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Invariant subspaces of elliptic systems II: Spectral theory

Matteo Capoferri and Dmitri Vassiliev

Abstract. Consider an elliptic self-adjoint pseudodifferential operator A acting on m-columns of half-densities on a closed manifold M, whose principal symbol is assumed to have simple eigenvalues. We show that the spectrum of A decomposes, up to an error with superpolynomial decay, into m distinct series, each associated with one of the eigenvalues of the principal symbol of A. These spectral results are then applied to the study of propagation of singularities in hyperbolic systems. The key technical ingredient is the use of the carefully devised pseudodifferential projections introduced in the first part of this work, which decompose $L^2(M)$ into almost-orthogonal almost-invariant subspaces under the action of both A and the hyperbolic evolution.

In memory of Misha Shubin,
whose support was invaluable to the second author at the beginning of his career

1. Statement of the problem

In this paper we continue the analysis of invariant subspaces of elliptic systems initiated in [12], focussing on the spectral-theoretic aspects of the problem.

Let M be a connected closed manifold of dimension $d \geq 2$. (When $d = 1$, the punctured cotangent bundle $T^*M \setminus \{0\}$ is not connected. Although this is not a fundamental obstacle, we assume the dimension of the manifold to be at least 2 to avoid repeated discussions on the difference between $d = 1$ and $d \geq 2$.) We denote by $(x^1, \dots, x^d)$ local coordinates on M.

As in [12], we denote by $C^\infty(M)$ the linear space of m-columns of smooth complex-valued half-densities over M and by $L^2(M)$ its closure with respect to the inner product

$$\langle v, w\rangle := \int_M v^* w \, dx,$$

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where $dx := dx^1 \dots dx^d$. Accordingly, we denote by $H^s(M)$, $s \in \mathbb{R}$, the corresponding Sobolev spaces. Here and further on, $*$ stands for Hermitian conjugation when applied to matrices and for adjunction when applied to operators.

Let Ψ^s be the space of classical pseudodifferential operators of order s acting from $H^s(M)$ to $L^2(M)$. For an operator $B \in \Psi^s$, we denote by B_{prin} and by B_{sub} its principal and subprincipal symbols, respectively.

Let $A \in \Psi^s$, $s \in \mathbb{R}$, $s > 0$, be an elliptic self-adjoint linear operator, where ellipticity means that

$$\det A_{\mathrm{prin}}(x,\xi) \neq 0 \quad \text{for all } (x,\xi) \in T^*M \setminus \{0\}.$$

Throughout this paper, we assume that the eigenvalues of A_{prin} are simple. We denote by m^+ (resp. m^-) the number of positive (resp. negative) eigenvalues of $A_{\mathrm{prin}}(x,\xi)$. We denote by $h^{(j)}(x,\xi)$ the eigenvalues of $A_{\mathrm{prin}}(x,\xi)$ and by $P^{(j)}(x,\xi)$ the corresponding eigenprojections. Eigenvalues are enumerated in increasing order, with positive index $j = 1, 2, \dots, m^+$ for positive $h^{(j)}(x,\xi)$ and negative index $j = -1, -2, \dots, -m^-$ for negative $h^{(j)}(x,\xi)$.

The spectrum of our operator $A \colon H^s(M) \to L^2(M)$ is discrete and accumulates to infinity. More precisely, if $m^+ \geq 1$, the spectrum accumulates to $+\infty$; if $m^- \geq 1$, the spectrum accumulates to $-\infty$; and if $m^+ \geq 1$ and $m^- \geq 1$, the spectrum accumulates to $\pm\infty$.

Let us recall a few results from [12] which will be useful later on.

Definition 1.1. We say that a symmetric pseudodifferential operator B is *nonnegative (resp. nonpositive) modulo* $\Psi^{-\infty}$ and write

$$B \geq 0 \mod \Psi^{-\infty} \quad (\text{resp. } B \leq 0 \mod \Psi^{-\infty})$$

if there exists a symmetric operator $C \in \Psi^{-\infty}$ such that $B + C \geq 0$ (resp. $B + C \leq 0$).

Theorem 1.2. *Let A be as above and let δ be the Kronecker symbol.*

(a) [12, Theorem 2.2] *There exist m pseudodifferential operators $P_j \in \Psi^0$ satisfying*

(i) $(P_j)_{\mathrm{prin}} = P^{(j)}$,

(ii) $P_j = P_j^* \mod \Psi^{-\infty}$,

(iii) $P_j P_l = \delta_{jl} P_j \mod \Psi^{-\infty}$,

(iv) $\sum_j P_j = \mathrm{Id} \mod \Psi^{-\infty}$,

(v) $[A, P_j] = 0 \mod \Psi^{-\infty}$.

These operators are uniquely determined, modulo $\Psi^{-\infty}$, by A.

(b) [12, Theorem 2.5] *We have*

$$P_j^* A P_j \geq 0 \mod \Psi^{-\infty} \quad \textit{for } j = 1, \dots, m^+,$$
$$P_j^* A P_j \leq 0 \mod \Psi^{-\infty} \quad \textit{for } j = -1, \dots, -m^-.$$

Theorem 1.2 tells us that, given an elliptic self-adjoint operator $A \in \Psi^s$, one can construct a unique orthonormal basis of pseudodifferential projections commuting with A. These projections partition $L^2(M)$ into m invariant subspaces under the action of A, modulo $C^\infty(M)$. Furthermore, they allow one to decompose A into precisely m (non-elliptic) sign definite operators $P_j^* A P_j \in \Psi^s$.

In the light of the above results, one would be led to think that the decomposition

$$A = \sum_j P_j^* A P_j \mod \Psi^{-\infty}$$

could be used, somehow, to obtain a similar decomposition at the level of the spectrum of A, at least in the limit $|\lambda| \to +\infty$. It is well known that, asymptotically, positive eigenvalues of A_{prin} account for the positive spectrum of A and negative eigenvalues of A_{prin} for the negative spectrum of A. One's hope would be to use pseudodifferential projections to achieve a finer partition of the spectrum of A into m distinct families, singling out the contribution of each individual eigenvalue of A_{prin}.

Now, the naive approach of looking at the spectra of the operators $P_j^* A P_j$ does not look very promising, in that the latter are *not* elliptic, hence the standard spectral-theoretic and asymptotic techniques cannot be applied. If one is to succeed in achieving the above spectral decomposition without abandoning completely the realm of elliptic operators, a more clever strategy is needed.

Remark 1.3. There are at least two other rather natural approaches to the problem at hand.

(1) The first approach would involve working with negative order operators. For fixed λ in the resolvent set of A, one can consider the negative order operators

$$R_j := P_j^* (A - \lambda \, \mathrm{Id})^{-1} P_j, \quad j = -m^-, \dots, -1, 1, \dots, m^+,$$

and study the asymptotics of their counting functions as the spectral parameter tends to zero, in the spirit of [3, 4]. We decided not to pursue this approach, which presents nontrivial technical obstacles, but to develop a novel strategy instead. The latter will have the advantage of allowing us to obtain our results without the need to work with non-elliptic operators.

(2) The second approach would involve microlocally diagonalizing the operator A, i.e. constructing an almost-unitary operator B such that $B^* A B$ is a diagonal matrix operator, modulo $\Psi^{-\infty}$; see, for example, [7, 15, 23, 24, 26, 33]. In the context of Dirac

operators, such a B is sometimes referred to as the *Foldy–Wouthuysen transform.* In constructing the operator B one encounters the issue that the almost-unitary operator B is not defined uniquely, not even at the level of the principal symbol, but only up to gauge transformations (see also [13, Section 5]). Neglecting to account for these gauge transformations and the curvatures that they bring about has led to mistakes in some publications, see [13, Section 11]. Furthermore, there may be topological obstructions to the existence of a global diagonalization: the issue here is that it is not always possible to adjust the gauge so that the eigenvectors of the principal symbol become (globally defined) smooth m-columns on $T^*M \setminus \{0\}$. Finally, the diagonalization procedure affects the asymptotics of the local counting function, in that it involves conjugation by a pseudodifferential operator. This makes diagonalization, in a sense, a less natural approach when it comes to studying the spectral properties of A. We refer the reader to [8] for a detailed analysis of the almost-unitary operator B. The use of pseudodifferential projections has the advantage of circumventing these issues altogether.

All in all, the results from Theorem 1.2 warrant the following natural questions.

Question 1. *Can we exploit the pseudodifferential projections P_j to achieve a partition of the spectrum of A into m disjoint families of eigenvalues?*

Question 2. *Can we exploit the pseudodifferential projections P_j to advance the current understanding of spectral asymptotics for elliptic systems?*

Question 3. *Can we exploit the pseudodifferential projections P_j to advance the current understanding of propagation of singularities for hyperbolic systems?*

The goal of this paper is to provide a rigorous affirmative answer to Questions 1, 2, and 3.

2. Main results

Our main results can be summarised in the form of five theorems stated in this section.

We will assume that $m^+ \geq 1$ and will be dealing with the asymptotics of the positive eigenvalues of A. The case of negative eigenvalues can be handled by replacing A with $-A$.

Let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty \tag{2.1}$$

be the positive eigenvalues of A enumerated in increasing order with account of multiplicity. The task at hand is to partition the eigenvalues (2.1) into m^+ separate series corresponding to the m^+ different positive eigenvalues of A_{prin}. Hence, we will assume that $m^+ \geq 2$.

In order to partition eigenvalues (2.1) into series we introduce the operators

$$A_j := A - 2\sum_{\substack{l=1,\dots,m^+ \\ l\neq j}} P_l^* A P_l, \quad j = 1, \dots, m^+. \tag{2.2}$$

Each operator A_j is "simpler" than our original operator A in that the principal symbol of A_j

$$(A_j)_{\mathrm{prin}} = h^{(j)} P^{(j)} - \sum_{l\neq j} |h^{(l)}| P^{(l)} \tag{2.3}$$

has only one positive eigenvalue, namely, $h^{(j)}(x, \xi)$. Note also that formula (2.2) implies

$$[A_j, A_l] = 0 \mod \Psi^{-\infty}, \quad j, l = 1, \dots, m^+. \tag{2.4}$$

Let θ be the Heaviside function. For a self-adjoint operator B we denote by

$$B^+ := B\theta(B) = \frac{1}{2}(B + |B|) \tag{2.5}$$

its nonnegative part. Then, [12, Theorem 2.7] implies

$$A^+ = \sum_{j=1}^{m^+} A_j^+ \mod \Psi^{-\infty}, \tag{2.6}$$

$$A_j^+ A_l^+ = 0 \mod \Psi^{-\infty}, \quad j, l = 1, \dots, m^+, \quad j \neq l. \tag{2.7}$$

Examination of formulae (2.4), (2.6) and (2.7) suggests that there should be a relation between the positive spectra of A and A_j, see Appendix A, and Remark A.9 therein in particular, for a compelling argument to this effect. The precise nature of this relation is established by the three theorems given below.

For a self-adjoint operator B we denote its spectrum by $\sigma(B)$, and we denote by

$$\sigma^+(B) := \sigma(B) \cap (0, +\infty) = \sigma(B^+) \setminus \{0\} \tag{2.8}$$

its positive part. Let

$$0 < \lambda_1^{(j)} \leq \lambda_2^{(j)} \leq \dots \leq \lambda_k^{(j)} \leq \dots \to +\infty \tag{2.9}$$

be the positive eigenvalues of A_j enumerated in increasing order with account of multiplicity. The following two theorems show that the positive eigenvalues of the operators A_j, $j = 1, \dots, m^+$, approximate the positive eigenvalues of the operator A, and vice versa.

Theorem 2.1. *For each $j = 1, \dots, m^+$ we have*

$$\operatorname{dist}(\lambda_k^{(j)}, \sigma^+(A)) = O(k^{-\infty}) \quad \textit{as} \quad k \to +\infty. \tag{2.10}$$

Theorem 2.2. *We have*

$$\operatorname{dist}\Big(\lambda_k, \bigcup_{j=1}^{m^+} \sigma^+(A_j)\Big) = O(k^{-\infty}) \quad \textit{as} \quad k \to +\infty. \tag{2.11}$$

Theorems 2.1 and 2.2 do not quite achieve the sought after partition of the spectrum (2.1) in that they do not establish a one-to-one correspondence between the positive eigenvalues of the operator A and the positive eigenvalues of the operators A_j, $j = 1, \dots, m^+$. The issue here is that formulae (2.10) and (2.11) establish asymptotic closeness of the spectra but do not provide sufficient information on the closeness of individual eigenvalues enumerated in our particular way. The following theorem addresses this issue and shows that the above construction is indeed "precise".

Let us combine the sequences (2.9), $j = 1, \dots, m^+$, into one sequence and denote it by

$$0 < \mu_1 \le \mu_2 \le \dots \le \mu_k \le \dots \to +\infty. \tag{2.12}$$

Here we combine them with account of multiplicities.

Theorem 2.3. *For any $\alpha > 0$ there exists an $r_\alpha \in \mathbb{Z}$ such that*

$$\lambda_k = \mu_{k+r_\alpha} + O(k^{-\alpha}) \quad \textit{as} \quad k \to +\infty. \tag{2.13}$$

Theorem 2.3 will allow us to derive two-term asymptotic formulae for the eigenvalue counting function of A refining previous results [1, 2, 13], see Section 5. Of course, Theorems 2.1 and 2.2 follow from Theorem 2.3, but we listed them as separate results for the sake of logical clarity.

Our last major result is an application of the above technology to first order hyperbolic systems. Let A be first order, $s = 1$, without any restrictions on m^+ and m^-. Consider the associated hyperbolic initial value problem

$$\Big(-i\frac{\partial}{\partial t} + A\Big)v = 0, \quad v|_{t=0} = v_0. \tag{2.14}$$

We call *propagator* the solution operator of (2.14), namely, the time-dependent unitary operator

$$U(t) := e^{-itA}. \tag{2.15}$$

It was shown in [11, 13] that $U(t)$ can be approximated, modulo $C^\infty(\mathbb{R}; \Psi^{-\infty})$ (i.e. modulo an integral operator with infinitely smooth time-dependent integral kernel), by the sum of precisely m invariantly defined oscillatory integrals $U^{(j)}(t)$ global in space and in time. Each oscillatory integral $U^{(j)}(t)$ is a Fourier integral operator whose Schwartz kernel is a Lagrangian distribution associated with the Lagrangian submanifold of $T^*\mathbb{R} \times T^*M \times T^*M$ generated by the Hamiltonian flow of $h^{(j)}$. These are m distinct smooth manifolds which encode information on the propagation of singularities in the hyperbolic system (2.14).

Theorem 2.4. *Let $A \in \Psi^1$ be an elliptic self-adjoint first order $m \times m$ operator. Suppose that the eigenvalues of its principal symbol are simple. Then*

$$U^{(j)}(t) = P_j U(t) = U(t) P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}),$$
$$j \in \{-m^-, \dots, -1, 1, \dots, m^+\}. \tag{2.16}$$

In fact, we will prove a stronger result, see Corollary 4.3.

Another important special case is that of nonnegative second order operators. For example, the operator of linear elasticity (Lamé operator) falls into this category, see [12, Section 8.2] for details. For such operators, we have $m = m^+$ and the propagator is defined as

$$U(t) := e^{-it\sqrt{A}}. \tag{2.17}$$

By means of a suitable modification of techniques from [11, 13], it will be shown in Section 4 that in this case the propagator can also be approximated, modulo $C^\infty(\mathbb{R}; \Psi^{-\infty})$, by the sum of precisely m invariantly defined oscillatory integrals $U^{(j)}(t)$ global in space and in time. This leads to the following analogue of Theorem 2.4.

Theorem 2.5. *Let $A \in \Psi^2$ be a nonnegative elliptic self-adjoint second order $m \times m$ operator. Suppose that the eigenvalues of its principal symbol are simple. Then*

$$U^{(j)}(t) = P_j U(t) = U(t) P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}), \quad j \in \{1, \dots, m\}. \tag{2.18}$$

Remark 2.6. Theorem 2.5 admits a further generalisation to the case when A is a nonnegative operator of positive even order $2n$. In this case, the propagator is defined as

$$U(t) = e^{-itA^{1/2n}}, \tag{2.19}$$

compare with (2.17). Proving Theorem (2.5) in this more general case does not present any additional difficulties: one can retrace the arguments given in Section 4.2 replacing $\sqrt{A}$ with $A^{1/2n}$, as appropriate. In particular, as explained in Section 4.2, one does not need to actually compute $A^{1/2n}$ in order to construct the operator (2.19). The reason why we state our main result for second order operators is twofold. On the one hand, it simplifies the presentation; on the other hand, the case $n = 1$ has a clearer physical meaning.

Note that Theorems 2.4 and 2.5 cannot be obtained by elementary functional-analytic arguments involving an expansion over eigenvalues and eigenfunctions of the operator A. Formulae (2.16) and (2.18) are to do with the propagation of singularities, a phenomenon which is not detected by the Spectral Theorem.

The paper is structured as follows.

Section 3 is the core of our paper: it contains the proofs of Theorems 2.1–2.3. In Section 3.1 we show that the positive spectrum of A is approximated by the union of the spectra of the A_j, $j = 1, \dots, m^+$, and *vice versa*, up to an error of order $O(\lambda^{-\infty})$. In Section 3.2 we demonstrate that our construction is asymptotically precise, namely, that when performing the above approximation no eigenvalue is missed. A key ingredient is a carefully devised partition of the positive semi-axis, provided in Section 3.2.1.

Section 4 is concerned with the analysis of hyperbolic systems. In Section 4.1 we focus on first order systems: after briefly recalling the construction of the propagator (wave group), we analyse the relation between the representation of the latter in terms of oscillatory integrals and our pseudodifferential projections, thus proving Theorem 2.4. In Section 4.2 we perform a similar analysis for nonnegative second order operators, proving Theorem 2.5.

Finally, in Section 5 we show how results from Section 3, Section 4 and the first part of this work [12] can be used to refine our understanding of asymptotic distribution of eigenvalues for first order systems.

The paper is complemented by two appendices.

3. Spectral analysis: partitioning the spectrum

3.1. Separating positive eigenvalues into m^+ distinct series

The goal of this section is to show that the positive spectrum of the operators A_j defined by (2.2) and the positive spectrum of A are mutually close, so as to prove Theorems 2.1 and 2.2.

Further on in this section, all estimates are to be understood as asymptotic estimates as $k \to +\infty$, unless otherwise specified.

Proof of Theorem 2.1. Let $u_k^{(j)}$ be a normalised eigenfunction of A_j corresponding to the eigenvalue $\lambda_k^{(j)}$, i.e.

$$A_j u_k^{(j)} = \lambda_k^{(j)} u_k^{(j)}, \tag{3.1}$$

$$\|u_k^{(j)}\|_{L^2} = 1. \tag{3.2}$$

For every $S \in \Psi^{-\infty}$, in view of the identity

$$S u_k^{(j)} = (\lambda_k^{(j)})^{-n} S (A_j)^n u_k^{(j)}, \quad n = 1, 2, \ldots,$$

and Weyl's law

$$\lambda_k^{(j)} = \Big(\frac{1}{(2\pi)^d} \int\limits_{h^{(j)}(x,\xi)<1} \mathrm{dVol}_{T^*M} \Big)^{-s/d} k^{s/d} + o(k^{s/d}), \tag{3.3}$$

see Theorem B.1, we have

$$S u_k^{(j)} = O(k^{-\infty}). \tag{3.4}$$

The above asymptotic estimate (as well as similar estimates in subsequent formulae) is understood in the strongest possible sense: any given partial derivative is estimated by any given negative power of k uniformly over M.

We claim that

$$\| P_l u_k^{(j)} \|_{L^2} = O(k^{-\infty}) \quad \text{for } l \neq j. \tag{3.5}$$

Indeed, taking into account (2.2) and using (3.4), for $l \neq j$ we have

$$-\operatorname{sgn}(h^{(l)}) P_l^* A P_l u_k^{(j)} = \lambda_k^{(j)} P_l u_k^{(j)} + O(k^{-\infty}),$$

which implies

$$-\operatorname{sgn}(h^{(l)}) \langle P_l u_k^{(j)}, P_l^* A P_l u_k^{(j)} \rangle = \lambda_k^{(j)} \| P_l u_k^{(j)} \|_{L^2}^2 + O(k^{-\infty}). \tag{3.6}$$

Combining (3.6) with Theorem 1.2(b) and using once again (3.4), we obtain

$$\lambda_k^{(j)} \| P_l u_k^{(j)} \|_{L^2}^2 \le O(k^{-\infty}),$$

which is equivalent to (3.5).

Formulae (3.1)–(3.4) imply

$$A_j P_l u_k^{(j)} = \lambda_k^{(j)} P_l u_k^{(j)} + O(k^{-\infty}),$$

which combined with (3.3) and (3.5) yields

$$\| A_j P_l u_k^{(j)} \|_{L^2} = O(k^{-\infty}) \quad \text{for } l \neq j. \tag{3.7}$$

By elliptic regularity formulae (3.5) and (3.7) give us

$$P_l u_k^{(j)} = O(k^{-\infty}) \quad \text{for } l \neq j. \tag{3.8}$$

Now, (2.2) and (3.1) imply

$$A u_k^{(j)} = \lambda_k^{(j)} u_k^{(j)} + 2 \sum_{\substack{l=1,\ldots,m^+ \\ l \neq j}} P_l^* A P_l u_k^{(j)},$$

which, on account of (3.8), can be recast as

$$Au_k^{(j)} = \lambda_k^{(j)} u_k^{(j)} + O(k^{-\infty}). \tag{3.9}$$

Formulae (3.2) and (3.9) yield (2.10). ∎

Proof of Theorem 2.2. Let $u_k \in L^2(M)$ be a normalised eigenfunction of A corresponding to the eigenvalue $\lambda_k > 0$, i.e.

$$Au_k = \lambda_k u_k, \tag{3.10}$$

$$\|u_k\|_{L^2} = 1. \tag{3.11}$$

The task at hand is to show that there exists a $j \in \{1, \dots, m^+\}$ such that

$$A_j v^{(j)} = \lambda_k v^{(j)} + O(k^{-\infty}) \tag{3.12}$$

for some smooth $v^{(j)}$ with $\|v^{(j)}\|_{L^2} = 1$. Indeed, formula (3.12) and the fact that $\|v^{(j)}\|_{L^2} = 1$ imply (2.11).

Arguing as in the proof of Theorem 2.1, one can show that for every $S \in \Psi^{-\infty}$ we have

$$Su_k = O(k^{-\infty}), \tag{3.13}$$

where the asymptotic estimate (as well as similar estimates in subsequent formulae) is understood in the strongest possible sense: any given partial derivative is estimated by any given negative power of k uniformly over M.

We claim that

$$P_l u_k = O(k^{-\infty}) \quad \text{for every } l \in \{-1, \dots, -m^-\}. \tag{3.14}$$

Indeed, formula (3.10), Theorem 1.2(a), and formula (3.13) imply

$$P_l^* A P_l u_k = \lambda_k P_l u_k + O(k^{-\infty}) \tag{3.15}$$

for every l, which, in turn, implies

$$\langle u_k, P_l^* A P_l u_k \rangle = \lambda_k \|P_l u_k\|_{L^2}^2 + O(k^{-\infty}). \tag{3.16}$$

Combining (3.16) with Theorem 1.2(b) and using once again (3.13) we obtain, for $l < 0$,

$$\lambda_k \|P_l u_k\|_{L^2}^2 \le O(k^{-\infty})$$

and hence

$$\|P_l u_k\|_{L^2} = O(k^{-\infty}). \tag{3.17}$$

By elliptic regularity, (3.15), (3.17), and (3.11) give us (3.14) (recall that P_l and A commute modulo $\Psi^{-\infty}$).

Now, in view of properties of differential projections, (3.13) and (3.14), we have

$$u_k = \sum_{l=1}^{m^+} P_l u_k + O(k^{-\infty}). \tag{3.18}$$

Formulae (3.18) and (3.11) imply that

$$\|P_j u_k\|_{L^2} \geq \frac{1}{m^+ + 1} \tag{3.19}$$

for some $j \in \{1, \dots, m^+\}$. By direct inspection we have

$$A_j P_j u_k = \lambda_k P_j u_k + O(k^{-\infty}), \tag{3.20}$$

see (2.2), (3.10), and (3.13).

Formulae (3.19) and (3.20) give us (3.12) with $v^{(j)} = P_j u_k / \|P_j u_k\|_{L^2}$. ■

We summarise below in the form of a proposition some of the results obtained along the way in the above proofs, as they will be useful later on.

Proposition 3.1. (a) *Let u_k be a normalised eigenfunction of A corresponding to the eigenvalue λ_k. Then*

$$P_l u_k = O(k^{-\infty}) \quad \textit{for every } l \in \{-1, \dots, -m^-\}. \tag{3.21}$$

(b) *Let $u_k^{(j)}$ be a normalised eigenfunction of A_j corresponding to the eigenvalue $\lambda_k^{(j)}$. Then*

$$P_l u_k^{(j)} = O(k^{-\infty}) \quad \textit{for every } l \neq j. \tag{3.22}$$

(c) *Under the same assumptions of part* (b), *we have*

$$A u_k^{(j)} = \lambda_k^{(j)} u_k^{(j)} + O(k^{-\infty}). \tag{3.23}$$

3.2. Spectral completeness

The goal of this section is to prove Theorem 2.3. The first steps in this direction were Theorems 2.1 and 2.2 which we proved in Section 3.1. The missing ingredient is taking account of the enumeration of eigenvalues, i.e. showing that none were missed when approximating the positive spectrum of A by the positive spectra of the A_j, $j = 1, \dots, m^+$.

Remark 3.2. Note that if (2.13) from Theorem 2.3 holds for some $\alpha > 0$, then it holds for all $0 < \tilde{\alpha} \leq \alpha$ with $r_{\tilde{\alpha}} = r_\alpha$. We will make use of this fact at various points of forthcoming arguments: whenever required, we will assume, without loss of generality, that α is as large as needed.

The proof of Theorem 2.3 is more sophisticated than that of Theorems 2.1 and 2.2. It requires devising a carefully chosen partition of the positive semi-axis and a number of preparatory results which will be given in Sections 3.2.1 and 3.2.2 respectively, before addressing the actual proof in Section 3.2.3.

Throughout this section we adopt the following notation:

$$N(\lambda;\rho) := \#\{\lambda_k \mid \lambda - \rho \leq \lambda_k \leq \lambda + \rho\},$$

$$\widetilde{N}(\lambda;\rho) := \#\{\mu_k \mid \lambda - \rho \leq \mu_k \leq \lambda + \rho\}.$$

We will use the capital letter C for denoting some positive constants, the precise values of which are unimportant and may change from line to line.

3.2.1. Partition of the positive semi-axis. We seek a partition of the positive semi-axis $(0, +\infty)$ into subintervals $(\nu_n, \nu_{n+1}]$, $n = 0, 1, 2, \dots$, satisfying the following properties:

(a) $\lim_{n\to\infty} \nu_n = +\infty$,

(b) the length of these intervals, $\nu_{n+1} - \nu_n$, tends to zero in such a way that it would allow us to achieve the required remainder term estimate in (2.13),

(c) each ν_n, $n = 1, 2, \dots$, is at a distance $\gtrsim n^{-\gamma}$ from the set of all eigenvalues (2.1) and (2.12), for some sufficiently large $\gamma > 1$.

Let $\alpha > 0$ be the exponent from Theorem 2.3. Put

$$\nu_0 := 0, \tag{3.24}$$

$$\nu_n := n^\beta + c_n n^{-1}, \quad n = 1, 2, \dots, \tag{3.25}$$

where

$$\beta := \frac{1}{1 + \frac{\alpha d}{s}} \tag{3.26}$$

and the c_n are some real numbers. Note that $\beta \in (0, 1)$ and it can be made arbitrarily small by choosing α sufficiently large.

Lemma 3.3. *If*

$$c_n \in [-\beta/4, \beta/4] \quad \textit{for all } n = 1, 2, \dots, \tag{3.27}$$

then the ν_n form a strictly increasing sequence.

Proof. Our choice of β guarantees $\nu_0 < \nu_1$. Therefore, it is enough to show that

$$n^\beta + \frac{\beta}{4n} < (n+1)^\beta - \frac{\beta}{4n} \tag{3.28}$$

for all $n = 1, 2, \ldots$. Clearly, (3.28) implies $\nu_n < \nu_{n+1}$, cf. (3.25), because $n^{-1} > (n+1)^{-1}$.

The inequality (3.28) is an immediate consequence of the Mean Value Theorem. ■

Lemma 3.3 tells us that the sequence ν_n constructed in accordance with equations (3.24)–(3.27) yields a partition of the positive semi-axis, thus establishing property (a).

Property (b) is established by the following Lemma.

Lemma 3.4. *We have*

$$\nu_{n+1} - \nu_n = O(\nu_n^{-\frac{\alpha d}{s}}) \quad \textit{as } n \to +\infty. \tag{3.29}$$

Proof. Formula (3.25) implies

$$\nu_{n+1} - \nu_n = O(n^{\beta-1}) = O((\nu_n^{1/\beta})^{\beta-1}) = O(\nu_n^{1-\frac{1}{\beta}}) \quad \text{as } n \to +\infty. \tag{3.30}$$

Combining (3.30) and (3.26) we obtain (3.29). ■

Suppose that $\lambda_k \in (\nu_n, \nu_{n+1}]$. Using Theorem B.1, we obtain

$$\nu_n = b^{-s/d} k^{s/d} + o(k^{s/d}) \quad \text{as } k \to +\infty.$$

This gives us a different take on the statement of Lemma 3.4 in that it allows us to equivalently recast (3.29) as

$$\nu_{n+1} - \nu_n = O(k^{-\alpha}) \quad \text{as } k \to +\infty. \tag{3.31}$$

Finally, the following Lemma establishes that the c_n can be chosen in such a way that our partition possesses property (c).

Lemma 3.5. *There exist constants $\gamma > 1$ and $C > 0$ such that, for a suitable choice of c_n in (3.25) compatible with condition (3.27), we have*

$$\operatorname{dist}\Big(\nu_n, \bigcup_{j=1}^{m^+} \sigma^+(A_j) \cup \sigma^+(A)\Big) \ge C n^{-\gamma} \tag{3.32}$$

for all $n = 1, 2, \ldots$.

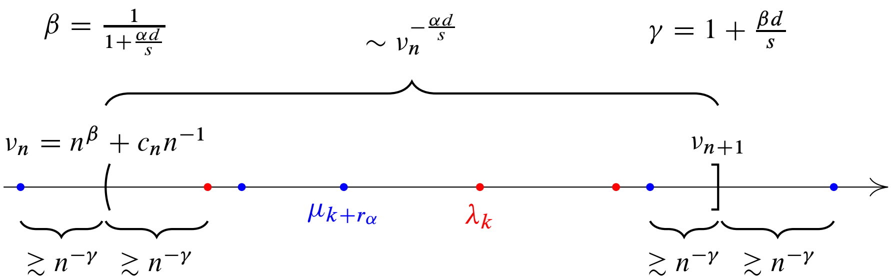

Figure 1. Construction of the partition of the positive semi-axis

Proof. In order to prove the lemma, it is enough to estimate from below the size of the largest gap in the set

$$\Big(\bigcup_{j=1}^{m^+}\sigma^+(A_j)\cup\sigma^+(A)\Big)\cap\Big[\nu_n-\frac{\beta}{4n},\nu_n+\frac{\beta}{4n}\Big]. \tag{3.33}$$

To begin with, let us estimate from above the number of eigenvalues we can have in the interval $[\nu_n-\frac{\beta}{4n},\nu_n+\frac{\beta}{4n}]$.

Theorem B.1 tells us that

$$N\Big(\nu_n;\frac{\beta}{4n}\Big)+\widetilde{N}\Big(\nu_n;\frac{\beta}{4n}\Big)=o(n^{\frac{\beta d}{s}}). \tag{3.34}$$

The quantity

$$\sup_{x\in[\nu_n-\frac{\beta}{4n},\nu_n+\frac{\beta}{4n}]}\operatorname{dist}\Big(x,\Big(\bigcup_{j=1}^{m^+}\sigma^+(A_j)\cup\sigma^+(A)\Big)\cap\Big[\nu_n-\frac{\beta}{4n},\nu_n+\frac{\beta}{4n}\Big]\Big)$$

is minimised when the eigenvalues λ_k and μ_k are equidistributed in $[\nu_n-\frac{\beta}{4n},\nu_n+\frac{\beta}{4n}]$. As we are looking at an interval of length $\frac{\beta}{2n}$, formula (3.34) implies that we can choose c_n such that (3.32) holds for

$$\gamma=1+\frac{d\beta}{s}. \tag{3.35}$$

■

The construction of our partition is summarised in Figure 1.

3.2.2. Preparatory lemmata. We will now state and prove a few simple lemmata.

Lemma 3.6. *Let $\mu_1 \le \mu_2 \le \cdots \le \mu_r$ be real numbers and let u_k, $k = 1, \ldots, r$, be an orthonormal set in $L^2(M)$. Suppose that*

$$\|(A - \mu_k)u_k\|_{L^2} \le \varepsilon, \quad k = 1, \ldots, r. \tag{3.36}$$

Then

$$\#\{\lambda \mid \lambda \in \sigma(A) \cap [\mu_1 - \sqrt{r}\varepsilon, \mu_r + \sqrt{r}\varepsilon]\} \ge r. \tag{3.37}$$

Proof. Without loss of generality, we can assume that $\mu_r \ge 0$ and $\mu_1 = -\mu_r$. Arguing by contradiction, suppose (3.37) is not true. Then one can choose a $u \in L^2(M)$ such that

$$u = \sum_{k=1}^{r} a_k u_k, \tag{3.38}$$

$$\sum_{k=1}^{r} |a_k|^2 = 1, \tag{3.39}$$

orthogonal to the eigenfunctions of A corresponding to eigenvalues in $[-\mu_r - \sqrt{r}\varepsilon, \mu_r + \sqrt{r}\varepsilon]$.

On the one hand, the Spectral Theorem implies

$$\|Au\|_{L^2} > \mu_r + \sqrt{r}\varepsilon. \tag{3.40}$$

On the other hand, using formulae (3.36), (3.38), (3.39), the triangle inequality in $L^2(M)$, and the Cauchy–Schwarz inequality in $\mathbb{C}^r$, we obtain

$$\|Au\|_{L^2} \le \left\|\sum_{k=1}^{r} a_k(A - \mu_k)u_k\right\|_{L^2} + \left\|\sum_{k=1}^{r} a_k \mu_k u_k\right\|_{L^2} \le \sqrt{r}\varepsilon + \mu_r. \tag{3.41}$$

Formulae (3.40) and (3.41) give us a contradiction. ■

Lemma 3.7. *Let $u_k^{(j)}$ be a normalised eigenfunction of A_j corresponding to the eigenvalue $\lambda_k^{(j)}$. Then*

$$\langle u_k^{(j)}, u_{k'}^{(l)} \rangle = O(n^{-\infty}) \tag{3.42}$$

for $l \ne j$ and for all k, k' such that $\lambda_k^{(j)}, \lambda_{k'}^{(l)} \in (\nu_n, \nu_{n+1}]$.

Proof. Using Proposition 3.1(b) and properties of pseudodifferential projections we get

$$\begin{aligned}\langle u_k^{(j)}, u_{k'}^{(l)} \rangle &= \langle P_j u_k^{(j)}, P_l u_{k'}^{(l)} \rangle + O(n^{-\infty}) \\ &= \langle P_l P_j u_k^{(j)}, u_{k'}^{(l)} \rangle + O(n^{-\infty}) = O(n^{-\infty}).\end{aligned}$$

■

Lemma 3.8. *Let a_k, b_k, $k = 1, \dots, r$, be nonnegative real numbers. Suppose that*

$$\sum_{k=1}^{r} a_k \le C \tag{3.43}$$

and

$$\sum_{k=1}^{r} b_n = 1. \tag{3.44}$$

Then there exists a $\tilde{k} \in \{1, \dots, r\}$ such that

$$a_{\tilde{k}} \le C b_{\tilde{k}}. \tag{3.45}$$

Proof. Suppose

$$a_k > C b_k \quad \text{for all } k = 1, \dots, r. \tag{3.46}$$

Then summing both sides of (3.46) over k and using (3.44) we obtain $\sum_{k=1}^{r} a_k > C$, which contradicts (3.43). ■

In the proof of Theorem 2.3 we will need to construct a set of orthonormal functions out of a set of r functions which are only approximately orthonormal, up to an error that decays superpolynomially. The Gram–Schmidt process, as well as its standard modifications, yields a number of terms growing factorially with r, thus resulting in an overall error that is too big for our purposes. The following lemma will give us an orthonormalisation procedure which circumvents this shortcoming.

Lemma 3.9. *Let F be an Hermitian $r \times r$ matrix such that*

$$\|F - I\|_{\max} \le \frac{1}{3r^2}, \tag{3.47}$$

where I is the $r \times r$ identity matrix and $\|F\|_{\max} := \max_{1 \le j,k \le r} |F_{jk}|$ is the max matrix norm. Then there exists an Hermitian matrix G such that

$$GFG = I \tag{3.48}$$

and

$$\|G - I\|_{\max} \le \|F - I\|_{\max}. \tag{3.49}$$

Proof. It is easy to see that under the condition (3.47) the matrix F is positive definite. Put

$$R := F - I \tag{3.50}$$

and define

$$G := F^{-1/2} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} R^k. \tag{3.51}$$

The series on the right-hand side of (3.51) converges in the max matrix norm as soon as $\|R\|_{\max} < r^{-1}$, which is guaranteed by (3.47). Of course, (3.51) implies (3.48).

Formulae (3.51), (3.50) and (3.47) imply

$$\|G - I\|_{\max} \le \sum_{k=1}^{\infty} \left| \binom{-\frac{1}{2}}{k} \right| \|R^k\|_{\max} \le \frac{1}{2}\|R\|_{\max} + \sum_{k=2}^{\infty} r^k \|R\|_{\max}^k$$
$$\le \frac{1}{2}\|R\|_{\max} + \frac{r^2\|R\|_{\max}^2}{1 - r\|R\|_{\max}} \le \frac{1}{2}\|R\|_{\max} + \frac{\frac{1}{3}\|R\|_{\max}}{1 - \frac{1}{3r}} \le \|R\|_{\max}.$$

In the above calculation we used the weighted submultiplicative property of the max matrix norm, $\|R^k\|_{\max} \le r^k \|R\|_{\max}^k$. ∎

Lastly, we recall for the reader's convenience a fact from elementary functional analysis.

Lemma 3.10. *Let B be a self-adjoint operator in a Hilbert space $(H, \|\cdot\|_H)$ with discrete spectrum. Let R be a positive number. Let ξ_k, $k = 1, \dots, r$, be all the eigenvalues of B on the closed interval $[-R, R]$, and let v_k be the corresponding eigenfunctions. Then for any $v \in D(B)$, $v \neq 0$, satisfying*

$$\langle v_k, v\rangle_H = 0, \quad k = 1, \dots, n, \tag{3.52}$$

we have

$$\|Bv\|_H > R\|v\|_H. \tag{3.53}$$

3.2.3. Proof of Theorem 2.3. We start by proving two propositions establishing that, for sufficiently large n, we have the same number of eigenvalues λ_k and μ_k in each interval $(\nu_n, \nu_{n+1}]$.

Proposition 3.11. *There exists a natural number K such that for all $n > K$ we have*

$$N\left(\frac{\nu_n + \nu_{n+1}}{2}; \frac{\nu_{n+1} - \nu_n}{2}\right) \ge \widetilde{N}\left(\frac{\nu_n + \nu_{n+1}}{2}; \frac{\nu_{n+1} - \nu_n}{2}\right). \tag{3.54}$$

Proof. Suppose $\widetilde{N}(\frac{\nu_n+\nu_{n+1}}{2}; \frac{\nu_{n+1}-\nu_n}{2}) = r$. This means that the interval $(\nu_n, \nu_{n+1}]$ contains precisely r elements $\mu_{p+1} \le \cdots \le \mu_{p+r}$ from the sequence (2.12) (recall that we have no eigenvalues in a neighbourhood of the endpoints of our partition).

Each μ_{p+k}, $k = 1, \dots, r$, is an eigenvalue of A_j for some j. Let us denote by v_k, $k = 1, \dots, r$, the corresponding normalised eigenfunctions. Here we assume that eigenfunctions corresponding to the same operator A_j are chosen to be orthogonal.

In view of Lemma 3.7, we have

$$\langle v_k, v_{k'}\rangle = \delta_{kk'} + O(n^{-\infty}), \quad k, k' = 1, \dots, r, \tag{3.55}$$

where $\delta_{kk'}$ is the Kronecker delta.

Let F be the $r \times r$ matrix whose entries are defined in accordance with

$$F_{kk'} := \langle v_k, v_{k'} \rangle, \quad 1 \le k, k' \le r, \tag{3.56}$$

and let G be the matrix given by Lemma 3.9. Then, formula (3.48) implies that the functions

$$\tilde{v}_k := \sum_{q=1}^{r} G_{qk} v_q, \quad k = 1, \dots, r, \tag{3.57}$$

satisfy

$$\langle \tilde{v}_k, \tilde{v}_{k'} \rangle = \delta_{kk'}, \quad k, k' = 1, \dots, r. \tag{3.58}$$

Furthermore, as $\|F - I\|_{\max} = O(n^{-\infty})$ in view of (3.55)–(3.56), formula (3.49) and elliptic regularity imply

$$\tilde{v}_k = v_k + O(n^{-\infty}), \quad k = 1, \dots, r. \tag{3.59}$$

Formula (3.59) and Proposition 3.1(c) give us $\|(A - \mu_{p+k})\tilde{v}_k\|_{L^2} = O(n^{-\infty})$, $k = 1, \dots, r$. In particular, we have

$$\|(A - \mu_{p+k})\tilde{v}_k\|_{L^2} \le C n^{-\tilde{\gamma}}, \quad k = 1, \dots, r, \tag{3.60}$$

for some $C > 0$ independent of k and n, and $\tilde{\gamma} := \frac{\beta d}{2s} + \gamma + 1$.

In light of (3.58) and (3.60), we can apply Lemma 3.6 to obtain

$$\#\{\lambda_k \mid \lambda_k \in [\mu_{p+1} - C\sqrt{r} n^{-\tilde{\gamma}}, \mu_{p+r} + C\sqrt{r} n^{-\tilde{\gamma}}]\} \ge r. \tag{3.61}$$

As $\sqrt{r} = o(n^{\frac{\beta d}{2s}})$, see Theorem B.1, we have $\sqrt{r} n^{-\tilde{\gamma}} = o(n^{-\gamma-1})$. Hence, Lemma 3.5 tells us that $[\mu_{p+1} - C\sqrt{r} n^{-\tilde{\gamma}}, \mu_{p+r} + C\sqrt{r} n^{-\tilde{\gamma}}] \subset (\nu_n, \nu_{n+1}]$ for sufficiently large n, so that (3.61) implies (3.54). ■

Proposition 3.12. *There exists a natural number K such that for all $n > K$ we have*

$$\tilde{N}\left(\frac{\nu_n + \nu_{n+1}}{2}; \frac{\nu_{n+1} - \nu_n}{2}\right) \ge N\left(\frac{\nu_n + \nu_{n+1}}{2}; \frac{\nu_{n+1} - \nu_n}{2}\right). \tag{3.62}$$

Proof. Let u_k be orthonormal eigenfunctions of A corresponding to λ_k and, for each j, let $u_k^{(j)}$ be orthonormal eigenfunctions of A_j corresponding to $\lambda_k^{(j)}$.

Let us define

$$E_n := \operatorname{span}(u_k \mid \lambda_k \in (\nu_n, \nu_{n+1}]),$$
$$\tilde{E}_n := \operatorname{span}(u_k^{(j)} \mid \lambda_k^{(j)} \in (\nu_n, \nu_{n+1}], j = 1, \dots, m^+).$$

Arguing by contradiction, suppose

$$\widetilde{N}\Big(\frac{\nu_n+\nu_{n+1}}{2};\frac{\nu_{n+1}-\nu_n}{2}\Big) < N\Big(\frac{\nu_n+\nu_{n+1}}{2};\frac{\nu_{n+1}-\nu_n}{2}\Big).$$

Then there exists a $u\in E_n$, $\|u\|_{L^2}=1$, such that

$$\langle u,v\rangle = 0 \quad \text{for all } v\in \widetilde{E}_n. \tag{3.63}$$

Proposition 3.1(a) implies

$$u=\sum_{j=1}^{m^+} P_j u + O(n^{-\infty}), \tag{3.64}$$

whereas Proposition 3.1(b) implies

$$u_k^{(j)} = P_j u_k^{(j)} + O(n^{-\infty}) \tag{3.65}$$

for $j=1,\dots,m^+$ and all k such that $u_k^{(j)}\in\widetilde{E}_n$.

We claim that

$$\langle u_k^{(j')}, P_j u\rangle = O(n^{-\infty}) \tag{3.66}$$

for all $j,j'=1,\dots,m^+$ and all k such that $u_k^{(j)}\in\widetilde{E}_n$. Indeed, for $j\neq j'$ (3.66) follows from (3.65) and Theorem 1.2(a). For $j=j'$, using Theorem 1.2(a), (3.65) and (3.63), we obtain

$$\langle u_k^{(j)}, P_j u\rangle = \langle P_j u_k^{(j)}, u\rangle + O(n^{-\infty}) = \langle u_k^{(j)}, u\rangle + O(n^{-\infty}) = O(n^{-\infty}).$$

By the Spectral Theorem, we have

$$\Big\|\Big(A-\frac{\nu_{n+1}+\nu_n}{2}\Big)u\Big\|_{L^2} \le \frac{\nu_{n+1}-\nu_n}{2}. \tag{3.67}$$

Squaring (3.67), substituting (3.64) in and using Theorem 1.2(a), we get

$$\sum_{j=1}^{m^+}\Big\|\Big(A-\frac{\nu_{n+1}+\nu_n}{2}\Big)P_j u\Big\|_{L^2}^2 \le \frac{(\nu_{n+1}-\nu_n)^2}{4} + O(n^{-\infty}). \tag{3.68}$$

Put

$$\rho := \beta+\gamma \tag{3.69}$$

and let $J := \{j\in\{1,\dots,m^+\} \mid \|P_j u\|_{L^2}\ge n^{-\rho}\}$. As

$$\sum_{j=1}^{m^+}\|P_j u\|_{L^2}^2 = 1 + O(n^{-\infty}), \tag{3.70}$$

the set J is nonempty.

When we restrict the summation to indices from J, formula (3.70) reads

$$\sum_{j\in J}\|P_j u\|^2_{L^2} = 1 + O(n^{-\rho}).$$

By suitably rescaling the function u, let us define a function $\tilde{u}$ such that

$$\sum_{j\in J}\|P_j \tilde{u}\|^2_{L^2} = 1. \tag{3.71}$$

By restricting the summation to indices from the set J only and expressing the result in terms of $\tilde{u}$, formula (3.68) turns into

$$\sum_{j\in J}\left\|\left(A-\frac{\nu_{n+1}+\nu_n}{2}\right)P_j\tilde{u}\right\|^2_{L^2}\le\frac{(\nu_{n+1}-\nu_n)^2}{4}(1+Cn^{-\rho}). \tag{3.72}$$

On account of (3.71) and (3.72), Lemma 3.8 implies that there exists an $l\in J$ such that

$$\left\|\left(A_l-\frac{\nu_{n+1}+\nu_n}{2}\right)P_l\tilde{u}\right\|^2_{L^2}\le\frac{(\nu_{n+1}-\nu_n)^2}{4}(1+Cn^{-\rho})\|P_l\tilde{u}\|^2_{L_2}. \tag{3.73}$$

In (3.73) we were able to replace A with A_l by resorting to formula (2.2), which implies

$$AP_l = A_l P_l \quad \mod \Psi^{-\infty}. \tag{3.74}$$

Formula (3.66) tells us that

$$\langle u_k^{(l)}, P_l\tilde{u}\rangle = O(n^{-\infty}) \tag{3.75}$$

for all k such that $u_k^{(l)}\in\widetilde{E}_n$. Put

$$\hat{u}_l := P_l\tilde{u} - \sum_{k:\lambda_k^{(l)}\in(\nu_n,\nu_{n+1}]}\langle u_k^{(l)}, P_l\tilde{u}\rangle u_k^{(l)}, \tag{3.76}$$

so that

$$\langle u_k^{(l)}, \hat{u}_l\rangle = 0 \quad \text{for all } u_k^{(l)}\in\widetilde{E}_n. \tag{3.77}$$

Formulae (3.76) and (3.75) give us $\hat{u}_l = P_l\tilde{u} + O(n^{-\infty})$, so (3.73) implies

$$\left\|\left(A_l-\frac{\nu_{n+1}+\nu_n}{2}\right)\hat{u}_l\right\|_{L^2}\le\frac{\nu_{n+1}-\nu_n}{2}(1+Cn^{-\rho})\|\hat{u}_l\|_{L_2}. \tag{3.78}$$

Formula (3.69) and Lemma 3.4 imply $(\nu_{n+1}-\nu_n)n^{-\rho} = O(n^{-\gamma-1})$. Therefore, in view of Lemma 3.5, for sufficiently large n we have

$$\widetilde{N}\left(\frac{\nu_n+\nu_{n+1}}{2};\frac{\nu_{n+1}-\nu_n}{2}\right) = \widetilde{N}\left(\frac{\nu_n+\nu_{n+1}}{2};\frac{(\nu_{n+1}-\nu_n)(1+Cn^{-\rho})}{2}\right). \tag{3.79}$$

But, on account of Lemma 3.10, formulae (3.78) and (3.77) imply

$$\tilde{N}\Big(\frac{\nu_n+\nu_{n+1}}{2};\frac{(\nu_{n+1}-\nu_n)(1+Cn^{-\rho})}{2}\Big) > \tilde{N}\Big(\frac{\nu_n+\nu_{n+1}}{2};\frac{\nu_{n+1}-\nu_n}{2}\Big),$$

which contradicts (3.79). ∎

Theorem 2.3 now follows easily.

Proof of Theorem 2.3. Propositions 3.11 and 3.12 imply that there exists a natural K such that

$$N\Big(\frac{\nu_n+\nu_{n+1}}{2};\frac{\nu_{n+1}-\nu_n}{2}\Big) = \tilde{N}\Big(\frac{\nu_n+\nu_{n+1}}{2};\frac{\nu_{n+1}-\nu_n}{2}\Big)$$

for all $n > K$. This means that for all $n > K$ each interval $(\nu_n, \nu_{n+1}]$ contains the same number of eigenvalues λ_k from (2.1) and eigenvalues μ_k from (2.12).

Let

$$k' := \min\{k \mid \lambda_k \in (\nu_{K+1}, \nu_{K+2}]\}, \quad k'' := \min\{k \mid \mu_k \in (\nu_{K+1}, \nu_{K+2}]\},$$

and put $r_\alpha := k'' - k'$. Then, using formula (3.31), we arrive at

$$\lambda_k - \mu_{k+r_\alpha} = O(k^{-\alpha}) \quad \text{as } k \to +\infty.$$ ∎

Remark 3.13. Observe that in the proofs of Theorems 2.1–2.3 choosing the operators A_j precisely in accordance with formula (2.2) is not crucial. One can, for instance, replace (2.2) with

$$A_j := A - \sum_{\substack{l=1,\dots,m^+\\ l\neq j}} c_{j,l} P_l^* A P_l, \quad j = 1, \dots, m^+,$$

where $c_{j,l} > 1$. What we mainly relied upon is the fact that the operators A_j satisfy the properties

$$P_j^* A_j P_j = P_j^* A P_j \quad \text{mod } \Psi^{-\infty},$$
$$[A_j, P_l] = 0 \quad \text{mod } \Psi^{-\infty} \quad \text{for all } l,$$
$$P_l^* A_j P_l \le 0 \quad \text{mod } \Psi^{-\infty} \quad \text{for } l \neq j.$$

4. Invariant subspaces in hyperbolic systems

In this section we will apply our results to the study of first and second order hyperbolic systems.

4.1. First order operators

Before addressing the proof of Theorem 2.4, let us recall, in an abridged manner and for the convenience of the reader, the propagator construction from [11], which builds upon [13, 21, 28] and is an extension to first order systems of earlier results for scalar operators [9, 10].

Let $A \in \Psi^1$ be an operator as in Section 1. For each $j \in \{-m^-, \dots, -1, 1, \dots, m^+\}$ let us denote by $(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))$ the Hamiltonian flow in the cotangent bundle generated by the Hamiltonian $h^{(j)}(x, \xi)$, namely, the solution to Hamilton's equations

$$\begin{cases} \dot{x}^{(j)} = h_\xi(x^{(j)}, \xi^{(j)}), \\ \dot{\xi}^{(j)} = -h_x(x^{(j)}, \xi^{(j)}) \end{cases} \tag{4.1}$$

with initial condition $(x^{(j)}(0; y, \eta), \xi^{(j)}(0; y, \eta)) = (y, \eta)$. Here and further on the dot denotes differentiation with respect to t and subscripts denote partial differentiation.

For each j choose a function $\varphi^{(j)}(t, x; y, \eta) \in C^\infty(\mathbb{R} \times M \times T'M; \mathbb{C})$ positively homogeneous in η of degree 1 satisfying

(i) $\varphi^{(j)}|_{x=x^{(j)}} = 0$,

(ii) $\varphi^{(j)}_{x^\alpha}|_{x=x^{(j)}} = \xi^{(j)}_\alpha$,

(iii) $\det \varphi^{(j)}_{x^\alpha \eta_\beta}|_{x=x^{(j)}} \neq 0$,

(iv) $\operatorname{Im} \varphi^{(j)} \geq 0$.

Such functions are called *phase functions* and they always exist [21, Lemma 1.4].

Then the propagator $U(t) := e^{-itA}$ can be written, modulo an infinitely smoothing operator, as the sum of precisely m oscillatory integrals

$$U(t) = \sum_j U^{(j)}(t) \quad \mod C^\infty(\mathbb{R}; \Psi^{-\infty}) \tag{4.2}$$

where

$$[U^{(j)}(t)u](x) = \frac{1}{(2\pi)^d} \int\limits_{T'M} e^{i\varphi^{(j)}(t,x;y,\eta)} \mathfrak{a}^{(j)}(t; y, \eta) \chi^{(j)}(t, x; y, \eta) w^{(j)}(t, x; y, \eta) u(y)\, dy\, d\eta \tag{4.3}$$

and

- the function $\chi^{(j)} \in C^\infty(\mathbb{R} \times M \times T'M)$ is a cut-off satisfying
  (a) $\chi^{(j)}(t, x; y, \eta) = 0$ on $\{(t, x; y, \eta) \mid |h^{(j)}(y, \eta)| \leq 1/2\}$,
  (b) $\chi^{(j)}(t, x; y, \eta) = 1$ on the intersection of $\{(t, x; y, \eta) \mid |h^{(j)}(y, \eta)| \geq 1\}$ with some conical neighbourhood of $\{(t, x^{(j)}(t; y, \eta); y, \eta)\}$,
  (c) $\chi^{(j)}(t, x; y, \alpha\eta) = \chi^{(j)}(t, x; y, \eta)$ for $\alpha \geq 1$ on $\{(t, x; y, \eta) \mid |h^{(j)}(y, \eta)| \geq 1\}$;

- the weight $w^{(j)}$ is defined by the phase function $\varphi^{(j)}$ in accordance with

$$w^{(j)}(t,x;y,\eta) := [\det{}^2(\varphi^{(j)}_{x^\alpha \eta_\beta})]^{\frac{1}{4}},$$

with the smooth branch of the complex root chosen in such a way that

$$w^{(j)}(0,y;y,\eta) = 1.$$

The smooth matrix-function $\mathfrak{a}^{(j)} \in \mathrm{S}^0_{\mathrm{ph}}(\mathbb{R} \times T'M; \mathrm{Mat}(m;\mathbb{C}))$ appearing in (4.3) is the unknown in the algorithm for the construction of $U^{(j)}(t)$. It is an element in the class of polyhomogeneous symbols of order zero with values in $m \times m$ complex matrices, which means that $\mathfrak{a}^{(j)}$ admits an asymptotic expansion in components positively homogeneous in momentum,

$$\mathfrak{a}^{(j)}(t;y,\eta) \sim \sum_{k=0}^{+\infty} \mathfrak{a}^{(j)}_{-k}(t;y,\eta), \quad \mathfrak{a}^{(j)}_{-k}(t;y,\alpha\eta) = \alpha^{-k}\mathfrak{a}^{(j)}_{-k}(t;y,\eta) \quad \text{for all } \alpha > 0.$$

The symbol $\mathfrak{a}^{(j)}$ is determined by the requirement that $U^{(j)}(t)$ satisfies, in a distributional sense, the hyperbolic equation

$$(-i\partial_t + A)U^{(j)}(t) = 0 \mod C^\infty(\mathbb{R};\Psi^{-\infty}). \tag{4.4}$$

Note that $\mathfrak{a}^{(j)}$ does not depend on x: this is achieved by means of a procedure called *reduction of the amplitude*, which turns the partial differential equations brought about by (4.4) into a hierarchy of transport equations for the homogeneous components of $\mathfrak{a}^{(j)}$ – ordinary differential equations in the variable t – which can be solved iteratively. We refer the reader to [11, Section 3] for further details.

The initial conditions for the transport equations are obtained from the initial condition for the propagator itself:

$$\sum_j U^{(j)}(0) = \mathrm{Id} \mod \Psi^{-\infty}. \tag{4.5}$$

Clearly, the oscillatory integrals $U^{(j)}(0)$ define pseudodifferential operators in Ψ^0. Furthermore, formula (4.5) tells us that the oscillatory integrals $U^{(j)}(t)$ for different j's are not independent, but they are related to one another via the initial condition.

By examining formulae (4.3)–(4.5) it is not difficult to see that

$$(U^{(j)}(0))_{\mathrm{prin}} = P^{(j)}.$$

This turns out not to be a coincidence: the following theorem will allow us to establish a relation between the pseudodifferential operators $U^{(j)}(0)$ and our pseudodifferential projections P_j.

Theorem 4.1. *Put*

$$V_{lj}(t) := P_l U^{(j)}(t). \tag{4.6}$$

Then for all $j, l \in \{-m^-, \dots, -1, 1, \dots, m^+\}$, $j \neq l$, *we have*

$$V_{lj}(t) = 0 \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.7}$$

Proof. By the definition of $U^{(j)}(t)$ and the fact that $[A, P_l] \in \Psi^{-\infty}$, the operator (4.6) satisfies

$$(-i\partial_t + A)V_{lj}(t) = 0 \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.8}$$

Arguing by contradiction, suppose that (4.7) is false. Then there exists an integer $k \geq 0$ such that

$$V_{lj}(t) \in C^\infty(\mathbb{R}; \Psi^{-k}), \quad V_{lj}(t) \notin C^\infty(\mathbb{R}; \Psi^{-k-1}). \tag{4.9}$$

Let $(V_{lj})_{\mathrm{prin,k}}(t; y, \eta)$ be the principal symbol of $V_{lj}(t)$ as an operator in $C^\infty(\mathbb{R}; \Psi^{-k})$, cf. [11, Definition 3.7].

A simple analysis of the leading transport equation for the homogeneous components of the symbol of $V_{lj}(t)$ arising from (4.8) – see, e.g., [25, Section 3.3.4] – tells us that (4.8) can be satisfied only if

$$[P^{(j)}(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))][(V_{lj})_{\mathrm{prin,k}}(t; y, \eta)] = [(V_{lj})_{\mathrm{prin,k}}(t; y, \eta)]. \tag{4.10}$$

Now, formula (4.6) and the idempotency property of pseudodifferential projections imply

$$P_l V_{lj}(t) = V_{lj}(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.11}$$

Computing the principal symbol of the left-hand side of (4.11), we conclude that (4.11) can only be satisfied if

$$[P^{(l)}(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))][(V_{lj})_{\mathrm{prin,k}}(t; y, \eta)] = [(V_{lj})_{\mathrm{prin,k}}(t; y, \eta)]. \tag{4.12}$$

In writing (4.12), we used the standard formula for the action of a pseudodifferential operator on an exponent [32, Section 18] and the fact that $V_{lj}(t)$ is an oscillatory integral with phase function $\varphi^{(j)}$.

As $P^{(j)} P^{(l)} = 0$ pointwise in $T^*M \setminus \{0\}$, formulae (4.10) and (4.12) imply

$$(V_{lj})_{\mathrm{prin,k}}(t; y, \eta) = 0 \quad \text{for all } t \in \mathbb{R}, \ (y, \eta) \in T'M \setminus \{0\},$$

which contradicts (4.9). ∎

Corollary 4.2. *For every* $j \in \{-m^-, \dots, -1, 1, \dots, m^+\}$ *we have*

$$U^{(j)}(0) = P_j \mod \Psi^{-\infty}. \tag{4.13}$$

Proof. Theorem 1.2(a)(iv) and formula (4.5) imply

$$\sum_j U^{(j)}(0) = \sum_j P_j \mod \Psi^{-\infty}, \tag{4.14}$$

whereas Theorem 1.2(a)(iv) and Theorem 4.1 imply

$$U^{(j)}(0) = P_j U^{(j)}(0) \mod \Psi^{-\infty}. \tag{4.15}$$

Acting with P_l on the left in (4.14) and using, once again, Theorem 4.1, we obtain

$$P_l U^{(l)}(0) = P_l \mod \Psi^{-\infty}. \tag{4.16}$$

Combining (4.15) and (4.16) we arrive at (4.13). ∎

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. The first equality, namely

$$U^{(j)}(t) = P_j U(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty}),$$

follows immediately from (4.2) and Theorem 4.1.

Let us prove the second equality, namely

$$P_j U(t) = U(t) P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.17}$$

Put $R(t) := U(-t) P_j U(t) - P_j$. Then, in view of (2.15), $R(t)$ satisfies

$$\partial_t R(t) = -i\big(-AU(-t)P_j U(t) + U(-t) P_j A U(t)\big), \tag{4.18}$$

$$R(0) = 0. \tag{4.19}$$

As A commutes with $U(-t)$ and, modulo $\Psi^{-\infty}$, with P_j, formula (4.18) implies

$$\partial_t R(t) = i U(-t)[A, P_j] U(t) = 0 \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.20}$$

Combining (4.20) and (4.19) we obtain

$$R(t) = 0 \mod C^\infty(\mathbb{R}; \Psi^{-\infty}),$$

which gives us (4.17). ∎

In fact, Theorem 4.1 and Theorem 2.4 imply the following stronger result.

Corollary 4.3. *We have*

$$P_j U^{(j)}(t) = U^{(j)}(t) P_j = U^{(j)}(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty}) \quad \textit{for all } j \tag{4.21}$$

and

$$P_l U^{(j)}(t) = U^{(j)}(t) P_l = 0 \mod C^\infty(\mathbb{R}; \Psi^{-\infty}) \quad \textit{for } l \neq j. \tag{4.22}$$

Proof. Theorem 4.1 implies

$$U^{(j)}(t) = P_j U^{(j)}(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.23}$$

Substituting (4.2) into (4.17) and using (4.23) we obtain

$$U^{(j)}(t) = \sum_l P_l U^{(l)}(t) P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.24}$$

Multiplying (4.24) by P_j on the left and using (4.23) once again we get

$$P_j U^{(j)}(t) = U^{(j)}(t) P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.25}$$

Formulae (4.23) and (4.25) give us (4.21).

Now, formula (4.21) implies that for $l \neq j$ we have

$$U^{(j)}(t) P_l = U^{(j)}(t) P_j P_l = 0 \mod C^\infty(\mathbb{R}; \Psi^{-\infty}). \tag{4.26}$$

Formula (4.26) and Theorem 4.1 give us (4.22). ∎

Remark 4.4. Note that a weaker version of Theorem 2.4 was, effectively, obtained in [11, Section 3]. More precisely, it was shown that

$$\sum_{j=1}^{m^+} U^{(j)}(t) = \sum_{j=1}^{m^+} P_j U(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty}),$$

$$\sum_{j=-m^-}^{-1} U^{(j)}(t) = \sum_{j=-m^-}^{-1} P_j U(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty}).$$

To see that this is the case, one needs to combine [11, Theorem 3.3] with [12, Theorem 2.7].

Note also that in [5, Section 5] the authors analysed, in a similar spirit, the localisation of the propagator in a given spectral window of the operator A, albeit in a somewhat different setting. The use of pseudodifferential projections in the study of the unitary evolution for matrix operators was employed in [6] as well, in the semiclassical setting and under additional assumptions on A, in the context of Egorov-type theorems. See also [14].

Theorem 2.4 tells us that pseudodifferential projections decompose $L^2(M)$ into almost-orthogonal almost-invariant subspaces under the unitary time evolution. That is, if $v \in P_j L^2(M)$ then

$$U(t)v = U^{(j)}(t)v \mod C^\infty(\mathbb{R} \times M),$$

$$U^{(l)}(t)v = 0 \mod C^\infty(\mathbb{R} \times M) \quad \text{for } l \neq j.$$

4.2. Nonnegative second order operators

In this section we will show that one can obtain results analogous to those from Section 4.1 for nonnegative second order operators.

Let $A \in \Psi^2$ be a nonnegative self-adjoint elliptic operator and suppose that its principal symbol has simple eigenvalues. As explained in Section 2, we define its propagator to be

$$U(t) := e^{-it\sqrt{A}}. \tag{4.27}$$

The fact that $\sqrt{A}$ is a well-defined pseudodifferential operator follows, for example, from [31]. The unitary operator (4.27) is the solution operator of the first order hyperbolic pseudodifferential system $(-i\partial_t + \sqrt{A})f = 0$, subject to the initial condition $f|_{t=0} = f_0$. Of course, the knowledge of $U(t)$ is sufficient for the construction of the general solution of the second order hyperbolic system $(\partial_t^2 - A)f = 0$, subject to initial conditions $f|_{t=0} = f_0$, $(\partial_t f)|_{t=0} = f_1$. Indeed, we have

$$f = \cos(t\sqrt{A})f_0 + A^{-1/2}\sin(t\sqrt{A})f_1 + t\sum_{k:\lambda_k=0}\langle u_k, f_1\rangle,$$

where $A^{-1/2}$ is the pseudoinverse of $\sqrt{A}$ (see [27, Chapter 2, Section 2]),

$$2\cos(t\sqrt{A}) = U(t) + U(t)^* \quad \text{and} \quad 2i\sin(t\sqrt{A}) = U(t)^* - U(t).$$

Let $h^{(j)}(x,\xi)$, $j = 1,\dots,m$, be the eigenvalues of $(\sqrt{A})_{\mathrm{prin}} = \sqrt{A_{\mathrm{prin}}}$. Clearly, the $h^{(j)}$'s are positively homogeneous in momentum ξ of degree 1, strictly positive and distinct. This follows from the fact that A is nonnegative, elliptic and A_{prin} has simple eigenvalues. Let $\varphi^{(j)}$, $j = 1,\dots,m^+$, be phase functions satisfying conditions (i)–(iv) from Section 4.1. The operator $U(t)$ can be constructed explicitly, modulo $C^\infty(\mathbb{R};\Psi^{-\infty})$, as the sum of m oscillatory integrals $U^{(j)}(t)$ of the form (4.3). Remarkably, the amplitude of the $U^{(j)}(t)$ can be determined without the need of extracting the square root of A. Indeed, one can retrace the construction algorithm outlined in Section 4.1 replacing (4.4) with

$$(\partial_t^2 - A)U^{(j)}(t) = 0 \mod C^\infty(\mathbb{R};\Psi^{-\infty}). \tag{4.28}$$

The use in (4.28) of the second order operator $\partial_t^2 - A$ as opposed to its "half-wave" version $-i\partial_t + \sqrt{A}$ is justified by [28, Theorem 3.2.1].

Proposition 4.5. *Let P_j and $\widetilde{P}_j$, $j = 1,\dots,m$, be the pseudodifferential projections uniquely determined, modulo $\Psi^{-\infty}$, by A and $\sqrt{A}$ respectively, in accordance with Theorem 1.2* (a). *Then,*

$$P_j = \widetilde{P}_j \mod \Psi^{-\infty} \tag{4.29}$$

for all j.

Proof. We have

$$A_{\text{prin}} = \sum_{l=1}^{m} h^{(l)} P^{(l)} \tag{4.30}$$

and

$$(\sqrt{A})_{\text{prin}} = \sum_{l=1}^{m} \sqrt{h^{(l)}}\, P^{(l)},$$

therefore $(P_j)_{\text{prin}} = (\widetilde{P}_j)_{\text{prin}} = P^{(j)}$. As $[\widetilde{P}_j, \sqrt{A}] = 0 \mod \Psi^{-\infty}$, clearly $[\widetilde{P}_j, A] = 0 \mod \Psi^{-\infty}$. Then the identity (4.29) follows from [12, Theorem 4.1] and the uniqueness of pseudodifferential projections. ∎

It is worth remarking that the claim of Proposition 4.5 is a nontrivial property of our pseudodifferential projections which cannot be obtained by simply looking at the functional calculus of A and $\sqrt{A}$.

Proof of Theorem 2.5. Equation (2.18) follows immediately by applying Theorem 2.4 to $\sqrt{A}$ and using Proposition 4.5. ∎

5. Refined spectral asymptotics

Theorems 2.1–2.3 open the way to computing spectral asymptotics for each of the m families which the spectrum of A partitions into. We will provide here a brief description of how the results of this paper can be used to refine our understanding of results available in the literature, focussing on first order operators. Further on we assume that $A \in \Psi^1$.

Remark 5.1. One could, in principle, perform the forthcoming argument for nonnegative operators of even order, but this would require a lengthy discussion and would substantially increase the size of the paper. For this reason we decided to refrain from discussing refined spectral asymptotics in greater generality in the current paper.

Let $N^+(\lambda)$ be the positive counting function of A and let $N_j^+(\lambda)$ be the positive counting function of A_j, $j = 1, \dots, m^+$, defined in accordance with (B.1). Establishing a precise relation between $N^+(\lambda)$ and the $N_j^+(\lambda)$, $j = 1, \dots, m^+$, is a challenging task, and it is not *a priori* clear whether a simple quantitative relation can be established in the general case. The issue at hand is that we are dealing with discontinuous functions which can experience massive jumps in the presence of spectral clusters.

What one can do is to establish a relation between the Weyl coefficients of $N^+(\lambda)$ and those of the $N_j^+(\lambda)$.

Let

$$N^+(x;\lambda) := \begin{cases} 0 & \text{for } \lambda \le 0, \\ \sum_{k:0<\lambda_k<\lambda} [v_k(x)]^* v_k(x) & \text{for } \lambda > 0, \end{cases}$$

be the positive *local* counting function of A. In an analogous manner, we define positive *local* counting functions $N_j^+(x;\lambda)$ for each of the A_j, $j = 1,\dots,m^+$.

Let $\hat\mu\colon \mathbb{R} \to \mathbb{C}$ be a smooth function such that $\hat\mu = 1$ in some neighbourhood of the origin and $\operatorname{supp}\hat\mu \subset (-T_0, T_0)$, where T_0 is the infimum of lengths of all the Hamiltonian loops (see (4.1)) originating from all the points of the manifold. Let μ be the inverse Fourier transform of $\hat\mu$, where we adopt the convention

$$\mathcal{F}[\mu](t) = \hat\mu(t) = \int_{-\infty}^{+\infty} e^{-it\lambda}\mu(\lambda)\,d\lambda,$$

$$\mathcal{F}^{-1}[\hat\mu](\lambda) = \mu(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{it\lambda}\hat\mu(t)\,dt,$$

for the Fourier transform and inverse Fourier transform, respectively.

It is known [16, 18–20, 28] that the mollified derivative of the positive local counting function admits a complete asymptotic expansion in integer powers of λ:

$$((N^+)' * \mu)(x,\lambda) = a_{d-1}(x)\lambda^{d-1} + a_{d-2}(x)\lambda^{d-2} + \cdots \quad \text{as } \lambda \to +\infty, \tag{5.1}$$

$$((N_j^+)' * \mu)(x,\lambda) = a^{(j)}_{d-1}(x)\lambda^{d-1} + a^{(j)}_{d-2}(x)\lambda^{d-2} + \cdots \quad \text{as } \lambda \to +\infty. \tag{5.2}$$

Here $*$ stands for the convolution in the variable λ and the prime stands for differentiation with respect to λ. The functions appearing as coefficients of powers of λ in the asymptotic expansions (5.1) and (5.2) are called *Weyl coefficients*.

Proposition 5.2. *For* $j \in \{1,\dots,m^+\}$ *we have*

$$U^+_{A_j}(t) = U^{(j)}(t) \mod C^\infty(\mathbb{R};\Psi^{-\infty}), \tag{5.3}$$

where $U^+_{A_j}(t) := \theta(A_j)e^{-itA_j}$ *is the positive propagator of the operator* A_j.

Proof. Let P_l and $\widetilde{P}_l$ be the pseudodifferential projections associated with A and A_j respectively, in accordance with Theorem 1.2(a). Recalling (2.2) and arguing as in the proof of Proposition 4.5, it is easy to see that the $\widetilde{P}_l$ are just a reshuffling of the P_l. In particular, we have

$$P_j = \widetilde{P}_1 \mod \Psi^{-\infty}. \tag{5.4}$$

Indeed, $(A_j)_{\mathrm{prin}}$ has only one positive eigenvalue, $h^{(j)}$, see (2.3).

Then, formula (5.4) and Corollary 4.2 imply

$$U^+_{A_j}(0) = U^{(j)}(0) = P_j \mod \Psi^{-\infty}. \tag{5.5}$$

Substituting (4.21) into (4.4) we obtain

$$(-i\partial_t + AP_j)U^{(j)}(t) = 0 \mod C^\infty(\mathbb{R};\Psi^{-\infty}).$$

In view of formula (3.74), the above equation can be recast as

$$(-i\partial_t + A_j)U^{(j)}(t) = 0 \mod C^\infty(\mathbb{R};\Psi^{-\infty}). \tag{5.6}$$

Now, $U^+_{A_j}(t)$ also satisfies (5.6) by definition. Thus, $U^{(j)}(t)$ and $U^+_{A_j}(t)$ satisfy the same first order hyperbolic equation (5.6) with the same initial condition (5.5). This gives us (5.3). ■

Theorem 5.3. (a) *We have*

$$((N_j^+)' * \mu)(x,\lambda) = \mathcal{F}^{-1}[\operatorname{tr} u^{(j)}(t,x,x)\hat{\mu}(t)] + O(\lambda^{-\infty}) \quad \text{as } \lambda \to +\infty, \tag{5.7}$$

where $u(t,x,y)$ is the Schwartz kernel of $U^{(j)}(t)$ and tr *stands for the matrix trace.*

(b) *The Weyl coefficients of A and A_j, $j = 1,\dots,m^+$, are related as*

$$a_k(x) = \sum_{j=1}^{m^+} a_k^{(j)}(x), \quad k = d-1, d-2, \dots \tag{5.8}$$

(c) *The first two Weyl coefficients of A_j read*

$$a_{d-1}^{(j)}(x) = \frac{d}{(2\pi)^d} \int\limits_{h^{(j)}(x,\xi)<1} d\xi, \tag{5.9}$$

$$a_{d-2}^{(j)}(x) = -\frac{d(d-1)}{(2\pi)^d} \int\limits_{h^{(j)}(x,\xi)<1} \operatorname{tr}\Big(P^{(j)}A_{\mathrm{sub}} + \frac{i}{2}\{P^{(j)},P^{(j)}\}A_{\mathrm{prin}} - \frac{1}{d-1}h^{(j)}(P_j)_{\mathrm{sub}}\Big)(x,\xi)\,d\xi, \tag{5.10}$$

where curly brackets denote the Poisson bracket

$$\{B,C\} := \sum_{\alpha=1}^{d}(B_{x^\alpha}C_{\xi_\alpha} - B_{\xi_\alpha}C_{x^\alpha})$$

on matrix-functions on the cotangent bundle.

Proof. (a) Formula (5.7) follows immediately from Proposition 5.2 and [11, (8.2)].

(b) Formula (5.8) is an immediate consequence of [11, (8.2)], (4.2), and (5.7).

(c) Parts (a) and (b) imply that formulae (5.9) and (5.10) can be obtained from [13, (1.23)] and [13, (1.24)], respectively, by dropping the summation over j. There is an additional factor d in the right-hand side of (5.9) and an additional factor $(d-1)$ in the right-hand side (5.10): this accounts for the somewhat nonstandard definition of Weyl coefficients adopted in this paper, compare (5.1) and [13, (1.6)].

Finally, in recasting [13, (1.24)] as (5.10) we used the identities

$$\{[v^{(j)}]^*, v^{(j)}\} = i\,\mathrm{tr}((P_j)_{\mathrm{sub}}) \tag{5.11}$$

and

$$\{[v^{(j)}]^*, A_{\mathrm{prin}}, v^{(j)}\} = -\,\mathrm{tr}(\{P^{(j)}, P^{(j)}\}A_{\mathrm{prin}}) + h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}, \tag{5.12}$$

where $v^{(j)}(x,\xi)$ denotes the normalised eigenvector of A_{prin} corresponding to the eigenvalue $h^{(j)}$ and $\{B, C, D\} := \sum_{\alpha=1}^{d}(B_{x^\alpha} C D_{\xi_\alpha} - B_{\xi_\alpha} C D_{x^\alpha})$ is the generalised Poisson bracket. Formula (5.11) follows from [13, formula (1.20)] and Corollary 4.2 (see also [12, Theorem 2.3]), whereas formula (5.12) is obtained via a lengthy but straightforward calculation involving (4.30) and properties of pseudodifferential projections. ■

In plain English, Theorem 5.3 tells us that, when applying Levitan's wave method [22] to the computation of spectral asymptotics for first order systems, the oscillatory integral $U^{(j)}(t)$ accounts for precisely the j-th of the m sequences of eigenvalues into which the spectrum of A was partitioned in Section 3.

A. Simultaneous diagonalization of unbounded operators

In this appendix we present some results from functional analysis. The purpose is to provide motivation for Theorems 2.1–2.3 in the main text of the paper. All operators in this appendix are assumed to be linear.

For the sake of clarity, let us start with the finite-dimensional setting. Let H be an n-dimensional complex inner product space. Given a self-adjoint operator $A\colon H \to H$ and a number $\lambda \in \mathbb{R}$ we denote by $\Pi^+(A;\lambda)$ the orthogonal projection onto the eigenspaces of A corresponding to eigenvalues greater than zero and less than λ. We also employ the notation (2.5).

Theorem A.1. *Let A and A_j, $j = 1, \dots, p$, be self-adjoint operators. Suppose that*

$$A^+ = \sum_{j=1}^{p} A_j^+, \tag{A.1}$$

$$A_j^+ A_l^+ = 0, \quad j, l = 1, \dots, p, \ j \neq l. \tag{A.2}$$

Then

$$\Pi^+(A; \lambda) = \sum_{j=1}^{p} \Pi^+(A_j; \lambda). \tag{A.3}$$

Proof. The self-adjoint operators A_j^+, $j = 1, \dots, p$, commute, hence one can choose a basis which simultaneously diagonalizes them [17, Theorem 2.3.3]. The diagonal entries in the matrix representations of the A_j^+s are either zeros or positive numbers, and formula (A.2) tells us that for different j the positive elements in the matrix representations of the A_j^+s are in different positions. This immediately implies (A.3). ∎

Let us now proceed to the infinite-dimensional setting. In what follows H is a separable complex Hilbert space. We will be dealing with self-adjoint operators which are not necessarily bounded and this leads to a number of difficulties. Indeed, generalising Theorem A.1 to infinite-dimensional spaces turns out to be a delicate matter.

Let us introduce the following definitions.

Definition A.2. Let $A \colon D \to H$ be a self-adjoint operator and $V \subseteq D$ be a vector subspace. We say that V is an *invariant subspace* of the operator A if $A(V) \subseteq V$.

Definition A.3. We say that an invariant subspace V of the self-adjoint operator A is *proper* if, for some λ in the resolvent set $\rho(A)$, the map $A - \lambda \, \mathrm{Id} \colon V \to V$ is surjective, and, hence, bijective.

The above definition can be equivalently recast as follows.

Definition A.4. We say that an invariant subspace V of the self-adjoint operator A is *proper* if, for some $\lambda \in \rho(A)$, V is an invariant subspace of the resolvent $(A - \lambda \, \mathrm{Id})^{-1}$.

Example A.5. (a) Any finite-dimensional invariant subspace is proper.

(b) Let $H = L^2(M)$ and let $A \in \Psi^s$, $s \in \mathbb{R}$, $s > 0$, be an elliptic self-adjoint operator (see Section 1 for notation). Then $C^\infty(M)$ is a proper invariant subspace.

(c) Let $H = l^2$, the Hilbert space of square summable sequences $x = (x_1, x_2, \dots)$. Let $S \colon (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$ be the right shift operator. Put $A := S + S^*$ and let $V = c_{00}$, the vector subspace of sequences that are eventually zero. Then V is an invariant subspace of the operator A but it is not proper. Indeed, take $y =$

$(1, 0, 0, \dots) \in V$. It is easy to see that for any $\lambda \in \mathbb{C}$ there does not exist an $x \in V$ satisfying $(A - \lambda \operatorname{Id})x = y$.

The following theorem gives sufficient conditions for the simultaneous diagonalizability of a family of unbounded commuting self-adjoint operators.

Theorem A.6. *Let H be an infinite-dimensional separable complex Hilbert space and let A_j, $j = 1, \dots, p$, be self-adjoint operators with discrete spectra, which admit a common proper invariant subspace V dense in H. If*

$$[A_j, A_l] = 0 \quad \text{on } V,\ j, l = 1, \dots, p, \tag{A.4}$$

then there exists an orthonormal basis $\{u_k\}$ such that each basis element u_k is an eigenvector of A_j for every $j = 1, \dots, p$.

Proof. Consider the pair of operators A_j and A_l for some $j \neq l$. Definitions A.2, A.3, and formula (A.4) imply that there exist $\lambda \in \rho(A_j)$ and $\mu \in \rho(A_l)$ such that the resolvents $(A_j - \lambda \operatorname{Id})^{-1}$ and $(A_l - \mu \operatorname{Id})^{-1}$ commute on V. Resolvents are bounded operators and V is dense in H, hence $[(A_j - \lambda \operatorname{Id})^{-1}, (A_l - \mu \operatorname{Id})^{-1}] = 0$ on H. This, in turn, implies that the operators A_j and A_l *strongly commute* in the sense of [30, Definition 5.2] in view of [30, Proposition 5.27]. The result now follows from [30, Theorem 5.21]. ■

Arguing along the lines of the proof of Theorem A.1, we see that Theorem A.6 immediately implies the following.

Theorem A.7. *Let H be an infinite-dimensional separable complex Hilbert space and let A and A_j, $j = 1, \dots, p$, be self-adjoint operators with the same domain D. Suppose that the operators A_j, $j = 1, \dots, p$, have discrete spectra and admit a common proper invariant subspace V dense in H. Furthermore, suppose that conditions (A.1) and (A.4) are fulfilled as well as*

$$\langle A_j^+(\,\cdot\,), A_l^+(\,\cdot\,)\rangle = 0 \quad \text{on } D \times D,\ j, l = 1, \dots, p, \quad j \neq l. \tag{A.5}$$

Then we have (A.3).

Of course, taking the trace in (A.3) one obtains an analogous result for the counting functions: $N^+(A; \lambda) = \sum_{j=1}^{p} N^+(A_j; \lambda)$. Here by $N^+(\,\cdot\,; \lambda)$ we denote the number of eigenvalues, with account of multiplicity, greater than zero and less than λ.

Examination of Theorem A.7 and Example A.5 (b) leads to the following corollary.

Corollary A.8. *Let A and A_j, $j = 1, \dots, p$, be elliptic self-adjoint operators from the class Ψ^s, $s \in \mathbb{R}$, $s > 0$. Suppose that conditions (A.1) and (A.2) are fulfilled and that the A_j, $j = 1, \dots, p$, commute. Then we have (A.3).*

For the sake of clarity, let us point out that the proper invariant subspace underpinning the above corollary is $C^\infty(M)$, because

- elliptic self-adjoint pseudodifferential operators and their resolvents map $C^\infty(M)$ to $C^\infty(M)$ and
- pseudodifferential operators form an algebra.

Remark A.9. Corollary A.8 connects with the arguments presented in Section 2 in that it provided strong motivation for our original conjecture on the structure of the spectrum of the operator A, compare formulae (A.1), (A.2), and $[A_j, A_l] = 0$ with (2.6), (2.7), and (2.4). Note, however, that in the main text the three conditions are not satisfied precisely but only modulo $\Psi^{-\infty}$. This calls for a more delicate spectral theoretic analysis than that given in this appendix and is ultimately responsible for the appearance of remainders in our main results, Theorems 2.1–2.3.

B. Weyl asymptotics for elliptic systems

In this appendix we provide, for the sake of completeness, a short proof of the Weyl law (one-term asymptotics with rough remainder estimate) for elliptic systems of arbitrary positive order. Though obtaining this result does not pose significant challenges, we were unable to find a rigorous proof for it in the literature.

Note that

(a) we are dealing with a *system* as opposed to a scalar operator,
(b) we allow the order of the operator to be *any positive real number* and
(c) the operator is *not necessarily semi-bounded.*

Spectral theory for elliptic systems has a long and troubled history, see [13, Section 11] for a review. Two-term asymptotic formulae for the counting function of a first order system ($s = 1$) were recently obtained by Chervova, Downes, and Vassiliev [13], see also [1, 2]. Some results are available for particular special cases, e.g. two-term asymptotics for nonnegative (pseudo)differential operators of even order are given in [29, 34], but we are unaware of general results.

Let $A \in \Psi^s$, $s > 0$, be an operator as in Section 1 and let

$$N^+(\lambda) := \begin{cases} 0 & \text{for } \lambda \le 0, \\ \sum_{k:0<\lambda_k<\lambda} 1 & \text{for } \lambda > 0 \end{cases} \tag{B.1}$$

be its positive counting function.

Theorem B.1. *We have*

$$N^+(\lambda) = b\lambda^{d/s} + o(\lambda^{d/s}) \quad \textit{as } \lambda \to +\infty, \tag{B.2}$$

where

$$b = \frac{1}{(2\pi)^d} \sum_{j=1}^{m^+} \int\limits_{h^{(j)}(x,\xi)<1} \mathrm{dVol}_{T^*M} \,. \tag{B.3}$$

Proof. Consider the function

$$f\colon (0,+\infty) \to [0,+\infty), \quad f(t) := \operatorname{Tr}(\theta(A)e^{-t|A|}) = \int\limits_{-\infty}^{+\infty} e^{-t\lambda}\, dN^+(\lambda).$$

The operators $|A|$ and $\theta(A)$ are pseudodifferential operators and this allows us to apply to the operator $\theta(A)e^{-t|A|}$ the standard technique from [31], giving us the asymptotic formula

$$f(t) = b\Gamma\Big(\frac{d}{s}+1\Big)t^{-d/s} + o(t^{-d/s}) \quad \text{as } t \to 0^+, \tag{B.4}$$

where Γ is the Gamma function. Karamata's Tauberian theorem [32, Problem 14.2] tells us that (B.4) implies (B.2). ∎

Remark B.2. Of course, for $s = 1$ the coefficient b appearing in Theorem B.1 is related to the coefficient $a_{d-1}(x)$ appearing in formula (5.1) as

$$b = \frac{1}{d} \int\limits_M a_{d-1}(x)\, dx.$$

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Matteo Capoferri
School of Mathematics, Cardiff University, Senghennydd Rd, Cardiff CF24 4AG, UK;
CapoferriM@cardiff.ac.uk

Dmitri Vassiliev
Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK; D.Vassiliev@ucl.ac.uk