation and their averaging. *Mat. Zametki* **49** (1991), no. 6, 42–58, 158; English translation: *Math. Notes* **49** (1991), no. 5–6, 583–594 Zbl 0752.35058 MR 1135514

[10] P. Gérard and T. Kappeler, On the integrability of the Benjamin–Ono equation on the torus. *Comm. Pure Appl. Math.* **74** (2021), no. 8, 1685–1747 Zbl 1471.35354 MR 4275336

[11] P. Gérard, T. Kappeler, and P. Topalov, On the spectrum of the Lax operator of the Benjamin–Ono equation on the torus. *J. Funct. Anal.* **279** (2020), no. 12, article no. 108762 Zbl 1452.37070 MR 4155287

[12] P. Gérard, T. Kappeler, and P. Topalov, Sharp well-posedness results of the Benjamin–Ono equation in $H^s(\mathbb{T}, \mathbb{R})$ and qualitative properties of its solutions. 2020, arXiv:2004.04857. To appear in *Acta Math.*

[13] P. Gérard, T. Kappeler, and P. Topalov, On the analyticity of the Birkhoff map of the Benjamin–Ono equation on $\mathbb{T}$. 2021, arXiv:2109.08988

[14] P. Gérard, T. Kappeler, and P. Topalov, On the analytic Birkhoff normal form of the Benjamin–Ono equation and applications. *Nonlinear Anal.* **216** (2022), article no. 112687 MR 4348313

[15] H. Hochstadt, Functiontheoretic properties of the discriminant of Hill's equation. *Math. Z.* **82** (1963), 237–242 Zbl 0127.04203 MR 156022

[16] T. Kappeler, A. Maspero, J. Molnar, and P. Topalov, On the convexity of the KdV Hamiltonian. *Comm. Math. Phys.* **346** (2016), no. 1, 191–236 Zbl 1353.35251 MR 3528420

[17] H. P. McKean and E. Trubowitz, Hill's operator and hyperelliptic function theory in the presence of infinitely many branch points. *Comm. Pure Appl. Math.* **29** (1976), no. 2, 143–226 Zbl 0339.34024 MR 427731

[18] L. Molinet, Global well-posedness in L^2 for the periodic Benjamin–Ono equation. *Amer. J. Math.* **130** (2008), no. 3, 635–683 Zbl 1284.35377 MR 2418924

[19] L. Molinet, private communication

[20] L. Molinet and D. Pilod, The Cauchy problem for the Benjamin–Ono equation in L^2 revisited. *Anal. PDE* **5** (2012), no. 2, 365–395 Zbl 1273.35096 MR 2970711

[21] J. Molnar, *Features of the nonlinear Fourier transform for the dNLS equation.* Ph.D. thesis, University of Zurich, Zurich, 2016

[22] A. Nakamura, Bäcklund transform and conservation laws of the Benjamin–Ono equation. *J. Phys. Soc. Japan* **47** (1979), no. 4, 1335–1340 Zbl 1334.35178 MR 550203

[23] J. Satsuma and Y. Ishimori, Periodic wave and rational soliton solutions of the Benjamin–Ono equation. *J. Phys. Soc. Japan* **46** (1979), 681–687

[24] J.-C. Saut, Sur quelques généralisations de l'équation de Korteweg-de Vries. *J. Math. Pures Appl. (9)* **58** (1979), no. 1, 21–61 Zbl 0449.35083 MR 533234

[25] J.-C. Saut, Benjamin–Ono and intermediate long wave equations: modeling, IST and PDE. In *Nonlinear dispersive partial differential equations and inverse scattering*, pp. 95–160, Fields Inst. Commun. 83, Springer, New York, 2019 Zbl 1442.35355 MR 3931835

[26] B. Talbut, Low regularity conservation laws for the Benjamin–Ono equation. *Math. Res. Lett.* **28** (2021), no. 3, 889–905 Zbl 1471.34034 MR 4270277

[27] H. Weyl, Über die Gleichverteilung von Zahlen mod. Eins. *Math. Ann.* **77** (1916), no. 3, 313–352 Zbl 46.0278.06 MR 1511862

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Patrick Gérard
Laboratoire de Mathématiques d'Orsay, CNRS, Université Paris-Saclay, 91405 Orsay, France;
patrick.gerard@math.u-psud.fr

Thomas Kappeler
Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, 8057 Zurich,
Switzerland; thomas.kappeler@math.uzh.ch

Petar Topalov
Department of Mathematics, Northeastern University, Boston, MA 02115, USA;
p.topalov@northeastern.edu