

Berezin–Toeplitz quantization associated with higher Landau levels of the Bochner Laplacian

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Abstract. In this paper, we construct a family of Berezin–Toeplitz type quantizations of a compact symplectic manifold. For this, we choose a Riemannian metric on the manifold such that the associated Bochner Laplacian has the same local model at each point (this is slightly more general than in almost-Kähler quantization). Then the spectrum of the Bochner Laplacian on high tensor powers L^p of the prequantum line bundle L asymptotically splits into clusters of size $\mathcal{O}(p^{3/4})$ around the points $p\Lambda$, where Λ is an eigenvalue of the model operator (which can be naturally called a Landau level). We develop the Toeplitz operator calculus with the quantum space, which is the eigenspace of the Bochner Laplacian corresponding to the eigenvalues from the cluster. We show that it provides a Berezin–Toeplitz quantization. If the cluster corresponds to a Landau level of multiplicity one, we obtain an algebra of Toeplitz operators and a formal star-product. For the lowest Landau level, it recovers the almost Kähler quantization.

Dedicated to the memory of Misha Shubin

1. Introduction

The main goal of our paper is to construct a family of Berezin–Toeplitz quantizations based on appropriate eigenspaces of the Bochner Laplacian under a certain condition on the Riemannian metric on the symplectic manifold (which is slightly more general than the almost-Kähler one). More precisely, let (X, \mathbf{B}) be a closed symplectic manifold of dimension $2n$. Assume that there exists a Hermitian line bundle (L, h^L) on X with a Hermitian connection ∇^L such that

$$\mathbf{B} = iR^L, \tag{1}$$

where R^L is the curvature of the connection ∇^L defined as $R^L = (\nabla^L)^2$.

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Let g be a Riemannian metric on X and (E, h^E) be a Hermitian vector bundle of rank r on X with a Hermitian connection ∇^E . For any $p \in \mathbb{N}$, let $L^p := L^{\otimes p}$ be the p th tensor power of L and let

$$\nabla^{L^p \otimes E}: C^\infty(X, L^p \otimes E) \rightarrow C^\infty(X, T^*X \otimes L^p \otimes E)$$

be the Hermitian connection on $L^p \otimes E$ induced by ∇^L and ∇^E . Consider the induced Bochner Laplacian $\Delta^{L^p \otimes E}$ acting on $C^\infty(X, L^p \otimes E)$ by

$$\Delta^{L^p \otimes E} = (\nabla^{L^p \otimes E})^* \nabla^{L^p \otimes E}, \tag{2}$$

where $(\nabla^{L^p \otimes E})^*: C^\infty(X, T^*X \otimes L^p \otimes E) \rightarrow C^\infty(X, L^p \otimes E)$ is the formal adjoint of $\nabla^{L^p \otimes E}$.

For an arbitrary $x \in X$, one can introduce a second order differential operator acting on $C^\infty(T_x X, E_x)$ (the model operator), which is obtained from the Bochner Laplacian $\Delta^{L^p \otimes E}$ by freezing coefficients at x (see (19) below and [22] for more details). It is the Bochner Laplacian on a constant curvature Hermitian line bundle over the Euclidean space $T_x X$. It can be also considered as the magnetic Laplacian with constant magnetic field. We consider the skew-adjoint operator $B_x: T_x X \rightarrow T_x X$ such that

$$\mathbf{B}_x(u, v) = g(B_x u, v), \quad u, v \in T_x X.$$

Its eigenvalues have the form $\pm i a_j(x)$, $j = 1, \dots, n$, with $a_j(x) > 0$. The spectrum of the model operator consists of eigenvalues of the form $\sum_{j=1}^n (2k_j + 1) a_j(x)$ with $(k_1, \dots, k_n) \in \mathbb{Z}_+^n$. Each eigenvalue has infinite multiplicity and is called a *Landau level*.

We assume that the functions a_j can be chosen to be constants:

$$a_j(x) \equiv a_j, \quad x \in X, \quad j = 1, \dots, n. \tag{3}$$

This is a condition on the Riemannian metric g , which can be satisfied for any symplectic manifold X . In this case, the spectrum of the model operator is independent of x and coincides with the countable discrete set

$$\Sigma := \left\{ \Lambda_{\mathbf{k}} := \sum_{j=1}^n (2k_j + 1) a_j: \mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n \right\}. \tag{4}$$

If $J = \frac{1}{2\pi} B$ is an almost-complex structure (the almost Kähler case), then $a_j = 2\pi$, $j = 1, \dots, n$, and

$$\Sigma = \{2\pi(2k + n): k \in \mathbb{Z}_+\}. \tag{5}$$

As shown in [22] (see also [14]), for any $K > 0$, there exists $c > 0$ such that for any $p \in \mathbb{N}$ the spectrum of $\Delta^{L^p \otimes E}$ in the interval $[0, K]$ is contained in the $c p^{3/4}$ -neighborhood of $p\Sigma$. In other words, the spectrum of $\Delta^{L^p \otimes E}$ asymptotically splits into clusters around $p\Sigma$ of size $\mathcal{O}(p^{3/4})$.

Next, we fix one of these clusters associated with $\Lambda \in \Sigma$ and develop the Toeplitz operator calculus associated with the eigenspace of the Bochner Laplacian corresponding to eigenvalues from this cluster. Consider an interval $I = (\alpha, \beta)$ such that $(\alpha, \beta) \cap \Sigma = \{\Lambda\}$. By the above mentioned fact, there exist $\mu_0 > 0$ and $p_0 \in \mathbb{N}$ such that, for any $p > p_0$,

$$\sigma(\Delta^{L^p \otimes E}) \subset (-\infty, p(\Lambda - \mu_0)) \cup (p\alpha, p\beta) \cup (p(\Lambda + \mu_0), \infty).$$

The spectral projection of the operator $\Delta^{L^p \otimes E}$ associated with $(p\alpha, p\beta)$ is independent of the choice of I and will be denoted by $P_{p,\Lambda}$.

For $f \in C^\infty(X, \text{End}(E))$, we define the associated Toeplitz operator to be the sequence of bounded linear operators

$$T_{f,p} = P_{p,\Lambda} f P_{p,\Lambda} : L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E), \quad p \in \mathbb{N}.$$

Theorem 1.1. *Let $f, g \in C^\infty(X, \text{End}(E))$. Then, for the product of the Toeplitz operators $\{T_{f,p}\}$ and $\{T_{g,p}\}$, we have*

$$T_{f,p} T_{g,p} = T_{fg,p} + \mathcal{O}(p^{-1}). \tag{6}$$

Moreover, if $f, g \in C^\infty(X)$, then, for the commutator of the operators $\{T_{f,p}\}$ and $\{T_{g,p}\}$, we have

$$[T_{f,p}, T_{g,p}] = ip^{-1} T_{\{f,g\},p} + \mathcal{O}(p^{-1/2}), \tag{7}$$

where $\{f, g\}$ is the Poisson bracket on the symplectic manifold (X, \mathbf{B}) .

Thus, the Toeplitz operators provide a Berezin–Toeplitz quantization for the compact symplectic manifold (X, \mathbf{B}) . The limit $p \rightarrow +\infty$ for Toeplitz operators can be thought of as a semiclassical limit, with semiclassical parameter $\hbar = \frac{1}{p} \rightarrow 0$. Theorem 1.3 shows that this quantization has a correct semiclassical limit.

In the case when the set $\mathcal{K}_\Lambda := \{\mathbf{k} \in \mathbb{Z}_+^n : \Lambda_{\mathbf{k}} = \Lambda\}$ consists of a single element, we construct the algebra of Toeplitz operators associated with Λ .

Definition 1.2. A Toeplitz operator is a sequence $\{T_p\} = \{T_p\}_{p \in \mathbb{N}}$ of bounded linear operators $T_p : L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E)$, satisfying the following conditions.

(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_{p,\Lambda} T_p P_{p,\Lambda}.$$

(ii) There exists a sequence $g_l \in C^\infty(X, \text{End}(E))$ such that

$$T_p = P_{p,\Lambda} \left(\sum_{l=0}^{\infty} p^{-l} g_l \right) P_{p,\Lambda} + \mathcal{O}(p^{-\infty}),$$

i.e. for any natural k there exists $C_k > 0$ such that

$$\left\| T_p - P_{p,\Lambda} \left(\sum_{l=0}^k p^{-l} g_l \right) P_{p,\Lambda} \right\| \leq C_k p^{-k-1}.$$

Theorem 1.3. *Assume that \mathcal{K}_Λ consists of a single element. Then, for any $f, g \in C^\infty(X, \text{End}(E))$, the product of the Toeplitz operators $\{T_{f,p}\}$ and $\{T_{g,p}\}$ is a Toeplitz operator in the sense of Definition 1.2. More precisely, it admits the asymptotic expansion*

$$T_{f,p} T_{g,p} = \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} + \mathcal{O}(p^{-\infty}), \tag{8}$$

with some $C_r(f, g) \in C^\infty(X, \text{End}(E))$, where the C_r are bidifferential operators. In particular, $C_0(f, g) = fg$ and, for $f, g \in C^\infty(X)$, we have

$$C_1(f, g) - C_1(g, f) = i\{f, g\}. \tag{9}$$

The idea to use Toeplitz operators for quantization of Kähler manifolds was suggested by Berezin in [4]. We refer the reader to [3, 12, 24, 35] for some recent surveys on Berezin–Toeplitz and geometric quantization. For a general compact Kähler manifold, the Berezin–Toeplitz quantization was constructed by Bordemann, Meinrenken, and Schlichenmaier [5], using the theory of Toeplitz structures of Boutet de Monvel and Guillemin [6]. In this case, the quantum space is the space of holomorphic sections of tensor powers of the prequantum line bundle over the Kähler manifold. For an arbitrary symplectic manifold, Guillemin and Vergne suggested to use the kernel of the spin^c Dirac operator as a quantum space. The corresponding Berezin–Toeplitz quantization was developed by Ma and Marinescu [25, 26]. It is based on the asymptotic expansion of the Bergman kernel outside the diagonal obtained by Dai, Liu, and Ma [11]. Another candidate for the quantum space was suggested by Guillemin and Uribe [17]. It is the space of eigensections of the renormalized Bochner Laplacian corresponding to eigenvalues localized near the origin. In this case, the Berezin–Toeplitz quantization was recently constructed in [19, 21], based on Ma and Marinescu’s work: the Bergman kernel expansion from [27] and Toeplitz calculus developed in [26] for spin^c Dirac operator and Kähler case (also with an auxiliary bundle). We note also that Charles [8] proposed recently another approach to quantization of symplectic manifolds and Hsiao and Marinescu [18] constructed a Berezin–Toeplitz quantization for eigensections of small eigenvalues in the case of complex manifolds.

In our paper, we follow the approach to Toeplitz operator calculus developed in [19, 21, 25, 26]. Asymptotic expansions of the kernels of spectral projections, which we need in our case, are proved in [22]. When $\Lambda = \Lambda_0$ is the lowest Landau level, our results are reduced to the results obtained in [19, 21], which hold for any Riemannian

metric g (not necessarily satisfying the condition (3)). We mention that, in two simultaneous papers [9, 10], Charles studies the same subject, using the methods of [8].

There are several papers devoted to Toeplitz operators acting on spectral subspaces of the Landau Hamiltonian in \mathbb{R}^{2n} (see, for instance, [7, 15, 28–30, 32, 33] and references therein). For constant magnetic fields, such operators are related with the Toeplitz operators acting on Bargmann–Fock type spaces of polyanalytic functions (see, for instance, [1, 2, 13, 16, 20, 34, 36] and references therein). In particular, in [20], quantization schemes defined by polyanalytic Toeplitz operators are discussed.

The paper is organized as follows. In Section 2, we introduce an algebra \mathfrak{A} of integral operators on X defined in terms of conditions on their smooth Schwartz kernels. In Section 3, we show that Toeplitz operators in the sense of Definition 1.2 belong to \mathfrak{A} . In Section 4, using the results of the previous sections, we prove the first part of Theorem 1.1 and reduce the proof of its second part to a similar statement in the Euclidean case. The proof of this statement is given in Section 5. In Section 6, we prove that the set of Toeplitz operators coincides with the algebra \mathfrak{A} in the case when \mathcal{K}_Λ consists of a single element, which gives a characterization of Toeplitz operators in terms of their Schwartz kernels in the form introduced in [26, Theorem 4.9]. Using Theorems 6.1 and 1.1, we easily complete the proof of Theorem 1.3.

2. Algebra of integral operators

In this section, we introduce an algebra \mathfrak{A} of integral operators on X defined in terms of conditions on their smooth Schwartz kernels. Our motivation comes from the description of Toeplitz operators in terms of their Schwartz kernels introduced in [25, 26]. Later, we will show that Toeplitz operators in the sense of Definition 1.2 belong to this algebra, and, if \mathcal{K}_Λ consists of a single element, the set of Toeplitz operators coincides with \mathfrak{A} .

We introduce normal coordinates near an arbitrary point $x_0 \in X$. We denote by $B^X(x_0, r)$ and $B^{T_{x_0}X}(0, r)$ the open balls in X and $T_{x_0}X$ with center x_0 and radius r , respectively. Let $r_X > 0$ be the injectivity radius of X . We identify $B^{T_{x_0}X}(0, r_X)$ with $B^X(x_0, r_X)$ via the exponential map $\exp_{x_0}^X: T_{x_0}X \rightarrow X$. Furthermore, we choose trivializations of the bundles L and E over $B^X(x_0, r_X)$, identifying their fibers L_Z and E_Z at $Z \in B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X)$ with the spaces L_{x_0} and E_{x_0} by parallel transport with respect to the connections ∇^L and ∇^E along the curve $\gamma_Z: [0, 1] \ni u \rightarrow \exp_{x_0}^X(uZ)$. Denote by $\nabla^{L^p \otimes E}$ and $h^{L^p \otimes E}$ the connection and the Hermitian metric on the trivial bundle with fiber $(L^p \otimes E)_{x_0}$ induced by these trivializations.

We choose an orthonormal base $\{e_j: j = 1, \dots, 2n\}$ in $T_{x_0}X$ such that

$$B_{x_0}e_{2k-1} = a_k e_{2k}, \quad B_{x_0}e_{2k} = -a_k e_{2k-1}, \quad k = 1, \dots, n. \tag{10}$$

Thus, we have

$$\mathbf{B}_{x_0} = \sum_{k=1}^n a_k dZ_{2k-1} \wedge dZ_{2k}. \tag{11}$$

We introduce a coordinate chart $\gamma_{x_0}: B(0, c) \subset \mathbb{R}^{2n} \rightarrow X$ defined on the ball $B(0, c) := \{Z \in \mathbb{R}^{2n} : |Z| < c\}$ with some $c \in (0, r_X)$, which is given by the restriction of the exponential map $\exp_{x_0}^X$ composed with the linear isomorphism $\mathbb{R}^{2n} \rightarrow T_{x_0}X$ determined by the base $\{e_j\}$.

Let dv_{TX} denote the Riemannian volume form of the Euclidean space $(T_{x_0}X, g_{x_0})$. We define a smooth function κ on $B^{T_{x_0}X}(0, r_X) \cong B^X(x_0, r_X)$ by the equation

$$dv_X(Z) = \kappa(Z)dv_{TX}(Z), \quad Z \in B^{T_{x_0}X}(0, r_X).$$

Let $\{\Xi_p\}$ be a sequence of linear operators

$$\Xi_p: L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E)$$

with smooth kernel $\Xi_p(x, x')$ with respect to dv_X . Consider the fiberwise product $TX \times_X TX = \{(Z, Z') \in T_{x_0}X \times T_{x_0}X : x_0 \in X\}$. Let $\pi: TX \times_X TX \rightarrow X$ be the natural projection given by $\pi(Z, Z') = x_0$. The kernel $\Xi_p(x, x')$ induces a smooth section $\Xi_{p, x_0}(Z, Z')$ of the vector bundle $\pi^*(\text{End}(E))$ on $TX \times_X TX$ defined for all $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with $|Z|, |Z'| < r_X$.

Denote by \mathcal{P} the Bergman kernel in \mathbb{R}^{2n} given by

$$\mathcal{P}(Z, Z') = \frac{1}{(2\pi)^n} \prod_{j=1}^n a_j \exp\left(-\frac{1}{4} \sum_{k=1}^n a_k (|z_k|^2 + |z'_k|^2 - 2z_k \bar{z}'_k)\right). \tag{12}$$

We will use the same notation for the corresponding scalar function $\mathcal{P}(Z, Z') = \mathcal{P}(Z, Z') \text{Id}_{E_{x_0}}$ on $T_{x_0}X \times T_{x_0}X$ with values in $\text{End}(E_{x_0})$.

Definition 2.1 ([25, 26]). We say that

$$p^{-n} \Xi_{p, x_0}(Z, Z') \cong \sum_{r=0}^k (Q_{r, x_0} \mathcal{P})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}) \tag{13}$$

with some $Q_{r, x_0} \in \text{End}(E_{x_0})[Z, Z']$, $0 \leq r \leq k$, depending smoothly on $x_0 \in X$, if there exist $\varepsilon' \in (0, r_X]$ and $C_0 > 0$ with the following property: for any $l \in \mathbb{N}$, there exist $C > 0$ and $M > 0$ such that for any $x_0 \in X$, $p \geq 1$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$, we have

$$\begin{aligned} & \left| p^{-n} \Xi_{p, x_0}(Z, Z') \kappa^{\frac{1}{2}}(Z) \kappa^{\frac{1}{2}}(Z') - \sum_{r=0}^k (Q_{r, x_0} \mathcal{P})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} \right|_{\mathcal{E}^l(X)} \\ & \leq Cp^{-\frac{k+1}{2}} (1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \exp(-\sqrt{C_0 p}|Z - Z'|) + \mathcal{O}(p^{-\infty}). \end{aligned}$$

Here $C^{m'}(X)$ is the $C^{m'}$ -norm for the parameter $x_0 \in X$. We say that $G_p = \mathcal{O}(p^{-\infty})$ if for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that \mathcal{C}^{l_1} -norm of G_p is estimated from above by $C_{l,l_1} p^{-l}$.

The expansion (13) will be called the *full off-diagonal expansion* for the kernel of Ξ_p .

Definition 2.2. We introduce the class \mathfrak{A} , which consists of sequences of linear operators $\{T_p: L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E)\}$, satisfying the following conditions.

(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_{p,\Lambda} T_p P_{p,\Lambda}.$$

(ii) For any $\varepsilon_0 > 0$ and $l \in \mathbb{N}$, there exists $C > 0$ such that

$$|T_p(x, x')| \leq C p^{-l}$$

for any $p \in \mathbb{N}$ and $(x, x') \in X \times X$ with $d(x, x') > \varepsilon_0$. (Here $d(x, x')$ is the geodesic distance.)

(iii) The kernel of T_p admits the full off-diagonal expansion

$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^k K_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}})$$

for any $k \in \mathbb{N}$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$ with some $\varepsilon' \in (0, r_X/4)$, with

$$K_{r,x_0}(Z, Z') = (\mathcal{Q}_{r,x_0} \mathcal{P})(Z, Z'),$$

where $\mathcal{Q}_{r,x_0} \in \text{End}(E_{x_0})[Z, Z']$ is a family of polynomials, depending smoothly on x_0 , of the same parity as r .

One can easily check the following properties of \mathfrak{A} .

Proposition 2.3. *The set \mathfrak{A} is an involutive algebra. For any $\{T_p\} \in \mathfrak{A}$, the operator T_p is bounded in $L^2(X, L^p \otimes E)$ with the norm, uniformly bounded in p .*

For any $F, G \in C^\infty(T_{x_0}X \times T_{x_0}X, \text{End}(E_{x_0}))$, exponentially decreasing away the diagonal, we denote by $F * G \in C^\infty(T_{x_0}X \times T_{x_0}X, \text{End}(E_{x_0}))$ the smooth kernel of the composition of the corresponding integral operators in $L^2(T_{x_0}X, E_{x_0})$:

$$(F * G)(Z, Z') = \int_{T_{x_0}X} F(Z, Z'') G(Z'', Z') dZ''.$$

Proposition 2.4. *For any $\{T'_p\}, \{T''_p\} \in \mathfrak{A}$, the coefficients $K_{r,x_0}(Z, Z')$ in the full off-diagonal expansion for the kernel of the composition $\{T'_p \circ T''_p\}$ are related with*

the analogous coefficients $K'_{r,x_0}(Z, Z')$ and $K''_{r,x_0}(Z, Z')$ for $\{T'_p\}$ and $\{T''_p\}$, respectively, by

$$K_{r,x_0} = \sum_{r_1+r_2=r} K'_{r_1,x_0} * K''_{r_2,x_0}. \tag{14}$$

3. Description of the kernels of Toeplitz operators

In this section, we show that any Toeplitz operator $\{T_p\}$ in the sense of Definition 1.2 belongs to the algebra \mathfrak{A} introduced in the previous section. It is easy to see that it suffices to do this for the operator $\{T_{f,p}\}$ determined by $f \in C^\infty(X, \text{End}(E))$.

Let $P_{p,\Lambda}(x, x')$, $x, x' \in X$, be the smooth kernel of $P_{p,\Lambda}$ with respect to the Riemannian volume form dv_X . The Schwartz kernel of $T_{f,p}$ is given by

$$T_{f,p}(x, x') = \int_X P_{p,\Lambda}(x, y) f(y) P_{p,\Lambda}(y, x') dv_X(y). \tag{15}$$

Lemma 3.1. *For any $\varepsilon > 0$ and $l, m \in \mathbb{N}$, there exists $C > 0$ such that for any $p \geq 1$ and $(x, x') \in X \times X$ with $d(x, x') > \varepsilon$ we have*

$$|T_{f,p}(x, x')|_{C^m} \leq Cp^{-l}.$$

Here $|T_{f,p}(x, x')|_{C^k}$ denotes the pointwise C^k -seminorm of the section $T_{f,p}$ at a point $(x, x') \in X \times X$, which is the sum of the norms induced by h^L, h^E and g of the derivatives up to order k of $T_{f,p}$ with respect to the connection $\nabla^{L^p \otimes E}$ and the Levi-Civita connection ∇^{TX} evaluated at (x, x') .

Proof. The proof follows from (15) and the off-diagonal exponential estimate for $P_{p,\Lambda}(x, x')$ (see [22]) as in [27, Lemma 4.2]. ■

By [22], for any $k \in \mathbb{N}$, the kernel of $P_{p,\Lambda}$ admits the full off-diagonal expansion

$$p^{-n} P_{p,\Lambda,x_0}(Z, Z') \cong \sum_{r=0}^k F_{r,x_0}(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}). \tag{16}$$

Here, for any $r \geq 0$, the coefficient $F_{r,x_0} \in C^\infty(T_{x_0}X \times T_{x_0}X, \text{End}(E_{x_0}))$ has the form

$$F_{r,x_0}(Z, Z') = J_{r,x_0}(Z, Z') \mathcal{P}(Z, Z'), \tag{17}$$

where $J_{r,x_0}(Z, Z')$ is a polynomial in Z, Z' , depending smoothly on x_0 , of the same parity as r and $\deg J_{r,x_0} \leq \kappa(\Lambda) + 3r$, where $\kappa(\Lambda) = \max\{|\mathbf{k}|: \Lambda_{\mathbf{k}} = \Lambda\}$.

Recall the description of the leading coefficient $F_{0,x_0}(Z, Z')$. We introduce the connection on the trivial line bundle on $T_{x_0}X \cong \mathbb{R}^{2n}$, with the connection one-form α given by

$$\alpha = \sum_{k=1}^n \frac{1}{2} a_k (Z_{2k-1} dZ_{2k} - Z_{2k} dZ_{2k-1}). \tag{18}$$

Its curvature is constant: $d\alpha = \mathbf{B}_{x_0}$.

Let $\mathcal{H}^{(0)}$ be the associated Bochner Laplacian on $C^\infty(\mathbb{R}^{2n})$:

$$\mathcal{H}^{(0)} = (d - i\alpha)^*(d - i\alpha). \tag{19}$$

The spectrum of $\mathcal{H}^{(0)}$ coincides with Σ and consists of eigenvalues of infinite multiplicity. Considered as an operator on $C^\infty(T_{x_0}X, E_{x_0}) \cong C^\infty(\mathbb{R}^{2n}, E_{x_0})$, the operator $\mathcal{H}^{(0)} \otimes \text{id}_{E_{x_0}}$ is exactly the model operator at x_0 mentioned in the introduction. So, the assumption (3) guarantees that, in a suitable coordinates, the model operator is independent of x_0 . Let \mathcal{P}_Λ be the orthogonal projection on the eigenspace of $\mathcal{H}^{(0)}$ with the eigenvalue Λ (see Section 5 for more information on \mathcal{P}_Λ).

The leading coefficient in (16) is given by

$$F_{0,x_0}(Z, Z') = \mathcal{P}_\Lambda(Z, Z'). \tag{20}$$

As in [27, Lemma 4.7], using an explicit formula for F_{1,x_0} given in [22], one can show that, for any $Z, Z' \in T_{x_0}X$, $F_{1,x_0}(Z, Z')$ is a scalar operator in E_{x_0} , and, therefore, commutes with any operator in E_{x_0} .

Since $P_{p,\Lambda}$ is a projection, we have

$$\mathcal{P}_\Lambda * F_{1,x_0} + F_{1,x_0} * \mathcal{P}_\Lambda = F_{1,x_0}, \tag{21}$$

$$\mathcal{P}_\Lambda * F_{2,x_0} + F_{1,x_0} * F_{1,x_0} + F_{2,x_0} * \mathcal{P}_\Lambda = F_{2,x_0}. \tag{22}$$

Lemma 3.2. *For any $f \in C^\infty(X, \text{End}(E))$, the operator $\{T_{f,p}\}$ belongs to \mathfrak{A} . The coefficients $K_{r,x_0}(f) \in C^\infty(T_{x_0}X \times T_{x_0}X, \text{End}(E_{x_0}))$ of the full off-diagonal expansion for the kernel of $T_{f,p}$ are given by*

$$K_{r,x_0}(f) = \sum_{r_1+r_2+k=r} F_{r_1,x_0} * \frac{1}{k!} (d^k f_{x_0})_0 \cdot F_{r_2,x_0}, \tag{23}$$

where

$$(d^k f_{x_0})_0(Z) = \sum_{|\alpha|=k} \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0) Z^\alpha$$

and we denote

$$(d^k f_{x_0})_0 \cdot F_{r_2,x_0}(Z, Z') = (d^k f_{x_0})_0(Z) F_{r_2,x_0}(Z, Z').$$

In particular, we have

$$K_{0,x_0}(f) = f(x_0)\mathcal{P}_\Lambda, \tag{24}$$

$$K_{1,x_0}(f) = f(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda, \tag{25}$$

and, for $f \in C^\infty(X)$,

$$\begin{aligned} K_{2,x_0}(f) &= f(x_0)F_{2,x_0} + F_{1,x_0} * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda \\ &\quad + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot F_{1,x_0} + \mathcal{P}_\Lambda * \frac{1}{2}(d^2 f_{x_0})_0 \cdot \mathcal{P}_\Lambda. \end{aligned} \tag{26}$$

Proof. The proof goes along the same lines as the proof of [27, Lemma 4.6], so we just highlight the main points.

The fact that $\{T_{f,p}\}$ belongs to \mathfrak{A} follows easily from (15), (16), and Proposition 2.4. By (15) and (16), we also get

$$p^{-n}T_{f,p,x_0}(Z, Z') \cong \sum_{r=0}^{\infty} p^{-\frac{r}{2}} \sum_{r_1+r_2=r} \int F_{r_1,x_0}(\sqrt{p}Z, \sqrt{p}W) f_{x_0}(W) \times F_{r_2,x_0}(\sqrt{p}W, \sqrt{p}Z') dW.$$

Now we write the Taylor expansion for f_{x_0} at 0:

$$f_{x_0}(W) = \sum_{\alpha \in \mathbb{Z}_+^{2n}} \frac{\partial^\alpha f_{x_0}(0) W^\alpha}{\alpha!},$$

We infer that

$$\begin{aligned} p^{-n}T_{f,p,x_0}(Z, Z') &\cong \sum_{\alpha \in \mathbb{Z}_+^{2n}} \sum_{r=0}^{\infty} p^{-\frac{r}{2}} \sum_{r_1+r_2+|\alpha|=r} \int F_{r_1,x_0}(\sqrt{p}Z, \sqrt{p}Z') \\ &\quad \times \frac{\partial^\alpha f_{x_0}(0) (\sqrt{p}W)^\alpha}{\alpha!} F_{r_2,x_0}(\sqrt{p}W, \sqrt{p}Z') dW, \end{aligned}$$

which proves (23).

Using (23), (21), (22), and the fact that $F_{1,x_0}(Z, Z')$ commutes with any operator in E_{x_0} , one can easily derive (24), (25), and (26). ■

4. The composition theorem

Now we will use the results of the previous sections to prove Theorem 1.1. In this section, we will prove the first part of the theorem and reduce the proof of the second part to the proof of a similar statement in the Euclidean case, which will be given in the next section.

Let $f, g \in C^\infty(X, \text{End}(E))$. By Proposition 2.4, the operator $T_{f,p}T_{g,p}$ belongs to \mathfrak{A} , and, by (14), the coefficients $K_{r,x_0}(f, g) \in C^\infty(T_{x_0}X \times T_{x_0}X, \text{End}(E_{x_0}))$ of the full off-diagonal expansion for the kernel of $T_{f,p}T_{g,p}$ are given by

$$K_{r,x_0}(f, g) = \sum_{r_1+r_2=r} K_{r_1,x_0}(f) * K_{r_2,x_0}(g). \quad (27)$$

To prove (6), it is sufficient to show that

$$K_{r,x_0}(f, g) = K_{r,x_0}(fg), \quad r = 0, 1.$$

By (27), (24), and the fact that \mathcal{P}_Λ commutes with $g(x_0)$, we get

$$\begin{aligned} K_{0,x_0}(f, g) &= K_{0,x_0}(f) * K_{0,x_0}(g) \\ &= f(x_0)\mathcal{P}_\Lambda * g(x_0)\mathcal{P}_\Lambda = f(x_0)g(x_0)\mathcal{P}_\Lambda = K_{0,x_0}(fg). \end{aligned}$$

Next, by (27), (24), (25), and (21), we have

$$\begin{aligned} K_{1,x_0}(f, g) &= K_{1,x_0}(f) * K_{0,x_0}(g) + K_{0,x_0}(f) * K_{1,x_0}(g) \\ &= (f(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda) * g(x_0)\mathcal{P}_\Lambda \\ &\quad + f(x_0)\mathcal{P}_\Lambda * (g(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda) \\ &= f(x_0)g(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (df_{x_0})_0 g(x_0) \cdot \mathcal{P}_\Lambda \\ &\quad + \mathcal{P}_\Lambda * f(x_0)(dg_{x_0})_0 \cdot \mathcal{P}_\Lambda \\ &= f(x_0)g(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (d(fg)_{x_0})_0 \cdot \mathcal{P}_\Lambda \\ &= K_{1,x_0}(fg), \end{aligned}$$

since \mathcal{P}_Λ and F_{1,x_0} commute with $g(x_0)$ and $f(x_0)$ and $h \cdot \mathcal{P}_\Lambda * \mathcal{P}_\Lambda = h \cdot \mathcal{P}_\Lambda$ for any h . This proves (6).

Now suppose that $f, g \in C^\infty(X)$. Then we have

$$K_{r,x_0}(f, g) - K_{r,x_0}(g, f) = 0, \quad r = 0, 1. \quad (28)$$

Let us compute $K_{2,x_0}(f, g)$. By (27), (24), (25), and (26), we have

$$\begin{aligned} K_{2,x_0}(f, g) &= \left(f(x_0)F_{2,x_0} + F_{1,x_0} * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot F_{1,x_0} \right. \\ &\quad \left. + \mathcal{P}_\Lambda * \frac{1}{2}(d^2 f_{x_0})_0 \cdot \mathcal{P}_\Lambda \right) * g(x_0)\mathcal{P}_\Lambda \\ &\quad + \left(f(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda \right) \\ &\quad * \left(g(x_0)F_{1,x_0} + \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda \right) \\ &\quad + f(x_0)\mathcal{P}_\Lambda * \left(g(x_0)F_{2,x_0} + F_{1,x_0} * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda \right. \\ &\quad \left. + \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot F_{1,x_0} + \mathcal{P}_\Lambda * \frac{1}{2}(d^2 g_{x_0})_0 \cdot \mathcal{P}_\Lambda \right). \end{aligned}$$

Using (22), we collect the terms with $f(x_0)g(x_0)$. Since \mathcal{P}_Λ and F_{1,x_0} commute with $g(x_0)$ and $f(x_0)$, $h \cdot \mathcal{P}_\Lambda * \mathcal{P}_\Lambda = h \cdot \mathcal{P}_\Lambda$ for any h , we get

$$\begin{aligned}
K_{2,x_0}(f, g) &= f(x_0)g(x_0)F_{2,x_0} \\
&\quad + F_{1,x_0} * (df_{x_0})_0 g(x_0) \cdot \mathcal{P}_\Lambda + \mathcal{P}_\Lambda * (df_{x_0})_0 g(x_0) \cdot F_{1,x_0} * \mathcal{P}_\Lambda \\
&\quad + \mathcal{P}_\Lambda * \frac{1}{2}(d^2 f_{x_0})_0 g(x_0) \cdot \mathcal{P}_\Lambda + F_{1,x_0} * f(x_0)(dg_{x_0})_0 \cdot \mathcal{P}_\Lambda \\
&\quad + \mathcal{P}_\Lambda * (df_{x_0})_0 g(x_0) \cdot \mathcal{P}_\Lambda * F_{1,x_0} \\
&\quad + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda \\
&\quad + \mathcal{P}_\Lambda * f(x_0)(dg_{x_0})_0 \cdot F_{1,x_0} + \mathcal{P}_\Lambda * \frac{1}{2}f(x_0)(d^2 g_{x_0})_0 \cdot \mathcal{P}_\Lambda \\
&= f(x_0)g(x_0)F_{2,x_0} + F_{1,x_0} * (d(fg)_{x_0})_0 \cdot \mathcal{P}_\Lambda \\
&\quad + \mathcal{P}_\Lambda * (d(fg)_{x_0})_0 \cdot F_{1,x_0} \\
&\quad + \mathcal{P}_\Lambda * \frac{1}{2}(d^2 f_{x_0})_0 g(x_0) \cdot \mathcal{P}_\Lambda \\
&\quad + \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda \\
&\quad + \mathcal{P}_\Lambda * \frac{1}{2}f(x_0)(d^2 g_{x_0})_0 \cdot \mathcal{P}_\Lambda.
\end{aligned}$$

Finally, we see that

$$\begin{aligned}
&K_{2,x_0}(f, g) - K_{2,x_0}(g, f) \\
&= \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda - \mathcal{P}_\Lambda * (dg_{x_0})_0 \cdot \mathcal{P}_\Lambda * (df_{x_0})_0 \cdot \mathcal{P}_\Lambda.
\end{aligned} \tag{29}$$

Thus, we have reduced the proof of (7) to the linear model.

We consider the linear space \mathbb{R}^{2n} equipped with the symplectic form

$$\omega_a = \sum_{k=1}^n a_k dZ_{2k-1} \wedge dZ_{2k}$$

(cf. (11)).

Proposition 4.1. *For linear functions F and G on \mathbb{R}^{2n} , we have*

$$[P_\Lambda F \mathcal{P}_\Lambda, P_\Lambda G \mathcal{P}_\Lambda] = \{F, G\}_a \mathcal{P}_\Lambda,$$

where $\{F, G\}_a$ is the Poisson bracket on the symplectic manifold $(\mathbb{R}^{2n}, \omega_a)$

A proof of Proposition 4.1 will be given in the next section. Now we demonstrate how it allows us to complete the proof of (7).

Observe that, for linear functions F and G , the Poisson bracket $\{F, G\}_a$ is a constant function on \mathbb{R}^{2n} , and, by (11), it is easy to see that, for any $f, g \in C^\infty(X)$,

$$\{f, g\}(x_0) = \{(df_{x_0})_0, (dg_{x_0})_0\}_a.$$

Therefore, identity (7) follows immediately from (28), (29), and Propositions 4.1 and 2.3.

5. The model case

In this section, we prove Proposition 4.1, thus completing the proof of Theorem 1.1. First, we need to recall some information on the spectral theory of the model operator $\mathcal{H}^{(0)}$, see [27].

We will use the complex coordinates $z \in \mathbb{C}^n \cong \mathbb{R}^{2n}$, $z_j = Z_{2j-1} + iZ_{2j}$, with $j = 1, \dots, n$, in the linear space \mathbb{R}^{2n} . Put

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial Z_{2j-1}} - i \frac{\partial}{\partial Z_{2j}} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial Z_{2j-1}} + i \frac{\partial}{\partial Z_{2j}} \right).$$

Define first order differential operators $b_j, b_j^+, j = 1, \dots, n$, on $C^\infty(\mathbb{R}^{2n}, E_{x_0})$ by the formulas

$$b_j = -2 \frac{\partial}{\partial z_j} + \frac{1}{2} a_j \bar{z}_j, \quad b_j^+ = 2 \frac{\partial}{\partial \bar{z}_j} + \frac{1}{2} a_j z_j, \quad j = 1, \dots, n.$$

Then b_j^+ is the formal adjoint of b_j on $L^2(\mathbb{R}^{2n}, E_{x_0})$, and

$$\mathcal{H}^{(0)} = \sum_{j=1}^n b_j b_j^+ + \Lambda_0.$$

We have the commutation relations

$$[b_i, b_j^+] = b_i b_j^+ - b_j^+ b_i = -2a_i \delta_{ij}, \quad [b_i, b_j] = [b_i^+, b_j^+] = 0, \tag{30}$$

and, for any polynomial $g(z, \bar{z})$ on z and \bar{z} ,

$$[g(z, \bar{z}), b_j] = 2 \frac{\partial}{\partial z_j} g(z, \bar{z}), \quad [g(z, \bar{z}), b_j^+] = -2 \frac{\partial}{\partial \bar{z}_j} g(z, \bar{z}). \tag{31}$$

By [26, (1.98)], we have

$$(b_j^+ \mathcal{P})(Z, Z') = 0, \quad (b_j \mathcal{P})(Z, Z') = a_j (\bar{z}_j - \bar{z}'_j) \mathcal{P}(Z, Z'). \tag{32}$$

We introduce first order differential operators $\bar{b}_j, \bar{b}_j^+, j = 1, \dots, n$, on $C^\infty(\mathbb{R}^{2n}, E_{x_0})$ by the formulas

$$\bar{b}_j = -2\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}a_j z_j, \quad \bar{b}_j^+ = 2\frac{\partial}{\partial z_j} + \frac{1}{2}a_j \bar{z}_j, \quad j = 1, \dots, n.$$

They commute with the operators $b_j, b_j^+, j = 1, \dots, n$, and satisfy the same commutation relations (30) as b_j, b_j^+ . We have

$$(\bar{b}_j^+ \mathcal{P})(Z, Z') = a_j \bar{z}'_j \mathcal{P}(Z, Z'), \quad (\bar{b}_j \mathcal{P})(Z, Z') = a_j z_j \mathcal{P}(Z, Z'). \quad (33)$$

Recall that any function $\Phi \in L^2(\mathbb{R}^{2n}, E_{x_0})$ of the form

$$\Phi = b^{\mathbf{k}} \left(f(z) \exp\left(-\frac{1}{4} \sum_{j=1}^n a_j |z_j|^2\right) \right), \quad (34)$$

where f is an analytic function in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and $\mathbf{k} \in \mathbb{Z}_+^n$, is an eigenfunction of the operator $\mathcal{H}^{(0)}$ with the eigenvalue $\Lambda_{\mathbf{k}} = \sum_{j=1}^n (2k_j + 1)a_j$. In particular, the eigenspace of $\mathcal{H}^{(0)}$ associated with an eigenvalue Λ consists of functions Φ given by (34) with $\mathbf{k} \in \mathcal{K}_\Lambda$.

In the case when $E_{x_0} = \mathbb{C}$, an orthonormal basis of the eigenspace of $\mathcal{H}^{(0)}$ associated with the lowest eigenvalue Λ_0 is formed by the functions

$$\varphi_\beta(Z) = \left(\frac{a^\beta}{(2\pi)^n 2^{|\beta|} \beta!} \prod_{i=1}^n a_i \right)^{1/2} z^\beta \exp\left(-\frac{1}{4} \sum_{j=1}^n a_j |z_j|^2\right), \quad \beta \in \mathbb{Z}_+^n. \quad (35)$$

An orthonormal basis of the eigenspace associated with the eigenvalue Λ is given by (see, for instance, [7])

$$\varphi_{\mathbf{k}, \beta} = \frac{1}{(2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!)^{1/2}} b^{\mathbf{k}} \varphi_\beta, \quad \beta \in \mathbb{Z}_+^n, \mathbf{k} \in \mathcal{K}_\Lambda. \quad (36)$$

Thus, the spectral projection \mathcal{P}_Λ in $L^2(\mathbb{R}^{2n}, E_{x_0})$ is given by

$$\mathcal{P}_\Lambda = \sum_{\mathbf{k} \in \mathcal{K}_\Lambda} \mathcal{P}_{\Lambda_{\mathbf{k}}},$$

where $\mathcal{P}_{\Lambda_{\mathbf{k}}}$ is the smoothing operator with the kernel

$$\mathcal{P}_{\Lambda_{\mathbf{k}}}(Z, Z') = \sum_{\beta \in \mathbb{Z}_+^n} \varphi_{\mathbf{k}, \beta}(Z) \overline{\varphi_{\mathbf{k}, \beta}(Z')} = \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} b_z^{\mathbf{k}} \bar{b}_{z'}^{\mathbf{k}} \mathcal{P}(Z, Z') \quad (37)$$

(see also (55) below for an explicit formula). We can also write the following formula for the operator $\mathcal{P}_{\Lambda_{\mathbf{k}}}$ itself:

$$\mathcal{P}_{\Lambda_{\mathbf{k}}} = \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} b^{\mathbf{k}} \mathcal{P}(b^+)^{\mathbf{k}}. \quad (38)$$

Observe that

$$b_j^+ \mathcal{P}_{\Lambda_{\mathbf{k}}} = \mathcal{P}_{\Lambda_{\mathbf{k}-e_j}} b_j^+, \quad b_j \mathcal{P}_{\Lambda_{\mathbf{k}}} = \mathcal{P}_{\Lambda_{\mathbf{k}+e_j}} b_j, \quad j = 1, \dots, n,$$

where (e_1, \dots, e_n) is the standard basis in \mathbb{Z}^n . Indeed, using (38) and (30), we get

$$b_j^+ \mathcal{P}_{\Lambda_{\mathbf{k}}} = \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} [b_j^+, b^{\mathbf{k}}] \mathcal{P}(b^+)^{\mathbf{k}} = \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} 2a_j k_j b^{\mathbf{k}-e_j} \mathcal{P}(b^+)^{\mathbf{k}} = \mathcal{P}_{\Lambda_{\mathbf{k}-e_j}} b_j^+.$$

The second identity follows by taking adjoints.

Next, we show that, for a linear function

$$F(z, \bar{z}) = \sum_{j=1}^n F_{z_j} z_j + F_{\bar{z}_j} \bar{z}_j,$$

we have

$$[F, \mathcal{P}_{\Lambda_{\mathbf{k}}}] = \sum_{j=1}^n \frac{1}{a_j} [F_{z_j} b_j^+ (\mathcal{P}_{\Lambda_{\mathbf{k}}} - \mathcal{P}_{\Lambda_{\mathbf{k}+e_j}}) + F_{\bar{z}_j} b_j (\mathcal{P}_{\Lambda_{\mathbf{k}}} - \mathcal{P}_{\Lambda_{\mathbf{k}-e_j}})]. \quad (39)$$

Observe that

$$[z_j, \mathcal{P}_{\Lambda_{\mathbf{k}}}] = \frac{1}{a_j} (\mathcal{P}_{\Lambda_{\mathbf{k}-e_j}} - \mathcal{P}_{\Lambda_{\mathbf{k}}}) b_j^+ = \frac{1}{a_j} b_j^+ (\mathcal{P}_{\Lambda_{\mathbf{k}}} - \mathcal{P}_{\Lambda_{\mathbf{k}+e_j}}). \quad (40)$$

Indeed, using (38), (31), and (32), we get

$$\begin{aligned} [z_j, \mathcal{P}_{\Lambda_{\mathbf{k}}}] &= \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} ([z_j, b^{\mathbf{k}}] \mathcal{P}(b^+)^{\mathbf{k}} + b^{\mathbf{k}} [z_j, \mathcal{P}](b^+)^{\mathbf{k}}) \\ &= \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} \left(2k_j b^{\mathbf{k}-e_j} \mathcal{P}(b^+)^{\mathbf{k}} - \frac{1}{a_j} b^{\mathbf{k}} \mathcal{P}(b^+)^{\mathbf{k}+e_j} \right) \\ &= \frac{1}{a_j} (\mathcal{P}_{\Lambda_{\mathbf{k}-e_j}} - \mathcal{P}_{\Lambda_{\mathbf{k}}}) b_j^+. \end{aligned}$$

Taking adjoints, we infer that

$$[\bar{z}_j, \mathcal{P}_{\Lambda_{\mathbf{k}}}] = \frac{1}{a_j} b_j (\mathcal{P}_{\Lambda_{\mathbf{k}}} - \mathcal{P}_{\Lambda_{\mathbf{k}-e_j}}) = \frac{1}{a_j} (\mathcal{P}_{\Lambda_{\mathbf{k}+e_j}} - \mathcal{P}_{\Lambda_{\mathbf{k}}}) b_j. \quad (41)$$

From (40) and (41), we get (39).

Using (39), we compute

$$\begin{aligned} P_{\Lambda_{\mathbf{k}_1}} F \mathcal{P}_{\Lambda_{\mathbf{k}_2}} &= \delta_{\mathbf{k}_1, \mathbf{k}_2} F \mathcal{P}_{\Lambda_{\mathbf{k}_2}} + [P_{\Lambda_{\mathbf{k}_1}}, F] \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &= \delta_{\mathbf{k}_1, \mathbf{k}_2} F \mathcal{P}_{\Lambda_{\mathbf{k}_2}} + \sum_{\ell=1}^n \frac{1}{a_{\ell}} [(\delta_{\mathbf{k}_1-e_{\ell}, \mathbf{k}_2} - \delta_{\mathbf{k}_1, \mathbf{k}_2}) F_{\bar{z}_{\ell}} b_{\ell} \\ &\quad - (\delta_{\mathbf{k}_1, \mathbf{k}_2} - \delta_{\mathbf{k}_1+e_{\ell}, \mathbf{k}_2}) F_{z_{\ell}} b_{\ell}^+] \mathcal{P}_{\Lambda_{\mathbf{k}_2}}. \end{aligned} \quad (42)$$

Now we are ready to complete the proof of Proposition 4.1. For linear functions F and G , using (42), we get

$$\begin{aligned} & P_{\Lambda_{\mathbf{k}_1}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &= \delta_{\mathbf{k}_1, \mathbf{k}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &+ \sum_{\ell=1}^n \frac{1}{a_\ell} [(\delta_{\mathbf{k}_1 - e_\ell, \mathbf{k}} - \delta_{\mathbf{k}_1, \mathbf{k}}) F_{\bar{z}_\ell} b_\ell - (\delta_{\mathbf{k}_1, \mathbf{k}} - \delta_{\mathbf{k}_1 + e_\ell, \mathbf{k}}) F_{z_\ell} b_\ell^+] \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}_2}}. \end{aligned}$$

Next, we transpose $\mathcal{P}_{\Lambda_{\mathbf{k}}}$ and G and apply (39):

$$\begin{aligned} & P_{\Lambda_{\mathbf{k}_1}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &= \delta_{\mathbf{k}_1, \mathbf{k}} \delta_{\mathbf{k}, \mathbf{k}_2} F G \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &+ \sum_{\ell=1}^n \frac{1}{a_\ell} \delta_{\mathbf{k}, \mathbf{k}_2} [(\delta_{\mathbf{k}_1 - e_\ell, \mathbf{k}} - \delta_{\mathbf{k}_1, \mathbf{k}}) F_{\bar{z}_\ell} b_\ell - (\delta_{\mathbf{k}_1, \mathbf{k}} - \delta_{\mathbf{k}_1 + e_\ell, \mathbf{k}}) F_{z_\ell} b_\ell^+] G \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &+ \sum_{j=1}^n \frac{1}{a_j} \delta_{\mathbf{k}_1, \mathbf{k}} F [(\delta_{\mathbf{k} + e_j, \mathbf{k}_2} - \delta_{\mathbf{k}, \mathbf{k}_2}) G_{z_j} b_j^+ + (\delta_{\mathbf{k} - e_j, \mathbf{k}_2} - \delta_{\mathbf{k}, \mathbf{k}_2}) G_{\bar{z}_j} b_j] \mathcal{P}_{\Lambda_{\mathbf{k}_2}} \\ &+ \sum_{j, \ell=1}^n \frac{1}{a_j a_\ell} [(\delta_{\mathbf{k}_1 - e_\ell, \mathbf{k}} - \delta_{\mathbf{k}_1, \mathbf{k}}) F_{\bar{z}_\ell} b_\ell - (\delta_{\mathbf{k}_1, \mathbf{k}} - \delta_{\mathbf{k}_1 + e_\ell, \mathbf{k}}) F_{z_\ell} b_\ell^+] \\ &\quad \times [(\delta_{\mathbf{k} + e_j, \mathbf{k}_2} - \delta_{\mathbf{k}, \mathbf{k}_2}) G_{z_j} b_j^+ + (\delta_{\mathbf{k} - e_j, \mathbf{k}_2} - \delta_{\mathbf{k}, \mathbf{k}_2}) G_{\bar{z}_j} b_j] \mathcal{P}_{\Lambda_{\mathbf{k}_2}}. \end{aligned}$$

In the case when $\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2 \in \mathcal{K}_\Lambda$, we necessarily have $\mathbf{k}_1 \pm e_\ell \notin \mathcal{K}_\Lambda, \mathbf{k} \pm e_j \notin \mathcal{K}_\Lambda$. Thus, from the last formula, we conclude that $P_{\Lambda_{\mathbf{k}_1}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}_2}} = 0$, unless $\mathbf{k}_1 = \mathbf{k}_2 = \mathbf{k}$, and in the latter case, we have

$$\begin{aligned} & P_{\Lambda_{\mathbf{k}}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}}} \\ &= F G \mathcal{P}_{\Lambda_{\mathbf{k}}} + \sum_{\ell=1}^n \frac{1}{a_\ell} [-F_{\bar{z}_\ell} b_\ell - F_{z_\ell} b_\ell^+] G \mathcal{P}_{\Lambda_{\mathbf{k}}} \\ &+ \sum_{j=1}^n \frac{1}{a_j} F [-G_{z_j} b_j^+ - G_{\bar{z}_j} b_j] \mathcal{P}_{\Lambda_{\mathbf{k}}} \\ &+ \sum_{j, \ell=1}^n \frac{1}{a_j a_\ell} [-F_{\bar{z}_\ell} b_\ell - F_{z_\ell} b_\ell^+] [-G_{z_j} b_j^+ - G_{\bar{z}_j} b_j] \mathcal{P}_{\Lambda_{\mathbf{k}}}. \end{aligned}$$

We see that

$$\begin{aligned} & P_{\Lambda_{\mathbf{k}}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}}} - P_{\Lambda_{\mathbf{k}}} G \mathcal{P}_{\Lambda_{\mathbf{k}}} F \mathcal{P}_{\Lambda_{\mathbf{k}}} \\ &= \sum_{j=1}^n \frac{1}{a_j^2} (F_{z_j} G_{\bar{z}_j} - F_{\bar{z}_j} G_{z_j}) (b_j^+ b_j - b_j b_j^+) \mathcal{P}_{\Lambda_{\mathbf{k}}} \end{aligned}$$

$$= \sum_{j=1}^n \frac{2}{a_j} (F_{z_j} G_{\bar{z}_j} - F_{\bar{z}_j} G_{z_j}) \mathcal{P}_{\Lambda_k} = \{F, G\}_a \mathcal{P}_{\Lambda_k},$$

which completes the proof of Proposition 4.1.

6. Characterization of Toeplitz operators

In the case when \mathcal{K}_Λ consists of a single element, we prove that the set of Toeplitz operators coincides with the algebra \mathfrak{A} , which gives a characterization of Toeplitz operators in terms of their Schwartz kernels, This type of characterization was introduced in [26, Theorem 4.9]. Using this result and Theorem 1.1, we easily complete the proof of Theorem 1.3.

Theorem 6.1. *Assume that \mathcal{K}_Λ consists of a single element. A sequence of bounded linear operators $\{T_p: L^2(X, L^p \otimes E) \rightarrow L^2(X, L^p \otimes E)\}$ is a Toeplitz operator in the sense of Definition 1.2 if and only if it belongs to \mathfrak{A} .*

The fact that any Toeplitz operator in the sense of Definition 1.2 belongs to \mathfrak{A} is proved in Section 3 and holds without any assumption on Λ . Thus, we assume that $\{T_p\}$ belongs to \mathfrak{A} and prove that it is a Toeplitz operator in the sense of Definition 1.2. The proof is divided in several steps and in the beginning we don't assume that \mathcal{K}_Λ consists of a single element.

The following is an analog of [27, Lemma 4.12]. Recall that $K_{0,x_0}(Z, Z')$ denotes the leading coefficient in the full off-diagonal expansion for the kernel of T_p ,

Proposition 6.2. *The coefficient $K_{0,x_0}(Z, Z')$ has the form*

$$K_{0,x_0}(Z, Z') = \sum_{\mathbf{k}, \mathbf{k}' \in \mathcal{K}_\Lambda} b_z^{\mathbf{k}} \bar{b}_{z'}^{\mathbf{k}'} [Q_{\mathbf{k}\mathbf{k}', x_0} \mathcal{P}]$$

with some polynomials $Q_{\mathbf{k}\mathbf{k}', x_0}(z, \bar{z}')$ for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$.

Proof. By Definition 2.2 (i) and (14), we get

$$K_{0,x_0} = \mathcal{P}_\Lambda \circ K_{0,x_0} \circ \mathcal{P}_\Lambda. \tag{43}$$

By (43) and (38), it follows that

$$K_{0,x_0} = \mathcal{P}_\Lambda \circ K_{0,x_0} = \sum_{\mathbf{k} \in \mathcal{K}_\Lambda} \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} b^{\mathbf{k}} \circ \mathcal{P} \circ (b^+)^{\mathbf{k}} \circ K_{0,x_0}. \tag{44}$$

By [27, (2.12)], there exists $F_k \in \mathbb{C}[z, Z']$ such that

$$(\mathcal{P} \circ (b^+)^k \circ K_{0,x_0})(Z, Z') = F_k \cdot \mathcal{P}(Z, Z').$$

Plugging this in (44), we get

$$K_{0,x_0}(Z, Z') = \sum_{k \in \mathcal{K}_\Lambda} \frac{1}{2^{|\mathbf{k}|} a^{\mathbf{k}} \mathbf{k}!} b_z^{\mathbf{k}} F_k(z, Z') \mathcal{P}(Z, Z'). \quad (45)$$

Similarly, using (43), (38), and (45), we can write

$$\begin{aligned} K_{0,x_0} &= K_{0,x_0} \circ \mathcal{P}_\Lambda = \sum_{k' \in \mathcal{K}_\Lambda} \frac{1}{2^{|\mathbf{k}'|} a^{\mathbf{k}'} \mathbf{k}'!} K_{0,x_0} \circ b^{\mathbf{k}'} \circ \mathcal{P} \circ (b^+)^{\mathbf{k}'} \\ &= \sum_{k, k' \in \mathcal{K}_\Lambda} \frac{1}{2^{|\mathbf{k}|+|\mathbf{k}'|} a^{\mathbf{k}+\mathbf{k}'} \mathbf{k}! \mathbf{k}'!} b_z^{\mathbf{k}} F_k \mathcal{P} \circ b^{\mathbf{k}'} \circ \mathcal{P} \circ (b^+)^{\mathbf{k}'}. \end{aligned} \quad (46)$$

Now we proceed as follows:

$$\begin{aligned} (F_k \mathcal{P} \circ b^{\mathbf{k}'})(Z, Z') &= (\bar{b}_z^+)^{\mathbf{k}'} (F_k \mathcal{P})(Z, Z') \\ &= \sum_{\mathbf{l} \leq \mathbf{k}'} \binom{\mathbf{k}'}{\mathbf{l}} 2^{|\mathbf{l}|} \frac{\partial^{|\mathbf{l}|}}{\partial z^{\mathbf{l}}} F_k(z, Z') (\bar{b}_z^+)^{\mathbf{k}'-\mathbf{l}} \mathcal{P}(Z, Z') \\ &= \sum_{\mathbf{l} \leq \mathbf{k}'} \binom{\mathbf{k}'}{\mathbf{l}} 2^{|\mathbf{l}|} \frac{\partial^{|\mathbf{l}|}}{\partial z^{\mathbf{l}}} F_k(z, Z') (a \bar{z}')^{\mathbf{k}'-\mathbf{l}} \mathcal{P}(Z, Z') \\ &= F_{\mathbf{k}\mathbf{k}'} \mathcal{P}(Z, Z') \end{aligned} \quad (47)$$

with some $F_{\mathbf{k}\mathbf{k}'} \in \mathbb{C}[z, Z']$.

By [27, Proof of Lemma 2.2], there exists $Q_{\mathbf{k}\mathbf{k}'} \in \mathbb{C}[z, \bar{z}']$ such that

$$(F_{\mathbf{k}\mathbf{k}'} \mathcal{P} \circ \mathcal{P})(Z, Z') = 2^{|\mathbf{k}|+|\mathbf{k}'|} a^{\mathbf{k}+\mathbf{k}'} \mathbf{k}! \mathbf{k}'! Q_{\mathbf{k}\mathbf{k}'} \mathcal{P}(Z, Z'). \quad (48)$$

Combining (46), (47), and (48), we complete the proof. ■

The following is an analog of the well-known upper estimate for the Wick symbol.

Proposition 6.3. *We have*

$$|K_{0,x_0}(Z, Z)| \leq \limsup_{p \rightarrow \infty} \|T_p\|$$

for any $x_0 \in X$ and $Z \in T_{x_0} X$.

Proof. By Definition 1.2 (i), we get

$$\begin{aligned} T_p(x, x') &= (P_{p,\Lambda} T_p P_{p,\Lambda})(x, x') \\ &= \int P_{p,\Lambda}(x, y) T_p(y, y') P_{p,\Lambda}(y', x') dv_X(y) dv_X(y'). \end{aligned} \quad (49)$$

For any $x, y \in X$, $P_{p,\Lambda}(y, x)$ is a linear map from $(L^p \otimes E)_x$ to $(L^p \otimes E)_y$. For $x \in X$ and $v \in (L^p \otimes E)_x$, introduce $S_{x,v}^p \in C^\infty(X, L^p \otimes E)$ by

$$S_{x,v}^p(y) = P_{p,\Lambda}(y, x)v, \quad y \in X.$$

Observe that

$$\begin{aligned} \langle S_{x,v}^p, S_{x',v'}^p \rangle &= \int_X \langle P_{p,\Lambda}(y, x)v, P_{p,\Lambda}(y, x')v' \rangle dv_X(y) \\ &= \int_X \langle v, P_{p,\Lambda}(x, y)P_{p,\Lambda}(y, x')v' \rangle dv_X(y) \\ &= \langle v, P_{p,\Lambda}(x, x')v' \rangle. \end{aligned}$$

In particular, we have

$$\langle S_{x,v}^p, S_{x,v}^p \rangle = \langle v, P_{p,\Lambda}(x, x)v \rangle = p^n |v|^2 (\mathcal{P}_\Lambda(0, 0) + \mathcal{O}(p^{-\frac{1}{2}})),$$

and

$$\|S_{x,v}^p\| = p^{n/2} |v| ((\mathcal{P}_\Lambda(0, 0))^{1/2} + \mathcal{O}(p^{-\frac{1}{2}})).$$

By (49), we infer that for $x, x' \in X$, $v \in (L^p \otimes E)_x$ and $v' \in (L^p \otimes E)_{x'}$,

$$\begin{aligned} \langle v, T_p(x, x')v' \rangle &= \int_X \langle S_{x,v}^p(y), T_p(y, y')S_{x',v'}^p(y') \rangle dv_X(y) dv_X(y') \\ &= \langle S_{x,v}^p, T_p S_{x',v'}^p \rangle. \end{aligned}$$

It follows that

$$p^{-n} |T_p(x, x')| \leq \|T_p\| (\mathcal{P}_\Lambda(0, 0) + \mathcal{O}(p^{-\frac{1}{2}})). \quad (50)$$

Fix $x_0 \in X$ and write $x = \exp_{x_0}^X(Z)$ and $x' = \exp_{x_0}^X(Z')$. Then, by (iii), we get

$$p^{-n} T_p(x, x') = p^{-n} T_{p,x_0}(Z, Z') \cong K_{0,x_0}(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-\frac{1}{2}})$$

for $|Z|, |Z'| < \varepsilon$. It follows that, for $|Z|, |Z'| < \varepsilon\sqrt{p}$,

$$\begin{aligned} K_{0,x_0}(Z, Z') &= p^{-n} T_{p,x_0}\left(\frac{1}{\sqrt{p}}Z, \frac{1}{\sqrt{p}}Z'\right) + \mathcal{O}(p^{-\frac{1}{2}}) \\ &= p^{-n} T_p\left(\exp_{x_0}^X\left(\frac{1}{\sqrt{p}}Z\right), \exp_{x_0}^X\left(\frac{1}{\sqrt{p}}Z'\right)\right) + \mathcal{O}(p^{-\frac{1}{2}}) \\ &\leq \|T_p\| (1 + \mathcal{O}(p^{-\frac{1}{2}})), \end{aligned}$$

which completes the proof. ■

From now on, we will assume that \mathcal{K}_Λ consists of a single element. The following is an analog of [27, Proposition 4.11]. We give a different proof, which is shorter than in [27] and based on Proposition 6.3.

Proposition 6.4. *Assume that \mathcal{K}_Λ consists of a single element $\mathbf{k} \in \mathbb{Z}_+^n$. Then*

$$K_{0,x_0}(Z, Z') = Q_{x_0} \mathcal{P}_{\Lambda_{\mathbf{k}}}(Z, Z')$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with some $Q_{x_0} \in \text{End}(E_{x_0})$.

Proof. By Proposition 6.2, using Leibniz rule, we get

$$K_{0,x_0}(Z, Z') = \sum_{1 \leq \mathbf{k}} \sum_{1' \leq \mathbf{k}} \binom{\mathbf{k}}{\mathbf{1}} \binom{\mathbf{k}}{\mathbf{1}'} \frac{\partial^{2\mathbf{k}-\mathbf{1}-\mathbf{1}'}}{\partial Z^{\mathbf{k}-\mathbf{1}} \partial Z'^{\mathbf{k}-\mathbf{1}'}} Q_{\mathbf{k}\mathbf{k},x_0}(z, \bar{z}') b_z^{\mathbf{1}} \bar{b}_{z'}^{\mathbf{1}'}, \mathcal{P}(Z, Z'). \quad (51)$$

To compute $b_z^{\mathbf{1}} \bar{b}_{z'}^{\mathbf{1}'}, \mathcal{P}$, we write \mathcal{P} as the product

$$\mathcal{P}(Z, Z') = \prod_{j=1}^n \mathcal{P}_j(Z_j, Z'_j)$$

and treat each factor separately. Then we get, for $l_j \geq l'_j$,

$$b_{j,z_j}^{l_j} \bar{b}_{j,z'_j}^{l'_j} \mathcal{P}_j(Z_j, Z'_j) = 2^{l'_j} a_j^{l_j} l'_j! (\bar{z}_j - \bar{z}'_j)^{l_j-l'_j} L_{l'_j}^{(l_j-l'_j)} \left(\frac{a_j |z_j - z'_j|^2}{2} \right) \mathcal{P}_j(Z_j, Z'_j),$$

where $L_k^{(m)}$, $k, m \in \mathbb{Z}_+$, is the generalized Laguerre polynomial:

$$L_k^{(m)}(x) = \frac{x^{-m} e^x}{k!} \frac{d^k}{dx^k} (e^{-x} x^{m+k}) = \sum_{j=0}^k \binom{k+m}{k-j} \frac{(-x)^j}{j!}, \quad x \geq 0,$$

The formula for $l_j \leq l'_j$ is obtained by considering the adjoints.

In particular, for $l_j = l'_j$, we have

$$b_{j,z_j}^{l_j} \bar{b}_{j,z'_j}^{l'_j} \mathcal{P}_j(Z_j, Z'_j) = 2^{l_j} a_j^{l_j} l_j! L_{l_j} \left(\frac{a_j |z_j - z'_j|^2}{2} \right) \mathcal{P}_j(Z_j, Z'_j), \quad (52)$$

where $L_k = L_k^{(0)}$, $k \in \mathbb{Z}_+$, is the Laguerre polynomial, and

$$b_{j,z_j}^{l_j} \bar{b}_{j,z'_j}^{l'_j} \mathcal{P}_j(Z_j, Z_j) = 2^{l_j} a_j^{l_j} l_j! \mathcal{P}_j(0, 0). \quad (53)$$

For $l_j \neq l'_j$, we get

$$b_{j,z_j}^{l_j} \bar{b}_{j,z'_j}^{l'_j} \mathcal{P}_j(Z_j, Z_j) = 0. \quad (54)$$

By (52) and (37), we derive an explicit formula for $\mathcal{P}_{\Lambda_{\mathbf{k}}}(Z, Z')$:

$$\begin{aligned} \mathcal{P}_{\Lambda_{\mathbf{k}}}(Z, Z') &= \frac{1}{(2\pi)^n} \prod_{j=1}^n a_j L_{k_j} \left(\frac{a_j |z_j - z'_j|^2}{2} \right) \\ &\quad \times \exp \left(-\frac{1}{4} \sum_{k=1}^n a_k (|z_k|^2 + |z'_k|^2 - 2z_k \bar{z}'_k) \right). \end{aligned} \tag{55}$$

Taking into account (53) and (54), the equality (51) for $Z = Z'$ takes the form

$$K_{0,x_0}(Z, Z) = \sum_{\mathbf{l} \leq \mathbf{k}} 2^{|\mathbf{l}|} a! \mathbf{l}! \binom{\mathbf{k}}{\mathbf{l}}^2 \frac{\partial^{2\mathbf{k}-2\mathbf{l}}}{\partial z^{\mathbf{k}-\mathbf{l}} \partial z'^{\mathbf{k}-\mathbf{l}}} Q_{\mathbf{k}\mathbf{k},x_0}(z, \bar{z}) \mathcal{P}(0, 0).$$

By Proposition 6.3, we get

$$\sum_{\mathbf{l} \leq \mathbf{k}} 2^{|\mathbf{l}|} a! \mathbf{l}! \binom{\mathbf{k}}{\mathbf{l}}^2 \frac{\partial^{2\mathbf{k}-2\mathbf{l}}}{\partial z^{\mathbf{k}-\mathbf{l}} \partial z'^{\mathbf{k}-\mathbf{l}}} Q_{\mathbf{k}\mathbf{k},x_0}(z, \bar{z}) = \text{const.}$$

Comparing the top degree coefficients in both sides of the last identity, one can easily see that $Q_{\mathbf{k}\mathbf{k},x_0}(z, \bar{z}) = Q_{x_0} = \text{const.}$ Since $Q_{\mathbf{k}\mathbf{k},x_0}$ is a polynomial of z and \bar{z}' , this implies $Q_{\mathbf{k}\mathbf{k},x_0}(z, \bar{z}') = Q_{x_0}$. ■

From now on, we will closely follow the arguments of the proof of [27, Theorem 4.9]. So we will be brief.

Define a section $g_0 \in C^\infty(X, \text{End}(E))$, setting

$$g_0(x_0) = Q_{x_0},$$

where $Q_{x_0} \in \text{End}(E_{x_0})$ is given by Proposition 6.4.

The following is an analog of [27, Proposition 4.17]. Its proof is based on Proposition 6.4 and therefore a little bit shorter than the proof of [27, Proposition 4.17].

Proposition 6.5. *We have $p^{-n}(T_p - T_{g_0,p})(Z, Z') \cong \mathcal{O}(p^{-1})$.*

Proof. Consider the sequence of operators

$$R_p = p^{1/2}(T_p - T_{g_0,p}), \quad p \in \mathbb{N}.$$

It is easy to see that it is in \mathfrak{A} . Moreover, computing the full off-diagonal expansions for the kernels of T_p and $T_{g_0,p}$, we get

$$p^{-n} R_{p,x_0}(Z, Z') \cong (K_{1,x_0} - K_{1,x_0}(g_0))(\sqrt{p}Z, \sqrt{p}Z') + \mathcal{O}(p^{-1/2}).$$

We apply Proposition 6.4 to $\{R_p\}$ and infer that

$$(K_{1,x_0} - K_{1,x_0}(g_0))(Z, Z') = S_{x_0} \mathcal{P}_{\Lambda_{\mathbf{k}}}(Z, Z')$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with some $S_{x_0} \in \text{End}(E_{x_0})$.

Since $K_{1,x_0} - K_{1,x_0}(g_0) = (\mathcal{Q}_{1,x_0} - \mathcal{Q}_{1,x_0}(g_0))\mathcal{P}$, where \mathcal{Q}_{1,x_0} and $\mathcal{Q}_{1,x_0}(g_0)$ are odd, we conclude that

$$(K_{1,x_0} - K_{1,x_0}(g_0))(Z, Z') = 0$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$, which completes the proof. ■

By Proposition 6.5, we have $T_p = P_{p,\Lambda}g_0P_{p,\Lambda} + \mathcal{O}(p^{-1})$. Consider the operator $p(T_p - P_{p,\Lambda}g_0P_{p,\Lambda})$. It is easy to see that it belongs to \mathfrak{A} . By Proposition 6.4, the leading coefficient $K'_{0,x_0}(Z, Z')$ in the full off-diagonal expansion for the kernel of $p(T_p - P_{p,\Lambda}g_0P_{p,\Lambda})$ has the form

$$K'_{0,x_0}(Z, Z') = Q'_{x_0}\mathcal{P}_{\Lambda_k}(Z, Z')$$

for any $x_0 \in X$ and $Z, Z' \in T_{x_0}X$ with some $Q'_{x_0} \in \text{End}(E_{x_0})$. Setting

$$g_1(x_0) = Q'_{x_0},$$

we get a section $g_1 \in C^\infty(X, \text{End}(E))$, and, as in Proposition 6.5, we will show that

$$p(T_p - P_{p,\Lambda}g_0P_{p,\Lambda}) = P_{p,\Lambda}g_1P_{p,\Lambda} + \mathcal{O}(p^{-1}),$$

which implies

$$T_p = P_{p,\Lambda}(g_0 + p^{-1}g_1)P_{p,\Lambda} + \mathcal{O}(p^{-2}).$$

So we can proceed by induction to complete the proof of Theorem 6.1.

Using Theorems 6.1 and 1.1, one can easily complete the proof of Theorem 1.3.

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