On the spectral properties
of the Hilbert transform operator on multi-intervals

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Abstract. Let $J, E \subset \mathbb{R}$ be two multi-intervals with non-intersecting interiors. Consider the operator

$$A: L^2(J) \to L^2(E), \quad (Af)(x) = \frac{1}{\pi} \int_J \frac{f(y)\,\mathrm{d}y}{y - x},$$

and let $A^\dagger$ be its adjoint. We introduce a self-adjoint operator $\mathcal{K}$ acting on $L^2(E) \oplus L^2(J)$, whose off-diagonal blocks consist of A and $A^\dagger$. In this paper we study the spectral properties of $\mathcal{K}$ and the operators $A^\dagger A$ and $AA^\dagger$. Our main tool is to obtain the resolvent of $\mathcal{K}$, which is denoted by $\mathcal{R}$, using an appropriate Riemann–Hilbert problem, and then compute the jump and poles of $\mathcal{R}$ in the spectral parameter λ. We show that the spectrum of $\mathcal{K}$ has an absolutely continuous component $[0, 1]$ if and only if J and E have common endpoints, and its multiplicity equals to their number. If there are no common endpoints, the spectrum of $\mathcal{K}$ consists only of eigenvalues and 0. If there are common endpoints, then $\mathcal{K}$ may have eigenvalues imbedded in the continuous spectrum, each of them has a finite multiplicity, and the eigenvalues may accumulate only at 0. In all cases, $\mathcal{K}$ does not have a singular continuous spectrum. The spectral properties of $A^\dagger A$ and $AA^\dagger$, which are very similar to those of $\mathcal{K}$, are obtained as well.

1. Introduction

Let J, E be two Lebesgue measurable subsets of $\mathbb{R}$. Consider the following Finite Hilbert Transform (FHT) operator

$$A: L^2(J) \to L^2(E), \quad (Af)(x) = \frac{1}{\pi} \int_J \frac{f(y)\,\mathrm{d}y}{y - x}, \qquad (1.1)$$

whose adjoint is

$$(A^\dagger g)(w) = \frac{1}{\pi} \int_E \frac{g(x)\,\mathrm{d}x}{w - x} : L^2(E) \to L^2(J). \qquad (1.2)$$

An important and classical problem is to determine the nature of the spectrum of A, e.g., find its discrete and/or continuous parts and their multiplicities. When $J = E$, A acts on the Hilbert space $L^2(J)$, and one can talk about the spectrum of A. In this setting the spectrum of A for different sets J was thoroughly studied starting in the 50's and 60's, see, e.g., [22–24, 26, 28, 30, 31]. For example, in the case where $J = E = \mathbb{R}$, the operator A is the usual Hilbert transform. The latter is well known to be anti-self-adjoint, and its spectrum consists of two eigenvalues $\pm i$. In particular, the spectrum of $A^\dagger A$ and $A A^\dagger$ is $+1$ (because the two operators are equal to the identity operator). This is easily seen by conjugating A with the Fourier transform, which maps A to the multiplication operator by $i \, \mathrm{sgn}(\xi)$, where ξ is the Fourier variable and sgn is the signum function, see, for example, [21]. Here and throughout the paper, the Fourier transform and its inverse are defined as follows:

$$\tilde{\phi}(\xi) := (\mathcal{F}\phi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \phi(t) e^{i\xi t} \, \mathrm{d}t, \tag{1.3a}$$

$$\phi(t) = (\mathcal{F}^{-1}\tilde{\phi})(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{\phi}(\xi) e^{-i\xi t} \, d\xi. \tag{1.3b}$$

The operator A is thus rather simple from the spectral point of view. In another known case [25], where $J = E$ is a finite interval, the operator A in L^2 is not even a Fredholm operator (the range is dense, but not closed). In this case, the spectrum of A is absolutely continuous, of multiplicity 1, and coincides with the interval $[-i, i]$ of the imaginary axis [24].

More recently, the problem was investigated in a number of new settings when $J \neq E$. Here, J and E can be intervals or multi-intervals. A multi-interval is defined as a union of finitely many non-intersecting, closed, possibly unbounded, intervals with non-empty interior each. The analysis in [2, 3, 19, 20] is based on the existence of a differential operator that commutes with the finite Hilbert transform, which was found in [15, 16].

Starting with [5], the authors initiated the program of investigating the cases where J and E are multi-intervals subject to the restriction that the interiors of J and E are disjoint. In [5] we consider an arrangement, in which E consists of two compact intervals, J consists of any finite number of intervals that are all located between the two E intervals, and $\mathrm{dist}(E, J) > 0$. Since the use of commuting differential operator no longer applies when either E or J consists of more than one subinterval, the main tool in this paper is based on a matrix Riemann–Hilbert problem (RHP) approach to integral operators with integrable kernels in the sense of [12]. The main findings of [5] include that the singular values of A (ordered in decreasing order) tend to zero exponentially fast and an explicit expression for the leading term of the asymptotics. Let $\mathcal{K}$ be the self-adjoint operator acting on $L^2(E) \oplus L^2(J)$, whose

off-diagonal blocks consist of A and $A^\dagger$ (see (2.2) below). In [5] we also showed that all the eigenvalues of $\mathcal{K}$ are simple and calculated the leading order behavior of its eigenvectors (in terms of Riemann Theta functions) as the spectral parameter $\lambda \to 0$. This operator is very convenient to work with as $\mathcal{K}^2$ is a block diagonal operator with the blocks $AA^\dagger$ and $A^\dagger A$. Speaking more generally, if J and E are arbitrary multi-intervals and $\mathrm{dist}(E, J) > 0$, it is easy to see that the operator $\mathcal{K}$ is compact, and the spectra of $A^\dagger A$ and $AA^\dagger$ are purely discrete. In fact, in the present paper we establish that $\mathcal{K}$ is of trace-class (that it is of Hilbert–Schmidt class is a simple exercise).

Before going forward, let us fix some terminology. The term *endpoint* denotes an endpoint of any of the intervals that make up E or J. An endpoint is called *simple* if it belongs only to one interval. An endpoint z is called *double* if it belongs to two adjacent intervals of different types, that is, if $z \in E \cap J$.

The approach of [5] works well when $\mathrm{dist}(E, J) > 0$, that is, when the integral operator $\mathcal{K}$ is not singular. However, the case when $\mathrm{dist}(E, J) = 0$ leads to some technical difficulties, like, for example, construction of parametrices for the asymptotic solution of the RHP. These type of problems were overcome in [4], where we used the RHP approach in the case where $J = [a, 0]$ and $E = [0, b]$ for $a < 0 < b$, i.e., when 0 is the only double endpoint. The results of [4] match with and in some instances generalize those of [20]. An arrangement where J and E have multiple common endpoints is considered in [17]. We assume there that J and E are multi-intervals, and their union is the whole line: $J \cup E = \mathbb{R}$. In this case, the corresponding RHP can be solved explicitly. Just as in [4], the spectrum is the segment $[0, 1]$, the spectrum is purely absolutely continuous, and its multiplicity equals to the number of double endpoints. Additionally, in [17] we find an explicit diagonalization of the two operators.

In this paper we build on the results of [5, 17] and extend the RHP approach further by allowing J and E to be general multi-intervals that can touch at any number of points, that is, J and E can have multiple double endpoints. Our goal is to perform a qualitative analysis of the spectrum of $\mathcal{K}$ as well as of $A^\dagger A$ and $AA^\dagger$, which includes determining what spectral components it has and their multiplicities. It is quite interesting that without performing an explicit asymptotic analysis of the RHP when $\lambda \to 0$ that is similar to the one in [5], and without access to an explicit solution of the RHP as in [17], much information can still be obtained by investigating $\mathcal{K}$ and the related RHP. Our main results are formulated in Section 2 below. In addition to the RHP analysis, our main tools include the Kato–Rosenblum theorem on the stability of the absolutely continuous spectrum of a self-adjoint operator with respect to trace class perturbations.

In Section 6 we consider an example when $U = \mathbb{R}$. In this case, an explicit diagonalization of $\mathcal{K}$ is found similarly to [17], and the properties of its spectrum turn out to be in agreement with our general qualitative analysis.

2. Main results

As is stated in the introduction, the goal of the paper is to obtain the properties of the spectrum of $A^\dagger A$ and $AA^\dagger$ when E and J are closed multi-intervals with disjoint interior $\mathring{J} \cap \mathring{E} = \varnothing$. Here and in what follows, $\mathring{U}$ denotes the interior of the set U. The operator A commutes with Möbius transformations mapping $\mathbb{R}$ onto $\mathbb{R}$ (Lemma 3.1) and, hence, it matters whether the sets E, J have common points on the extended line. More precisely, if both E and J extend to infinity, we should consider ∞ as a common endpoint. The stated goal is essentially equivalent to studying the spectral properties of the self-adjoint operator $\mathcal{K} = A \oplus A^\dagger \colon L^2(U) \to L^2(U)$ with $U = E \cup J$. The latter is an operator with the kernel

$$K(x, y) = \frac{\chi_J(x)\chi_E(y) - \chi_J(y)\chi_E(x)}{\pi(x - y)}, \tag{2.1}$$

where, for a subset $S \subset \mathbb{R}$, we denote by χ_S its indicator function. Here the convention is that

$$\mathcal{K}[f](x) = \int_U K(x, y) f(y)\, d y.$$

In matrix form, we can represent $\mathcal{K}$ as follows:

$$\mathcal{K} = \begin{bmatrix} 0 & A \\ A^\dagger & 0 \end{bmatrix} \colon L^2(E) \oplus L^2(J) \to L^2(E) \oplus L^2(J). \tag{2.2}$$

The operator $\mathcal{K}$ is a convenient object to study because it is clearly self–adjoint and

$$\mathcal{K}^2 = AA^\dagger \oplus A^\dagger A = \begin{bmatrix} AA^\dagger & 0 \\ 0 & A^\dagger A \end{bmatrix} \colon L^2(E) \oplus L^2(J) \to L^2(E) \oplus L^2(J). \tag{2.3}$$

Thus, knowing $\mathfrak{Sp}(\mathcal{K})$, the spectrum of $\mathcal{K}$, it is easy to find the spectrum of $AA^\dagger$ and $A^\dagger A$. Hence, analysing $\mathfrak{Sp}(\mathcal{K})$ is also an important goal of this paper.

It is well known that

$$\mathfrak{Sp}(\mathcal{K}) = \mathfrak{Sp}_{\mathrm{ac}}(\mathcal{K}) \cup \mathfrak{Sp}_{\mathrm{sc}}(\mathcal{K}) \cup \mathfrak{Sp}_{\mathrm{p}}(\mathcal{K}), \tag{2.4}$$

where $\mathfrak{Sp}_{\mathrm{ac}}, \mathfrak{Sp}_{\mathrm{sc}}, \mathfrak{Sp}_{\mathrm{p}}$ denote the absolutely continuous, singular continuous, and point spectra of $\mathcal{K}$, respectively. The main result of this paper, Theorem 2.1, describes the connection between the geometry of the multi-intervals E, J and the spectral components of $\mathcal{K}$.

The main tool to studying the spectrum of $\mathcal{K}$ is to construct the (nonsingular) resolvent operator $\mathcal{R}(\lambda) = \frac{1}{\lambda}\mathcal{K}(\mathrm{Id} - \frac{1}{\lambda}\mathcal{K})^{-1} = (\mathrm{Id} - \frac{1}{\lambda}\mathcal{K})^{-1} - \mathrm{Id}$.

It is a matter of inspection to ascertain that $\mathcal{K}$ is a Hilbert–Schmidt operator if E and J have no common endpoints (which implies that either E or J is bounded).

In such case, therefore, the spectrum is purely discrete, and each eigenvalue has finite multiplicity.

Naturally, two adjacent intervals of the *same* type are considered belonging to one interval. Our main theorem below provides a detailed description of $\mathfrak{Sp}(\mathcal{K})$.

Theorem 2.1. *Let* $\mathcal{K} = A \oplus A^\dagger \colon L^2(U) \to L^2(U)$ *be the operator with the kernel* (2.1)*. Here* $U = E \cup J$*, and* $J, E \subset \mathbb{R}$ *are multi-intervals with disjoint interiors.*

1. $\mathfrak{Sp}(\mathcal{K}) \subseteq [-1, 1]$.

2. *The points* $\lambda = \pm 1$ *and* $\lambda = 0$ *are not eigenvalues of* $\mathcal{K}$*. The eigenvalues of* $\mathcal{K}$*, if they exist, are symmetric with respect to* $\lambda = 0$ *and have finite multiplicities. Moreover, they can accumulate only at* $\lambda = 0$.

3. *If there are* $n \geq 1$ *($n \in \mathbb{N}$) double endpoints, then* $\mathfrak{Sp}_{ac}(\mathcal{K}) = [-1, 1]$*, and the multiplicity of* $\mathfrak{Sp}_{ac}(\mathcal{K})$ *equals n.*

4. *If there are no double endpoints, then* $\mathcal{K}$ *is of trace class. In this case,* $\mathfrak{Sp}(\mathcal{K})$ *consists only of eigenvalues and* $\lambda = 0$*, which is the accumulation point of the eigenvalues.*

5. *The singular continuous component is empty, i.e.,* $\mathfrak{Sp}_{sc}(\mathcal{K}) = \varnothing$.

Remark 2.2. According to (1), all spectral components in (2.4) are subsets of $[-1, 1]$, i.e., the eigenvalues of $\mathcal{K}$ are embedded in the absolutely continuous spectrum $\mathfrak{Sp}_{ac}(\mathcal{K})$ provided that both components are not empty.

Remark 2.3. When there is at least one double endpoint, the presence of eigenvalues is not guaranteed. For example, it is shown in [4, Proposition 4] that $A^\dagger A$ and $A A^\dagger$ do not have eigenvalues when $J = [b_L, 0]$ and $E = [0, b_R]$ (i.e., 0 is a double endpoint). Here $b_L < 0 < b_R$. This implies that $\mathcal{K}$ does not have eigenvalues, because otherwise $\mathcal{K}^2$ (and $A^\dagger A$, $A A^\dagger$) would have had eigenvalues as well (see (2.3)). The analysis of embedded point spectrum is generally quite complicated (see, e.g., [1]). At this time, the authors do not know whether the presence of a gap implies the existence of eigenvalues when at least one double endpoint is present. In this case, if eigenvalues exist, they are necessarily embedded.

The proofs of (1), (3), (4) and part of (2) are given in Section 3. They are based on the known facts about the spectrum of multi-interval Hilbert transforms, see [4,5,17], and the spectral trace class perturbation theorem by Kato and Rosenblum [14]. An important part of our argument is Theorem 3.5, which states that the operator A is of trace class provided that there are no double endpoints. The proof of (5) requires a deep study of the solution $\Gamma(z; \lambda)$ of a Riemann–Hilbert problem (RHP), which is associated with $\mathcal{K}$ on a certain (infinite-sheeted) Riemann surface $\lambda \in \mathcal{R}$, see Section 4. We also add more details about $\Gamma(z; \lambda)$ there. The remaining part of (2) (finite

multiplicities of the eigenvalues), completing the proof of Theorem 2.1, is given in Section 5.

In addition, in Section 6 we obtain an explicit diagonalization of the operator $\mathcal{K}$ in the case when $U = \mathbb{R}$ (see (6.20)). This is done by following an approach analogous to the one in [17], which is based on finding $\Gamma(z; \lambda)$ explicitly (see [17] for more details). Results in this section are consistent with the assertions of Theorem 2.1. They also show that in this case $\mathcal{K}$ does not have eigenvalues (see Theorem 6.1).

Results from Theorem 2.1 can be naturally extended from the operator $\mathcal{K}$ to $\mathcal{K}^2$ (cf. (2.3)), thereby allowing us to obtain the analogue of Theorem 2.1 for $AA^{\dagger}$ and $A^{\dagger}A$. This is also done in Section 5.

Theorem 2.4. *Consider the operators A and $A^{\dagger}$ defined by (1.1) and (1.2), respectively. The operators $AA^{\dagger}$ and $A^{\dagger}A$ are unitarily equivalent. Also, the following assertions hold when $B = AA^{\dagger}$ or $A^{\dagger}A$.*

1. $\mathfrak{Sp}(B) \subseteq [0, 1]$.

2. $\lambda = 0, 1$ *are not eigenvalues of B. The eigenvalues of B have finite multiplicities, and they can accumulate only at $\lambda = 0$.*

3. *If there are $n \geq 1$ ($n \in \mathbb{N}$) double endpoints, then $\mathfrak{Sp}_{\mathrm{ac}}(B) = [0, 1]$, and the multiplicity of $\mathfrak{Sp}_{\mathrm{ac}}(B)$ equals n.*

4. *If there are no double endpoints, then B is of trace class. In this case, $\mathfrak{Sp}(B)$ consists only of eigenvalues and $\lambda = 0$, which is the accumulation point of the eigenvalues.*

5. *The singular continuous spectrum of B is empty, i.e., $\mathfrak{Sp}_{\mathrm{sc}}(B) = \varnothing$.*

3. Proof of Theorem 2.1 (1, 3, 4 and part of 2)

3.1. Theorem 2.1 (1): spectral interval

Denote by $\mathcal{H}$ the Hilbert transform on $\mathbb{R}$. Then $A = \Pi_E \circ \mathcal{H} \circ \Pi_J$ where Π_E, Π_J are the projectors on $L^2(E), L^2(J)$, respectively. It is well known that $\|\mathcal{H}\| = 1$ (see e.g., [21, Section 4.6]), hence $\|A\| \leq 1$. Consequently, the spectral radius of $\mathcal{K}$ is also bounded by 1, and Theorem 2.1 (1) is proven.

3.2. Theorem 2.1 (4), first part

We have already commented that if there are no double endpoints, i.e., the sets J, E are separated (in the extended line), then the self-adjoint operator $\mathcal{K}$ is a Hilbert–Schmidt operator, see also [5]. Then its spectrum is purely discrete and the eigenvalues

(counted with multiplicity) form an ℓ^2 sequence. In fact, it will be shown below that the operator $\mathcal{K}$ is of trace class if there are no double endpoints.

3.3. Multiplicity of the continuous spectrum

In this section we need a more detailed description of the multi-intervals E, J. Let

$$E = \bigcup_{j=1}^{r} E_j, \qquad J = \bigcup_{j=1}^{r} J_j, \tag{3.1}$$

be the representations of E and J as unions of $r < \infty$ multi-intervals. Since the interiors of E and J do not intersect, we can arrange for the following properties to hold (see Figure 1 for an illustration):

1. $J_i < J_j$ for $i < j$, i.e., the sets are ordered (ditto for the E collection);
2. $\text{dist}(J_j, J_k) > 0$ for $j \neq k$ (ditto for E);
3. for all $j = 1 \ldots r$ the set $U_j := J_j \cup E_j$ is a single interval;
4. for all $i < j$ we have $E_i < J_j$ and $J_i < E_j$ and in particular the distance $\text{dist}(E_i, J_j) > 0$ for $i \neq j$;
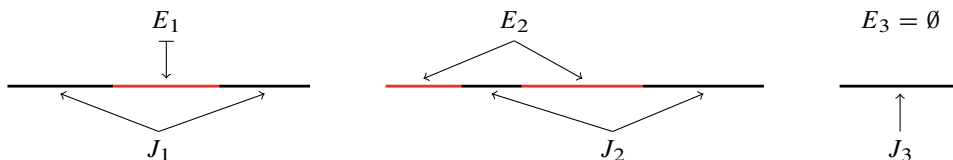5. for every $j = 1, \ldots, r$ the intersection $J_j \cap E_j$ consists of n_j endpoints of the sub-intervals.

Figure 1. An example of arrangement of J, E. The intervals U_j are $U_j = E_j \cup J_j$. Note that $U_3 = J_3$ in this example.

Lemma 3.1. *Let* $m(x) = \frac{ax+b}{cx+d}$ *with* $a, b, c, d \in \mathbb{R}$ *and* $ad - bc = 1$ *be a Möbius tranformation mapping* $\mathbb{R}$ *to* $\mathbb{R}$*; let* $\mathcal{U} : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$ *be the corresponding unitary tranformation defined by*

$$\mathcal{U}(f)(x) = \frac{f(m(x))}{(cx + d)}. \tag{3.2}$$

Then the Hilbert transform $\mathcal{H}$ *on* $\mathbb{R}$ *commutes with* $\mathcal{U}$: $\mathcal{H} \circ \mathcal{U} = \mathcal{U} \circ \mathcal{H}$.

Proof. Let $g = \mathcal{H} f$. Then, using $\mathrm{d}\, m / \mathrm{d}\, x = \frac{1}{(cx+d)^2}$ we obtain

$$
\pi \frac{g(m(y))}{cy + d} = \int_{\mathbb{R}} \frac{f(\zeta)\, \mathrm{d}\, \zeta}{(cy + d)(\zeta - m(y))}
$$

$$
= \int_{\mathbb{R}} \frac{f(m(x))\frac{\mathrm{d}\, x}{cx+d}}{(cy + d)(cx + d)(m(x) - m(y))}
$$

$$
= \int_{\mathbb{R}} \frac{f(m(x))\frac{\mathrm{d}\, x}{cx+d}}{x - y}, \tag{3.3}
$$

where $\zeta = m(x)$. Note also that (3.2) preserves the L^2 norm of f. ∎

Lemma 3.1 implies that the spectral properties of $\mathcal{H}$ and all its possible restrictions are invariant under Möbius transformations. In particular, our operator A is $A = \Pi_E \mathcal{H} \Pi_J$, where Π_U is the projection operator on the multi-interval U. In this case, if we define $\widetilde{A} = \Pi_{m^{-1}(E)} \mathcal{H} \Pi_{m^{-1}(J)}$, then $\widetilde{A}\mathcal{U} = \mathcal{U}A$.

Lemma 3.2. *Let $A\colon L^2(J) \to L^2(E)$ be the operator* (1.1) *and $\mathcal{K} = A \oplus A^\dagger$. If $\mathrm{dist}(J, E) > 0$, then $\mathcal{K}$ is of trace-class.*

Proof. Let γ be a union of Jordan curves separating J from E, see Figure 2. Recall that $U = E \cup J$ and $U = \bigcup_j U_j$. Note that, under the assumption that $\mathrm{dist}(J, E) > 0$, it follows from the definition that for each j either E_j or J_j is empty, and U_j is a single interval that coincides with whichever of the two is nonempty, see Figure 1.

Consider the Hilbert space $\mathcal{L} = L^2(U \cup \gamma, |\, \mathrm{d}\, z|) \simeq L^2(J) \oplus L^2(E) \oplus L^2(\gamma)$. Let $A_{\mathrm{ext}}\colon \mathcal{L} \to \mathcal{L}$ be the operator with the kernel $\chi_J(y)\chi_E(x)/(\pi(x - y))$. Thus, A_{ext} coincides with A when the former is restricted to $L^2(J) \to L^2(E)$. We show that A_{ext} is the product of two Hilbert–Schmidt operators, which immediately implies that A is of trace class. Indeed, let $T_k\colon \mathcal{L} \to \mathcal{L}, k = 1, 2$, be the following operators:

$$
T_1(f)(w) = \frac{\chi_\gamma(w)}{\pi} \int_J \frac{f(y)\, \mathrm{d}\, y}{w - y}, \quad T_2(g)(x) = \frac{\chi_E(x)}{2i\pi} \int_\gamma \frac{g(w)\, \mathrm{d}\, w}{x - w}. \tag{3.4}
$$

The orientation of the contour γ is chosen so that all points of E are on the positive side. By construction, it follows immediately that both T_1, T_2 are Hilbert–Schmidt. Consider the composition

$$
T_2 \circ T_1(f)(x) = \frac{\chi_E(x)}{2i\pi^2} \int_\gamma \int_J \frac{f(y)\, \mathrm{d}\, y}{w - y} \frac{\mathrm{d}\, w}{x - w}. \tag{3.5}
$$

An application of Cauchy's residue theorem shows that $T_2 \circ T_1 = A_{\mathrm{ext}}$. Thus, A_{ext}, $A_{\mathrm{ext}}^\dagger$, and $A_{\mathrm{ext}} + A_{\mathrm{ext}}^\dagger$ are all of trace class. Since $A_{\mathrm{ext}} + A_{\mathrm{ext}}^\dagger = \mathcal{K} \oplus \mathcal{O}$, where $\mathcal{O}\colon L^2(\gamma) \to L^2(\gamma)$ is the zero operator, we prove that $\mathcal{K}$ is of trace class. ∎

Figure 2. An example of an arrangement of J, E and γ when there are no common endpoints. Note that J or E may have an unbounded component (in the picture it is E), but not both simultaneously.

Remark 3.3. Iterating the argument in the proof of Lemma 3.2 one can represent A_{ext} as a product of an arbitrary number of Hilbert–Schmidt operators. This means that the eigenvalues λ_j (counted with multiplicity) of $\mathcal{K}$ form a sequence in ℓ^p for all $p \in (0, 1]$, namely,

$$\sum_{j \geq 1} \lambda_j^p < \infty, \quad \text{for all } p \text{ such that } 0 < p \leq 1. \tag{3.6}$$

Lemma 3.4. *Suppose* $r = 1$ *in* (3.1), *i.e.,* $J = J_1$, $E = E_1$, *and* $U = U_1$. *Suppose* $U = E \cup J$ *is a single compact interval* $[a, b]$, *and* J *and* E *have* n *endpoints in common (i.e., there are* n *double endpoints). Then* $\mathcal{K} : L^2(U) \to L^2(U)$ *has absolutely continuous spectrum* $[-1, 1]$ *of multiplicity* n.

Proof. There are two cases that need to be considered.

1. The leftmost and rightmost sub-intervals in U are of the same type: either both are parts of J or both are parts of E.

2. The leftmost and rightmost sub-intervals in U are of opposite types (e.g., the one on the left is a part of J, and the one on the right is a part of E).

First case. For definiteness, suppose that both leftmost and rightmost intervals are part of E. Let $\mathcal{K}_{\text{ext}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the operator with the same kernel as $\mathcal{K}$ (see (2.1)). The two operators act in a similar way, but $\mathcal{K}_{\text{ext}}$ acts on functions defined on all of $\mathbb{R}$. Consider $\mathcal{K}_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined the same way as $\mathcal{K}_{\text{ext}}$ (i.e., with the kernel (2.1)), but with E replaced by $\widehat{E} = E \cup U^c$. Here $U^c = \mathbb{R} \setminus U$, i.e., $\widehat{E}$ "extends" E to infinity.

The number n of common endpoints between J and $\widehat{E}$ is the same as between J and E. It is shown in Theorem 6.1 that $\mathcal{K}_0$ has absolutely continuous spectrum $[-1, 1]$ with multiplicity n. We also have

$$\mathcal{K}_0 = \mathcal{K}_{\text{ext}} + \mathcal{S}, \tag{3.7}$$

where $\mathcal{S}$ is the operator with the kernel

$$S(x, y) = \frac{\chi_J(x)\chi_{U^c}(y) - \chi_{U^c}(y)\chi_J(x)}{\pi(x - y)}. \tag{3.8}$$

Since $\text{dist}(J, U^c) > 0$, this operator is of trace class by Lemma 3.2 and, hence, $\mathcal{K}_0$ is a trace–class perturbation of $\mathcal{K}_{\text{ext}}$. By the Kato and Rosenblum [14, Theorem 4.4 of Chapter X], they have the same absolutely continuous spectrum with the same multiplicity.

Finally, $\mathcal{K}_{\text{ext}}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ coincides with the direct sum $\mathcal{O} \oplus \mathcal{K}$, where $\mathcal{K}: L^2(U) \to L^2(U)$ is the original operator with the kernel (2.1), and $\mathcal{O}: L^2(U^c) \to L^2(U^c)$ is the zero operator. Therefore, the multiplicities of $\mathfrak{Sp}_{\text{ac}}(\mathcal{K})$ and $\mathfrak{Sp}_{\text{ac}}(\mathcal{K}_{\text{ext}})$ are the same.

Second case. Let $U = J \cup E = [a, b]$. For definiteness, suppose that the leftmost subinterval is part of J, and the other is part of E so that $a \in J, b \in E$. Define $\hat{J}$ and $\hat{E}$ by extending the corresponding two sub-intervals up to infinity. Let $\mathcal{K}_0$ be defined as $\mathcal{K}_{\text{ext}}$ with the replacements $J \to \hat{J}$ and $E \to \hat{E}$. Let $E_\infty := \hat{E} \setminus E = [b, \infty)$ and $J_\infty = \hat{J} \setminus J = (-\infty, a]$. Similarly to the previous case, we have

$$\mathcal{K}_0 = \mathcal{K}_{\text{ext}} + \mathcal{K}_\infty + \mathcal{S}, \tag{3.9}$$

where the two operators $\mathcal{S}, \mathcal{K}_\infty$ have kernels, respectively,

$$S(x, y) = \frac{\chi_{J_\infty}(x)\chi_E(y) - \chi_{E_\infty}(y)\chi_J(x)}{\pi(x - y)}$$
$$+ \frac{\chi_J(x)\chi_{E_\infty}(y) - \chi_E(y)\chi_{J_\infty}(x)}{\pi(x - y)}, \tag{3.10}$$

$$K_\infty(x, y) = \frac{\chi_{J_\infty}(x)\chi_{E_\infty}(y) - \chi_{E_\infty}(y)\chi_{J_\infty}(x)}{\pi(x - y)}. \tag{3.11}$$

Since $\text{dist}(J_\infty, E) > 0$ and $\text{dist}(E_\infty, J) > 0$, it follows that $\mathcal{S}$ is of trace class as shown earlier.

However $\mathcal{K}_\infty$ is not trace-class because E_∞, J_∞ are both unbounded and "meet" at infinity. Indeed, by Lemma 3.1 the spectral properties of the Hilbert transform are invariant under Möbius transformations that preserve the real line (i.e., $\text{SL}_2(\mathbb{R})$).

Thus, the spectral properties of $\mathcal{K}_\infty$ are equivalent to those of $\hat{\mathcal{K}}$ defined with $J = [-1, 0]$ and $E = [0, 1]$. This case was analyzed in [4] where it was shown to have (only) absolutely continuous spectrum on $[-1, 1]$ of multiplicity one.

On the other hand, Theorem 6.1 and Lemma 3.1 show that $\mathfrak{Sp}(\mathcal{K}_0) = [-1, 1]$, with the absolutely continuous part of multiplicity $n + 1$, where n is the number of common endpoints between J and E, and the additional $+1$ multiplicity is due to the fact that $\hat{J}$ and $\hat{E}$ meet at infinity (which can be mapped to a finite point by a Möbius transformation).

Since $\mathcal{K}_0$ is now a trace–class perturbation of $\mathcal{K}_{\text{ext}} + \mathcal{K}_\infty$ as per (3.9), the multiplicity of $\mathfrak{Sp}_{\text{ac}}(\mathcal{K}_0)$ must be the sum of the multiplicities of $\mathfrak{Sp}_{\text{ac}}(\mathcal{K})$ and $\mathfrak{Sp}_{\text{ac}}(\mathcal{K}_\infty)$.

The last statement follows, because the operator

$$\mathcal{K}_{\text{ext}} + \mathcal{K}_{\infty} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

coincides with the direct sum $\mathcal{K} \oplus \mathcal{K}_{\infty}$, where $\mathcal{K} \colon L^2(E \cup J) \to L^2(E \cup J)$ and $\mathcal{K}_{\infty} \colon L^2(E_{\infty} \cup J_{\infty}) \to L^2(E_{\infty} \cup J_{\infty})$ (we used here the same notation $\mathcal{K}_{\infty}$ for the original and restricted operators with a slight abuse of notation). It then follows that the multiplicity of $\mathfrak{Sp}_{\text{ac}}(\mathcal{K})$ equals n. ∎

The following theorem completes the proof of Theorem 2.1 (3).

Theorem 3.5. *Let $A \colon L^2(J) \to L^2(E)$ be the operator (1.1) and $\mathcal{K} = A \oplus A^{\dagger}$. Let $n = \sum_{j=1}^{r} n_j$ be the total number of double endpoints in U. If $n > 0$, then the operator $\mathcal{K}$ has absolutely continuous spectrum $[-1, 1]$ with multiplicity n.*

Proof. Let $\mathcal{K}$ be the operator discussed above with the kernel (2.1). Consider the operators $\mathcal{K}_j \colon L^2(U_j) \to L^2(U_j)$ defined by the kernels

$$K_j(x, y) = \frac{\chi_{J_j}(x)\chi_{E_j}(y) - \chi_{J_j}(y)\chi_{E_j}(x)}{\pi(x - y)}. \tag{3.12}$$

Consider also the operators $\mathcal{H}_{jk} \colon L^2(U) \to L^2(U)$ that are given by the kernels

$$H_{jk}(x, y) = \frac{\chi_{J_j}(x)\chi_{E_k}(y) - \chi_{I_k}(y)\chi_{E_j}(x)}{\pi(x - y)}, \quad j \neq k. \tag{3.13}$$

Since $\text{dist}(J_j, E_k) > 0$ for $j \neq k$, all the operators $\mathcal{H}_{jk}$ are trace class by Lemma 3.2. Using the two families of operators, represent the full operator $\mathcal{K}$ as follows

$$\mathcal{K} = \bigoplus_{j=1}^{r} \mathcal{K}_j + \sum_{j < k} \mathcal{H}_{jk}. \tag{3.14}$$

Therefore, $\mathcal{K}$ is a trace-class perturbation of the self-adjoint operator

$$\mathcal{K}_{\oplus} := \bigoplus_{j=1}^{r} \mathcal{K}_j.$$

The $\mathcal{K}_j$ are endomorphisms of the collection of orthogonal subspaces $\{L^2(U_j)\}_{j=1}^{r}$ in $L^2(U)$. The spectrum of $\mathcal{K}_{\oplus}$ is the disjoint union of the spectra of each $\mathcal{K}_j$. By Lemma 3.4, each $\mathcal{K}_j$ has absolutely continuous spectrum $[-1, 1]$ with multiplicity n_j. Hence, $\mathcal{K}_{\oplus}$ has absolutely continuous spectrum on $[-1, 1]$ of multiplicity $n = \sum n_j$. By [14, Theorem 4.4 and p. 542], the absolutely continuous parts of $\mathcal{K}$ and $\mathcal{K}_{\oplus}$ are unitarily equivalent, and the theorem is proven. ∎

3.4. Theorem 2.1 (2 (first part), 4)

In this section we prove that $\lambda = \pm 1$ and 0 are not eigenvalues of $\mathcal{K}$, and the eigenvalues accumulate at 0 if there are no double endpoints. We begin by proving that $\lambda = \pm 1$ are not eigenvalues. Assume, by contrapositive, that, for example, $\lambda = 1$ is an eigenvalue. Then, there exists $\phi \in L^2(U)$ with $\mathcal{K}\phi = \phi$. Let ϕ_J and ϕ_E denote the restrictions of ϕ onto J and E, respectively. Whenever necessary, they are extended by zero to all of $\mathbb{R}$. The kernel of A is the usual Hilbert transform kernel. Using that $\|\mathcal{H}\|_{L^2(\mathbb{R})} = 1$, where $\mathcal{H}$ is the conventional Hilbert transform on all of $\mathbb{R}$, we obtain a contradiction

$$\begin{aligned}
\|\mathcal{K}\phi\|_{L^2(U)}^2 &= \|\mathcal{H}\phi_J\|_{L^2(E)}^2 + \|\mathcal{H}\phi_E\|_{L^2(J)}^2 \\
&< \|\mathcal{H}\phi_J\|_{L^2(\mathbb{R})}^2 + \|\mathcal{H}\phi_E\|_{L^2(\mathbb{R})}^2 \\
&= \|\phi_J\|_{L^2(J)}^2 + \|\phi_E\|_{L^2(E)}^2 \\
&= \|\phi\|_{L^2(U)}^2.
\end{aligned} \tag{3.15}$$

Here $\|\mathcal{H}\phi_J\|_{L^2(E)}$ is the L^2 norm of $\mathcal{H}\phi_J$ restricted to E, $\|\mathcal{H}\phi_E\|_{L^2(J)}$ is defined similarly, and we have used that $\mathcal{H}\phi_J$ and $\mathcal{H}\phi_E$ are analytic and, thus, nonzero in $\mathbb{R} \setminus J$ and $\mathbb{R} \setminus E$, respectively.

Next, if $\phi \in L^2(J)$, then $A\phi \equiv 0$ if and only if $\phi \equiv 0$, since $A\phi$ is analytic in the interior of E. Similarly, $A^\dagger \psi \not\equiv 0$ if $\psi \not\equiv 0$, where $\psi \in L^2(E)$. Therefore, $\lambda = 0$ is not an eigenvalue of $\mathcal{K}$.

Finally, by Lemma 3.2 and Theorem 3.5, $\mathcal{K}$ is a trace class operator if and only if there are no double endpoints. In the latter case, $\lambda = 0$ is not an eigenvalue implies that $\mathcal{K}$ has a sequence of eigenvalues convergent to $\lambda = 0$. Thus, we proved Theorem 2.1 (4).

4. Proof of Theorem 2.1 (5)

To prove the remaining items of Theorem 2.1, we need to introduce the following RHP 4.1 that is closely related with the resolvent of $\mathcal{K}$. This approach goes back to [12], see also [6]. A good introduction to RHPs can be found, for example, in [7,8]. In the rest of the paper we will use the following three Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4.1}$$

4.1. Riemann–Hilbert problem and the resolvent of $\mathcal{K}$

Let us consider the following RHP.

RHP 4.1. *Find a matrix–valued function $\Gamma(z; \lambda)$ such that for any fixed parameter $\lambda \in \mathbb{C} \setminus [-1, 1]$,*

  a. *the matrix $\Gamma(z; \lambda)$ is analytic together with its inverse in $z \in \overline{\mathbb{C}} \setminus U$;*

  b. *$\Gamma(z; \lambda)$ satisfies the jump condition*

$$\Gamma_+(z; \lambda) = \Gamma_-(z; \lambda)\left(\mathbf{1} - \frac{2i}{\lambda} f(z) g^T(z)\right), \quad z \in U = J \cup E, \qquad (4.2)$$

  where

$$f^T(x) = [\chi_E(x), \chi_J(x)], \quad g^T(x) = [-\chi_J(x), \chi_E(x)]; \qquad (4.3)$$

  c. *$\Gamma(\infty; \lambda) = \mathbf{1}$;*

  d. *the limiting values $\Gamma_\pm(z; \lambda)$ are in L^2_{loc} near the endpoints of the intervals.*

The jump condition (4.2) equivalently reads (the jump being understood at all interior points of the respective intervals)

$$\Gamma_+(z; \lambda) = \Gamma_-(z; \lambda)\begin{bmatrix} 1 & 0 \\ \frac{2i}{\lambda} & 1 \end{bmatrix}, \quad z \in J, \qquad (4.4)$$

$$\Gamma_+(z; \lambda) = \Gamma_-(z; \lambda)\begin{bmatrix} 1 & -\frac{2i}{\lambda} \\ 0 & 1 \end{bmatrix}, \quad z \in E. \qquad (4.5)$$

Remark 4.2. Using standard arguments, one can show that the requirement (d) in the RHP 4.1 implies the uniqueness of solution $\Gamma(z; \lambda)$ of the RHP 4.1. The existence of a solution in $\mathbb{C} \setminus [-1, 1]$ will be proven in Theorem 4.7 below. Moreover, the solution has the following symmetries

$$\overline{\Gamma(\bar{z}; \bar{\lambda})} = \Gamma(z; \lambda), \quad \Gamma(z; -\lambda) = \sigma_3 \Gamma(z; \lambda)\sigma_3. \qquad (4.6)$$

Remark 4.3. Let $\widetilde{\Gamma}(z; \lambda)$ denote the solution of the RHP 4.2 were the jump matrices on the multi intervals J, E are interchanged, that is, (4.4) is the jump relation on E and (4.5) is the jump relation on J. Then it is straightforward to check that the solutions of the two RHPs are related by

$$\widetilde{\Gamma}(z; \lambda) = \sigma_2 \Gamma(z; \lambda)\sigma_2. \qquad (4.7)$$

For example, the first symmetry follows from the fact that the matrix $V(z; \lambda) = \overline{\Gamma(\bar{z}; \bar{\lambda})}$ satisfies the same RHP 4.1. The proof of the second symmetry is also straightforward. Additionally, it can be shown that if the solution to the RHP 4.1 exists, then it must satisfy

$$\det \Gamma(z; \lambda) \equiv 1. \qquad (4.8)$$

Remark 4.4. We can now observe that the RHP 4.1 is related with the operator $\mathcal{K}$ since its kernel $K(x, y)$ given by (2.1) can be represented as

$$K(x, y) = \frac{f^T(x)g(y)}{\pi(x - y)}, \tag{4.9}$$

with f, g given by (4.3).

Let us now study the local behavior of $\Gamma(z; \lambda)$ near the endpoints. Consider for example a simple right endpoint $z = a$ of E. Denote by $\mathbf{A}, \mathbf{B}$ the first and second columns of the matrix Γ, respectively. Then (4.4)–(4.5) imply that $\mathbf{A}(z)$ is analytic at $z = a$, and $\mathbf{B}(z) + \frac{2i}{\lambda} \frac{\ln(z-a)}{2i\pi} \mathbf{A}(z)$ is analytic in the punctured neighborhood of $z = a$. The requirement (d) of the RHP 4.1 forces us to conclude that the latter expression is actually analytic (no pole). In other words,

$$\Gamma(z; \lambda) = \mathcal{O}(1) \begin{bmatrix} 1 & -\frac{2i}{\lambda} \frac{\ln(z-a)}{2i\pi} \\ 0 & 1 \end{bmatrix}. \tag{4.10}$$

Here and henceforth, $\mathcal{O}(1)$ denotes a matrix-valued function which is locally analytic in z and invertible. A similar argument applies to all simple endpoints of J, E.

Consider a double endpoint. Without loss of generality, according to Lemma 3.1 and the Remark 4.3, we can place it at $z = 0$ with E locally on the right of $z = 0$ and J on the left. The first issue is the type of growth behavior that the entries of Γ have near $z = 0$. To this end, we observe that the jump matrices in (4.4), (4.5) are constant in z and, therefore, we can analytically continue $\Gamma(z)$ on the universal cover of a punctured neighborhood of $z = 0$. Such analytic continuation has the following multivaluedness:

$$\Gamma(z) = \Gamma(ze^{2i\pi}) \begin{bmatrix} 1 & \frac{2i}{\lambda} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{2i}{\lambda} & 1 \end{bmatrix} = \Gamma(ze^{2i\pi}) \begin{bmatrix} 1 - \frac{4}{\lambda^2} & \frac{2i}{\lambda} \\ \frac{2i}{\lambda} & 1 \end{bmatrix} = \Gamma(ze^{2i\pi}) M_0, \tag{4.11}$$

provided $\Im z > 0$. Similar calculations show that $\Gamma(z) = \Gamma(ze^{2i\pi}) M_0$ is valid for $\Im z < 0$ as well.

The matrix M_0 plays an important role in the analysis below. To calculate its eigenvalues and eigenvectors, it is convenient to introduce a new variable ρ, which is related to λ as follows:

$$\rho(\lambda) = -\frac{1}{2} + \frac{1}{i\pi} \ln\left(\frac{1 - \sqrt{1 - \lambda^2}}{\lambda}\right), \quad \lambda(\rho) = -\frac{1}{\sin(\pi\rho)}. \tag{4.12}$$

We choose the branch of the square root so that $\frac{1 - \sqrt{1-\lambda^2}}{\lambda} = i + O(\lambda^{-1})$ as $\lambda \to \infty$ and the branch of logarithm in (4.12) so that $\rho(\lambda)$ is a conformal mapping of $\mathbb{C} \setminus [-1, 1]$ into the vertical strip $|\Re(\rho)| < \frac{1}{2}$. Note that $\rho(\infty) = 0$. We will also
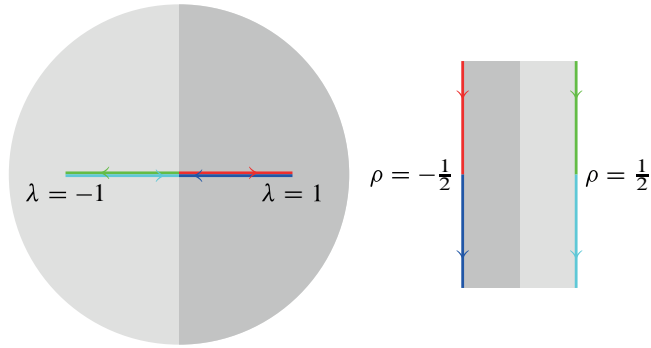
Figure 3. The slit λ-plane is mapped to the strip $|\Re\rho| < \frac{1}{2}$. The other strips $|\Re\rho - k| < \frac{1}{2}$ in the ρ-plane are mapped to the same slit λ-plane and represent the various sheets of the branched map $\lambda(\rho)$.

consider the analytic continuation of this map as a map from the Riemann surface $\mathcal{R}$ of $\rho(\lambda)$ onto $\mathbb{C}$. Figure 3 provides a visualization of the map (4.12) between the main sheet of $\mathcal{R}$, both shores of the branch cut $[-1, 1]$ included, and the vertical strip $|\Re\rho| \leq \frac{1}{2}$. In general, each sheet of $\mathcal{R}$ is mapped onto the corresponding integer-shifted vertical strip $|\Re\rho| \leq \frac{1}{2}$, so that ρ becomes a global coordinate on $\mathcal{R}$. Note that $\lambda(\rho)$ is a single-valued meromorphic function on $\mathbb{C}$. The determination of the logarithm in (4.12) is consistent with condition (d) from RHP 4.1.

Direct calculations show that $e^{\pm 2\pi i \rho(\lambda)}$ are the eigenvalues of M_0, and

$$C_+^{-1} e^{2i\pi\rho\sigma_3} = M_0 C_+^{-1}, \tag{4.13}$$

where

$$C_+(\rho) := \begin{bmatrix} 1 & -e^{-i\pi\rho} \\ 1 & e^{i\pi\rho} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-i\lambda}{1-\sqrt{1-\lambda^2}} \\ 1 & \frac{-i\lambda}{1+\sqrt{1-\lambda^2}} \end{bmatrix}, \quad \det C_+ = 2\cos(\pi\rho). \tag{4.14}$$

To simplify notations, here and henceforth we often use ρ instead of $\rho(\lambda)$. We also introduce

$$C_-(\rho) := C_+(\rho) \begin{bmatrix} 1 & \frac{2i}{\lambda} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -e^{i\pi\rho} \\ 1 & e^{-i\pi\rho} \end{bmatrix}. \tag{4.15}$$

In the following proposition we derive the local behavior of $\Gamma(z; \lambda)$ for z near a double endpoint.

Proposition 4.5. *Let $\lambda \in \mathbb{C} \setminus [-1, 1]$. If $z = 0$ is a double endpoint with E adjacent to the right of $z = 0$, then any matrix valued function $\Gamma(z; \lambda)$ satisfying conditions* (a) *and* (b) *of the RHP 4.1 that is also L^2_{loc} in a small disk $\mathbb{D}$ centered at $z = 0$ can be*

written in the form

$$\Gamma(z;\lambda) = Y(z;\lambda)z^{\rho(\lambda)\sigma_3}C_{\pm}(\rho(\lambda)), \quad z \in \mathbb{D} \cap \mathbb{C}^{\pm}, \tag{4.16}$$

where $\mathbb{C}^{\pm}$ denote the upper/lower complex half plane and $\rho(\lambda)$, $C_{\pm}$ are given by (4.12), (4.14), and (4.15) respectively. Here $Y(z;\lambda)$ denotes a matrix valued function analytic near $z = 0$ and with $\det Y(0;\lambda) \neq 0$.

Proof. Define the matrix function

$$P(z;\lambda) := z^{\rho\sigma_3}C_{\pm}(\rho), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{4.17}$$

where the subscript "+" is for $\Im(z) > 0$, and "−" is for $\Im(z) < 0$. Note also that z^{ρ} is defined as the principal power, with a branch-cut along the negative real axis. We note that $\det P(z;\lambda) = 2\cos\pi\rho$ is constant in z. A direct computation shows that P satisfies the jump condition:

$$P_+(z;\lambda) = P_-(z;\lambda)\begin{bmatrix} 1 & \frac{-2i}{\lambda} \\ 0 & 1 \end{bmatrix}, \quad z \in \mathbb{R}^+, \tag{4.18}$$

$$P_+(z;\lambda) = P_-(z;\lambda)\begin{bmatrix} 1 & 0 \\ \frac{2i}{\lambda} & 1 \end{bmatrix}, \quad z \in \mathbb{R}^-, \tag{4.19}$$

where $P_{\pm}(z;\lambda)$ are to be understood as the boundary values of the respective matrices at a point $z \in \mathbb{R}$. Indeed, the first equation follows from (4.15). The second equation becomes $e^{2i\pi\rho\sigma_3}C_+ = C_+M_0$ as it takes into account the jump of $z^{\rho\sigma_3}$ on $\mathbb{R}^-$. Now, (4.19) follows from (4.13).

For $\lambda \notin [-1, 1]$, $\rho(\lambda)$ in (4.12) satisfies $|\Re\rho(\lambda)| < \frac{1}{2}$, with $\Re(\rho(\lambda)) = \frac{1}{2}$ for $\lambda \in (-1, 0)$ and $\Re(\rho(\lambda)) = -\frac{1}{2}$ for $\lambda \in (0, 1)$. So, the required inequality is a consequence of the maximum principle for harmonic functions, see Figure 3. Hence, the matrix entries of $P(z;\lambda)$ are all in L_{loc}^2 near the origin for $\lambda \notin [-1, 1]$.

Now, let $\Gamma(z;\rho)$ satisfy conditions (a) and (b) of the RHP 4.1 with entries in L_{loc}^2 near $z = 0$. Then ΓP^{-1} has no jumps in a neighborhood of the origin and, hence, it may only have an isolated singularity at $z = 0$. The L_{loc}^2 condition together with $|\Re\rho(\lambda)| < \frac{1}{2}$ implies that the singularity is removable. Thus, the matrix Γ has precisely the proposed representation (4.16). ∎

Remark 4.6. We should also point out that the solution $\Gamma(z;\lambda)$ of the RHP 4.1, if it exists, solves a Fuchsian differential equation in z of the form

$$\Gamma'(z;\lambda) = \left(\sum_{z_j \in \partial E \cup \partial J} \frac{A_j}{z - z_j} \right)\Gamma(z;\lambda),$$

where the matrices A_j are independent of z. These matrices, for a fixed λ, depend on the position of the endpoints according to the so–called *Schlesinger equations* [13],

which express the fact that the monodromy representation induced by a fundamental solution of this ODE is independent of the position of the endpoints of the multi-intervals J, E.

The main tool for the analysis of the remaining Theorem 2.1 (4,5) is the following theorem for the so-called *regularized resolvent* defined by $\text{Id} + \mathcal{R}(\lambda) = (\text{Id} - \frac{1}{\lambda}\mathcal{K})^{-1}$.

Theorem 4.7. *The resolvent $\mathcal{R}(\lambda)$ of $\mathcal{K}$ is an integral operator with the kernel*

$$R(x, y; \lambda) := \frac{1}{\lambda} \frac{f^T(x)\Gamma^T(x; \lambda)\Gamma^{-T}(y; \lambda)g(y)}{\pi(x - y)}, \tag{4.20}$$

where $\Gamma(z; \lambda)$ is the solution of the RHP 4.1. Moreover, the operator $\text{Id} - \frac{1}{\lambda}\mathcal{K}$ has bounded inverse if and only if the RHP 4.1 is solvable, and the solution is given by

$$\Gamma(z; \lambda) = \mathbf{1} - \int_U \frac{F(x; \lambda) \cdot g^T(x)\,\mathrm{d}x}{x - z}, \tag{4.21}$$

where $F(x; \lambda) = (\text{Id} + \mathcal{R}(\lambda))[f](x)$.

Proof. Assume that $\Gamma(z; \lambda)$ is the solution of the RHP 4.1. We first show that the integral operator $\mathcal{R}(\lambda)$ with the kernel (4.20) is bounded from $L^2(U)$ into $L^2(U)$. Fix some $\lambda \in \mathbb{C} \setminus [-1, 1]$. We first note that, according to Remark 4.6, $\Gamma_{\pm}(z; \lambda)$ is analytic on U with the exception of the endpoints, where the local behavior of $\Gamma_{\pm}(z; \lambda)$ is given either by (4.10) (simple endpoint) or by Proposition 4.5 (double endpoint). Thus, the task of proving that $\mathcal{R}$ is bounded requires only a local analysis in a neighborhood of each endpoint. If z is a simple endpoint, the result is established in [5]. Even though the geometry of the intervals in [5] is slightly less general than the one here, the argument is purely local and applies in our situation as well. Suppose now that $z = 0$ is a double endpoint. Since the problem is local, we can assume that y is confined to a small neighborhood of $z = 0$. Obviously, $\int_{U \setminus D_\varepsilon} R(x, y; \lambda)\phi(y)\,\mathrm{d}y \in L^2(U)$ for any $\phi \in L^2(U)$, where D_ε is an ε-neighborhood of the origin, $\varepsilon > 0$. Consider now the integral over D_ε. Using the analyticity of $Y(z; \lambda)$ from Proposition 4.5, we obtain

$$\Gamma^{-1}(y; \lambda)\Gamma(x; \lambda) = C_{\pm}^{-1}y^{-\rho\sigma_3}\left[\mathbf{1} + \mathcal{O}(x - y)\right]x^{\rho\sigma_3}C_{\pm}, \quad x \to y, \tag{4.22}$$

where, according to Proposition 4.5, the subindices of C, C^{-1} are determined by inequalities $\pm\Im x > 0$, $\pm\Im y > 0$ respectively. In (4.22), $\mathcal{O}(x - y)$ denotes a matrix whose coefficients vanish to at least first order at $x = y$. This matrix is uniform for $x, y \in D_\varepsilon$ with a sufficiently small ε because of formula (4.16). Since the integral operator corresponding to the $\mathcal{O}(x - y)$ term is nonsingular and $|\Re\rho(\lambda)| < \frac{1}{2}$, it

remains to show that the integral operator with the kernel

$$R_0(x, y; \lambda) = \frac{g^T(y) C_+^{-1} (\frac{x}{y})^{\rho \sigma_3} C_+ f(x)}{\lambda \pi (x - y)} \tag{4.23}$$

is a bounded operator in $L^2(U)$. According to (4.3), the kernel R_0 is a linear combination of $(\frac{x}{y})^{\pm \rho}$ and characteristic functions of E, J, so we can restrict our attention to the integral operator

$$\mathbf{r}[\phi](y) = \int\limits_U \frac{(\frac{x}{y})^\rho \chi(x) \phi(x)}{x - y} \, dx, \tag{4.24}$$

where χ is either χ_E or χ_J, and $\phi \in L^2(U)$. Using again that $|\Re \rho(\lambda)| < \frac{1}{2}$ and appealing to [11, Lemma 4.2, p. 32], it is straightforward to conclude that $\mathbf{r} \colon L^2(U) \to L^2(U)$ is a bounded operator. Thus, we proved that $\mathcal{R}(\lambda)$ is a bounded operator in $L^2(U)$.

Let us now prove that the integral operator $\mathcal{R}(\lambda)$ with the kernel (4.20) is the resolvent of $\mathcal{K}$. The equation for the resolvent is

$$(\mathrm{Id} + \mathcal{R}) \circ \left(\mathrm{Id} - \frac{1}{\lambda} \mathcal{K} \right) = \mathrm{Id} \iff \mathcal{R} - \frac{1}{\lambda} \mathcal{K} = \frac{1}{\lambda} \mathcal{R} \circ \mathcal{K}. \tag{4.25}$$

As it was shown above, the kernel (4.20) defines a bounded integral operator $\mathcal{R}$ in $L^2(U)$, and we now verify that it satisfies (4.25). Indeed, the kernel $R \circ K$ of $\mathcal{R} \circ \mathcal{K}$ is

$$\frac{1}{\lambda} R \circ K(z, w) = \frac{1}{(\lambda \pi)^2} \int\limits_U \frac{f^T(z) \Gamma^T(z; \lambda) \Gamma^{-T}(x; \lambda) g(x)}{z - x} \frac{f^T(x) g(w)}{x - w} \, dx. \tag{4.26}$$

Note that Γ^{-T} solves

$$\Gamma_+^{-T} = \Gamma_-^{-T} \left(1 + \frac{2i}{\lambda} g f^T \right), \quad z \in U, \tag{4.27}$$

so that for $x \in U$

$$\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x) = \frac{2i}{\lambda} \Gamma_-^{-T}(x) g(x) f^T(x), \tag{4.28}$$

and the right-hand side does not depend on the side of the boundary value being taken because

$$\Gamma_+^{-T}(x) g(x) f^T(x) = \Gamma_-^{-T}(x) \left(1 + \frac{2i}{\lambda} g(x) f^T(x) \right) g(x) f^T(x)$$

and $f^T(x)g(x) \equiv 0$. Thus, (4.26) yields

$$\int\limits_U f^T(z)\Gamma^T(z)\left(\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)\right) g(w) \frac{1}{(z-x)(x-w)} \frac{\mathrm{d}\,x}{2i\pi^2\lambda}$$

$$= \int\limits_U f^T(z)\Gamma^T(z)\left(\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)\right) g(w) \frac{1}{z-w}\left(\frac{1}{z-x} + \frac{1}{x-w}\right)\frac{\mathrm{d}\,x}{2i\pi^2\lambda}.$$

$$(4.29)$$

To simplify notations, we drop the λ dependence in $\Gamma(x;\lambda)$ here and in the rest of the proof. We show that (4.29) splits into two essentially equal integrals. Indeed, choose $z \notin U$ and, using Cauchy's theorem together with the fact that $\Gamma(\infty) = \mathbf{1}$, we have

$$\int\limits_U \frac{\Gamma_+^{-T}(x) - \Gamma_-^{-T}(x)}{z-x} \frac{\mathrm{d}\,x}{2i\pi} = \mathbf{1} - \Gamma^{-T}(z). \tag{4.30}$$

Substituting (4.30) into (4.29) we finally obtain

$$\frac{1}{\lambda\pi(z-w)} f^T(z)\Gamma^T(z)\left[\overbrace{\int\limits_U \frac{\left(\Gamma_-^{-T}(x) - \Gamma_+^{-T}(x)\right)}{z-x} \frac{\mathrm{d}\,x}{2i\pi}}^{\mathbf{1}-\Gamma^{-T}(z)}\right.$$

$$\left. + \overbrace{\int\limits_U \frac{\left(\Gamma_-^{-T}(x) - \Gamma_+^{-T}(x)\right)}{x-w} \frac{\mathrm{d}\,x}{2i\pi}}^{\Gamma^{-T}(w)-\mathbf{1}}\right] g(w) \tag{4.31}$$

$$= R(z,w) - \frac{1}{\lambda}K(z,w). \tag{4.32}$$

Thus, we have shown that $\mathcal{R}\mathcal{K} = \mathcal{R} - \frac{1}{\lambda}\mathcal{K}$. Hence, the integral operator $\mathcal{R}$ with the kernel R given by (4.20) is the regularized resolvent.

Vice versa, suppose now that $\mathrm{Id} - \frac{1}{\lambda}\mathcal{K}$ is invertible, and denote by $\mathrm{Id} + \mathcal{R}$ its resolvent. Define

$$F(z) = (\mathrm{Id} + \mathcal{R})[f](z). \tag{4.33}$$

Here the equation is understood as applying the resolvent in an entry-wise fashion to each of the two entries of the vector f defined in (4.3). Then define

$$\Gamma(z;\lambda) := \mathbf{1} - \int\limits_U \frac{F(x)g^T(x)}{\lambda\pi(x-z)}\mathrm{d}\,x. \tag{4.34}$$

We then observe that $(\Gamma_+(x) - \Gamma_-(x)) f(x) = 0$ for $x \in U$, so that we have

$$\Gamma_\pm(z; \lambda) f(z) = f(z) - \int_U \frac{F(x) g^T(x) f(z)}{\lambda \pi (x - z)} \, dx$$

$$= f + \frac{1}{\lambda} \mathcal{K}[F] = f + F - \left(\mathrm{Id} - \frac{1}{\lambda} \mathcal{K} \right)[F]$$

$$= f + F - \left(\mathrm{Id} - \frac{1}{\lambda} \mathcal{K} \right) \left[\left(\mathrm{Id} - \frac{1}{\lambda} \mathcal{K} \right)^{-1} [f] \right]$$

$$= f + F - f = F. \tag{4.35}$$

To verify that (4.34) satisfies the RHP 4.1, we see that the condition $\Gamma(\infty) = \mathbf{1}$ is immediate. Regarding the jump condition we use the Sokhotskii–Plemelj formula for $x \in U$:

$$\Gamma_+(x; \lambda) - \Gamma_-(x; \lambda) = -2i\pi \frac{F(x) g^T(x)}{\lambda \pi} \overset{(4.35)}{=} -\frac{2i}{\lambda} \Gamma_-(x; \lambda) f(x) g^T(x).$$

This latter equation is equivalent to the jump condition of the RHP 4.1. Condition (d) from the RHP 4.1 follows from the fact that $F(z)$ is given by (4.33) where the resolvent operator is now, by hypothesis, a bounded operator in $L^2(U)$.

Thus, for any λ in the resolvent set of $\mathcal{K}$, there exists $\Gamma(z; \lambda)$ given by (4.34) that solves the RHP 4.1. ∎

Using Remark 4.2, one can show that the RHP 4.1 is uniquely solvable if and only if the operator $\mathrm{Id} - \frac{1}{\lambda} \mathcal{K}$ has a bounded inverse, i.e., when $\lambda \notin \mathfrak{Sp}(\mathcal{K})$, where $\mathfrak{Sp}(\mathcal{K}) \subset [-1, 1]$.

Remark 4.8. It is easy to show using the identity $f^T(z) g(z) \equiv 0$, $z \in U$, that the kernel $R(x, y; \lambda)$ in (4.20) does not have a jump across U. One can then combine this fact with the first equation of (4.6) to prove that $R(x, y; \bar{\lambda}) = \overline{R(x, y; \lambda)}$ when $x, y \in U$.

4.2. Study of the spectrum of $\mathcal{K}$ by means of analytic continuation of the RHP solution across the spectral interval $(-1, 1)$

According to Theorems 3.5 and 4.7, in the case of double endpoints the RHP 4.1 does not have a solution for any $\lambda \in [-1, 1]$. In this section we discuss the meromorphic continuation of the solution $\Gamma(z; \lambda)$ to the RHP 4.1 over the segment $\lambda \in [-1, 1]$ to the Riemann surface $\mathfrak{R}$ of $\rho(\lambda)$ beyond this segment. We will then use this continuation to analyze the resolvent $\mathcal{R}$ on $[-1, 1]$. Since ρ is a global coordinate on $\mathfrak{R}$, it will be convenient to introduce the notation $\Gamma(z; \rho) := \Gamma(z; \lambda(\rho))$, where $\lambda(\rho)$ is defined by (4.12).

RHP 4.9. *Let $z_1, \ldots, z_N$, $N \in \mathbb{N}$, be the double endpoints (common endpoints $J \cap E$). For a point $\rho \in \mathbb{C} \setminus (\frac{1}{2} + \mathbb{Z})$ we are looking for a matrix function $\Gamma(z; \rho)$ with the following properties:*

1. *the matrix $\Gamma(z; \rho)$ is analytic together with its inverse in $z \in \overline{\mathbb{C}} \setminus U$;*

2. *$\Gamma(z; \rho)$ satisfies the jump condition*

$$\Gamma_+(z; \rho) = \Gamma_-(z; \rho) \begin{bmatrix} 1 & 0 \\ -2i \sin(\pi\rho) & 1 \end{bmatrix}, \quad z \in J, \tag{4.36}$$

$$\Gamma_+(z; \rho) = \Gamma_-(z; \rho) \begin{bmatrix} 1 & 2i \sin(\pi\rho) \\ 0 & 1 \end{bmatrix}, \quad z \in E; \tag{4.37}$$

3. *$\Gamma(\infty; \rho) = \mathbf{1}$;*

4. *the limiting values $\Gamma_\pm(z; \rho)$ are in L^2_{loc} for any $z \in U \setminus \{z_1, \ldots, z_N\}$;*

5. *the local behavior of $\Gamma(z; \rho)$ near the double endpoints is given by (4.16) with $C_\pm$ given by (4.14) and (4.15).*

Corollary 4.10. *For any ρ satisfying $|\Re\rho| < \frac{1}{2}$ the solution $\Gamma(z; \rho)$ of the RHP 4.9 exists and coincides with the solution $\Gamma(z; \lambda) = \Gamma(z; \lambda(\rho))$ of the RHP 4.1.*

Proof. According to (4.12), conditions (a)–(c) of the RHPs 4.1 and 4.9 are the same. Moreover, conditions (d) and (e) of the RHP 4.9 imply condition (d) of the RHP 4.1 provided $|\Re\rho| < \frac{1}{2}$. Now, the statement of the corollary follows from Theorem 4.7 and Proposition 4.5. ∎

We now aim to show that the solution $\Gamma(z; \rho)$ of the RHP 4.1 admits an extension to a meromorphic function of ρ in the whole ρ-plane. The proof proceeds in two steps.

• First, we prove that Γ admits an extension to a meromorphic function of ρ in $\mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$. Observe that the points $\rho = \frac{1}{2} + 2\mathbb{Z}$ are all mapped to $\lambda = -1$, while the points $\rho = -\frac{1}{2} + 2\mathbb{Z}$ are mapped to $\lambda = 1$. This implies that, in addition to $\rho = \infty$, the poles of $\Gamma(z; \rho)$ can possibly accumulate at half integer ρ.

• We then prove that near each of the points $\rho = \frac{1}{2} + k$, $k \in \mathbb{Z}$, $\Gamma(z; \rho)$ is also meromorphic (i.e., has only finitely many poles).

The first point is proven in the next lemma.

Lemma 4.11. *The solution of the RHP 4.9 admits a meromorphic extension to the domain $\rho \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$.*

Proof. Let c be an endpoint of E or J, and let $\mathbb{D}_c$ be a small disk centered at $z = c$. We choose these disks centered at every endpoint of E and J small enough so that

they are disjoint. Define $\Phi(z;\rho) := \Gamma(z;\rho)$ outside of these disks, and

$$\Phi(z;\rho) := \begin{cases} \Gamma(z;\rho)V^{\pm 1}(\pm(z-e);\rho), & z \in \mathbb{D}_e, \\ \Gamma(z;\rho)W^{\pm 1}(\pm(z-f);\rho), & z \in \mathbb{D}_f, \\ \Gamma(z;\rho)P_R^{-1}(z-q;\rho), & z \in \mathbb{D}_q, \\ \Gamma(z;\rho)P_L^{-1}(z-p;\rho), & z \in \mathbb{D}_p, \end{cases} \tag{4.38}$$

where

$$V(z;\rho) := \begin{bmatrix} 1 & \frac{-\sin(\pi\rho)\ln(z)}{\pi} \\ 0 & 1 \end{bmatrix}, \quad W(z;\rho) := \begin{bmatrix} 1 & \\ \frac{\sin(\pi\rho)\ln(z)}{\pi} & 1 \end{bmatrix}, \tag{4.39}$$

$P_R(z;\rho) = P(z;\rho)$ is the parametrix given by (4.17), and $P_L = \sigma_2 P_R \sigma_2$. Here e is a simple endpoint of E, f is a simple endpoint of J, q is a double endpoint having E adjacent on the right, and p is a double endpoint having E adjacent on the left, see Figure 4. The sign "+" in (4.38) is for the case that e (respectively, f) is a right endpoint of E (respectively, J), and the sign "−" is for the left endpoints.

The results of Proposition 4.5 and the discussion immediately preceding it show that the matrix $\Phi(z;\rho)$ is a piecewise analytic matrix-valued function on the complement of the contour Σ that consists of the disks around the endpoints together with the part of U outside of these disks (see Figure 4), and Φ is uniformly bounded with respect to $z \in \mathbb{C}$. It is the solution of a RHP with jumps on Σ, where the jump matrices on the circles $\partial\mathbb{D}_c$ depend *analytically* on $\rho \in \mathbb{C} \setminus \mathbb{Z} + \frac{1}{2}$. Moreover, the product of all the jump matrices at an intersection of any disk $\mathbb{D}_c$ taken according to the orientation of each jump contour equals to the identity matrix $\mathbf{1}$.

Under these circumstances, it is known, see [10, Proposition 3.2] and also the original paper [32], that the solution $\Phi(z;\rho)$ of the corresponding RHP either never exists or is meromorphic in ρ, with poles at an exceptional locus of points that may accumulate only at the boundary of the domain of analyticity in the parameter space ρ. The first option is not possible due to Corollary 4.10. Therefore, $\Phi(z;\rho)$ is meromorphic with respect to $\rho \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$, which implies the statement of the lemma. ∎

We now need to prove that the solution $\Gamma(z;\rho)$ of the RHP 4.9 is also meromorphic near $\rho = \frac{1}{2} + k$, $k \in \mathbb{Z}$. To this end we first prove the lemma where, according to Remark 4.3, it is sufficient to consider E to the right of a double endpoint.

Lemma 4.12. *Let z_j, $j = 1, \ldots, N$, be a double endpoint with E adjacent from the right. Define*

• *for $k = -1, -2, \ldots,$*

$$Q_k(z;\rho) := \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2\cos\pi\rho} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(z-z_j)^{2|k|-1} & 1 \end{bmatrix} (z-z_j)^{\rho\sigma_3} C_\pm(\rho); \tag{4.40a}$$
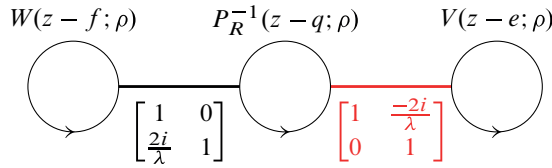
Figure 4. An example of the contour Σ supporting the jump discontinuities of the RHP for the matrix Φ. Indicated near each arc is the corresponding jump matrix. The black line segment represents J, and the red line segment represents E.

- *for $k = 0, 1, 2, \ldots$,*

$$Q_k(z; \rho) := \begin{bmatrix} \frac{1}{2\cos\pi\rho} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -(z - z_j)^{2k+1} \\ 0 & 1 \end{bmatrix} (z - z_j)^{\rho\sigma_3} C_\pm(\rho), \qquad (4.40b)$$

where $\pm\Im z > 0$, respectively. Then the local behaviour of the solution $\Gamma(z; \rho)$ near $z = z_j$ and near the points $\rho \in \frac{1}{2} + k$, where $k \in \mathbb{Z}$, can be represented by

$$\Gamma(z; \rho) = Y_k^{(j)}(z; \rho) Q_k(z; \rho), \quad j = 1, \ldots, N. \qquad (4.41)$$

Moreover, $\det Y_k^{(j)} \equiv 1$, and the matrix functions $Y_k^{(j)}(z; \rho)$ are analytic in z in ρ-independent disks centered at z_j.

Proof. We consider the case when E is adjacent to z_j from the right. The fact that $\det Y_k^{(j)} \equiv 1$ follows from (4.14). The analyticity of $Y_k^{(j)}(z; \rho)$ with respect to z in a ρ-independent neighborhood of the double endpoint z_j follows from the fact that $\Gamma(z; \rho) Q_k^{-1}(z; \rho)$ is analytic near z_j. The other case when J is adjacent to z_j from the right can be considered analogously. ∎

Proposition 4.13. *Assume $z_j = 0$ in (4.40). Then the matrices $Q_k(z; \rho)$ with $z \neq 0$ are analytic in some neighborhoods of $\rho = \frac{1}{2} + k$ for all $k \in \mathbb{Z}$. For $\rho = \frac{1}{2} + k$, where $k = -1, -2, \ldots$, we have*

$$\lim_{\varepsilon \to 0} Q_k\left(z; \frac{1}{2} - |k| + \varepsilon\right) = \begin{bmatrix} z^{\frac{1}{2} - |k|} & (-1)^k i z^{\frac{1}{2} - |k|} \\ \frac{(-1)^k z^{-\frac{1}{2} + |k|} \ln z}{\pi} & z^{-\frac{1}{2} + |k|} - \frac{1}{\pi i} z^{-\frac{1}{2} + |k|} \ln z \end{bmatrix}, \qquad (4.42)$$

where the expression is for z in the upper half plane. Similarly, for $\rho = \frac{1}{2} + k$ and $k = 0, 1, \ldots$ we have

$$\lim_{\varepsilon \to 0} Q_k\left(z; \frac{1}{2} + k + \varepsilon\right) = \begin{bmatrix} \frac{(-1)^k z^{\frac{1}{2} + k} \ln z}{\pi} & -z^{\frac{1}{2} + k} + \frac{1}{\pi i} z^{\frac{1}{2} + k} \ln z \\ z^{-\frac{1}{2} - k} & (-1)^k i z^{-\frac{1}{2} - k} \end{bmatrix}. \qquad (4.43)$$

These equations can be easily modified for $\Im z < 0$ using (4.15) and (4.40).

Proof. Multiplication of all the factors in the first case of (4.40) yields

$$Q_k(z;\rho) = \begin{bmatrix} z^\rho & -e^{-i\pi\rho}z^\rho \\ \dfrac{-z^{\rho+2|k|-1}+z^{-\rho}}{2\cos\pi\rho} & -\dfrac{z^{\rho+2|k|-1}e^{-i\pi\rho}+z^{-\rho}e^{i\pi\rho}}{2\cos\pi\rho} \end{bmatrix}. \qquad (4.44)$$

Substituting $\rho = \frac{1}{2} + k + \varepsilon$ into (4.44) we derive (4.42) after some algebra. All other cases can be considered analogously. ∎

With these preparations we can finally prove the meromorphic continuation of $\Gamma(z;\rho)$ onto the whole ρ-plane.

Lemma 4.14. *The solution of the RHP* 4.9 *admits an extension to a meromorphic function of $\rho \in \mathbb{C}$.*

Proof. We know from Lemma 4.11 that the matrix $\Gamma(z;\rho)$ admits a meromorphic extension to $\rho \in \mathbb{C} \setminus (\mathbb{Z} + \frac{1}{2})$. In principle, that lemma does not rule out an accumulation of poles near the points $\mathbb{Z} + \frac{1}{2}$. Therefore, we still need to prove that $\Gamma(z;\rho)$ is meromorphic also in neighborhoods of each of the points $\rho_k = \frac{1}{2} + k$, $k \in \mathbb{Z}$. This part of the proof is now only a minor revision of Lemma 4.11 and, therefore, we use the same notation from that proof. Fix $k \in \mathbb{Z}$ and define

$$\Phi(z;\rho) = \begin{cases} \Gamma(z;\rho)V^{\pm 1}(\pm(z-e);\rho), & z \in \mathbb{D}_e, \\ \Gamma(z;\rho)W^{\pm 1}(\pm(z-f);\rho), & z \in \mathbb{D}_f, \\ \Gamma(z;\rho)Q_k^{-1}(z-q;\rho), & z \in \mathbb{D}_q, \\ \Gamma(z;\rho)\sigma_2 Q_k^{-1}(z-p;\rho)\sigma_2, & z \in \mathbb{D}_p, \end{cases} \qquad (4.45)$$

where V, W are defined in (4.38), and Q_k – in (4.40). Define $\Phi(z;\rho) = \Gamma(z;\rho)$ outside of the disks from (4.45). Here, as in Lemma 4.11, e is a simple endpoint of E, f is a simple endpoint of J, q is a double endpoint having E adjacent on the right, whereas p is a double endpoint having E adjacent on the left. The sign "+" is for the case when e (respectively, f) is a right endpoint of E (respectively, J), and the sign "−" for the left endpoints. Choose a small neighborhood S of $\rho = \frac{1}{2} + k$. The same reasoning used in Lemma 4.11 now applies to $\rho \in S$ due to Lemma 4.12. Thus, we conclude that $\Gamma(z;\rho)$ is meromorphic in a neighborhood of $\rho = \frac{1}{2} + k$. ∎

4.3. Absence of singular continuous spectrum, Theorem 2.1 (5)

Before proceeding we briefly summarize the consequences of Lemma 4.14, see also (4.12) and Figure 3. Since the main strip $\Re\rho \in (-\frac{1}{2}, \frac{1}{2})$ corresponds to $\lambda \notin [-1, 1]$, Theorem 4.7 implies that none of the poles of $\Gamma(z;\rho)$ (which is the solution to the RHP 4.9) occurs in that strip. Thus, if any, the only poles in the closure of the main strip may occur on the lines $\Re\rho = \pm\frac{1}{2}$, that is, on the shores of the segment $(-1, 1)$

of the spectral λ plane. If $\mathcal{K}$ has a continuous spectrum, that is, if there is at least one double endpoint, then the poles of $\Gamma(z; \rho)$, as we are going to show in Section 5, correspond to the embedded point spectrum of $\mathcal{K}$. Thus, to complete the spectral description of $\mathcal{K}$, in the following theorem we prove the absence of singular continuous spectrum. This will prove Theorem 2.1 (5).

Theorem 4.15. *The singular continuous component of* $\mathfrak{Sp}(\mathcal{K})$ *is empty, i.e.,*

$$\mathfrak{Sp}_{sc}(\mathcal{K}) = \varnothing.$$

Proof. Let $\Gamma(z; \lambda)$ be the solution of the RHP 4.1. It is clear that a pole $\rho_0, .\Re \rho_0| = \frac{1}{2}$, of the solution $\Gamma(z; \rho)$ of the RHP 4.9 corresponds to the pole $\lambda_0 = \lambda(\rho_0), \lambda_0 \in (-1, 1)$ of $\Gamma(z; \lambda)$. Since $\det(\Gamma(z; \lambda)) \equiv 1$, the poles (in λ) of $\Gamma(z; \lambda)$ and $\Gamma^{-1}(z; \lambda)$ coincide. Thus, the kernel $R(x, y; \lambda)$ of the resolvent operator of $\mathcal{K}$, given by (4.20), is meromorphic in λ. Then so is the jump $\Delta_\lambda R(x, y; \lambda) := R(x, y; \lambda_+) - R(x, y; \lambda_-)$ over $\lambda \in (-1, 0) \cup (0, 1)$. In particular, $\Delta_\lambda R(x, y; \lambda)$ has no more than finitely many poles on any closed subinterval of $(-1, 1) \setminus \{0\}$.

Pick any $\phi \in C_0^\infty(\mathring{J} \cup \mathring{E})$. Let $\phi_1 = \mathcal{P}\phi$ be the projection of ϕ onto the direct sum of all the eigenspaces of $\mathcal{K}$ (i.e., the subspace of discontinuity with respect to $\mathcal{K}$, see, e.g., [14, Section X.1.1]). Set $\phi_2 := \phi - \phi_1$. Let $\mathcal{E}_\lambda$ denote the resolution of the identity associated with $\mathcal{K}$. Using the properties of $\mathcal{E}_\lambda$ (i.e., $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{E}_\lambda \mathcal{P} = \mathcal{P}\mathcal{E}_\lambda$, see, e.g., [14, Sections VI.5.1 and X.1.1]), we have by starting with $\mathcal{P}\phi_1 = \phi_1$:

$$\sigma_{\phi_2}(\lambda) := (\mathcal{E}_\lambda(\phi - \phi_1), \phi - \phi_1) = (\mathcal{E}_\lambda\phi, \phi) - (\mathcal{E}_\lambda\phi_1, \phi_1). \qquad (4.46)$$

We want to prove that $\sigma_{\phi_2}(\lambda)$ is smooth for any $\lambda \neq 0$ not in the point spectrum of $\mathcal{K}$. Let $[\lambda_1, \lambda_2]$ be any interval that does not contain any eigenvalue of $\mathcal{K}$ such that $\lambda_1 < \lambda < \lambda_2$. Without loss of generality we may assume $\lambda_1 > 0$. The case $\lambda_2 < 0$ can be considered analogously. Then $\mathcal{E}_\lambda\phi_1 = \mathcal{E}_{\lambda_1}\phi_1$, and the second term on the right in (4.46) is locally constant with respect to λ. Also, $\mathcal{E}_\lambda = \mathcal{E}_{\lambda_1} + \mathcal{E}_{[\lambda_1,\lambda]}$. According to [9, p. 921], $\mathcal{E}_\lambda$ is computed by the formula

$$\mathcal{E}_\lambda = \frac{-1}{2\pi i} \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\lambda} [\mathcal{R}_1(t + i\varepsilon) - \mathcal{R}_1(t - i\varepsilon)]\, dt, \qquad (4.47)$$

if λ is not an eigenvalue. Here

$$\mathcal{R}_1(\lambda) := (\lambda \, \mathrm{Id} - \mathcal{K})^{-1}.$$

Clearly, $\mathcal{R}_1(\lambda) = (1/\lambda)(\mathrm{Id} + \mathcal{R}(\lambda))$. Therefore,

$$(\mathcal{E}_{[\lambda_1,\lambda]}\phi, \phi) = \frac{-1}{2\pi i \lambda} \int_{\lambda_1}^{\lambda} \int_U \int_U \Delta_t R(x, y; t)\phi(x)\bar{\phi}(y)\, dx\, dy\, dt \qquad (4.48)$$

is a locally smooth function of λ because $R(x, y; t)$ is C^∞ on supp $\phi \times$ supp $\phi \times$ $[\lambda_1, \lambda_2]$. By construction, $\sigma_{\phi_2}(\lambda)$ is a continuous function of λ. We just proved that it may fail to be smooth only at the eigenvalues of $\mathcal{K}$ and at $\lambda = 0$. Since the number of eigenvalues of $\mathcal{K}$ is finite in any set $[-1, -\varepsilon) \cup (\varepsilon, 1]$, $\varepsilon > 0$, this implies that $\sigma_{\phi_2}(\lambda)$ is absolutely continuous. Since the span of $C_0^\infty(\mathring{J} \cup \mathring{E})$ is dense in $L^2(U)$, we see that $\mathcal{K}$ has no singular continuous spectrum. ∎

5. Proofs of Theorem 2.1 (2) and Theorem 2.4

5.1. End of proof of Theorem 2.1 (2)

We will now prove the remaining part of Theorem 2.1 (2), namely, that each eigenvalue $\lambda_0 \in (-1, 1)$ of $\mathcal{K}$ has a finite-dimensional eigenspace. The symmetry of the eigenvalues with respect to $\lambda = 0$ follows by noticing that if $\mathcal{K}(\phi, \psi)^T = \lambda(\phi, \psi)^T$, $\lambda \neq 0$, $\phi \in L^2(E)$, $\psi \in L^2(J)$, $(\phi, \psi) \not\equiv 0$, then

$$\mathcal{K}(-\phi, \psi)^T = -\lambda(-\phi, \psi)^T,$$

see (2.2). This also follows from the symmetry (4.6) of the solution $\Gamma(z; \lambda)$ of the RHP 4.1.

Proposition 5.1. *Any pole, in the spectral variable λ, of the solution $\Gamma(z; \lambda)$ to the RHP 4.1 is a simple pole.*

Proof. Since $\mathcal{K}$ is a bounded, self-adjoint operator, the resolvent $\mathcal{R}$ of $\mathcal{K}$ has only simple poles, and

$$|\lambda|^{-1} \| \operatorname{Id} + \mathcal{R}(\lambda)\| = [\operatorname{dist}(\lambda, \mathfrak{Sp}(\mathcal{K}))]^{-1}$$

(see [29, Example 2 on p. 224]). So, if λ_0 is a pole of $\mathcal{R}(\lambda)$, then $\|\mathcal{R}(\lambda)\| \leq \frac{c}{|\lambda - \lambda_0|}$, $\lambda \to \lambda_0$, for some $c > 0$. Then, according to (4.21), $\Gamma(z; \lambda)$ also has a pole at λ_0 whose order can not exceed one. ∎

Let $\lambda_0 \in (-1, 1)$ be a simple pole of $\Gamma(z; \lambda)$ with the Laurent expansion near λ_0 given by

$$\Gamma(z; \lambda) = \frac{\Gamma^0(z)}{\lambda - \lambda_0} + \Gamma^1(z) + \mathcal{O}(\lambda - \lambda_0), \tag{5.1}$$

where the term $\mathcal{O}(\lambda - \lambda_0)$ is uniform near any point of analyticity (in z) of $\Gamma(z; \lambda)$. The representation (5.1) can be modified in a natural way so that it works near simple and double endpoints, see (4.10) and Proposition 4.5 respectively.

Proposition 5.2. *The matrix $\Gamma^0(z)$ in (5.1) can be written as follows*

$$\Gamma^0(z) = \begin{bmatrix} a \\ b \end{bmatrix} \Psi(z), \tag{5.2}$$

where $a, b \in \mathbb{C}$ are constants that are not both zero, and the vector

$$\Psi(z) := [\psi_1(z), \psi_2(z)]$$

has the jump condition and asymptotics given by

$$\Psi_+(z) = \Psi_-(z)\left(1 - \frac{2i}{\lambda_0}\sigma_+ \chi_E(z) + \frac{2i}{\lambda_0}\sigma_- \chi_J(z)\right), \quad \Psi(z) = \mathcal{O}(z^{-k}) \text{ as } z \to \infty \tag{5.3}$$

with some $k = 1, 2, \ldots$. Here

$$\sigma_+ := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma_- := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{5.4}$$

Proof. The jump conditions in (5.3) follow immediately from equations (5.2) and (5.1) and RHP 4.1. Also, note that the matrix $\Gamma^0(z)$ is analytic at $z = \infty$ and vanishes because

$$\Gamma(z; \lambda) = \mathbf{1} + \mathcal{O}(z^{-1}).$$

This implies that there is $k \in \mathbb{N}$ such that

$$\Gamma^0(z) = \mathcal{O}(z^{-k}),$$

and this implies also the same bound for Ψ in (5.3).

Thus, it remains to prove (5.2). In view of (5.1) and the relation $\Gamma^{-1}(z; \lambda) = \sigma_2 \Gamma^T(z; \lambda)\sigma_2$, we can expand formula (4.20) for the resolvent in a neighbourhood of $\lambda = \lambda_0$ as follows:

$$
\begin{aligned}
R(x, &y; \lambda) \\
&= \frac{1}{\lambda_0(\lambda - \lambda_0)^2} \frac{g(y)^T \sigma_2(\Gamma^0)^T(y)\sigma_2 \Gamma^0(x) f(x)}{(x - y)} \\
&+ \frac{1}{(\lambda - \lambda_0)} \Bigg(\frac{g(y)^T \left(\sigma_2(\Gamma^0)^T(y)\sigma_2 \Gamma^1(x) + \sigma_2(\Gamma^1)^T(y)\sigma_2 \Gamma^0(x)\right) f(x)}{\lambda_0(x - y)} \\
&\qquad\qquad - \frac{g(y)^T \sigma_2(\Gamma^0)^T(y)\sigma_2 \Gamma^0(x) f(x)}{\lambda_0^2(x - y)} \Bigg) + \mathcal{O}(1). \tag{5.5}
\end{aligned}
$$

If follows from Proposition 5.1 that the kernel R should have a first order pole in λ, so that the coefficient of the second–order pole in (5.5) must be identically (in x, y)

zero. Now, the numerator of the second order pole equals

$$[-\Gamma^0_{22}(y)\chi_J(y)-\Gamma^0_{21}(y)\chi_E(y), \ +\Gamma^0_{12}(y)\chi_J(y)+\Gamma^0_{11}(y)\chi_E(y)]$$
$$\times \begin{bmatrix} \Gamma^0_{11}(x)\chi_E(x) + \Gamma^0_{12}(x)\chi_J(x) \\ \Gamma^0_{21}(x)\chi_E(x) + \Gamma^0_{22}(x)\chi_J(x) \end{bmatrix}$$
$$= \left(-\Gamma^0_{22}(y)\Gamma^0_{11}(x)+\Gamma^0_{12}(y)\Gamma^0_{21}(x)\right)\chi_J(y)\chi_E(x)$$
$$+ \left(-\Gamma^0_{21}(y)\Gamma^0_{12}(x)+\Gamma^0_{11}(y)\Gamma^0_{22}(x)\right)\chi_E(y)\chi_J(x)$$
$$+ \left(-\Gamma^0_{22}(y)\Gamma^0_{12}(x)+\Gamma^0_{12}(y)\Gamma^0_{22}(x)\right)\chi_J(y)\chi_J(x)$$
$$+ \left(-\Gamma^0_{21}(y)\Gamma^0_{11}(x)+\Gamma^0_{11}(y)\Gamma^0_{21}(x)\right)\chi_E(y)\chi_E(x). \tag{5.6}$$

This expression is identically zero if and only if the two rows of Γ^0 are proportional by a *constant*. This is so because, for example, $\Gamma^0_{22}(y)\Gamma^0_{12}(x) - \Gamma^0_{12}(y)\Gamma^0_{22}(x) \equiv 0$ implies $\Gamma^0_{22}(y)/\Gamma^0_{12}(y) = \Gamma^0_{22}(x)/\Gamma^0_{12}(x)$ and both sides must be constants because they depend on different variables. ∎

Corollary 5.3. *The leading order term* $R_{\text{sing}}(x, y; \lambda) := \frac{R_{\lambda_0}(x,y)}{\lambda-\lambda_0}$ *of* $R(x, y; \lambda)$ *near a pole* $\lambda = \lambda_0$ *does not have a jump across* $x, y \in U$, *where the residue* $R_{\lambda_0}(x, y)$ *of the resolvent kernel* $R(x, y; \lambda)$ *at* λ_0 *is given by*

$$R_{\lambda_0}(x, y) := \frac{g(y)^T \left(\sigma_2(\Gamma^0)^T(y)\sigma_2\Gamma^1(x) + \sigma_2(\Gamma^1)^T(y)\sigma_2\Gamma^0(x)\right) f(x)}{\lambda_0(x - y)}. \tag{5.7}$$

Proof. Equation (5.7) follows directly from (5.5), where the $O((\lambda - \lambda_0)^{-2})$ term is zero, and (5.2). Indeed, from (5.2) it follows that $(\Gamma^0)^T(y)\sigma_2\Gamma^0(x) \equiv 0$ and hence the last term in the simple-pole term of (5.5) is zero. Also, since $R_{\text{sing}}(x, y; \lambda)$ is the singular part in λ of $R(x, y; \lambda)$, it follows from Remark 4.8 that $R_{\lambda_0}(x, y)$ has no jump across U. ∎

Lemma 5.4. *The kernel* $R_{\lambda_0}(x, y)$ *is degenerate.*

Proof. Substituting (5.1) into the RHP 4.1 we obtain the following jump conditions and the asymptotics for the Laurent coefficients $\Gamma^{0,1}(z)$:

$$\Gamma^0_+(z) = \Gamma^0_-(z)\left(1 - \frac{2i}{\lambda_0}\sigma_+\chi_E(z) + \frac{2i}{\lambda_0}\sigma_-\chi_J(z)\right), \tag{5.8}$$

$$\Gamma^0(z) = \mathcal{O}(z^{-1}), \quad z \to \infty, \tag{5.9}$$

$$\Gamma^1_+(z) = \Gamma^1_-(z)\left(1 - \frac{2i}{\lambda_0}\sigma_+\chi_E(z) + \frac{2i}{\lambda_0}\sigma_-\chi_J(z)\right)$$
$$+ \Gamma^0_-(z)\left(\frac{2i}{\lambda_0^2}\sigma_+\chi_E(z) - \frac{2i}{\lambda_0^2}\sigma_-\chi_J(z)\right), \tag{5.10}$$

$$\Gamma^1(z) = 1 + \mathcal{O}(z^{-1}), \tag{5.11}$$

where

$$\det \Gamma^0 \equiv 0, \quad \mathrm{Tr}(\Gamma^0 \sigma_2 (\Gamma^1)^T \sigma_2) \equiv 0. \tag{5.12}$$

The determinant and trace conditions follow from the property $\det \Gamma(z; \lambda) \equiv 1$ (see equation (4.8)).

Inserting (5.2) into the trace condition (5.12) gives

$$0 \equiv [b, -a] \Gamma^1(z) \begin{bmatrix} \psi_2(z) \\ -\psi_1(z) \end{bmatrix} \implies [b, -a] \Gamma^1(z) = h(z) \Psi(z) \tag{5.13}$$

for some scalar function $h(z)$ to be identified. Next, our goal is to show that $h(z)$ is a rational function. This is done in three steps.

• First, multiplying (5.10) on the left by $[b, -a]$ and noticing that $[b, -a] \Gamma^0(z) \equiv 0$, we obtain that $h_+(z) = h_-(z)$ for $z \in U$. This means that $h(z)$ extends analytically across U.

• Second, let $z_0 \in \mathbb{C}$ be any point other than an endpoint of J or E where *both* components of Ψ vanish. Then $\Gamma^0(z_0) = 0$ and so, according to (5.1), $\det \Gamma(z_0; \lambda) = \det \Gamma^1(z_0) + \mathcal{O}(\lambda - \lambda_0)$. If a zero z_0 of the vector Ψ has multiplicity μ, then $h(z)$ may have a pole of order at most μ since the left side of (5.13) is bounded.

• Zeroes of Ψ cannot accumulate at any $z_* \in \mathbb{C}$. Assuming the opposite, let z^* be a point where the zeroes of Ψ accumulate. If z_* is not an endpoint, then $\Gamma(z; \lambda)$ and, consequently, $\Gamma^0(z)$ are analytic at z_* (see (5.1)). By (5.2), such accumulation implies that $\Gamma^0(z) \equiv 0$, and λ_0 is not a pole. Suppose z_* is an endpoint, for example, z_* is a double endpoint. Since $z^{\rho(\lambda)\sigma_3} C_\pm(\rho(\lambda))$ is analytic in λ in a neighborhood of λ_0, then (4.16) implies that $Y(z; \lambda)$ has a pole at λ_0. Thus, we can repeat the same argument with the matrix function $Y(z; \lambda)$ (which is analytic near z_*) and its residue $Y^0(z) = \Gamma^0(z) z^{-\rho(\lambda_0)\sigma_3} C_\pm^{-1}(\rho(\lambda_0))$. In the case when z_* is a simple endpoint one can use (4.10) instead of (4.16).

• Finally, observe that by (5.11) the left-hand side of the second equation in (5.13) tends to $[b, -a]$ at $z = \infty$. Hence, we conclude that $h(z)$ has polynomial growth of degree not exceeding k (see (5.3)) and, therefore, according to Liouville's theorem, it is a rational function.

We also observe that (5.13) and (5.2) imply

$$\sigma_2 (\Gamma^0)^T(y) \sigma_2 \Gamma^1(x) = h(x) \begin{bmatrix} \psi_2(y) \\ -\psi_1(y) \end{bmatrix} [\psi_1(x), \psi_2(x)]. \tag{5.14}$$

Substituting (5.14) into (5.7) we obtain

$$R_{\lambda_0}(x, y) = \frac{g(y)^T \big(\sigma_2(\Gamma^0)^T(y)\sigma_2\Gamma^1(x) + \sigma_2(\Gamma^1)^T(y)\sigma_2\Gamma^0(x)\big) f(x)}{\lambda_0(x - y)}$$

$$= i\frac{h(x) - h(y)}{\lambda_0(x - y)} g(y)^T \sigma_2 \Psi^T(y)\Psi(x)f(x). \tag{5.15}$$

The expression $\frac{h(x)-h(y)}{\lambda_0(x-y)}$ is a finite linear combination of products of rational functions in x and y separately with at most as many terms as the degree of the scalar rational function $h(z)$. Thus, R_{λ_0} is a degenerate kernel. ∎

To obtain a more explicit expression for $R_{\lambda_0}(x, y)$ we simplify (5.15). Suppose $h(x) = S_t(x)/S_b(x)$, where S_t and S_b are some polynomials without common factors. Then

$$\frac{h(x) - h(y)}{x - y} = \frac{S_t(x)S_b(y) - S_t(y)S_b(x)}{(x - y)S_b(x)S_b(y)} = \frac{\sum_{m,n} b_{mn}x^m y^n}{S_b(x)S_b(y)}, \quad b_{mn} = b_{nm}, \tag{5.16}$$

where (b_{mn}) is the Bézout matrix of the polynomials $S_t(x)$, $S_b(x)$. Using equations (4.1) and (4.3), we get

$$g(y)^T \sigma_2 \Psi^T(y)\Psi(x)f(x)$$

$$= [\chi_J(y) \quad +\chi_E(y)]\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \psi_1(y)\psi_1(x) & \psi_1(y)\psi_2(x) \\ \psi_2(y)\psi_1(x) & \psi_2(y)\psi_2(x) \end{bmatrix}\begin{bmatrix} \chi_E(x) \\ \chi_J(x) \end{bmatrix}$$

$$= +i\big(\psi_1(x)\chi_E(x) + \psi_2(x)\chi_J(x)\big)\big(\psi_1(y)\chi_E(y) + \psi_2(y)\chi_J(y)\big). \tag{5.17}$$

Combining (5.15)–(5.17) gives

$$R_{\lambda_0}(x, y) = B(x, y)H(x)H(y), \tag{5.18a}$$

where

$$B(x, y) := -\lambda_0^{-1} \sum_{m,n} b_{mn}x^m y^n, \quad H(x) := \frac{\psi_1(x)\chi_E(x) + \psi_2(x)\chi_J(x)}{S_b(x)}. \tag{5.18b}$$

From (5.3), (5.4), and (5.13) it follows that $H(z)$ is analytic in a neighborhood of any $z \in \mathring{E} \cup \mathring{J}$.

Let $R_+(x, y;\lambda)$ and $R_-(x, y;\lambda)$ be the analytic continuations in λ of the kernel of the resolvent across the cut $[-1, 1]$ from above and from below, respectively. It follows from Lemma 4.14 that $R_\pm(x, y;\lambda)$ are meromorphic functions of λ as long as neither x nor y coincides with an endpoint of E and J. The locations of the poles, of course, are independent of the choice of x, y. Let λ_0 be a pole of, say, $R_+(x, y;\lambda)$. Then, by symmetry (see Remark 4.8), λ_0 is also a pole of $R_-(x, y;\lambda)$. In what follows, with a

slight abuse of notation, we denote by $R(x, y; \lambda)$ the kernel, which is the average of $R_+(x, y; \lambda)$ and $R_-(x, y; \lambda)$. The residue of $R(x, y; \lambda)$ at λ_0 is then

$$\operatorname*{res}_{\lambda=\lambda_0} R(x, y; \lambda) = \frac{1}{2}(\operatorname*{res}_{\lambda=\lambda_0} R_+(x, y; \lambda) + \operatorname*{res}_{\lambda=\lambda_0} R_-(x, y; \lambda)). \qquad (5.19)$$

Applying (5.18) to $R_+(x, y; \lambda)$ and $R_-(x, y; \lambda)$, combining the two residues, and using the symmetry of $R_\pm$ (see Remark 4.8), we get that the residue of R defined in (5.19) equals

$$R_{\lambda_0}(x, y) = \Re\left(B(x, y)H(x)H(y)\right). \qquad (5.20)$$

Clearly, $R_{\lambda_0}(x, y)$ in (5.20) is real-valued and satisfies $R_{\lambda_0}(x, y) = R_{\lambda_0}(y, x)$.

Theorem 5.5. *Let λ_0 be an eigenvalue of $\mathcal{K}$ imbedded in the continuous spectrum. Then the corresponding eigenspace has a finite dimension bounded by twice the degree of the rational function $h(z)$ from Proposition 5.1.*

Proof. First, we show that the residue of the resolvent defined according to (5.19) defines the projector in $L^2(U)$ onto the corresponding eigenspace. A similar statement in the case of an isolated eigenvalue is well known [29]. Here our situation is a bit more complex, since all the eigenvalues are imbedded in the continuous spectrum. Nevertheless, the proof is fairly straightforward. We could not find a reference in any of the well-known texts on operator theory, so we decide to give it here for completeness. As is known, the projector onto the eigenspace of $\mathcal{K}$ corresponding to λ_0 can be computed as follows:

$$\mathcal{P}_{\lambda_0} = \lim_{\delta \to 0} \widehat{E}_{(\lambda_0 - \delta, \lambda_0 + \delta)} = \frac{-1}{2\pi i \lambda_0} \lim_{\delta \to 0} \lim_{\varepsilon \to 0^+} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} [\mathcal{R}(t + i\varepsilon) - \mathcal{R}(t - i\varepsilon)] \, dt,$$

$$(5.21)$$

where all the limits are in the sense of strong operator convergence. Pick any two functions $\phi_{1,2} \in C_0^\infty(\mathring{E} \cup \mathring{J})$. Using that the kernels of $\mathcal{R}_\pm$ are analytic with respect to x, y away from the endpoints of J and E and is a meromorphic function of λ, it is immediate that

$$\lim_{\varepsilon \to 0^+} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} (\mathcal{R}(t \pm i\varepsilon)\phi_1, \phi_2) \, dt = \lim_{\varepsilon \to 0^+} \int_{\lambda_0 - \delta}^{\lambda_0 + \delta} (\mathcal{R}_\pm(t \pm i\varepsilon)\phi_1, \phi_2) \, dt = \int_{B_\pm(\delta)} \mathcal{R}_\pm(\lambda) \, d\lambda$$

$$= \pm \pi i \operatorname*{res}_{\lambda=\lambda_0} (\mathcal{R}_\pm(\lambda)\phi_1, \phi_2) + O(\delta), \qquad (5.22)$$

where $B_\pm(\delta)$ is the half-circle centered at λ_0 with radius δ in the upper and lower half-planes, respectively, oriented in the counter clockwise direction. Substituting (5.22)

into (5.21) we obtain

$$(\mathcal{P}_{\lambda_0}\phi_1, \phi_2) = \frac{-1}{2\pi i \lambda_0} \pi i \left((\operatorname*{res}_{\lambda=\lambda_0} \mathcal{R}_+(\lambda) + \operatorname*{res}_{\lambda=\lambda_0} \mathcal{R}_-(\lambda))\phi_1, \phi_2 \right)$$

$$= \frac{-1}{\lambda_0} (\operatorname*{res}_{\lambda=\lambda_0} \mathcal{R}(\lambda)\phi_1, \phi_2), \qquad (5.23)$$

where the last equality follows from the definition (5.19). Therefore, the operators on the left and on the right in (5.23) act the same way on $C_0^\infty(\mathring{E} \cup \mathring{J})$. Comparing (5.23) and (5.19) implies that the kernel of $\mathcal{P}_{\lambda_0}$ is the expression in (5.20).

Since $R_{\lambda_0}(x, y)$ is self-adjoint, real-valued (cf. (5.20)) and degenerate (cf. (5.18)), we can represent it in the form

$$R_{\lambda_0}(x, y) = \sum_{m,n=1}^{N} a_{mn} f_m(x) f_n(y) = \sum_{n=1}^{N'} \alpha_n g_n(x) g_n(y) \qquad (5.24)$$

for some real, symmetric matrix (a_{mn}) and real-valued functions

$$f_n(x) \in C^\infty(\mathring{E} \cup \mathring{J}).$$

The latter are are analytic on $\mathring{E} \cup \mathring{J}$. Here α_n are non-zero eigenvalues of (a_{mn}) (hence, $N' < N$), and g_n's are obtained by a unitary transformation from the f_n's. Without loss of generality we can assume that the set of functions $\{f_n\}$ is linearly independent in $C^\infty(\mathring{E} \cup \mathring{J})$. Then the set $\{g_n\}$ is linearly independent as well. To prove that $\mathcal{P}_{\lambda_0}$ is of finite rank we just need to show that $g_n \in L^2(U)$ for all n. Clearly, we can find $\phi \in C_0^\infty(\mathring{E} \cup \mathring{J})$, such that $\int_U g_n(x)\phi(x) \, dx \neq 0$ if $n = 1$ and equals zero for all other n. Since $\mathcal{P}_{\lambda_0}\phi \in L^2(U)$, it follows that $g_1 \in L^2(U)$. Repeating the same argument for all n implies the desired result.

Finally, from (5.16), (5.18), (5.20), and (5.24) it follows that the dimension of the eigenspace of $\mathcal{K}$ corresponding to λ_0 does not exceed twice the degree of the rational function h, i.e., $2 \max(\deg S_b, \deg S_t)$. ∎

5.2. Proof of Theorem 2.4

The first half of (2) is proven similarly to Theorem 2.1 (2). Alternatively, this statement can be proven by showing that if λ^2 is an eigenvalue of B, then λ (and $-\lambda$ if $\lambda \neq 0$) is an eigenvalue of $\mathcal{K}$, and then invoking Theorem 2.1.

By changing variables

$$\mathcal{K}^2 = \int \lambda^2 \, dE_\lambda = \int_\infty^0 t \, dE_{-\sqrt{t}} + \int_0^\infty t \, dE_{\sqrt{t}} = \int_0^\infty t \, d(E_{\sqrt{t}} - E_{-\sqrt{t}}), \qquad (5.25)$$

it follows that the resolution of the identity associated with $\mathcal{K}^2$ is given by

$$V_\lambda = \begin{cases} E_{[-\sqrt{\lambda},\sqrt{\lambda}]}, & \lambda > 0, \\ 0, & \lambda \leq 0. \end{cases} \tag{5.26}$$

Here $E_{[a,b]} := \text{s-lim}_{\delta\to 0+} E_{(a-\delta,b]}$, and E_λ is assumed to be strongly continuous from the right, see [14, Section X.1.1]. The above definition ensures that V_λ is strongly continuous from the right as well. Now, (1) and (5), the rest of (2), as well as the first half of (3), follow directly from (5.26) and the corresponding assertions of Theorem 2.1.

To prove the second half of (3) we show that $AA^\dagger$ and $A^\dagger A$ are unitarily equivalent. Indeed, let $A = V(A^\dagger A)^{1/2}$ be the polar decomposition of A. Here V is a partial isometry with $\text{Ran } V = \overline{\text{Ran } A}$, which is uniquely defined by the condition $\text{Ker } V = \text{Ker } A$ (see [14, Section VI.7]). Using that $\text{Ker } A^\dagger = (\text{Ran } A)^\perp$ (see [14, eq. (5.10) Chapter III]) and that both A and $(-1)A^\dagger$ are Hilbert transforms (i.e., densely defined with zero kernels), it follows that $\text{Ker } V = \text{Ker } A = 0$ and $\text{Ran } V = \overline{\text{Ran } A} = L^2(E)$. Hence, $V: L^2(J) \to L^2(E)$ is an isometry. Then $AA^\dagger = VA^\dagger AV^\dagger$, and the result follows. Therefore, in particular, the absolutely continuous spectra of $AA^\dagger$ and $A^\dagger A$ have the same multiplicity, and the latter equals to half of the multiplicity of $\mathfrak{Sp}_{\text{ac}}(\mathcal{K}^2)$.

Consider now (4). If there are no double endpoints, Theorem 2.1 implies that $\mathcal{K}^2$ and, therefore, B is of trace class. The last statement of (4) follows from the standard operator theory.

Remark 5.6. In the proof of Theorem 2.4 we showed that

$$\mathfrak{Sp}_{\text{ac}}(AA^\dagger) = \mathfrak{Sp}_{\text{ac}}(A^\dagger A) = [0, 1]$$

with the same multiplicity n. It is instructive to prove this assertion directly by following the arguments of Theorem 3.5. Consider first the simple case of $r = 1$ in (3.1), with both endpoints of $U = U_1$ belonging to $E = E_1$ (Lemma 3.4, first case). Using (3.7), we obtain

$$\mathcal{K}_0^2 = \mathcal{K}_{\text{ext}}^2 + \mathcal{S}^2 + \mathcal{K}_{\text{ext}}\mathcal{S} + \mathcal{S}\mathcal{K}_{\text{ext}}. \tag{5.27}$$

According to the proof of Lemma 3.4, $\mathcal{S}$ and, therefore, $\mathcal{S}^2 + \mathcal{K}_{\text{ext}}\mathcal{S} + \mathcal{S}\mathcal{K}_{\text{ext}}$, are trace class operators. Hence, the absolutely continuous parts of $\mathcal{K}_0^2$ and $\mathcal{K}_{\text{ext}}^2$ are unitarily equivalent. Since (a) $\mathcal{K}_0^2$ and $\mathcal{K}_{\text{ext}}^2$ are block diagonal relative to the decomposition $L^2(\mathbb{R}) = L^2(\hat{E}) \oplus L^2(J)$; (b) diagonal blocks of a block-diagonal trace class operator are also trace class, and; (c) each diagonal block of $\mathcal{K}_0^2$ has absolutely continuous spectrum $[0, 1]$ of multiplicity n (see [17]), we conclude that the absolutely continuous spectrum of each block of $\mathcal{K}_{\text{ext}}^2$ is the interval $[0, 1]$, and its multiplicity equals n. Restricting the blocks of $\mathcal{K}_{\text{ext}}^2$ to the blocks of $\mathcal{K}^2$ similarly to how this is done at the end of the proof of the first case in Lemma 3.4, we obtain the desired assertion.

In a similar fashion, we use (3.9) and (3.14) to prove the assertion in all the remaining cases. The key observation is that all the cross terms not containing trace class operators are zero when the right-hand sides of (3.9) and (3.14) are squared. Consider, for example, (3.9). Now, $\mathcal{K}_0^2$ is a trace class perturbation of $(\mathcal{K}_{\text{ext}} + \mathcal{K}_\infty)^2 = \mathcal{K}_{\text{ext}}^2 + \mathcal{K}_\infty^2$, because, by construction, $\mathcal{K}_{\text{ext}}\mathcal{K}_\infty = \mathcal{K}_\infty\mathcal{K}_{\text{ext}} = 0$. In (3.14) we get from (3.12)

$$\left(\bigoplus_{j=1}^r \mathcal{K}_j \right)^2 = \bigoplus_{j=1}^r \mathcal{K}_j^2, \tag{5.28}$$

and the rest of the argument is analogous.

6. Spectrum and diagonalization of the operator $\mathcal{K}$ when $U = \mathbb{R}$

In this section we extend an approach, which was originally developed in [17], see also [18]. One is given a collection of $2n$ points $b_j \in \mathbb{R}$, $1 \le j \le 2n$ (i.e., all b_j are double endpoints). We assume that they are arranged in ascending order: $b_j < b_{j+1}$, $1 \le j < 2n$. Define

$$J := [b_1, b_2] \cup [b_3, b_4] \cup \cdots \cup [b_{2n-1}, b_{2n}], \tag{6.1a}$$

$$E := [b_2, b_3] \cup [b_4, b_5] \cup \cdots \cup [b_{2n}, b_1], \tag{6.1b}$$

where

$$[b_{2n}, b_1] := (-\infty, b_1] \cup [b_{2n}, \infty). \tag{6.1c}$$

For clarity, the interval $[b_{2n}, b_1]$ is the closed interval (on the extended line) from b_{2n} to $+\infty$ and from $-\infty$ to b_1. In terms of (3.1) this means that $r = 1$ and $U_1 = U = \mathbb{R}$. We have assumed that the point at infinity belongs to E, but this does not affect the generality of the argument due to Lemma 3.1. Define

$$\beta_{\text{od}}(z) = \prod_{j=1}^n (z - b_{2j-1}), \quad \beta_{\text{ev}}(z) = \prod_{j=1}^n (z - b_{2j}), \quad \beta(z) = \beta_{\text{ev}}(z)/\beta_{\text{od}}(z),$$

$$\tag{6.2a}$$

$$\overset{\circ}{\phi}(z) = \ln\beta(z), \quad \phi(z) = \Re\overset{\circ}{\phi}(z), \tag{6.2b}$$

where we choose the standard branch of the logarithm.

The following facts are proven for $x \in J$ in [17], and the proofs for $x \in E$ are analogous.

(1) One has

$$\Im\overset{\circ}{\phi}(x) = \pi, \quad x \in J \quad \text{and} \quad \Im\overset{\circ}{\phi}(x)(x) = 0, \quad x \in E. \tag{6.3}$$

(2) The behavior of ϕ on the subintervals $(b_{2j-1}, b_{2j}) \subset J$ and $(b_{2j}, b_{2j+1}) \subset E$ satisfies

$$\phi'(x) < 0, x \in J, \quad \phi(x) \to +\infty, \quad x \to b_{2j-1}^+, \quad \phi(x) \to -\infty, \quad x \to b_{2j}^-,$$
(6.4a)

$$\phi'(x) > 0, x \in E, \quad \phi(x) \to -\infty, \quad x \to b_{2j}^+, \quad \phi(x) \to +\infty, \quad x \to b_{2j+1}^-.$$
(6.4b)

Therefore, $\phi(x)$ is monotonic and invertible on each subinterval, and the range of $\phi(x)$ on each subinterval is $\mathbb{R}$.

(3) One has

$$\phi'(x) = \frac{Q(x)}{\beta_{\text{od}}(x)\beta_{\text{ev}}(x)}, \quad Q(x) := \beta_{\text{ev}}'(x)\beta_{\text{od}}(x) - \beta_{\text{ev}}(x)\beta_{\text{od}}'(x),$$
(6.5)

and $Q(x) > 0$ is bounded away from zero on $\mathbb{R}$.

Suppose that

$$s = \phi(z)/2, \quad z \in (b_{2m}, b_{2m+1}) \subset E$$

and

$$t = \phi(x)/2, \quad x \in (b_{2k-1}, b_{2k}) \subset J.$$

Then

$$\begin{aligned}
\cosh(s - t) &= \cosh\left(\frac{\phi(z) - \phi(x)}{2}\right) \\
&= \cosh\left(\frac{\mathring{\phi}(z) - (\mathring{\phi}(x) - i\pi)}{2}\right) \\
&= i \sinh\left(\frac{\mathring{\phi}(z) - \mathring{\phi}(x)}{2}\right).
\end{aligned}$$
(6.6)

Moreover,

$$\begin{aligned}
2\sinh\left(\frac{\mathring{\phi}(z) - \mathring{\phi}(x)}{2}\right) &= \sqrt{\frac{\beta(z)}{\beta(x)}} - \sqrt{\frac{\beta(x)}{\beta(z)}} \\
&= \frac{\beta_{\text{ev}}(z)\beta_{\text{od}}(x) - \beta_{\text{ev}}(x)\beta_{\text{od}}(z)}{D} \\
&= \frac{(z - x)\sum_{i,j=1}^{n} B_{ij} z^{i-1} x^{j-1}}{D} \\
&= \frac{(z - x)\sum_{j=1}^{2n} \rho_j P_j(z) P_j(x)}{D},
\end{aligned}$$
(6.7)

where $B := B(\beta_{\text{ev}}, \beta_{\text{od}}) = (B_{ij})$ is the Bézout matrix of the polynomials $\beta_{\text{ev}}(z)$, $\beta_{\text{od}}(z)$, and

$$D := \beta_{\text{od}}(z)\sqrt{\beta(z)}\beta_{\text{od}}(x)\sqrt{\beta(x)}.$$
(6.8)

Using (6.3), we find

$$D = \text{sgn}(\beta_{\text{od}}(z))i \sqrt{\prod_{j=1}^{2n} |z - b_j|} \, \text{sgn}(\beta_{\text{od}}(x)) \sqrt{\prod_{j=1}^{2n} |x - b_j|}$$

$$= i \, \text{sgn}(\beta_{\text{od}}(z)) \, \text{sgn}(\beta_{\text{od}}(x)) \sqrt{\prod_{j=1}^{2n} |x - b_j||z - b_j|}. \tag{6.9}$$

Introduce two isometries

$$T_{\text{in}} : L^2(J) \to L^2_n(\mathbb{R}), \quad T_{\text{ex}} : L^2(E) \to L^2_n(\mathbb{R}), \tag{6.10a}$$

such that

$$\check{\mathbf{h}}_{\text{in}}(t) := (T_{\text{in}}h)(t)$$
$$:= \sqrt{2}\left(\frac{\text{sgn}(\beta_{\text{od}}(x))h(x)}{\sqrt{|\phi'(x)|}} \bigg|_{x=\phi_1^{-1}(2t)}, \ldots, \frac{\text{sgn}(\beta_{\text{od}}(x))h(x)}{\sqrt{|\phi'(x)|}} \bigg|_{x=\phi_{2n-1}^{-1}(2t)} \right), \tag{6.10b}$$

$$\check{\mathbf{h}}_{\text{ex}}(s) := (T_{\text{ex}}h)(s)$$
$$:= \sqrt{2}\left(\frac{\text{sgn}(\beta_{\text{od}}(z))h(z)}{\sqrt{\phi'(z)}} \bigg|_{z=\phi_2^{-1}(2s)}, \ldots, \frac{\text{sgn}(\beta_{\text{od}}(x))h(z)}{\sqrt{\phi'(z)}} \bigg|_{z=\phi_{2n}^{-1}(2s)} \right), \tag{6.10c}$$

where $L^2_n(\mathbb{R})$ is the direct sum of n copies of $L^2(\mathbb{R})$, $L^2_n(\mathbb{R}) = \bigoplus_{j=1}^{n} L^2(\mathbb{R})$. Here we set $\|\check{\mathbf{h}}\|^2 = \|\check{h}_1\|^2 + \cdots + \|\check{h}_n\|^2$, where $\check{\mathbf{h}} = (\check{h}_1, \ldots, \check{h}_n) \in L^2_n(\mathbb{R})$ and $\|\check{h}_m\|$ is the conventional $L^2(I_m)$ norm. Also, in (6.10), ϕ_k^{-1} is the inverse of $\phi(x)$ on the k-th interval (b_k, b_{k+1}). By convention, the $2n$-th interval is $\mathbb{R} \setminus (b_1, b_{2k})$, i.e., it includes the point at infinity.

Changing variables in the definition of A gives

$$(T_{\text{ex}}AT_{\text{in}}^{-1}\check{\mathbf{h}}_{\text{in}})_m(s)$$
$$= \frac{\text{sgn}(\beta_{\text{od}}(z_m))}{\pi} \sqrt{\frac{2}{\phi'(z_m)}} \sum_{k=1}^{n} \int_{\mathbb{R}} \frac{\text{sgn}(\beta_{\text{od}}(x_k))\check{h}_k(t)}{\sqrt{|\phi'(x_k)|/2}(x_k - z_m)} \, dt$$
$$= \frac{2\,\text{sgn}(\beta_{\text{od}}(z_m))}{\pi} \sum_{k=1}^{n} \int_{\mathbb{R}} \frac{\text{sgn}(\beta_{\text{od}}(x_k))\check{h}_k(t)}{\sqrt{|\phi'(x_k)|\phi'(z_m)}(x_k - z_m)} \, dt, \tag{6.11a}$$

where

$$x_k := \phi_k^{-1}(2t), \quad z_m := \phi_m^{-1}(2s). \tag{6.11b}$$

Combining (6.11), (6.5), (6.6), and (6.7), we find

$$\frac{\operatorname{sgn}(\beta_{\mathrm{od}}(x_k)) \operatorname{sgn}(\beta_{\mathrm{od}}(z_m))}{\sqrt{|\phi'(x_k)||\phi'(z_m)|} . x_k - z_m)} = -\frac{1}{2\cosh(s-t)} \sum_{j=1}^{n} \frac{\rho_j P_j(x_k) P_j(z_m)}{\sqrt{Q(x_k)Q(z_m)}}. \tag{6.12}$$

Define two matrix functions

$$\mathcal{M}_{\mathrm{in}} := \{M_{jk}^{(\mathrm{in})}(t)\}, \quad M_{jk}(t) := P_j(x_k)\sqrt{\frac{\rho_j}{Q(x_k)}}, \quad x_k := \phi_k^{-1}(2t), \tag{6.13a}$$

$$\mathcal{M}_{\mathrm{ex}} := \{M_{jm}^{(\mathrm{ex})}(s)\}, \quad M_{jm}(s) := P_j(z_m)\sqrt{\frac{\rho_j}{Q(z_m)}}, \quad z_m := \phi_m^{-1}(2s). \tag{6.13b}$$

It is shown in [17] that $\{M_{jk}^{(\mathrm{in})}(t)\}$ is an orthogonal matrix for all $t \in \mathbb{R}$. The proof that $\{M_{jk}^{(\mathrm{in})}(s)\}$, $s \in \mathbb{R}$, is an orthogonal matrix is analogous. Substituting (6.12) and (6.13) into (6.11) gives

$$(T_{\mathrm{ex}} A T_{\mathrm{in}}^{-1} \check{\mathbf{h}}_{\mathrm{in}})_m(s) = -\sum_{j=1}^{n} M_{jm}^{(\mathrm{ex})}(s) \sum_{k=1}^{n} \int_{\mathbb{R}} \frac{M_{jk}^{(\mathrm{in})}(t) \check{h}_k(t)}{\pi \cosh(s-t)} \, dt. \tag{6.14}$$

In compact form, (6.14) can be written as follows

$$T_{\mathrm{ex}} A T_{\mathrm{in}}^{-1} \check{\mathbf{h}}_{\mathrm{in}} = \mathcal{M}_{\mathrm{ex}}^T K \mathcal{M}_{\mathrm{in}} \check{\mathbf{h}}_{\mathrm{in}}, \tag{6.15}$$

where K is the operator of component-wise convolution with $-1/(\pi \cosh(t))$.

Equation (6.14) matches with the results in [17] in the case $n = 1$ (see [17, (2.12)]). Indeed, suppose $J = [-b, b]$, and $E = (-\infty, b] \cup [b, \infty)$. Then (6.5) and (6.7) imply that $\rho_1 = 2b$ and $Q(x) \equiv 2b$, i.e., $\mathcal{M}_{\mathrm{ex}} \equiv \mathcal{M}_{\mathrm{in}} \equiv 1$ in (6.13). Observe also that there is one sign change between (6.14) and [17, (2.12)]. The sign change arises because T_{ex} in (6.10) is the negative of T_{ex} in [17, (2.11)].

Let $\mathcal{F} \colon L_n^2(\mathbb{R}) \to L_n^2(\mathbb{R})$ denote the map consisting of n component-wise one-dimensional Fourier transforms (cf. (1.3)). Using (6.14) and [27, integral 2.5.46.5], we get

$$K = \mathcal{F}^{-1}\left(\frac{-1}{\cosh(\pi\lambda/2)} \operatorname{Id}_n\right)\mathcal{F}, \tag{6.16}$$

where λ is the spectral (Fourier) variable, and Id_n is the $n \times n$ identity matrix. Therefore, (6.14) gives

$$Af = (\mathcal{F}\mathcal{M}_{\mathrm{ex}}T_{\mathrm{ex}})^{-1}\left(\frac{-1}{\cosh(\pi\lambda/2)} \operatorname{Id}_n\right)(\mathcal{F}\mathcal{M}_{\mathrm{in}}T_{\mathrm{in}})f. \tag{6.17}$$

Applying the adjoint to (6.17), we get that $\mathcal{K}$ satisfies

$$\mathcal{K} = \begin{bmatrix} 0 & A \\ A^\dagger & 0 \end{bmatrix} = U^{-1}\begin{bmatrix} 0 & \frac{-1}{\cosh(\pi\lambda/2)} \operatorname{Id}_n \\ \frac{-1}{\cosh(\pi\lambda/2)} \operatorname{Id}_n & 0 \end{bmatrix}U, \tag{6.18a}$$

where

$$U := \begin{bmatrix} \mathcal{F} \, \mathcal{M}_{\text{ex}} T_{\text{ex}} & 0 \\ 0 & \mathcal{F} \, \mathcal{M}_{\text{in}} T_{\text{in}} \end{bmatrix} : L^2_{2n}(\mathbb{R}) \to L^2_{2n}(\mathbb{R}), \qquad (6.18b)$$

is an isometry. As is easily checked, the following self-adjoint isometry diagonalizes the middle operator on the right in (6.18)

$$V := \frac{1}{\sqrt{2}} \begin{bmatrix} \text{Id}_n & \text{Id}_n \\ \text{Id}_n & -\text{Id}_n \end{bmatrix} : L^2_{2n}(\mathbb{R}) \to L^2_{2n}(\mathbb{R}), \qquad (6.19)$$

therefore

$$\mathcal{K} = U^{-1} V^{-1} \begin{bmatrix} -\frac{1}{\cosh(\pi\lambda/2)} \, \text{Id}_n & 0 \\ 0 & \frac{1}{\cosh(\pi\lambda/2)} \, \text{Id}_n \end{bmatrix} V U. \qquad (6.20)$$

The range of the function $(\cosh(\pi\lambda/2))^{-1}$ is $(0, 1]$, and each value is taken twice. Hence, we proved the following result.

Theorem 6.1. *Suppose $r = 1$ in (3.1), and $J_1 \cup E_1 = U_1 = \mathbb{R}$, i.e., U consists of only one interval and coincides with all of $\mathbb{R}$. In this case the spectral interval of $\mathcal{K}$ is $[-1, 1]$, the spectrum is absolutely continuous (i.e., there are no eigenvalues), and its multiplicity equals to the number of double endpoints (which is twice the number of subintervals in J_1 or E_1).*

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