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Integral representations of isotropic semiclassical functions and applications

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Abstract. In a previous paper, we introduced a class of "semiclassical functions of isotropic type," starting with a model case and applying Fourier integral operators associated with canonical transformations. These functions are a substantial generalization of the "oscillatory functions of Lagrangian type" that have played major role in semiclassical and microlocal analysis. In this paper we exhibit more clearly the nature of these isotropic functions by obtaining oscillatory integral expressions for them. Then, we use these to prove that the classes of isotropic functions are equivariant with respect to the action of general FIOs (under the usual clean-intersection hypothesis). The simplest examples of isotropic states are the "coherent states," a class of oscillatory functions that has played a pivotal role in mathematics and theoretical physics beginning with their introduction by of Schrödinger in the 1920's. We prove that every oscillatory function of isotropic type can be expressed as a superposition of coherent states, and examine some implications of that fact. We also show that certain functions of elliptic operators have isotropic functions for Schwartz kernels. This lead us to a result on an eigenvalue counting function that appears to be new (Corollary 4.5).

In memory of Mikhail Shubin

1. Introduction

The world of microlocal analysis is populated by objects of "Lagrangian type." For instance, pseudodifferential operators on a manifold X live microlocally on the diagonal in $T^*X \times (T^*X)^-$, and Fourier integral operators on a canonical relation

$$\Gamma\colon T^*X \rightrightarrows T^*Y,$$

which is a Lagrangian submanifold of $T^*X \times (T^*Y)^-$. The topic of this paper, however, will be objects in semiclassical analysis that live instead on *isotropic* submanifolds, which can be defined as any submanifold of a Lagrangian submanifold.

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Microlocal objects of this type are not entirely unfamiliar, and in complex analysis have been around since the 1970's. A prototypical example is the Szegö projector,

$$\Pi \colon L^2(X) \to H(X),$$

where X is the boundary of a smooth compact pseudoconvex domain, Ω, in $\mathbb{C}^n$, and $H(X)$ is the L^2-closure of $\mathcal{O}(\Omega)|_X$. More generally, from Π one gets a large class of operators,

$$T = \Pi P \Pi \colon H(X) \to H(X), \tag{1.1}$$

where P is a pseudodifferential operator on X. Operators of the form (1.1), known as *(generalized) Toeplitz operators,* were first defined and studied by Boutet de Monvel and the first named author of this paper in [1], and have played an important role in the theory of geometric quantization as well as in other areas. Their Schwartz kernels are isotropic distributions, associated to a particular submanifold of the diagonal (which we will not describe here).

In this paper we will consider objects of this type in the real C^∞ category rather than in the complex category, and within the framework of semiclassical rather than microlocal analysis. Furthermore, since we are working in the semiclassical world, we no longer need to assume that these isotropic objects are conic, as in the microlocal setting.

The prototypical example of such an isotropic function is

$$\Upsilon(x,\hbar) := \hbar^r e^{if(x)/\hbar}\varphi(x', \hbar^{-1/2}x'', \hbar), \tag{1.2}$$

where $\hbar$ is Planck's constant (or the semiclassical parameter) and

- $x = (x', x'')$ are coordinates on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$,
- $f \in C^\infty(\mathbb{R}^n)$ is a real-valued smooth function,
- $\varphi \in C^\infty(\mathbb{R}^n \times (0, \hbar_0), \mathbb{C})$ is a smooth function which is Schwartz in the variable x'' with estimates locally uniform in x' and admits an asymptotic expansion

$$\varphi(x', x'', \hbar) \sim \hbar^r \sum_{j=0}^{\infty} \varphi_j(x', x'')\hbar^{j/2}, \tag{1.3}$$

and where, for all j, $\varphi_j(x', x'')$ is a Schwartz function in the variable x'' satisfying Schwartz estimates that are locally uniform in x'.

As $\hbar$ tends to zero, the function Υ becomes more and more concentrated on the manifold

$$Z = \mathbb{R}^k \times \{0\}$$

in configuration space and more and more concentrated on the manifold

$$\Sigma = \{(x, df_x) \mid x \in Z\}$$

in phase space. Note that Σ is an isotropic submanifold inside the Lagrangian submanifold

$$\Lambda_f = \{(x, df_x) \mid x \in \mathbb{R}^n\}, \tag{1.4}$$

which has f as its generating function.

More generally, suppose that X is an n-dimensional manifold and $\Sigma \subset T^*X$ an isotropic submanifold of its cotangent space. In [4], we defined classes of $\hbar$-dependent smooth functions on X, $I^r(X, \Sigma)$ (r being a half integer), whose semiclassical wave front set is contained in Σ. Our general definition of $I^r(X, \Sigma)$ is as follows. We take, as a model case, the prototypical example above where $f = 0$. The resulting functions have wave-front set in the canonical isotropic Σ_0 given by

$$\Sigma_0 = \{(x, \xi)\colon x'' = 0, \xi' = 0, \xi'' = 0\} \subset T^*\mathbb{R}^n, \tag{1.5}$$

and we define a model class $I^r(\mathbb{R}^n, \Sigma_0)$ whose elements $\Upsilon \in I^r(\mathbb{R}^n, \Sigma_0)$ are of the form

$$\Upsilon(x, \hbar) = \varphi(x', \hbar^{-1/2}x'', \hbar) \tag{1.6}$$

where $\varphi(x', x'', \hbar)$, as $\hbar \to 0$, admits an asymptotic expansion

$$\varphi(x', x'', \hbar) \sim \hbar^r \sum_{j=0}^{\infty} \varphi_j(x', x'')\hbar^{j/2}, \tag{1.7}$$

and where, for all j, $\varphi_j(x', x'')$ is a Schwartz function in the variable x'' satisfying Schwartz estimates that are locally uniform in x'. After proving that the space $I^r(\mathbb{R}^n, \Sigma_0)$ is invariant under FIOs associated with canonical transformations preserving Σ_0 (as a set), we were able to define the general classes $I^r(X, \Sigma)$ microlocally as the images of these functions under FIOs associated to local canonical transformations mapping Σ_0 to Σ.

One of the main results of the present paper is that these classes are invariant under the action of arbitrary Fourier integral operators, provided the usual clean intersection condition is satisfied:

Theorem 1.1. *Let $\Sigma \subset T^*Y$ be an isotropic submanifold and $\Upsilon \in I^r(Y, \Sigma)$ an associated isotropic function. Let $F\colon C_0^\infty(Y) \to C^\infty(X)$ be a semiclassical FIO of order zero whose canonical relation, $\Gamma \subset T^*X \times (T^*Y)^-$, intersects Σ cleanly. Then,*

$$F(\Upsilon) \in I^{r+(\dim Y - \dim X - e)/2}(X, \Gamma(\Sigma)),$$

i.e. $F(\Upsilon)$ is an isotropic function associated with the image of Σ under Γ. Here, e is the excess of the clean composition (see Section 3).

It turns out that, in proving this FIO invariance result, the definition of $I^r(X,\Sigma)$ we alluded to above is not convenient to use. For this reason, and because of its general interest to this theory, we will give another equivalent characterization for elements in $I^r(X,\Sigma)$ in terms of local integral representations, generalizing the Lagrangian case as developed by Hörmander.

Before describing this Hörmander approach, we will first of all assume that Σ is a *horizontal* isotropic submanifold of T^*X, i.e. that there exist a submanifold X_Σ of X and a function $f \in C^\infty(X)$ such that

$$(x,\xi) \in \Sigma \iff x \in X_\Sigma \text{ and } \xi = df_x.$$

In this case, we will define $I^r(X,\Sigma)$ as the set of functions $\Upsilon(x,\hbar) \in C^\infty(X \times \mathbb{R})$ which, at $x \in X \setminus X_\Sigma$, vanish to order $O(\hbar^\infty)$ on a neighborhood of x and for $x \in X_\Sigma$ are of the form (1.2) on a coordinate patch $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$ with the properties above.

Coming back to the Hörmander's approach, let $\Lambda \subset T^*X$ be an arbitrary Lagrangian submanifold. Then, at least locally, one can define Λ as follows. One can find a fiber bundle

$$Z \xrightarrow{\pi} X$$

and a function $f \in C^\infty(Z)$ such that

(I) the manifold Λ_f (see (1.4)) intersects the horizontal subbundle

$$H^*Z := \pi^*T^*X$$

transversally;

(II) the canonical fiber bundle map

$$H^*Z \to T^*X \tag{1.8}$$

maps the intersection $\Lambda_f \cap H^*Z$ diffeomorphically onto Λ.

In fact, we will take $Z = X \times \mathbb{R}^N$ to be the trivial bundle, in which case

$$H^*Z = \{(x,s,\zeta,\sigma) \mid \sigma = 0\}$$

so that $\Lambda_f \cap H^*Z = C_f = \{(x,s,(d_x f)_{(x,s)},0);\ (d_s f)_{(x,s)} = 0\}$. Following Hörmander, we will call such an f a *generating function* for $\Lambda \subset T^*X$ with respect to the fibration π. The procedure of passing from Λ_f to Λ is the *reduction* of Λ_f with respect to the co-isotropic submanifold H^*Z of T^*Z.

Our adaption of this approach to the isotropic setting is as follows. We will show (Proposition 2.2) that if $\Sigma \subset T^*X$ is an isotropic submanifold, then one can (at least locally) find a Lagrangian submanifold $\Lambda \supset \Sigma$ in T^*X, a fibration $\pi\colon Z \to X$ and a function $f \in C^\infty(Z)$ with properties (I) and (II) above, and a submanifold $Z_\Sigma \subset Z$ such that

(III) the manifold

$$\Sigma_f = \{(z, df_z) \mid z \in Z_\Sigma\}$$

intersects the horizontal subbundle H^*Z transversally, and

(IV) the bundle map $H^*Z \to T^*X$ maps the intersection $\Sigma_f \cap H^*Z$ diffeomorphically onto Σ.

Such a function f will be called a *non-degenerate phase function parametrizing the pair* (Σ, Λ).

Our generalization to the isotropic case (and in the semiclassical setting) of Hörmander's characterization of $I^k(X, \Lambda)$ via generating functions is the following. We first equip Z and X with non-vanishing smooth measures, μ_Z and μ_X, and define a push forward operation

$$\pi_* \colon C_0^\infty(Z) \to C^\infty(X)$$

with the defining property

$$\pi_*(g)\mu_X = \pi_*(g\mu_Z) \tag{1.9}$$

for all $g \in C_0^\infty(Z)$, where π_* on the right-hand side is the fiber integral operation. Then we claim:

Theorem 1.2. *A function lies in* $I^r(X, \Sigma)$ *if and only if locally it is of the form*

$$\pi_*(\Upsilon_0), \quad \Upsilon_0 \in I^{r-\frac{N}{2}}(Z, \Sigma_f), \tag{1.10}$$

where f is a non-degenerate phase function parametrizing (Σ, Λ) *and* $N = \dim Z - \dim X$ *is the fiber dimension.*

Remark 1.3. (1) As we will see, this theorem translates in the integral representation (2.6).

(2) In particular, if Σ is a Lagrangian submanifold of T^*X, then these objects are semiclassical analogues of Hörmander's distribution of Lagrangian type, and were discussed at length in [3].

For x a point in X and $\Sigma = \{(x, \xi)\}$ a one-point set in T^*X, the associated isotropic functions have been around in the physics literature since the 1920's and are known as *coherent states*. There is a vast literature on coherent states both in mathematics and physics journals; we mention work by R. Littlejohn [7], G. Hagedorn [6], and particularly the monograph by M. Combescure and D. Robert [2] and references therein.

On coordinate patches centered at the point and if $\xi = df_0$, they have the simple form

$$e^{if(x)/\hbar}\varphi(\hbar^{-1/2}x, \hbar),$$

where φ is a Schwartz function in x that satisfies estimates of the form (1.7), with $x'' = x$. Thus, as $\hbar$ tends to zero, they become more and more concentrated at the point $x = 0$ in configuration space and at the point $(0, \xi)$ in phase space. In the last part of this paper, we will discuss isotropic functions from a more intuitive perspective as "superpositions of coherent states." In Section 5, we will show that every semiclassical oscillatory function of isotropic type can be defined as a superposition of coherent states, and we discuss some consequences of this fact.

For Hörmander, the main purpose of introducing distributions of Lagrangian type was that it gave him a very clean simple way of defining Fourier integral operators. Namely, let $\Gamma\colon T^*Y \Rightarrow T^*X$ be a canonical relation, and let $\Gamma^\# \subset T^*(X \times Y)$ be the associated Lagrangian submanifold, i.e.

$$((x,\xi),(y,\eta)) \in \Gamma \iff (x,y,\xi,-\eta) \in \Gamma^\#.$$

Then, the associated semiclassical Fourier integral operators $F\colon C_0^\infty(Y) \to C^\infty(X)$ are operators whose Schwartz kernels are elements in the space $I^k(X \times Y, \Gamma^\#)$ for some k. This definition works equally well for *isotropic* canonical relations, i.e. relations $\Gamma\colon T^*Y \Rightarrow T^*X$ for which $\Gamma^\#$ is an isotropic submanifold. Consequently, we will call the isotropic analogues of the operators above *Fourier integral operators of isotropic type*. As a consequence of Theorem 1.1, one can easily show that if F_1 and F_2 are Fourier integral operators of isotropic type with microsupports on canonical relations Γ_1 and Γ_2 and these relations are cleanly composable, then $F_2 \circ F_1$ is a Fourier integral operator of isotropic type with microsupport on $\Gamma_2 \circ \Gamma_1$. In Section 4 we prove that certain functions of elliptic operators are of this kind, which leads to a variation of the Weyl eigenvalue counting function that appears to be new (Corollary 4.5). We would like to think that Mikhail Shubin would have enjoyed this result, if he did not know it already.

We hope to do a more systematic study of isotropic FIOs in the future.

2. Local integral representations

2.1. Non-degenerate phase functions

The purpose of this section is to prove Theorem 1.2, which gives an alternative definition of $I^r(X, \Sigma)$. First, we introduce some terminology:

Definition 2.1. Let X be a smooth manifold of dimension n and $\Sigma \subset T^*X$ an isotropic submanifold. A *framing* of Σ is a Lagrangian submanifold $\Lambda \subset T^*X$ such that $\Sigma \subset \Lambda$. The pair (Σ, Λ) is called a *framed isotropic submanifold* of T^*X.

The existence of framings for a given Σ is not hard to establish; the proof is sketched in [5, Lemma 1]. In what follows, strictly speaking, we will work with germs of framed isotropics, by which we mean that we will not distinguish between (Σ, Λ) and (Σ, Λ') if $\Lambda \cap \Lambda'$ contains a relative open set in each. We will not use the language of germs explicitly, but germ-equivalence will be tacitly assumed.

We have explained Hörmander's extension of generating functions for arbitrary Lagrangian submanifolds in the introduction. The isotropic version of Hörmander's theorem is as follows.

Proposition 2.2. *Let (Σ, Λ) be a framed isotropic submanifold of T^*X. Then there exist, at least for a neighborhood of each point in Σ, a fiber bundle $\pi\colon Z \to X$, a horizontal Lagrangian submanifold, $\Lambda_f \subset T^*Z$, with generating function $f \in C^\infty(Z)$, and an isotropic submanifold, Σ_f, of Λ_f such that Λ_f and Σ_f intersect H^*Z transversally and the projection map (1.8) maps $\Lambda_f \cap H^*Z$ onto Λ and $\Sigma_f \cap H^*Z$ onto Σ.*

We now rephrase this proposition in the language of phase functions. For any Lagrangian submanifold $\Lambda \subset T^*X$, recall that a function

$$f = f(x, s) \in C^\infty(X \times \mathbb{R}^N)$$

is a *non-degenerate phase function parametrizing* Λ if

(1) zero is a regular value of the map

$$X \times \mathbb{R}^N \to \mathbb{R}^N, \quad (x, s) \mapsto d_s f$$

and

(2) the map

$$\Phi\colon X \times \mathbb{R}^N \to T^*X, \quad (x, s) \mapsto (x, (d_x f)_{(x,s)})$$

is an embedding from the critical set

$$C_f := \{(x, s) \mid d_s f = 0\}$$

onto Λ (it is automatically an immersion).

Then, Proposition 2.2 can be restated as:

Proposition 2.3. *Given a framed isotropic submanifold (Σ, Λ), there is a covering of Λ by relative open sets $\Gamma \subset \Lambda$ such that, for each Γ, there exist*

(1) *open sets $\mathcal{V} \subset X$ and $S \subset \mathbb{R}^N$,*

(2) *a splitting of the variables $S \ni s = (t, u) \in \mathbb{R}^K \times \mathbb{R}^l$,*

(3) *a function $f\colon \mathcal{V} \times S \to \mathbb{R}$*

such that

(a) *f is a non-degenerate phase function parametrizing Γ;*

(b) *the intersection*

$$C_f \cap \{u = 0\}$$

is transverse, and

(c) *Under the map Φ above, $C_f \cap \{u = 0\}$ maps onto $\Sigma \cap \Gamma$.*

We postpone the proof until after Proposition 2.5.

Definition 2.4. A function f satisfying (a), (b), and (c) above will be called a *non-degenerate phase function parametrizing the framed isotropic* (Σ, Λ).

It is obvious that such parametrizations are not unique. But it is easy to check that, in any parametrization, the number l of u variables must be the same, namely,

$$l = n - \dim(\Sigma).$$

Note that this l coincides with the notation used for the splitting of $\mathbb{R}^n$ in the introduction.

Now, let (Σ_1, Λ_1) be a framed isotropic submanifold of T^*X, and let the map $\phi: T^*X \to T^*Y$ be a symplectomorphism. Then $(\Sigma_2, \Lambda_2) = \phi(\Sigma_1, \Lambda_1)$ is a framed isotropic in T^*Y. Let $\Gamma \subset T^*(X \times Y)$ be the Lagrangian submanifold associated with ϕ, and let

$$g = g(x, y, s) \in C^\infty(X \times Y \times S)$$

be a non-degenerate phase function parametrizing Γ. Then, by definition, the map

$$\Phi_2: X \times Y \times S \to T^*(X \times Y), \quad (x, y, s) \mapsto (x, y, (d_{x,y}g)_{(x,y,s)})$$

maps the critical set

$$C_g = \{(x, y, s) \mid d_s g = 0\}$$

diffeomorphically onto the Γ. As a consequence,

$$\phi(x, -(d_x g)_{x,y,s}) = (y, (d_y g)_{(x,y,s)}), \quad \text{for all } (x, y, s) \in C_g. \tag{2.1}$$

In proving the main theorem of this section, we will need the following:

Proposition 2.5. *Let $f = f(x, t, u)$ be a non-degenerate phase function (with fiber variables t and u) parametrizing the framed isotropic submanifold (Σ_1, Λ_1) in T^*X, and let $g = g(y, x, s)$ be a non-degenerate phase function (with fiber variables s) parametrizing Γ, the Lagrangian submanifold associated with the symplectomorphism $\phi: T^*X \to T^*Y$. Then the function*

$$F(y, x, s, t, u) := f(x, t, u) + g(y, x, s)$$

is a non-degenerate phase function (with fiber variables x, s, t, and u) parametrizing the framed isotropic submanifold $(\Sigma_2, \Lambda_2) = \phi(\Sigma_1, \Lambda_1)$ *in* T^*Y.

Proof. Since ϕ is a symplectomorphism, the composition $\Gamma \circ \Lambda_1$ is transversal. As a consequence, F is a non-degenerate phase function parametrizing the Lagrangian submanifold $\Lambda_2 = \phi(\Lambda_1)$. In particular, the map

$$\Phi\colon Y \times (X \times S \times T \times U) \to T^*Y, \quad (y, x, s, t, u) \mapsto (y, d_y F)$$

is a diffeomorphism from the critical set

$$C_F := \{(y, x, s, t, u)\colon d_x F = 0, d_t F = 0, d_u F = 0, d_s F = 0\}$$

onto Λ_2. (For a proof of these two assertions, see, e.g. [3, §4.3 and Theorem 5.6.1]).

It remains to prove

(a) C_F intersects $\{(y, x, s, t, u)\colon u = 0\}$ transversally and

(b) the intersection $C_F \cap \{(y, x, s, t, u)\colon u = 0\}$ gets mapped onto Σ_2 under the map Φ.

Note that, by definition,

$$C_f = \{(x, t, u)\colon d_t f = 0, d_u f = 0\}$$

intersects the set $\{(x, t, u) \mid u = 0\}$ transversally and the intersection gets mapped diffeomorphically onto Σ_2 under the map

$$\Phi_1\colon X \times T \times U \to T^*X, \quad (x, t, u) \mapsto (x, d_x f).$$

Denote by $\pi\colon Y \times X \times T \times U \times S \to X \times T \times U$ the standard projection map. Then, by definition, $\pi(C_F) \subset C_f$. We can show that $\pi|_{C_F}\colon C_F \to C_f$ is a diffeomorphism. In fact, by definition, $(y, x, t, u, s) \in C_F$ implies $(x, y, s) \in C_g$ and $d_x f = -d_x g$. In view of (2.1), the following diagram commutes:

$$\begin{array}{ccc} Y \times X \times T \times U \times S \supset C_F & \xrightarrow{\Phi} & \Lambda_2 \subset T^*Y \\ \pi \downarrow & & \uparrow \phi \\ X \times T \times U \supset C_f & \xrightarrow{\Phi_1} & \Lambda_1 \subset T^*X \end{array} \tag{2.2}$$

So, π maps C_F diffeomorphically onto C_f. Since C_f intersects $\{(x, t, u) \mid u = 0\}$ transversally, we conclude that C_F intersects $\{(y, x, t, s, u)\colon u = 0\}$ transversally.

Finally, to prove assertion (b), it is enough to chase the commutative diagram (2.2) with C_F replaced by $C_F \cap \{(y, x, t, s, u)\colon u = 0\}$, to get

$$\begin{array}{ccc} C_F \cap \{(y, x, s, t, u)\colon u = 0\} & \xrightarrow{\Phi} & \Sigma_2 \\ \pi \downarrow & & \uparrow \phi \\ C_f \cap \{(x, t, u)\colon u = 0\} & \xrightarrow{\Phi_1} & \Sigma_1 \end{array}$$

■

Proof of Proposition 2.3. It is easy to see that, locally, any framed isotropic (Λ, Σ) can be mapped by a canonical transformation to the pair (Λ_0, Σ_0) where $\Lambda_0 \subset T^*\mathbb{R}^n$ is the zero section and Σ_0 the model isotropic (1.5) of the appropriate dimension. By Proposition 2.5, it suffices to show that there is a non-degenerate phase function parametrizing (Λ_0, Σ_0), and one readily can check that

$$f(x', x'', t, u) = t \cdot (x'' - u) \tag{2.3}$$

is such a phase function. ∎

2.2. Amplitudes and the local form

We now introduce smooth amplitudes

$$a(x, t, u, \hbar) \in C^\infty(\mathcal{V} \times \mathcal{T} \times \mathbb{R}^l \times (0, \hbar_0)), \quad \mathcal{V} \subset \mathbb{R}^n,\ \mathcal{T} \subset \mathbb{R}^K \text{ open},$$

which are Schwartz functions in the variable u, with estimates locally uniform in (x, t). We will also have to allow for $\hbar$-dependence, in the C^∞ topology in (x, t):

$$\begin{aligned}&\text{for all } \alpha, \beta, \gamma, m \text{ there exists } C \text{ such that}\\ &|\partial_x^\alpha \partial_t^\beta \partial_u^\gamma a(x, t, u, \hbar)| \le C(1 + \|u\|)^{-m} \quad \text{for all } (x, t, u) \in X \times \mathbb{R}^N.\end{aligned} \tag{2.4}$$

We will further assume that there exists a sequence $\{a_j(x, t, u)\}$ of smooth functions, compactly supported in the variable t and Schwartz in the variable u, and such that

$$a(x, t, u, \hbar) \sim \sum_{j=0}^{\infty} a_j(x, t, u) h^{j/2}. \tag{2.5}$$

We can now restate Theorem 1.2 as follows, which is the main result of this section:

Theorem 2.6. *Given a parametrization of* (Σ, Λ)*,* $\Lambda \subset T^*X$*, by a phase function* $f: X \times \mathcal{T} \times U \to \mathbb{R}$*, where* $U \subset \mathbb{R}^l$*,* $\mathcal{T} \in \mathbb{R}^K$*, and* $N = K + l$*, then for amplitudes given by* (2.5)*,*

$$\Upsilon(x, \hbar) = \hbar^{r - \frac{N}{2}} \int_{\mathcal{T} \times U} e^{\sqrt{-1}\hbar^{-1} f(x,t,u)} a(x, t, \hbar^{-1/2} u, \hbar)\, dt\, du \tag{2.6}$$

is in the class $I^r(X, \Sigma)$*. Conversely, if* $\Upsilon \in I^r(X, \Sigma)$ *and* $\sigma_0 \in \Sigma$*, then microlocally near* σ_0*,* Υ *is equal to an integral of the previous form.*

Note that the integration over u is only over the open set U. However, as a function of $u \in U$, the function a is defined on all of $\mathbb{R}^l$, which in particular implies that for any $\hbar$ the integrand above is well defined and the integral converges absolutely. Note also that a may be chosen to be compactly supported in the variable u.

A preliminary result is:

Lemma 2.7. *If Υ is as in (2.6), then its semiclassical wave-front set equals*

$$WF(\Upsilon) = \{(x, (d_x f)_{(x,t,0)});\ (d_s f)_{(x,t,0)} = 0 \text{ and } (x,t,0) \in \operatorname{supp}(a)\}, \tag{2.7}$$

where we have let $s = (t, u)$. In particular,

$$WF(\Upsilon) \subset \Sigma. \tag{2.8}$$

Proof. Let us begin by considering a function of the form

$$\Upsilon_0(x,u) = e^{i\hbar^{-1} f(x,u)} a(x, h^{-1}u) \tag{2.9}$$

on an open set of the form $X \times U$. (We will see below that in fact Υ_0 is one of the oscillatory functions associated to the pair (Σ, Λ), where Λ is the graph of df and Σ the part of Λ where $u = 0$.) It is clear that, for any compact set $D \subset X \times U$ such that $D \cap (X \times \{0\}) = \emptyset$, one has

$$\Upsilon_0|_D = O(\hbar^\infty)$$

in the C^∞ topology, by the fact that $a(x,u)$ is Schwartz in u. On the other hand, the wave-front set of $e^{i\hbar^{-1} f(x,u)}$ is just the graph of df. From this, it follows that

$$\mathrm{WF}(\Upsilon_0) = \{(x, u; df_{(x,u)}) \colon u = 0 \text{ and } (x,0) \in \operatorname{supp}(a)\}. \tag{2.10}$$

Consider next a function of the form

$$\Upsilon_1(x) = \int_U e^{i\hbar^{-1} f(x,u)} a(x, h^{-1}u)\, du. \tag{2.11}$$

This is simply the push-forward by the natural projection $X \times U \to X$ of a function Υ_0 of the form considered above. By the calculus of wave-front sets, we obtain for the wave-front set of (2.11)

$$\mathrm{WF}(\Upsilon_1) = \{(x, (d_x f)_{(x,u)}) \colon (d_u f)_{(x,u)} = 0 \text{ and } u = 0 \text{ and } (x,0) \in \operatorname{supp}(a)\}. \tag{2.12}$$

Consider now the general case (as in the statement of the Lemma). By (2.12), we know the wave-front set of

$$\Upsilon_2(x,t) = \int_U e^{i\hbar^{-1} f(x,t,u)} a(x,t,\hbar^{-1/2}u, \hbar)\, du, \quad (x,t) \in U \times \mathcal{T}.$$

But $\Upsilon = \pi_*(\Upsilon_2)$, where $\pi \colon X \times \mathcal{T} \to X$ is the natural projection. The general result follows, again by the calculus of wave-front sets. ∎

2.3. Proof of the first half of Theorem 2.6

To prove the first half of Theorem 2.6, we first prove a special case:

Lemma 2.8. *Let $x = (x', x'')$ denote the variables in $\mathbb{R}^n = \mathbb{R}^{k+l}$, $\Lambda_0 \subset T^*\mathbb{R}^{k+l}$ the zero section, and $\Sigma_0 = \Lambda_0 \cap \{x'' = 0\}$. Let $\mathcal{V} \subset \mathbb{R}^{k+l}$, $\mathcal{S} \subset \mathbb{R}^{K+l}$ be open sets and $f\colon \mathcal{V} \times \mathcal{S} \to \mathbb{R}$, $f = f(x,t,u)$ be a non-degenerate phase function parametrizing (Σ_0, Λ_0). Let $a(x,t,u,\hbar)$ be an amplitude as in (2.5). Then*

$$\Upsilon(x,\hbar) = \hbar^{r-\frac{N}{2}} \int e^{\sqrt{-1}\hbar^{-1} f(x,t,u)} a(x,t,u/\sqrt{\hbar},\hbar)\, dt\, du$$

is in the model class of isotropic functions $I^r(\mathbb{R}^n, \Sigma_0)$.

Proof. We will apply the method of stationary phase to the integral. For this, we need to understand the critical points of the phase with respect to the fiber variables $s = (t,u)$.

By definition, for the critical set $C_f = \{(x,s) \mid d_s f = 0\}$ we know that

- the Jacobian
$$\mathbb{J} = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial s} & \frac{\partial^2 f}{\partial s \partial s} \end{pmatrix}$$
has full rank at each point on C_f,
- the projection $\pi_1\colon C_f \to \mathcal{V}$, $(x,s) \mapsto x$ is a local diffeomorphism,
- C_f intersect transversally with the set $\{u = 0\}$,
- the intersection $C_f \cap \{u = 0\}$ is diffeomorphic to Σ_0.

Take any point p in the intersection $C_f \cap \{u = 0\}$. Then, as we know, the map

$$\pi_1\colon C_f \to \mathcal{V}, \quad (x', x'', t, u) \mapsto (x', x'')$$

is a local diffeomorphism near p. So, in particular, there exist functions f_1, f_2 so that on C_f, $t = f_1(x', x'')$ and $u = f_2(x', x'')$ near p. We also know that C_f intersects $\{u = 0\}$ transversally and the map

$$\Psi\colon C_f \cap \{u = 0\} \to \Sigma_0, \quad (x', x'', t, u) \mapsto (x', x'', t, 0) \mapsto (x', 0, 0, 0)$$

is a local diffeomorphism near p. As a consequence, we see that the matrix $\left(\frac{\partial f_2}{\partial x''}\right)$ is non-degenerate. Therefore, there is a smooth function g so that on C_f, $x'' = g(x', u)$ near p. Now, we define a new coordinate system of $\mathbb{R}^{n+N}$ near p,

$$(x', x'', t, u_{\mathrm{new}}) = (x', x'', t, g(x', u)),$$

and we denote the inverse coordinate transformation by

$$(x', x'', t, u) = (x', x'', t, \tilde{g}(x', u_{\mathrm{new}})).$$

Then, under this new coordinate system, the critical set C_f is contained in the set $\{x'' = u_{\text{new}}\}$. As a consequence, the phase function f can be written, in these new coordinates, as

$$f(x', x'', t, u) = f_0(x', t, u) + \sum_{|\alpha|=2} f_\alpha(x', x'', t, u)(x'' - u)^\alpha.$$

According to the condition that f parametrizes (Σ_0, Λ_0), we conclude that

$$f_0(x', t, u) = C_0 + \sum_{|\gamma|=2} h_\gamma(x', t)t^\alpha + \sum_{|\beta|=2} g_\beta(x', t, u)u^\beta.$$

It follows that

$$\Upsilon(x) = e^{\sqrt{-1}\hbar^{-1}C_0} \int e^{\sqrt{-1}\hbar^{-1}\sum h_\gamma(x',t)t^\alpha + g_\beta(x',t,u)u^\beta + f_\alpha(x',x'',t,u)(x''-u)^\alpha} \cdot \tilde{a}(x', x'', t, u, \hbar) \det J \, dt \, du,$$

where the summation is over $|\alpha| = 2, |\beta| = 2, |\gamma| = 2$,

$$\tilde{a}(x', x'', t, u, \hbar) = a(x', x'', t, \tilde{g}(x', u)/\sqrt{\hbar}, \hbar),$$

and $J = (\frac{\partial g}{\partial u})$. We want to show that the function $\Upsilon(x', \sqrt{\hbar}x'', \hbar)$ is Schwartz in x''. In fact, by changing variables $t \to \sqrt{\hbar}t$ and $u \to \sqrt{\hbar}u$, we have

$$\Upsilon(x', \sqrt{\hbar}x'', \hbar) = \hbar^{N/2} e^{\sqrt{-1}\hbar^{-1}C_0} \int e^{\mathfrak{S}} \tilde{a}(x', \sqrt{\hbar}x'', \sqrt{\hbar}t, \sqrt{\hbar}u, \hbar) \det J dt du,$$

where

$$\begin{aligned}\mathfrak{S} := \sqrt{-1} \sum h_\gamma(x', \sqrt{\hbar}t)t^\alpha + g_\beta(x', \sqrt{\hbar}t, \sqrt{\hbar}u)u^\beta \\ + f_\alpha(x', \sqrt{\hbar}x'', \sqrt{\hbar}t, \sqrt{\hbar}u)(x'' - u)^\alpha.\end{aligned}$$

Noting that $\tilde{g}(x', \sqrt{\hbar}u)/\sqrt{h}$ has an expansion in nonnegative powers of $\hbar$, we see that

$$\tilde{a}(x', \sqrt{\hbar}x'', \sqrt{\hbar}t, \sqrt{\hbar}u, \hbar) \sim \hbar^{r-\frac{N}{2}} \sum \tilde{a}_j(x, t, u)\hbar^{j/2},$$

where each $\tilde{a}_j$ is Schwartz in u. Now the conclusion follows. ∎

To prove the general case, we let $\phi: T^*X \to T^*\mathbb{R}^n$ be a symplectomorphism such that

$$\phi(\Sigma, \Lambda) = (\Sigma_0, \Lambda_0).$$

Let F be a semiclassical Fourier integral operator of degree zero quantizing ϕ (We are following the definition of order of an FIO of [3, §8.2].). Then by definition, there exists an amplitude b such that

$$F(\Upsilon)(y) = \hbar^{r-(N+n+N')/2} \int e^{\sqrt{-1}\hbar^{-1}(g(x,y,s)+f(x,t,u))} b(x, y, s, \hbar) \cdot a(x, t, u/\sqrt{\hbar}, \hbar) \, dx \, ds dt du, \quad (2.13)$$

where N' is the number of s-parameters. According to Proposition 2.5, the new phase function $g(x,y,s) + f(x,t,u)$ is a non-degenerate phase function (with parameters x,s,t,u) parametrizing (Σ_0, Λ_0). According to Lemma 2.8, $F(\Upsilon)$ is an element of $I^r(\mathbb{R}^n, \Sigma_0)$. It follows that $\Upsilon \in I^r(X, \Sigma)$.

2.4. Applying an FIO associated to a canonical transformation to the model case

Finally, we prove the second part of Theorem 2.6. As before we split $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$; let $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^{k+l}$ be open sets, and $F: C_0^\infty(\mathcal{V}) \to C^\infty(\mathcal{U} \times (0, \hbar_0))$ be a semiclassical FIO of degree zero associated to a canonical transformation Φ: $\mathcal{U} \times \mathbb{R}^n \to \mathcal{V} \times \mathbb{R}^n$. Let $f(x,y,s) \in C^\infty(\mathcal{U} \times \mathcal{V} \times \mathcal{S})$, $\mathcal{S} \subset \mathbb{R}^N$, be a non-degenerate phase function parametrizing the graph of the canonical transformation Φ. Let Υ be a function in our model space, (1.6). The result of applying F to a model isotropic state is an integral of the form

$$F(\Upsilon)(x) = \hbar^{-(n+N)/2} \int e^{\sqrt{-1}\hbar^{-1} f(x,y',y'',s)} b(x,y',y'',s,\hbar) \cdot \varphi(y', y''/\sqrt{\hbar})\, dy'\, dy''\, ds. \tag{2.14}$$

We will show that $F(\Upsilon)$ is locally of the form (2.6).

Lemma 2.9. *Let $f(x,y,s) \in C^\infty(\mathcal{U} \times \mathcal{V} \times \mathcal{S})$, with $\mathcal{S} \subset \mathbb{R}^N$, be a non-degenerate phase function parametrizing the graph of a canonical transformation Φ: $\mathcal{U} \times \mathbb{R}^n \to \mathcal{V} \times \mathbb{R}^n$. Then f, considered as a phase function with fiber variables (y,s), is a non-degenerate phase function parametrizing the image of the zero section $\mathcal{U} \times \{0\}$ under Φ. Moreover, if we split $y = (y', y'')$, then $f(x,y',y'',s)$ (where the fiber variables are (y',y'',s)) parametrizes the isotropic $\Phi(\Sigma_0)$.*

Proof. The assumption on f is that zero is a regular value of the map

$$(x,y,s) \mapsto d_s f(x,y,s),$$

and that the map

$$C_f := \{(x,y,s);\ d_s f(x,y,s) = 0\} \to (y, -d_y f;\ x, d_x f) \tag{2.15}$$

is a (local) diffeomorphism onto the graph of Φ. It follows that the map

$$C_f \to d_y f$$

is a submersion. From this, we will argue that zero is a regular value of the map

$$(x,y,s) \mapsto (d_y f(x,y,s), d_s f(x,y,s)),$$

which means that f is a non-degenerate phase function in the fiber variables (y,s).

Indeed, the assumption is that the Jacobian

$$\mathbb{J} = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial s} & \frac{\partial^2 f}{\partial y \partial s} & \frac{\partial^2 f}{\partial s \partial s} \end{pmatrix}$$

is surjective at each point of C_f, and we also know that the Jacobian

$$\mathbb{J}_y = \begin{pmatrix} \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y \partial y} & \frac{\partial^2 f}{\partial s \partial y} \end{pmatrix}$$

(differential of the map $(x, y, s) \mapsto d_y f(x, y, s)$) is surjective when restricted to the kernel of $\mathbb{J}$, at each point of $\widetilde{C}_f := C_f \cap \{d_y f = 0\}$. From this, it follows that the Jacobian

$$\widetilde{\mathbb{J}} = \begin{pmatrix} \mathbb{J}_y \\ \mathbb{J} \end{pmatrix}$$

is surjective when restricted to $\widetilde{C}_f$.

It is clear from (2.15) that

$$\widetilde{C}_f := C_f \cap \{d_y f = 0\} \mapsto (x, d_x f)$$

maps $\widetilde{C}_f$ locally diffeomorphically onto the image of the zero section under Φ.

The proof of the second assertion in the lemma is proved in a similar way, using that the map $\widetilde{C}_f \ni (x, y', y'', s) \mapsto u$ is a submersion. ■

Let us go back to (2.14). By the previous lemma, the phase is a non-degenerate phase function parametrizing the image of Σ_0 under the underlying canonical transformation, and we have the correct power of $\hbar$ since the fiber variable has dimension $n + N$ now. It remains to show that the amplitude

$$b(x, y', y'', s, \hbar)\varphi(y', y''/\sqrt{\hbar})$$

is of the appropriate kind. This follows by Taylor expanding the amplitude b in the variables y'', at $y'' = 0$:

$$b(x, y', y'', s, \hbar) \sim \sum_{\alpha} (y'')^{\alpha} b_{\alpha}(x, y', s, \hbar).$$

It then follows that

$$b(x, y', y'', s, \hbar)\varphi(y', y''/\sqrt{\hbar}) \sim \sum_{\alpha} b_{\alpha}(x, y', s, \hbar)\hbar^{|\alpha|/2}(y''/\sqrt{\hbar})^{\alpha}\varphi(y', y''/\sqrt{\hbar}),$$

and, since the functions $(y'')^{\alpha}\varphi(y', y'')$ are Schwartz in y'', we can conclude that (2.14) is an integral of the desired form.

2.5. Global existence and Bohr–Sommerfeld conditions

In the previous discussion (and in fact in our previous paper [4]), we have ignored the issue of patching together local isotropic functions to obtain a global one. Here, we discuss how. In order to have a global isotropic function, we need that the isotropic Σ satisfies a Bohr–Sommerfeld condition, which is a condition on the de Rham cohomology class of $\iota_\Sigma^*(p\,dx)$ where $\iota_\Sigma\colon \Sigma \hookrightarrow T^*X$ is the inclusion and $p\,dx$ the canonical one-form of T^*X.

We begin by reviewing the situation in the Lagrangian case. If f is a non-degenerate phase function parametrizing a Lagrangian submanifold $\Lambda \subset T^*X$, so is $f + c$ for any constant $c \in \mathbb{R}$. Adding a constant to the phase has the effect of multiplying an associated Lagrangian state, written using the phase f, by $e^{-i\hbar^{-1}c}$ (which does not change it as a quantum state). However, this additional phase factor can be important to keep track of in applications. To remedy this ambiguity, we proceed as in [3, Chapter 12] (an equivalent approach also appeared in [8]).

Let us begin by noticing that if $f \in C^\infty(U \times \mathbb{R}^N)$ is a non-degenerate phase function such that $C_f \to \Lambda$ is an embedding onto a relative open set $\Gamma \subset \Lambda$, then we can form the composition

$$\psi_f\colon \Gamma \to C_f \hookrightarrow U \times \mathbb{R}^N \xrightarrow{f} \mathbb{R}. \tag{2.16}$$

Lemma 2.10. *If $\iota_\Gamma\colon \Gamma \subset T^*X$ is the inclusion, then $d\psi_f = \iota_\Gamma^*(p\,dx)$, where $p\,dx$ is the canonical one-form of T^*X.*

Proof. Consider $C_f \to \Gamma \hookrightarrow T^*X$. The pull-back of $p\,dx$ by this map is the same as the pull-back of df by the inclusion $C_f \subset U \times \mathbb{R}^N$, because by definition the partial derivatives of f with respect to the fiber variables vanish on C_f. ∎

It follows from this that given two phase functions f, f' parametrizing relative open sets $\Gamma, \Gamma' \subset \Lambda$ one must have that $(\psi_f - \psi_{f'})|_{\Gamma\cap\Gamma'}$ is locally constant. To fix these constants globally, one needs that Λ satisfies a Bohr–Sommerfeld condition. The strongest one is the following:

Definition 2.11. A Lagrangian submanifold $\Lambda \subset T^*X$ is called *exact* (or *strongly admissible,* in the terminology of [8]) if there exists $\psi \in C^\infty(X)$ such that $d\psi = \iota_\Lambda^*(p\,dx)$ (that is, if the de Rham cohomology class of $\iota_\Lambda^*(p\,dx)$ is zero).

For such manifolds, one can fix ψ and use local integral expressions with phase functions f such that $\psi_f = \psi|_\Gamma$, using the notation (2.16). Using such phase functions one can glue locally-defined Lagrangian states, using a microlocal partition of unity.

The condition of being exact is too restrictive for some applications. One can relax the condition a bit, as follows:

Definition 2.12. The Lagrangian submanifold Λ satisfies the Bohr–Sommerfeld condition if and only if the cohomology class of $\iota_\Lambda^*(p\,dx)$ is integral. Equivalently, if and only if there exists a function $\chi = e^{2\pi i\psi}\colon \Lambda \to S^1$ such that

$$\frac{1}{2\pi i}\frac{d\chi}{\chi} = \iota_\Lambda^*(p\,dx).$$

Cohomologically, the condition is that $\iota_\Lambda^*(p\,dx)$ represents an integral cohomology class of Λ. Under this condition, the expression $e^{i\hbar^{-1}\psi}$ is a well-defined function on Λ, provided $\hbar = 1/k$ for some $k \in \mathbb{N}$.

As we have seen, given an isotropic Σ, to parametrize it we need to frame it first, and the previous considerations apply to any framing of Σ. Since it is enough to consider framings that lie in a small neighborhood of Σ, we can always assume that we have framed Σ by a Lagrangian Λ such that Σ is a deformation retract of Λ. Therefore, the Bohr–Sommerfeld condition is that $\iota_\Sigma^*(p\,dx)$ represents an integral cohomology class of Σ.

3. General FIO invariance

In this section we will prove the general FIO invariance of the class of isotropic functions under the standard clean composition condition (Theorem 1.1).

3.1. Geometric considerations

Let $\Sigma \subset T^*Y$ be an isotropic submanifold. We suppose $F\colon C_0^\infty(Y) \to C^\infty(X)$ is a semiclassical FIO whose canonical relation $\Gamma\colon T^*Y \rightrightarrows T^*X$ composes with Σ cleanly. Recall that this means that

$$\Gamma \text{ intersects } T^*X \times \Sigma \text{ cleanly.} \tag{C.I.}$$

If this is the case, the *excess* of the intersection is the number

$$e = \dim[\Gamma \cap (T^*X \times \Sigma)] + \dim(T^*X \times T^*Y) - (\dim\Gamma + \dim(T^*X \times \Sigma)), \tag{3.1}$$

which is zero precisely when the intersection is transverse. As a corollary of the clean intersection condition, the set

$$\Gamma(\Sigma) := \{p \in T^*X \mid \text{there exists } q \in \Sigma \text{ such that } (p,q) \in \Gamma\}$$

is an immersed isotropic submanifold of T^*X, see [1, Proposition 7.1].

Since the classes of isotropic functions are invariant under the action of FIOs associated to canonical transformations, without loss of generality, we can assume that Σ is contained in the zero section of T^*Y. Moreover, since the statement is microlocal, we will be making various simplifying assumptions along the course of the proof. In particular, we can assume that $Y = Y' \times Y''$ where $Y'' \subset \mathbb{R}^l$ and $Y' \subset \mathbb{R}^{\dim Y - l}$ are open sets, $0 \in Y''$, and that

$$\Sigma = Y' \times \{0\},$$

where we are identifying Y' with the zero section of T^*Y'. We will denote the coordinates on Y by

$$y = (y', y''), \quad y' = (y'_1, \dots, y'_{\dim Y - l}), \quad y'' = (y''_1, \dots, y''_l).$$

If $\eta = (\eta', \eta'')$ are the dual coordinates, then Σ is defined by the equations:

$$\Sigma\colon \quad y'' = 0, \quad \eta = 0. \tag{3.2}$$

Lemma 3.1. *Let $\Gamma \subset T^*X \times T^*Y$ be a Lagrangian submanifold. Then, there exists a non-degenerate generating function $f \in C^\infty(X \times Y \times S)$ of Γ, where $S \subset \mathbb{R}^N$ is open, such that the differentials*

$$dy''_1, \quad \dots, dy''_l, \quad d\Big(\frac{\partial f}{\partial y''_1}\Big), \quad \dots, \quad d\Big(\frac{\partial f}{\partial y''_l}\Big) \tag{3.3}$$

are linearly independent everywhere.

Proof. Start with an arbitrary non-degenerate generating function

$$f_0 \in C^\infty(X \times Y \times S_0).$$

Let $S := S_0 \times \mathbb{R}^l$, and define $f_1 \in C^\infty(X \times Y \times S)$ by

$$f_1(x, y', y'', s_0, s'') = f_0(x, y', y'', s_0) + \frac{1}{2}|s''|^2$$

where $s_0 \in S_0$ and $s'' \in \mathbb{R}^l$. Now let $g\colon S \to S$ be the diffeomorphism

$$g(x, y', y'', s_0, s'') = (x, y', y'', s_0, y'' + s'')$$

and define

$$f(x, y', y'', s_0, s'') = (f_1 \circ g)(x, y', y'', s_0, s'') = f_0(x, y', y'', s_0) + \frac{1}{2}|y'' + s''|^2.$$

The three functions f_0, f_1, f are equivalent: they are non-degenerate and parametrize Γ. Moreover,

$$d\Big(\frac{\partial f}{\partial y''_i}\Big) = d\Big(\frac{\partial f_0}{\partial y''_i}\Big) + dy''_i + ds''_i, \quad 1 \le i \le l.$$

The differentials

$$dy''_1, \quad \ldots, \quad dy''_l, \quad d\Big(\frac{\partial f}{\partial y''_1}\Big), \quad \ldots, \quad d\Big(\frac{\partial f}{\partial y''_l}\Big)$$

are linearly independent everywhere, because of the presence of the ds''_i in the second set of differentials. This completes the proof. ■

For the rest of this section we will assume that f is an *adapted phase function*, meaning as in the lemma above. We let, as usual,

$$\mathcal{C}_f := \Big\{(x,y,s) \in X \times Y \times S \;\Big|\; \frac{\partial f}{\partial s}(x,y,s) = 0\Big\}$$

and $\Phi\colon \mathcal{C}_f \to T^*X \times T^*Y$ be the map

$$\Phi(x,y,s) = \big(x, y, \partial_x f(x,y,s), \partial_y f(x,y,s)\big)$$

which we assume is a diffeomorphism onto Γ. Finally, we assume that (C.I.) holds.

Lemma 3.2. *Under the previous assumptions, at each point $\kappa \in \Gamma \cap (T^*X \times \Sigma)$, $\kappa = \Phi(x,y,s)$, the space*

$$d\Phi^{-1}_{(x,y,s)} T_\kappa\big(\Gamma \cap (T^*X \times \Sigma)\big) \subset \mathbb{R}^{\dim X} \times \mathbb{R}^{\dim Y} \times \mathbb{R}^N \tag{3.4}$$

is the solution set of the system of equations

$$d\Big(\frac{\partial f}{\partial s_i}\Big) = 0, \quad dy''_j = 0, \quad d\Big(\frac{\partial f}{\partial y''_j}\Big) = 0, \quad d\Big(\frac{\partial f}{\partial y'_a}\Big) = 0 \tag{3.5}$$

for $1 \le i \le N$, $1 \le j \le l$, $1 \le a \le \dim Y - l$.

Proof. By the clean intersection condition

$$T_\kappa\big(\Gamma \cap (T^*X \times \Sigma)\big) = T_\kappa\Gamma \cap T_\kappa(T^*X \times \Sigma).$$

Thus the space (3.4) consists of the vectors in $T_{(x,y,s)}\mathcal{C}_f$ that are mapped into the space $T_\kappa(T^*X \times \Sigma)$ by $d\Phi$. The first set of equations (3.5) carves out $T_{(x,y,s)}\mathcal{C}_f$, since f is non-degenerate, and the remaining equations correspond to the condition that the image is tangent to the intersection. ■

Next we extract a *maximal* set of linearly independent differentials among those appearing in (3.5). We will include the $2l$ differentials corresponding to the variables y''_j. It will be convenient to re-label the coordinates on $Y' \times S$ as follows.

We will write

$$t = (t', t'')$$

for a re-labeling of the variables $(y'_1, \ldots, y'_{\dim Y - l}, s_1, \ldots, s_N)$, where

(1) $t' = (t'_1, \dots, t'_m)$ is such that the differentials

$$dy''_j, \quad d\Big(\frac{\partial f}{\partial y''_j}\Big), \quad d\Big(\frac{\partial f}{\partial t'_b}\Big)$$

$1 \le j \le l$, $1 \le b \le m$, are linearly independent near (x, y, s), and

(2) m is maximal with respect to property (1).

(Since the intersection is a manifold it is clear that such re-labeling is possible.) The number m has the following interpretation:

Corollary 3.3. *The dimension of the intersection* $\Gamma \cap (T^*X \times \Sigma)$ *is*

$$\dim \Gamma \cap (T^*X \times \Sigma) = \dim X + \dim Y + N - (2l + m) \tag{3.6}$$

and the excess is

$$e = N + \dim Y - (l + m). \tag{3.7}$$

Proof. The dimension formula follows from the fact that the codimension of the space $\Phi^{-1}(\Gamma \cap (T^*X \times \Sigma))$ in $X \times Y \times S$ is $2l + m$. The excess formula follows from this and (3.1):

$$\begin{aligned} e &= \dim X + \dim Y + N - (2l + m) + 2\dim X + 2\dim Y \\ &\quad - (\dim X + \dim Y + 2\dim X + \dim Y - l) \\ &= N + \dim Y - (l + m). \end{aligned}$$

∎

3.2. End of the proof

It suffices to prove the theorem for an FIO of order zero, which we represent by an oscillatory integral with an adapted phase function. Therefore,

$$F(\Upsilon)(x) = \hbar^{r - (\dim X + N)/2} \int_{Y' \times Y'' \times \mathcal{S}} e^{\sqrt{-1}\hbar^{-1} f(x, y', y'', s)} a(x, y', \hbar^{-1/2} y'', s, \hbar) dy' dy'' ds. \tag{3.8}$$

Let $t = (t', t'') \in \mathcal{T}' \times \mathcal{T}''$ as above. By viewing t'' as parameters, we can write the integral (3.8) as

$$\hbar^{r - (\dim X + N)/2} \int_{\mathcal{T}''} \Bigg(\int_{\mathcal{T}' \times Y''} e^{\sqrt{-1} h^{-1} f(x, t', y''; t'')} a(x, t', h^{-1/2} y'', \hbar; t'') dt' dy'' \Bigg) dt''. \tag{3.9}$$

Note that, for each fixed t'', the phase function

$$\tilde{f}_{t''}(x, t', y'') := f(x, t', y'', t'')$$

(with fiber variables t', y'') is a non-degenerate phase function parametrizing the isotropic submanifold

$$\Sigma_{t''} = \left\{\left(x, \frac{\partial \tilde{f}}{\partial x}(x,t',0)\right) \,\Big|\, \frac{\partial \tilde{f}_{t''}}{\partial t'}(x,t',0) = 0, \frac{\partial \tilde{f}_{t''}}{\partial y''}(x,t',0) = 0\right\}$$

with framing

$$\Gamma_{t''} = \left\{\left(x, \frac{\partial \tilde{f}}{\partial x}(x,t',y'')\right) \,\Big|\, \frac{\partial \tilde{f}_{t''}}{\partial t'}(x,t',y'') = 0, \frac{\partial \tilde{f}_{t''}}{\partial z''}(x,t',z'') = 0\right\}.$$

Moreover,

$$\left(x, \frac{\partial \tilde{f}_{t''}}{\partial x}(x,t',y'')\right) = \left(x, \frac{\partial f}{\partial x}(x,t',t'',y'')\right).$$

So, all these $\Sigma_{t''}$ coincide within an open set inside $\Gamma(\Sigma)$, i.e. they are the same in the sense of germs. By Theorem 2.6, for each t'' the integrand in (3.9) is in the class $I^{r'}(X, \Gamma(\Sigma))$, where

$$r' = \frac{m+l}{2}.$$

Taking into account the powers of $\hbar$ in front of the integral, we obtain that

$$F(\Upsilon) \in I^{r+r'-(\dim X+N)/2}(X, \Gamma(\Sigma)).$$

However, by (3.7)

$$r' - N/2 = \frac{1}{2}(m+l-N) = \frac{1}{2}(\dim Y - e)$$

and therefore

$$F(\Upsilon) \in I^{r+(\dim Y - \dim X - e)/2}(X, \Gamma(\Sigma)),$$

which is what we had to prove.

4. An example of an FIO of isotropic type

Let X be a compact Riemannian manifold, Δ its Laplace–Beltrami operator, and let $P = \frac{1}{2}\hbar^2\Delta + V$ be a Schrödinger operator where $V \in C^\infty(X)$. Denote by

$$H : T^*X \to \mathbb{R}, \quad H(x,\xi) = \|\xi\|^2 + V(x)$$

its principal symbol, and let

$$P\psi_j = E_j\psi_j, \quad E_1 \leq E_2 \leq \cdots$$

where $\{\psi_j\}$ is an orthonormal basis of $L^2(X)$.

Note that the ψ_j and E_j depend on $\hbar$, but we suppress that dependence from the notation for simplicity.

Theorem 4.1. *Assume that zero is a regular value of H, and let $\Theta := H^{-1}(0)$. Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function, and define*

$$P_\varphi := \varphi\Big(\frac{1}{\sqrt{\hbar}}P\Big)$$

by the spectral theorem so that $P_\varphi(\psi_j) = \varphi(\frac{E_j}{\sqrt{\hbar}})\psi_j$ for all j. Then the Schwartz kernel of P_φ is an isotropic function associated with the following isotropic submanifold of $T^*X \times T^*X$:

$$\Theta \overset{\Delta}{\times} \Theta := \{(x, x, \xi, -\xi) \mid (x, \xi) \in \Theta\}. \tag{4.1}$$

Remark 4.2. This result is to be compared with the following:

(a) the kernels of operators of the form $\varphi(P)$ are $\hbar$-pseudodifferential operators, and

(b) operators of the form

$$\varphi\Big(\frac{1}{\hbar}P\Big) \quad \text{with } \hat{\varphi} \in C_0^\infty(\mathbb{R}) \tag{4.2}$$

are semiclassical Fourier integral operators associated to the null-leaf relation of Θ.

In the proof of the theorem we will need:

Lemma 4.3. *Let $\chi \in C_0^\infty(\mathbb{R})$ be a function that is identically equal to one in a neighborhood of the origin, and let $\chi_c(t) = 1 - \chi(t)$. Let $\rho \in \mathcal{S}(\mathbb{R})$, and consider*

$$\gamma(\lambda, \hbar) = \int e^{i\lambda s}\rho(s/\sqrt{\hbar})\chi_c(s)\, ds. \tag{4.3}$$

Then

$$\begin{gathered}\textit{for all } k \geq 0, N > 0 \textit{ there exists } C > 0 \textit{ such that}\\ |\gamma(\lambda, \hbar)| \leq C\lambda^{-k}\hbar^N \quad \textit{for all } \lambda > 0, \hbar \in (0, 1).\end{gathered} \tag{4.4}$$

Proof. The technique of the proof is standard. Let $D_s = \frac{1}{i}\partial_s$, and write $e^{is\lambda} = \lambda^{-k}D_s^k(e^{is\lambda})$. Integrating by parts one gets

$$\begin{aligned}\gamma(\lambda, \hbar) &= (-1)^k\lambda^{-k}\int e^{i\lambda s}D_s^k(\rho(s/\sqrt{\hbar})\chi_c(s))\, ds\\ &= i^k\lambda^{-k}\sum_{j=0}^{k}\binom{k}{j}\hbar^{-j/2}\int e^{i\lambda s}\rho^{(j)}(s/\sqrt{\hbar})\chi_c^{(k-j)}(s)\, ds.\end{aligned}$$

However, for each j there exists $C > 0$ such that

$$\left| \int e^{i\lambda s} \rho^{(j)}(s/\sqrt{\hbar}) \chi_c^{(k-j)}(s)\, ds \right| \leq C \int\limits_{|s|\geq 1} |\rho^{(j)}(s/\sqrt{\hbar})|\, ds$$

$$= \sqrt{h} C \int\limits_{|u|\geq \hbar^{-1/2}} |\rho^{(j)}(u)|\, du.$$

Moreover, since ρ is Schwartz, for any $N > 0$ there is a constant C such that for all u, $|u| \geq 1$ implies $|\rho^{(j)}(u)| \leq C|u|^{-N}$. Therefore, if $\hbar \in (0, 1)$,

$$\int\limits_{|u|\geq \hbar^{-1/2}} |\rho^{(j)}(u)|\, du \leq C \int\limits_{|u|\geq \hbar^{-1/2}} |u|^{-N}\, du = C' \hbar^{(N-1)/2}. \qquad \blacksquare$$

Proof of Theorem 4.1. One has

$$P_\varphi = \frac{1}{2\pi} \int\limits_{\mathbb{R}} e^{itP/\sqrt{\hbar}} \hat{\varphi}(t)\, dt,$$

where $\hat{\varphi}$ is the ordinary Fourier transform of φ. Making the substitution $t = s/\sqrt{\hbar}$ gives

$$P_\varphi = \frac{1}{2\pi\sqrt{\hbar}} \int\limits_{\mathbb{R}} e^{isP/\hbar} \hat{\varphi}(s/\sqrt{\hbar})\, ds.$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be identically equal to one in a neighborhood of the origin. Let us write

$$P_\varphi = Q + R \quad \text{where } Q = \frac{1}{2\pi\sqrt{\hbar}} \int\limits_{\mathbb{R}} e^{isP/\hbar} \hat{\varphi}(s/\sqrt{\hbar}) \chi(s)\, ds, \tag{4.5}$$

and

$$R = \frac{1}{2\pi\sqrt{\hbar}} \int\limits_{\mathbb{R}} e^{isP/\hbar}, \hat{\varphi}(s/\sqrt{\hbar}) \chi_c(s)\, ds \quad \text{where } \chi_c = 1 - \chi. \tag{4.6}$$

The Schwartz kernel of R is

$$\mathcal{K}_R(x, y) = \frac{1}{2\pi\sqrt{\hbar}} \sum_j \gamma(\hbar^{-1} E_j, \hbar) \psi_j(x) \overline{\psi_j}(y), \tag{4.7}$$

where $\gamma(\hbar^{-1} E_j, \hbar)$ is given by (4.3) with $\rho = \hat{\varphi}$. By the previous lemma,

for all k, N there exists C such that

$|\gamma(\hbar^{-1} E_j, \hbar)| \leq C |E_j|^{-k} \hbar^N \quad$ for all $j, \hbar \in (0, 1)$.

We want to show that R is a residual operator, that is, that $\mathcal{K}_R$ is $O(\hbar^\infty)$ with all its derivatives. To see this, let us break (4.7) into two sums,

$$\text{(I)} = \frac{1}{2\pi\sqrt{\hbar}} \sum_{\substack{j \\ E_j > |V_{\min}|}} \gamma(\hbar^{-1}E_j, \hbar)\psi_j(x)\overline{\psi_j}(y)$$

$$\text{(II)} = \mathcal{K}_R(x, y) - \text{(I)},$$

where $V_{\min} < 0$ is the minimum of the potential.

To estimate (I) we use that, by the min-max principle,

$$E_j \geq \hbar^2\lambda_j + V_{\min} \quad \text{for all } j, \hbar, \tag{4.8}$$

where $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of Δ listed with multiplicities. By the Weyl law for the Laplacian, there exists $A > 0$ such that $\#\{\ell \mid \lambda_\ell < \lambda\} \leq A\lambda^{n/2}$ for all λ. Let $j \in \mathbb{N}$ and let $\lambda = \lambda_j$ in this inequality to obtain that

$$j \leq A\lambda_j^{n/2} \quad \text{or} \quad \lambda_j \geq Bj^{2/n}$$

for some constant $B > 0$. Combining this with (4.8), we conclude that

$$E_j \geq \hbar^2 Bj^{2/n} + V_{\min} \quad \text{for all } j, \hbar.$$

Now, the summation in (I) is over indices j such that $E_j > -V_{\min}$. Adding these inequalities, we get that in all the terms in (I) one has

$$\hbar^{-1}E_j \geq \frac{B}{2}\hbar j^{2/n}$$

and, therefore, for all k, N, there exists C such that

$$|\gamma(\hbar^{-1}E_j, \hbar)| \leq C\hbar^N j^{-k}.$$

This implies that (I) is $O(\hbar^\infty)$ together with all its derivatives and can be neglected.

To estimate

$$\text{(II)} = \frac{1}{2\pi\sqrt{\hbar}} \sum_{\substack{j \\ E_j \leq |V_{\min}|}} \gamma(\hbar^{-1}E_j, \hbar)\psi_j(x)\overline{\psi_j}(y)$$

(whose number of summands is $O(\hbar^{-n})$) we use the L^∞ bounds on eigenfunctions by $O(\hbar^{-n})$, together with the previous lemma with $k = 0$ and N arbitrary. Again (II) can be neglected.

The Schwartz kernel, $\mathcal{U}_P$, of $e^{isP/\hbar}$ (regarded as a distribution on $\mathbb{R} \times X \times X$) is a Lagrangian distribution associated to

$$\Gamma := \{(s, \delta s; \phi_s(\tilde{x})'; \tilde{x}) \in T^*\mathbb{R} \times T^*X \times T^*X;\ \delta s = -H(\tilde{x})\},$$

where ϕ_s is the Hamilton flow of H and the prime denotes the usual map

$$(y, \delta y)' = (y, -\delta y) \in T^*X$$

(and we have separated points in different cotangent bundles by a semicolon). Equation (4.5) shows that the kernel of Q is the push-forward of $\varphi(s/\sqrt{\hbar})\chi(s)\mathcal{U}_P$ to $X \times X$. Therefore, by Theorem 1.1 the Schwartz kernel of Q is an isotropic function, and it is easy to check that the associated isotropic is (4.1). ∎

Taking the trace of P_φ leads to the following result, which can also be proved directly:

Proposition 4.4. *Under the previous assumptions, for any Schwartz function φ,*

$$\sum_j \varphi\Big(\frac{E_j}{\sqrt{\hbar}}\Big) \sim \frac{1}{(2\pi)^n}\hbar^{-n+\frac{1}{2}}|\Theta|\hat{\varphi}(0) + \hbar^{-n+\frac{1}{2}}\sum_{j=1}^{\infty}\hbar^{j/2}c_j, \tag{4.9}$$

as $\hbar \to 0$, where $|\Theta|$ is the Liouville measure of Θ.

We emphasize that the assumption that $\hat{\varphi}$ be compactly supported is not needed, in contrast to (4.2).

Proof. We first note that the left-hand side is the trace of P_φ, and the operation of taking the trace is a Fourier integral operator from $X \times X$ to a point. Therefore, by Theorem 1.1 the trace has an asymptotic expansion in powers of $\sqrt{\hbar}$.

To find the leading term, let us use (4.5) together with a known FIO approximation to the Schwartz kernel of $\mathcal{U}_P$ for small time, yields that the Schwartz kernel of P_φ is locally well approximated by integrals of the form

$$\mathcal{V}(x, y, \hbar) = \frac{1}{(2\pi)^{n+1}\hbar^{n+\frac{1}{2}}}\int e^{i\hbar^{-1}[S(s,x,p)-y\cdot p]} \cdot a(s, x, y, p, \hbar)\hat{\varphi}(s/\sqrt{\hbar})\chi(s)\,dp\,ds, \tag{4.10}$$

where S solves the Hamilton–Jacobi equation

$$H(x, \nabla_x S(s, x, p)) + \frac{\partial S}{\partial s}(s, x, p) = 0, \quad S(0, x, p) = x \cdot p$$

and a is an amplitude such that $a|_{s=0} = 1$. The result follows from a stationary phase expansion of the integrals

$$\int \mathcal{V}(x, y, \hbar)\,dx = \frac{1}{(2\pi)^{n+1}\hbar^{n+\frac{1}{2}}}\int e^{i\hbar^{-1}[S(s,x,p)-x\cdot p]} \cdot a(s, x, x, p, \hbar)\hat{\varphi}(s/\sqrt{\hbar})\chi(s)\,dp\,ds\,dx,$$

or, equivalently,

$$\int \mathcal{V}(x, x, \hbar)\,dx = \frac{1}{(2\pi)^{n+1}\hbar^{n}}\int e^{i\hbar^{-1}[S(t\sqrt{\hbar},x,p)-x\cdot p]} \cdot a(t\sqrt{\hbar}, x, x, p, \hbar)\hat{\varphi}(t)\chi(t\sqrt{\hbar})dp\,dt\,dx.$$

In order to achieve this, we need first to Taylor-expand the phase in the variable s at $s = 0$:

$$S(t\sqrt{\hbar}, x, p) = S(0, x, p) + t\sqrt{\hbar}\frac{\partial S}{\partial s}(0, x, p) + \hbar t^2 T(t, x, p, \hbar).$$

Then, noting that $S_s(0, x, p) = -H(x, \nabla_x S(0, x, p))$ and $\nabla_x S(0, x, p) = p$, we get

$$\int \mathcal{V}(x, x, \hbar)\, dx = \frac{1}{(2\pi)^{n+1}\hbar^n} \int e^{-i\hbar^{-1/2}tH(x,p)} e^{it^2 T(t,x,p,\hbar)} \cdot a(t\sqrt{\hbar}, x, x, p, \hbar)\hat{\varphi}(t)\chi(t\sqrt{\hbar})dp\, dt\, dx.$$

It is clear from the dt integral that we can cut-off the integrand away from $\Theta = \{H(x, p) = 0\}$. Since zero is a regular value of H, in a neighborhood of Θ we can use H as a coordinate in phase space. Let $(x, p) \mapsto (H, \zeta)$ be such a coordinate change, and let $dx\, dp = dH \wedge d\zeta$ where $d\zeta$ represents a form of degree $2n - 1$ that pull-backs to Liouville measure on Θ. Then the integral above is of the form

$$\int \mathcal{V}(x, x, \hbar)\, dx = \frac{1}{(2\pi)^{n+1}\hbar^n} \int e^{-i\hbar^{-1/2}tH} A(t, H, \zeta, \hbar)\hat{\varphi}(t)\chi(t\sqrt{\hbar})dH\, d\zeta\, dt + O(\hbar^\infty).$$

Applying the method of stationary phase in the $dt\, dH$ integral, which now has a quadratic phase, yields

$$\int \mathcal{V}(x, x, \hbar)\, dx \sim \hat{\varphi}(0)\frac{\hbar^{1/2}}{(2\pi\hbar)^n}\int A(0, 0, \zeta, 0)d\zeta. \qquad \blacksquare$$

From this result, a standard argument involving approximating the characteristic function χ_c of $[-c, c]$ by functions $\varphi_\varepsilon \in C_0^\infty(\mathbb{R})$ yields the following:

Corollary 4.5. *With the previous notation and assumption, for all $c > 0$,*

$$\#\{j\ ; |E_j| \leq c\hbar^{1/2}\} \sim 2c\frac{1}{(2\pi)^n}\hbar^{-n+\frac{1}{2}}|\Theta|. \tag{4.11}$$

Remark 4.6. It is straightforward to replace in the proof of Proposition 4.4 the exponent $1/2$ by any $\alpha \in (0, 1)$, yielding the estimate

$$\#\{j\ ; |E_j| \leq c\hbar^{\alpha}\} \sim 2c\frac{1}{(2\pi)^n}\hbar^{-n+\alpha}|\Theta|. \tag{4.12}$$

By the semiclassical trace formula [8], the result remains true for $\alpha = 1$ *provided* one assumes that the set of periodic trajectories has Liouville measure equal to zero (no such assumption is needed for $\alpha \in (0, 1)$).

5. Superpositions of coherent states

By definition, a coherent state is an isotropic function where the isotropic is a single point. In this section we show that *every* isotropic function is a "superposition of coherent states," in a sense to be made precise. We also prove that, conversely, certain superpositions of coherent states yield isotropic functions.

5.1. Every isotropic function is a superposition of coherent states

As mentioned, the elements of $I^k(X, \Sigma)$ are *coherent states* if the isotropic submanifold, Σ, of T^*X is just a single point (x_0, ξ_0), which will be referred to as the *center* of the coherent state. More precisely, a coherent state centered at $(x_0, \xi_0) \in T^*\mathbb{R}^n$ is a (semiclassical) function defined on $\mathbb{R}^n$ of the form

$$\Upsilon_{x_0,\xi_0}(x) = e^{\sqrt{-1} f(x)/\hbar} a(\hbar^{-1/2}(x - x_0), \hbar), \tag{5.1}$$

where f is a smooth function with the property that

$$df(x_0) = \xi_0,$$

and $a(x, \hbar)$ is a Schwartz function in x having an asymptotic expansion in $\hbar$ as in Section 2.2. The classical example are the Gaussian functions

$$e^{(-|x-x_0|^2 + i(x-x_0)\cdot\xi_0)/\hbar}.$$

If Υ is a coherent state centered at (x_0, ξ_0), then its semiclassical wave front set is $\mathrm{WF}_\hbar(\Psi_{x_0,\xi_0}) = \{(x_0, \xi_0)\}$. Moreover, if F is a semiclassical Fourier integral operator associated to a canonical transformation that maps (x_0, ξ_0) to (x_1, ξ_1), then $F(\Psi_{x_0,\xi_0})$ is a coherent state centered at (x_1, ξ_1). As a consequence, one can define coherent states on manifolds as follows. Let X be a smooth manifold, and $(x_0, \xi_0) \in T^*X$ a point in the cotangent space. A coherent state Ψ_{x_0,ξ_0} centered at (x_0, ξ_0) is a function on X so that

- in some coordinate neighborhood U of X near x_0, Ψ can be written as a function of the form (5.1);
- in some open set V of any $x \neq x_0$, Ψ and all its derivatives is of $O(\hbar^\infty)$, uniformly on compact sets.

It is immediate that the previously mentioned properties of coherent states in Euclidean space extend to manifolds.

We now prove:

Proposition 5.1. *If $\Sigma \subset T^*X$ is an isotropic submanifold, every oscillatory function, $\Upsilon \in I^r(X, \Sigma)$ is a superposition of coherent states supported at points $(x_0, \xi_0) \in \Sigma$. Specifically, given $\Upsilon \in I^r(X, \Sigma)$, there is a smooth family of coherent states Υ_τ, indexed by points $\tau \in \Sigma$, and a 1-density $d\tau$ on Σ such that*

$$\Upsilon = \int_\Sigma \Upsilon_\tau \, d\tau. \tag{5.2}$$

Proof. Using a microlocal partition of unit, to prove Proposition 5.1 it suffices to prove the result in the case when Σ is the model isotropic, Σ_0, in (1.6). Namely, there exist a neighborhood, $\mathcal{U}$, of p in T^*X, a neighborhood, $\mathcal{U}_0$, of p_0 in $T^*\mathbb{R}^n$ and a symplectomorphism, $\varphi\colon \mathcal{U} \to \mathcal{U}_0$, mapping $\Sigma \cap \mathcal{U}$ onto $\Sigma_0 \cap \mathcal{U}_0$. Moreover, by a microlocal partition of unity, we can assume, without loss of generality, that the Υ in Proposition 5.1 is supported in such a neighborhood and is, de facto, a Υ in $I^r(\mathbb{R}^n, \Sigma_0)$.

Let $\Upsilon(x, \hbar) = \varphi(x', \hbar^{-1/2}x'', \hbar)$ be an isotropic state as given in (1.6). For each $\tau \in \mathbb{R}^n$, let

$$\Upsilon_\tau := \varphi(x', \hbar^{-1/2}x'', \hbar)e^{-\|x'-\tau\|^2/\hbar}.$$

This is a coherent state centered at $(\tau, 0; 0, 0) \in \Sigma_0$, and

$$\Upsilon = \frac{1}{(\pi\hbar)^{n/2}} \int_{\Sigma_0} \Upsilon_\tau \, d\tau. \qquad \blacksquare$$

5.2. "Coherent" superposition of coherent states yield isotropic functions

Our goal is to make precise and prove the statement which is the title of this section. The Bohr–Sommerfeld condition plays a key role to define "coherent."

Definition 5.2. An isotropic submanifold $\Sigma \xrightarrow{\gamma} T^*X$ *satisfies the Bohr–Sommerfeld condition* if $\gamma^*(p\,dx)$, where $p\,dx$ is the natural one-form on T^*X, has periods in $2\pi\mathbb{Z}$.

Lemma 5.3. *Let $\gamma\colon \Sigma \to T^*X$ be an isotropic embedding. Assume that γ is horizontal, that is, assume that the composition*

$$\Sigma \xrightarrow{\gamma} T^*X \xrightarrow{\pi} X$$

is a diffeomorphism onto an embedded submanifold $M \subset X$. Then, there exists a horizontal Lagrangian $\Lambda \subset T^*X$ containing $\gamma(\Sigma)$ (a horizontal framing).

Proof. Let $s: M \to T^*X$ be the inverse of the projection $\gamma(\Sigma) \to M$. To each $x \in M$, s associates a covector to X, $s(x) \in T_x^*X$. Composing with the restriction maps

$$T_x^*X \to T_x^*M \quad \text{for all } x \in M,$$

s defines a one-form σ on M. One can check that the condition that the image of s is isotropic is equivalent to $d\sigma = 0$.

Claim. *There is a neighborhood $\mathcal{U}$ of M in X and a* closed *one-form $\beta \in \Omega^1(\mathcal{U})$ such that $s(x) = \beta_x$ for all $x \in M$.*

To prove the claim, we let $\mathcal{U}$ be a tubular neighborhood of M, with projection $\mathfrak{p}: \mathcal{U} \to M$. Let $\alpha = \mathfrak{p}^*\sigma$. Then α is closed and agrees with s on vectors tangent to M. We now show that one can modify α by adding the differential of a function $f: \mathcal{U} \to \mathbb{R}$ so that $\beta = \alpha + df$ has the desired property. It is enough to require that

$$f|_M \equiv 0 \quad \text{and} \quad df_x(v) = s_x(v) \quad \text{for all } x \in M, v \in \ker(d\mathfrak{p}_x). \tag{5.3}$$

Let $\{V_i\}$ be a locally finite cover of M such that there exist trivializations

$$\mathfrak{p}^{-1}(V_i) \cong V_i \times \mathbb{R}^\nu.$$

For each i, it is easy to construct $f_i: \pi^{-1}(V_i) \to \mathbb{R}$ solving (5.3) on V_i. Let $\{\chi_i\}$ be a partition of unit subordinate to $\{V_i\}$ and let

$$f = \sum_i \pi^*(\chi_i) f_i.$$

It is easy to see that f has the desired property. Now let Λ to be the image of the one-form β. ∎

Remark 5.4. If Σ satisfies the Bohr–Sommerfeld condition, so does the horizontal framing constructed above, since the Lagrangian retracts to the isotropic. The one-form β is locally of the form $\beta = d\psi$, where ψ is a function that is defined modulo $2\pi\mathbb{Z}$, so that $e^{i\psi}$ is a well-defined function near M.

Let $\gamma: \Sigma \to T^*X$ now be any isotropic embedding of a compact manifold. Let us define

$$\widetilde{\Sigma} := \{(t, 0; \gamma(t));\ t \in \Sigma\} \subset T^*(\Sigma \times X). \tag{5.4}$$

It is easy to see that this is an isotropic submanifold of $T^*(\Sigma \times X)$, and it is clearly horizontal. The image of the projection of $\widetilde{\Sigma}$ onto $\Sigma \times X$ is the graph

$$M = \{(t, x(t));\ t \in \Sigma,\ x(t) = \pi(\gamma(t))\} \tag{5.5}$$

of $\pi \circ \gamma\colon \Sigma \to X$. Moreover, Σ satisfies the Bohr–Sommerfeld condition if and only if $\tilde{\Sigma}$ does.

Assume from now on that Σ satisfies the Bohr–Sommerfeld condition, and let us restrict the values of $\hbar$ to the reciprocals of the natural numbers. Let $\psi(t,x)$ be a function defined on a neighborhood $\mathcal{U}$ of the graph M, such that the image of $d\psi$ is a framing of $\tilde{\Sigma}$. By the Bohr–Sommerfeld condition, ψ is defined only modulo $2\pi\mathbb{Z}$, but $e^{i\psi(t,x)}$ is well defined on $\mathcal{V}$.

Proposition 5.5. *Consider a family of coherent states in $L^2(X)$ along γ of the form*

$$\psi_{\gamma(t)}(x) = e^{i\hbar^{-1}\psi(t,x)} a\Big(t, \frac{x - x(t)}{\sqrt{\hbar}}\Big), \tag{5.6}$$

where $a(t,\cdot)$ satisfies the usual Schwartz estimates. Then the superposition of coherent states

$$\Upsilon := \int_{\Sigma} \psi_{\gamma(t)}\, dt \tag{5.7}$$

is an isotropic function on X associated with the image of γ.

Proof. Let us consider the entire family $\Psi = \{\psi_{\gamma(t)}\}_{t\in\Sigma}$ as a smooth function on $\Sigma \times X$. Then one has that

$$\Upsilon = p_*(\Psi), \quad \text{where } p\colon \Sigma \times X \to X \text{ is the projection.} \tag{5.8}$$

By inspecting (5.6) and taking into account (5.5), we can conclude that Ψ is an isotropic function associated to $\tilde{\Sigma}$. By functoriality of isotropic functions with respect to the action of Fourier integral operators (in this case the push-forward operator, p_*), Υ is an isotropic function given that $\tilde{\Sigma}$ is contained in the conormal bundle to the fibers of π. ∎

5.3. An application

We briefly sketch some applications of our previous results. Throughout, we let X be a compact manifold and P a self-adjoint $\hbar$-pseudodifferential operator of order zero, for example a Schrödinger operator, $P = \frac{1}{2}\hbar^2\Delta + V$. Let $H\colon T^*X \to \mathbb{R}$ be its principal symbol, $H(x,\xi) = \frac{1}{2}\|\xi\|_x^2 + V(x)$. We denote by $U(t) = e^{-it\hbar^{-1}P}$ the fundamental solution to the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}U(t)\psi_0 = PU(t)(\psi_0), \quad U(0) = I.$$

This is a semiclassical Fourier integral operator, in the strong sense that its Schwartz kernel $U(t,x,y)$ is a Lagrangian distribution on $\mathbb{R}\times X\times X$ associated with the canonical relation

$$\Gamma = \{(t,\tau;\ x,\xi;\ y,\eta) \in T^*(\mathbb{R} \times X \times X);\ (x,\xi) = \phi_t(y,\eta), \tau = H(x,\xi)\}, \quad (5.9)$$

where $\phi_t \colon T^*X \to T^*X$ is the Hamilton flow of H.

By the general theory, if $\psi_0 \in C^\infty(X)$ is a coherent state centered at $(x_0,\xi_0) \in T^*X$, then, for each $t \in \mathbb{R}$, $U(t)(\psi_0)$ is a coherent state centered at $\phi_t(x_0,\xi_0)$. Moreover, the estimates implied by this statement are uniform provided t, (x_0,ξ_0) take values in a compact set.

Consider now a compact isotropic (possibly Lagrangian) submanifold, $\Sigma \subset T^*X$, and let $\Upsilon \in I^r(X,\Sigma)$ be an associated isotropic function. By Proposition 5.1, there exists a smooth family Υ_σ of coherent states, indexed by points $\sigma \in \Sigma$, such that

$$\Upsilon = \int_\Sigma \Upsilon_\sigma \, d\sigma$$

for some density $d\sigma$ on Σ. Then we can write

$$U(t)(\Upsilon) = \int_\Sigma U(t)(\Upsilon_\sigma)\, d\sigma. \quad (5.10)$$

This expresses $U(t)(\Upsilon)$ as a superposition of coherent states. Thus, in principle, it suffices to propagate coherent states in order to compute the propagation of isotropic states.

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References

[1] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*. Ann. of Math. Stud. 99, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981 Zbl 0469.47021 MR 620794

[2] M. Combescure and D. Robert, *Coherent states and applications in mathematical physics*. Theoret. and Math. Phys., Springer, Dordrecht, 2012 Zbl 1243.81004 MR 2952171

[3] V. Guillemin and S. Sternberg, *Semi-classical analysis*. International Press, Boston, MA, 2013 Zbl 1298.58001 MR 3157301

[4] V. Guillemin, A. Uribe, and Z. Wang, Semiclassical states associated with isotropic submanifolds of phase space. *Lett. Math. Phys.* **106** (2016), no. 12, 1695–1728 Zbl 1362.58014 MR 3569643

[5] V. W. Guillemin, Symplectic spinors and partial differential equations. In *Géométrie symplectique et physique mathématique* (Colloq. Internat. C.N.R.S., Aix-en-Provence, 1974), pp. 217–252, Éditions Centre Nat. Recherche Sci., Paris, 1975 Zbl 0341.58014 MR 0461591

[6] G. A. Hagedorn, Raising and lowering operators for semiclassical wave packets. *Ann. Physics* **269** (1998), no. 1, 77–104 Zbl 0929.34067 MR 1650826

[7] R. G. Littlejohn, The semiclassical evolution of wave packets. *Phys. Rep.* **138** (1986), no. 4–5, 193–291 MR 845963

[8] T. Paul and A. Uribe, The semi-classical trace formula and propagation of wave packets. *J. Funct. Anal.* **132** (1995), no. 1, 192–249 Zbl 0837.35106 MR 1346223

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