

Propagation of polyhomogeneity, diffraction, and scattering on product cones

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Abstract. We consider diffraction of waves on a product cone. We first show that diffractive waves enjoy a one-step polyhomogeneous asymptotic expansion, which is an improvement of Cheeger and Taylor’s classical result of half-step polyhomogeneity of diffractive waves [Comm. Pure Appl. Math. 35 (1982), 275–331 and 487–529]. We also conclude that on product cones, the scattering matrix is the diffraction coefficient, which is the principal symbol of the diffractive half wave kernel, for strictly diffractively related points on the cross section. This generalizes the result of Ford, Hassell and Hillairet in 2-dimensional flat cone settings (Ford, Hassell, and Hillairet, 2018). In the last section, we also give a radiation field interpretation of the relationship between the scattering matrix and the diffraction coefficient.

1. Introduction

In this paper, we study the diffraction coefficient of the wave equation

$$\square u = 0$$

and the scattering matrix corresponding to the Helmholtz equation

$$(\Delta - \lambda^2)u = 0$$

on a cone $C(N)$. The diffraction refers to the effect that when a propagating wave encounters a corner of an obstacle or a slit, its wave front bends around the corner of the obstacle and propagates into the geometrical shadow region. When studying the wave equation on cones, we see that its singularities likewise split into two types after they encounter the cone point. One propagates along the natural geometric extensions of the incoming ray, while other singularities emerge at the cone point and start propagating along *all* outgoing directions as a spherical wave. The outgoing singularities are described to leading order by a diffraction coefficient, which is one of the central objects we study in this paper.

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The (stationary) scattering theory of the wave equation gives an approach to studying the continuous spectrum of the Laplacian on non-compact manifolds. The scattering matrix, which, intuitively speaking, maps the incoming solution at the infinity of the (stationary) wave equation to the outgoing solution, is a central object of study.

In this paper, we focus on the diffraction and the scattering of the wave equation on cones. For notational purposes, we denote our n -dimensional cone by $C(N)$, which is $\mathbf{R}_+ \times N^{n-1}$ with metric $dr^2 + r^2h(\theta, d\theta)$ where $h(\theta, d\theta)$ is the metric on the smooth manifold N^{n-1} . We consider the fundamental solution to the wave equation on $C(N)$ corresponding to the Friedrichs extension of the Laplacian. For t large enough, the singularities of the fundamental solutions consist of two parts by [4, 5]. One lies on a sphere (up to reflection by the boundary of N) of radius t to the initial point (r', θ') , while the other part lies on a sphere of radius $t - r'$ around the cone point and is conormal to $\{r = t - r'\}$. We refer the latter to the *diffractive wave front*, and it is the main object of our interest in this paper. The former notion will be called the *geometric wave front*. See Figure 2 in Section 2 for an example of the geometric and diffractive waves. The diffraction coefficient is therefore defined by comparing the principal symbol of the incoming wave to the principal symbol of the diffractive wave, or equivalently, reading off the principal symbol of the diffractive half wave kernel. On the other hand, the scattering matrix is defined by considering the leading order behavior to the stationary wave equation

$$(\Delta - \lambda^2)u = 0$$

under certain boundary/asymptotic conditions; the solution u then has the leading order behavior

$$u \sim a_+(\theta)r^{-\frac{n-1}{2}}e^{i\lambda r} + a_-(\theta)r^{-\frac{n-1}{2}}e^{-i\lambda r} + \mathcal{O}(r^{-\frac{n+1}{2}}) \quad \text{as } r \rightarrow \infty$$

where $a_+(\theta)$ is uniquely determined by $a_-(\theta)$; the scattering matrix $S(\lambda)$ is then defined by the unitary map from $a_+(\theta)$ to $a_-(\theta)$ for $\lambda \in \mathbf{R} \setminus \{0\}$.

The following theorem thus relates two central concepts of the theories of diffraction and scattering:

Theorem 1.1. *Away from the intersection of the geometric wave front and diffractive wave front, the kernel of the diffractive half wave propagator is a conormal distribution of the form:*

$$U_D(t) = (2\pi)^{-\frac{n+1}{2}} \int e^{i(r+r'-t)\cdot\lambda} K(r, \theta; r', \theta'; \lambda) d\lambda |r^{n-1} dr d\theta r'^{n-1} dr' d\theta'|^{\frac{1}{2}}.$$

The principal symbol $K_0(r, r', \theta, \theta')$ of the diffractive half wave kernel $U_D(t, r, r', \theta, \theta')$ is related to the kernel of the scattering matrix $S(\lambda, \theta, \theta')$ by

$$K_0(r, \theta, r', \theta') = (2\pi)^{-1} (rr')^{-\frac{n-1}{2}} S(\lambda, \theta, \theta'). \tag{1}$$

It is worthwhile to point out here that we are actually showing that the smooth part of the scattering matrix corresponds to the diffraction coefficient, while the singular part of the scattering matrix corresponds to the geometric wave. This was proved in a special case of 2-dimensional flat cones by Ford, Hassell and Hillairet [6].

We also give a finer description of the structure of the diffractive wave by showing that it is *one-step* polyhomogeneous thus improving the result of half-step polyhomogeneity of diffractive waves that appears in Cheeger and Taylor [4] and [5, Theorems 5.1 and 5.3]. Consequently, one half of the coefficients that appear in Cheeger–Taylor’s expansion must vanish. The proof of the following theorem is given in Section 3.

Theorem 1.2. *The symbol $K(r, r', \theta, \theta'; \lambda)$ of the diffractive half wave kernel $U_D(t)$ is one-step polyhomogeneous in λ for $\lambda > 0$, i.e., it admits the following asymptotic symbol expansion*

$$K(r, r', \theta, \theta'; \lambda) \sim \sum_{i=0}^{-\infty} K_i(r, r', \theta, \theta') \lambda^i. \tag{2}$$

Finally, we give an interpretation of the relation in Theorem 1.1 in terms of the radiation field. The radiation field was introduced by Friedlander [9] for smooth asymptotically Euclidean manifolds. Intuitively, it can be regarded as measuring the waves of different time delay that arrive at infinity. We define the forward radiation field \mathcal{R}_+ as the limit, as time goes to infinity, of the derivative of the forward fundamental solution of the wave equation along certain light rays. By reversing time, one can define the backward radiation field \mathcal{R}_- . The forward/backward radiation field \mathcal{R}_\pm is related to the scattering matrix $S(\lambda)$ in the following formula:

$$S(\lambda) = \mathcal{F} \circ \mathcal{R}_+ \circ \mathcal{R}_-^{-1} \circ \mathcal{F}^{-1}. \tag{3}$$

This was first introduced by Friedlander [8] in \mathbf{R}^n , and was proved later by Sá Barreto [18] for smooth asymptotically Euclidean manifolds. The intuition and motivation of Theorem 1.1 also come from the following facts. In principle, the scattering operator

$$\mathcal{S} := \mathcal{R}_+ \circ \mathcal{R}_-^{-1}$$

is given by the Fourier conjugation of the leading symbol to the forward fundamental solutions [9]; the scattering matrix is the Fourier conjugation of the scattering operator [18]. This suggests that the scattering matrix and the diffraction coefficient should be the same up to some constant or scaling in radial variables.

We combine Cheeger–Taylor’s functional calculus on cones and Melrose–Wunsch’s propagation of conormality to give a simpler calculation of the diffractive coefficient and the one-step polyhomogeneity. As for determining the scattering matrix, we consider it mode-by-mode to reduce the original equation to a Bessel equation.

The outline of this paper is as follows. In Section 2, we prove a characterization of the one-step polyhomogeneous solutions to wave equations on cones and the propagation of one-step polyhomogeneity for the diffractive wave. These will be used in Section 3 to determine the diffraction coefficient. Then in Section 3, we compute the diffraction coefficient using the functional calculus on cones and the propagation of conormality. In Section 4, we focus on computing the scattering matrix and give the diffraction-scattering relation, Theorem 1.1, in the end. Finally, in Section 5, we give an interpretation of this result using the radiation field.

2. Conic diffraction geometry

In this section we recall some basic notions on the geometry of product cones together with the geometric and diffractive wave on it.

Again, we denote the product cone with total dimension n by $C(N) \cong \mathbf{R}_+ \times N^{n-1}$ with metric $dr^2 + r^2h(\theta, d\theta)$, where N is a smooth manifold and h is the metric restricted to N . This is a particular case of general cones $(C(Y), \tilde{g})$ with metric $\tilde{g} = dr^2 + r^2\tilde{h}(r, dr, \theta, d\theta)$, where in our case the metric h does not have r and dr dependence. Sometimes we also use X to denote the product cone with boundary for simplicity when the cross section N is not involved explicitly in the discussion, then ∂X is its boundary and $X^\circ = X \setminus \partial X$ denotes the interior. Without loss of generality, we assume N has one connected component since otherwise we can restrict to a single component.

The Laplacian¹ on $C(N)$ is defined as

$$\Delta = D_r^2 - i\frac{n-1}{r}D_r + \frac{\Delta_\theta}{r^2},$$

where $D_r := \frac{1}{i} \frac{\partial}{\partial r}$ is the Fourier normalization of r -derivative and Δ_θ is the Laplacian on N . Following Friedrichs, we define the form domain

$$\begin{aligned} \mathcal{D} &:= \text{Dom}(\Delta^{\frac{1}{2}}) \\ &= \text{cl}\{u \in \mathcal{C}_c^\infty(X^\circ) \text{ with respect to the norm } \|du\|_{L_g^2(X)} + \|u\|_{L_g^2(X)}\}, \end{aligned}$$

where “cl” denotes the closure of the set. Hence, the Friedrichs domain is defined as

$$\text{Dom}(\Delta_{\text{Fr}}) = \{u \in \text{Dom}(\Delta^{\frac{1}{2}}); \Delta u \in L_g^2(X)\}$$

and \mathcal{D}_s denotes the corresponding domain of $\Delta^{s/2}$. Later in this paper, we will use $L^2(\mathbf{R}_t; \mathcal{D}_s)$ to denote the regularity on spacetime $\mathbf{R} \times X$. The following proposition from [14, Proposition 3.1] gives a characterization of domains of Friedrichs extension:

¹We use the positive Laplacian throughout this paper.

Proposition 2.1 (domains of the Friedrichs extension). *For $n > 4$,*

$$\text{Dom}(\Delta) = \{u \in r^w L_b^2(X); \Delta u \in L_g^2(X)\}$$

is independent of w in the range $w \in (-n + 2, -n/2 + 2)$, where $L_b^2(X)$ is the boundary weighted L^2 -space with $r^{-\frac{n}{2}} L_b^2(X) = L_g^2(X)$. For $n = 3$, the same is true for $w \in (-1, 0)$. For $n = 2$,

$$\text{Dom}(\Delta) = \{u \in L_g^2(X); u = c + u', c \in \mathbb{C}, u' \in r^w L_b^2(X), \Delta u \in L_g^2(X)\}$$

is independent of w for $w > 0$ sufficiently small. In all cases, Δ is an unbounded operator:

$$\Delta: \text{Dom}(\Delta) \rightarrow L_g^2(X)$$

and $\text{Dom}(\Delta)$ coincides with the domain of the Friedrichs extension $\text{Dom}(\Delta_{\text{Fr}})$.

Henceforth, we use Δ to denote the Friedrichs extension of the Laplacian from the domain $\mathcal{C}_c^\infty(X^\circ)$. For a more detailed discussion we refer to [13, 14], though it is worth pointing out the following observation from the proposition:

Corollary 2.2. *If $u \in \mathcal{E}'(X^\circ)$, i.e., a compactly supported distribution in the interior of the cone X , then $u \in \mathcal{D}_s$ is equivalent to $u \in H^s(X^\circ)$.*

The d'Alembertian acting on the spacetime $\mathbf{R} \times X$ is

$$\square = D_t^2 - \Delta$$

and we also define the *half-wave propagator* $U(t)$ as

$$U(t) := e^{-it\sqrt{\Delta}}.$$

We now consider the diffraction of waves with respect to the cone point, which has been studied in detail by Cheeger and Taylor in [4] and [5] for product cones. There are two different notions of geodesics on cones, one more restrictive than the other. We can see that these notions on product cones are special cases of [14] on general cones with metric $dr^2 + r^2 h(r, \theta, d\theta)$ within a small neighborhood of the cone point.

Definition 2.3. Suppose $\gamma: (-\varepsilon, +\varepsilon) \rightarrow C(N)$ is a piecewise *geodesic* on the cone $C(N)$ hitting the cone point only at time $t = 0$, then

- the curve γ is a *diffractive geodesic* if the intermediate terminal point $\gamma(0_-)$ and the initial point $\gamma(0_+)$ lie on the boundary $\{0\} \times N$;
- the curve γ is a *geometric geodesic* if it is a diffractive geodesic such that the intermediate terminal point $\gamma(0_-)$ and the initial point $\gamma(0_+)$ are connected by a geodesic of length π on the boundary $\{0\} \times N$ (with respect to the boundary metric h);

- the curve γ is a *strictly diffractive geodesic* if it is a diffractive geodesic but not geometric geodesic.

As pointed out in [14], the geometric geodesics are those that are locally realizable as limits of families of ordinary geodesics in the interior X° . Figure 1 gives geometric and diffractive geodesics at the cone point.

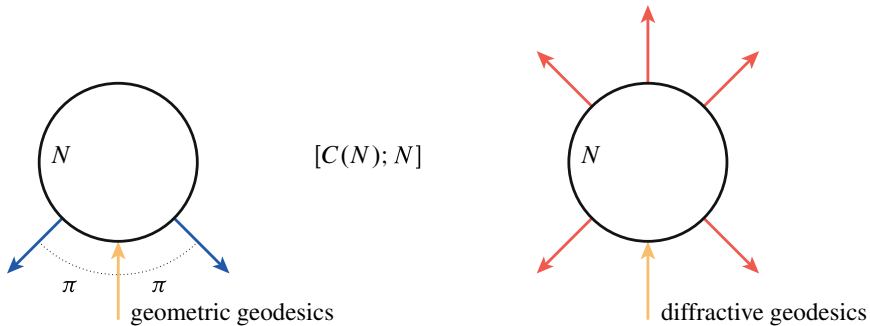


Figure 1. Diffractive and geometric geodesics. The blow up pictures of cone $C(N)$ at cone point $\{0\} \times N: [C(N); N]$. On the right are the diffractive geodesics, while on the left are geometric geodesics and they are connected by geodesics in the boundary with length π .

In this paper, we focus on the diffraction coefficient and the scattering matrix away from the points that are related by the geometric geodesics, i.e., we consider the pair (r, θ) and (r', θ') with $d_h(\theta, \theta') \neq \pi$ for the study of the diffraction coefficient and the smooth part of scattering matrix. At the intersection of the geometric and diffractive fronts, the structure of the singularities is more complicated. This can be seen intuitively from the following picture of diffraction by a slit in Figure 2, which is equivalent to a product cone of angle 4π . In the case of 2-dimensional flat cone, the wave kernel close to the intersection is then a singular Fourier integral operator in a calculus associated to two intersecting Lagrangian submanifolds. We refer to [6] for a detailed discussion.

3. Propagation of polyhomogeneity

In this section, we briefly discuss the propagation of one-step polyhomogeneity on product cones in order to prove our diffractive symbol estimates. This also gives a one-step polyhomogeneity of the diffractive wave. Prior to these, we give a characterization of one-step polyhomogeneous solution to the wave equation on cones. Without loss of generality, we assume that u is the spherical wave hitting the cone point at time $t = 0$.

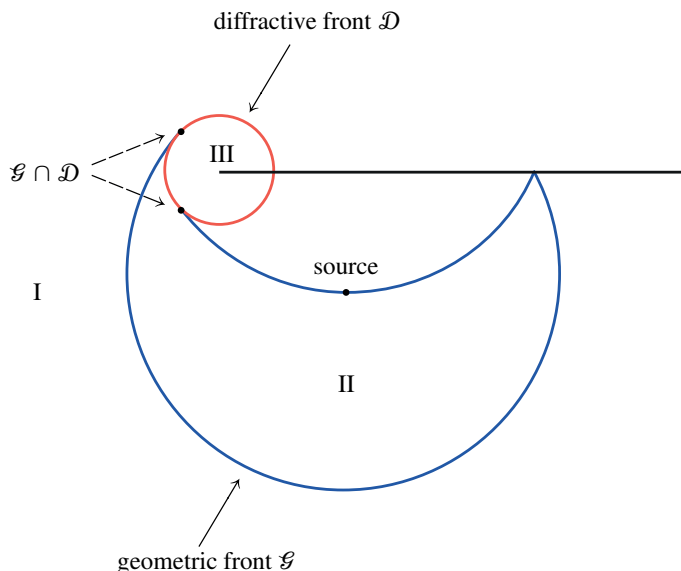


Figure 2. Geometric and diffractive front. Diffraction by a slit in \mathbf{R}^2 which is a cone with cross section $N = (0, 2\pi)$. The wave source is marked on the picture. The diffractive front \mathcal{D} is the boundary of region III (the red circle), while the geometric front \mathcal{G} is part of the boundary of region II (the blue arcs). In this picture, the intersection of these two wave fronts $\mathcal{G} \cap \mathcal{D}$ consists of two points.

We first introduce the polyhomogeneous symbols:

Definition 3.1 (polyhomogeneous symbols). A symbol $a(x; \lambda) \in C^\infty(X^o \times \mathbf{R}^l)$ is called (*one-step*) *polyhomogeneous* of order m if it admits an asymptotic symbol expansion:

$$a \sim \sum_{k=0}^{\infty} a_k(x, \lambda),$$

where a_k are homogeneous symbols of order $m - k$, i.e., $a_k(x, \lambda) \in S^{m-k}(X; \mathbf{R}^l)$ and $a_k(x, c\lambda) = c^{m-k}a_k(x, \lambda)$ for $c \in \mathbf{R}_+$. We denote the polyhomogeneous symbol class of order m by $S_{\text{phg}}^m(X; \mathbf{R}^l)$.

We introduce the radial and tangential operators for later reference. Let

$$R = tD_t + rD_r \tag{4}$$

denote the radial vector field on $\mathbf{R} \times X$. And we also define tangential operators on N :

$$Y_s = (I + \Delta_\theta)^{s/2}. \tag{5}$$

Note that for the above operators and the d'Alembertian \square , we have a group of commutator relations acting on $\mathcal{C}^{-\infty}(\mathbf{R} \times X)$:

$$[\square, R] = -2i\square \text{ and } [\square, Y_s] = 0 \tag{6}$$

which will be useful to prove the propagation of polyhomogeneity. Here $\mathcal{C}^{-\infty}(\mathbf{R} \times X)$ denotes the dual space of $\dot{\mathcal{C}}^\infty(\mathbf{R} \times X)$ which are functions vanishing to infinite order at the cone point. In particular, the space $\mathcal{C}^{-\infty}(\mathbf{R} \times X)$ contains the power domains \mathcal{D}_s for all s . These commutator relations are motivated by Melrose and Wunsch's argument on propagation of conormality: from [14, Theorem 4.8], these commutator relations imply that if $\square u = 0$ and u is conormal to $\{t + r = 0\}$ with respect to \mathcal{D}_s for $t < 0$ then u is conormal to $\{t - r = 0\}$ with respect to \mathcal{D}_s for $t > 0$. The conormality here is characterized by the operators Y_s, R and \square through the following definition:

Definition 3.2 (conormality on cones). (1) We define $u \in \mathcal{C}^{-\infty}(\mathbf{R} \times X)$ to be *conormal* with respect to $\{t + r = 0\}$ of regularity \mathcal{D}_s for $t < 0$ if u has the iterative regularity:

$$Y_i R^j \square^k u \in L^2_{\text{loc}}(\mathbf{R}_-; \mathcal{D}_s)$$

for all $i, j, k \in \mathbb{N}$, where $\mathbf{R}_- = (-\infty, 0)_t$.

(2) On the other hand, we define $u \in \mathcal{C}^{-\infty}(\mathbf{R} \times X)$ to be *conormal* with respect to $\{t - r = 0\}$ of regularity \mathcal{D}_s for $t > 0$ if u has the iterative regularity:

$$Y_i R^j \square^k u \in L^2_{\text{loc}}(\mathbf{R}_+; \mathcal{D}_s)$$

for all $i, j, k \in \mathbb{N}$, where $\mathbf{R}_+ = (0, +\infty)_t$. We use $I\mathcal{D}_s$ to denote both conormalities, where I stands for *iterative regularity*, and specify $t \lesseqgtr 0$ when necessary.

For a more detailed discussion on the conormal distribution, we refer to [11]. By the Hörmander–Melrose theory, on the product cone X of total dimension n , the iterative regularity $I\mathcal{D}_{-m-1/2-\varepsilon}$ for any $\varepsilon > 0$, with order that we will discuss later, is equivalent to the oscillatory integral definition of conormality of order $m - (n - 1)/4$ which is defined in the following sense:

Definition 3.3 (oscillatory integral definition of conormality). (1) We define $u \in \mathcal{C}^{-\infty}(\mathbf{R} \times X)$ to be *conormal* with respect to $\{t + r = 0\}$ for $t < 0$ of order k if it locally admits an oscillatory integral representation for $t < 0$:²

$$u = \int e^{i(t+r)\lambda} a(r, \theta; \lambda) d\lambda \text{ mod } \mathcal{D}_\infty$$

with Kohn–Nirenberg symbol $a(r, \theta; \lambda) \in S^{k+(n-1)/4}(X; \mathbf{R}_\lambda)$.

²Since away from the cone point the power domain \mathcal{D}_s agrees with the usual Sobolev space H^s , it is equivalent to use $\text{mod } \mathcal{C}^\infty$ here.

(2) On the other hand, we define $u \in \mathcal{C}^{-\infty}(\mathbf{R} \times X)$ to be *conormal* with respect to $\{t - r = 0\}$ for $t > 0$ of order k if it locally admits an oscillatory integral representation for $t > 0$:

$$u = \int e^{i(t-r)\lambda} a(r, \theta; \lambda) d\lambda \pmod{\mathcal{D}_\infty}$$

with Kohn–Nirenberg symbol $a(r, \theta; \lambda) \in S^{k+(n-1)/4}(X; \mathbf{R}_\lambda)$. Let

$$I^k(\mathbf{R} \times X; N^*\{t \pm r = 0\})$$

denote the space of all distributions on $\mathbf{R} \times X$ that are conormal with respect to $\{t \pm r = 0\}$ of order k , where $N^*\{t \pm r = 0\}$ is the conormal bundle of $\{t \pm r = 0\}$.

Remark 3.4. We use $S^m(X; \mathbf{R}_\lambda)$ to denote the Kohn–Nirenberg symbol class on the cone X of order m hereafter.

Following [11], we have the following equivalent relation between the previous two definitions of conormality.

Theorem 3.5. *The iterative regularity definition of conormality is equivalent to the oscillatory integral definition in the sense that for $t \leq 0$ we have the following inclusions of conormal distributions on $\mathbf{R} \times X$:*

$$I\mathcal{D}_{-m-1/2} \subseteq I^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t \pm r = 0\}) \subseteq I\mathcal{D}_{-m-1/2-\varepsilon}$$

for any $\varepsilon > 0$.

We also need the following interpolation lemma to raise the iterative conormal regularity from the Sobolev regularity and the lower iterative regularity. We refer the readers to [13, Lemma 12.2] for a detailed proof of the lemma. It is presented in the form of coisotropic regularity there but the essence is the same.

Lemma 3.6 (interpolation). *Suppose $u \in L^2_{\text{loc}}(\mathbf{R}_\mp; \mathcal{D}_s) \cap I\mathcal{D}_m$ for $s > m$, then $u \in I\mathcal{D}_{s-\varepsilon}$ for any $\varepsilon > 0$.*

Suppose now that $\square u = 0$, and for $t \leq 0$,

$$u \in I^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t \pm r = 0\}).$$

We can thus write, for $t \leq 0$,

$$u = \int e^{i(t\pm r)\lambda} a(r, \theta, \lambda) d\lambda \pmod{\mathcal{D}_\infty} \tag{7}$$

for some $a \in S^m(X; \mathbf{R}_\lambda)$. From this, recall also that $u \in L^2_{\text{loc}}(\mathbf{R}_\mp; \mathcal{D}_s)$ for all $s < -m - 1/2$. We employ the notation $u \in L^2_{\text{loc}}(\mathbf{R}_\mp; \mathcal{D}_{-m-1/2-0})$ to denote this type of space hereafter. For the propagation of conormality, note that the symbols of Y_1, \square

and R consist of the defining functions of $N^*\{t \pm r = 0\}$; by showing the iterative regularity:

$$Y_i R^j \square^k u \in L^2_{\text{loc}}(\mathbf{R}_{\mp}; \mathcal{D}_s)$$

for any $i, j, k \in \mathbb{N}$ and for some order s , we can show that the conormality is therefore preserved. For the propagation of polyhomogeneity, we need stronger conditions. In fact, in addition to the above preservation of iterative tangential regularity, we need that applying the radial vector field with particular shifts improves the regularity by one-step at each time. We show later that this actually leads to a complete characterization of polyhomogeneous distributions. Before proceeding to the proof of the complete characterization, we start by showing a characterization of the leading order polyhomogeneity. This characterization is due to Baskin and Wunsch [2]. From now on, we employ the notation $I\mathcal{D}_{s-0}$ to denote that u lies in the iterative regularity class $I\mathcal{D}_{s-\varepsilon}$ for any $\varepsilon > 0$.

Lemma 3.7 (characterization of leading polyhomogeneity [2]). *Assume $\square u \in \mathcal{C}^\infty$, u is conormal to $\{t \pm r = 0\}$ and takes the oscillatory integral form (7) for $t \lesssim 0$ (away from the cone point) with $a \in S^m(X; \mathbf{R}_\lambda)$. Then*

$$u \in I\mathcal{D}_{-m-1/2-0}.$$

Set

$$\alpha = m + \frac{n + 1}{2}.$$

Then

1. If a is polyhomogeneous of order m , then

$$(R - i\alpha)u \in I\mathcal{D}_{-m+1/2-0}.$$

2. Conversely, suppose that

$$(R - i\alpha)u \in L^2_{\text{loc}}(\mathbf{R}_{\mp}; \mathcal{D}_{-m+1/2-0});$$

then for $t \lesssim 0$

$$a = a_m + r_{m-1}$$

where a_m is homogeneous of degree m and $r_{m-1} \in S^{m-1+0}(X; \mathbf{R}_\lambda)$, where

$$S^{m-1+0}(X; \mathbf{R}_\lambda) := \bigcap_{r>m-1} S^r(X; \mathbf{R}_\lambda).$$

Proof. Applying the operator $R - i\alpha$ to (7) and integrating by parts, we obtain

$$\begin{aligned} (R - i\alpha)u &\equiv \int e^{i(t \pm r)\lambda} ((t \pm r)\lambda + rD_r - i\alpha)a(r, \theta, \lambda) d\lambda \\ &\equiv \int e^{i(t \pm r)\lambda} (-D_\lambda \lambda + rD_r - i\alpha)a(r, \theta, \lambda) d\lambda \end{aligned}$$

$$\equiv \int e^{i(t \pm r)\lambda} (-\lambda D_\lambda + r D_r - i(\alpha - 1)) a(r, \theta, \lambda) d\lambda. \tag{8}$$

Here we use \equiv to denote the equivalence modulo \mathcal{C}^∞ -errors. Now, we make use of the crucial fact that $\square u \in \mathcal{C}^\infty$. Since

$$\square = D_t^2 - D_r^2 + i \frac{n-1}{r} D_r - \frac{\Delta_\theta}{r^2},$$

applying \square to (7) yields

$$\int e^{i(t \pm r)\lambda} (\pm 2i\lambda \partial_r \pm i\lambda(n-1)/r + \Delta) a d\lambda \equiv 0 \pmod{\mathcal{C}^\infty};$$

thus a must satisfy the transport equation

$$\left(r \partial_r + \frac{n-1}{2} \mp \frac{ir}{2\lambda} \Delta \right) a \in S^{-\infty}(X; \mathbf{R}_\lambda), \tag{9}$$

where Δ is the Laplacian on cones; in particular, since $\frac{ir}{2\lambda} \Delta a \in S^{m-1}(X; \mathbf{R}_\lambda)$, this forces

$$\left(r \partial_r + \frac{n-1}{2} \right) a \in S^{m-1}(X; \mathbf{R}_\lambda).$$

Plug this into the right side of (8) yields

$$\begin{aligned} (R - i\alpha)u &= \int e^{i(t \pm r)\lambda} \left((-\lambda D_\lambda + i \frac{n-1}{2} - i(\alpha - 1)) a(r, \theta, \lambda) + e \right) d\lambda \\ &= \int e^{i(t \pm r)\lambda} ((-\lambda D_\lambda - im) a(r, \theta, \lambda) + e) d\lambda, \end{aligned}$$

where $e \in S^{m-1}(X; \mathbf{R}_\lambda)$ is the remainder term.

Thus, if $a \in S^m(X; \mathbf{R}_\lambda)$ is a polyhomogeneous symbol, then so is $(\lambda D_\lambda + im)a \in S^{m-1}(X; \mathbf{R}_\lambda)$ and we find that

$$(R - i\alpha)u \in I^{m-1-(n-1)/4}(\mathbf{R} \times X; N^*\{t \pm r = 0\}) \subset I\mathcal{D}_{-m+1/2-0}$$

by the equivalence of two definitions of conormality. This proves the first part of the lemma.

Conversely, if $(R - i\alpha)u \in L^2(\mathbf{R}_\mp; \mathcal{D}_{-m+1/2-0})$, by the commutator relations (6) and the fact that the symbol of R is one of the defining functions of $\{t \pm r = 0\}$, we have $(R - i\alpha)u \in I\mathcal{D}_{-m-1/2-0}$ by conormality of u . Thus, by Lemma 3.6 (cf. [13, Lemma 12.2]), we know $(R - i\alpha)u$ is also conormal with iterative regularity $I\mathcal{D}_{-m+1/2-0}$. Equivalently,

$$(R - i\alpha)u \in I^{m-1-(n-1)/4-0}(\mathbf{R} \times X; N^*\{t \pm r = 0\}).$$

This forces the symbol of $(R - i\alpha)u$ to be in the class $S^{m-1+0}(X; \mathbf{R}_\lambda)$. By the proof of the first part, consider the order of the symbol of $(R - i\alpha)u$ gives

$$(-\lambda D_\lambda - im)a \in S^{m-1+0}(X; \mathbf{R}_\lambda).$$

Equivalently,

$$D_\lambda(\lambda^{-m}a) \in \mathcal{O}(\lambda^{-2+0}).$$

Integrating it to infinity yields

$$\lim_{\lambda \rightarrow \infty} \lambda^{-m}a = a_m \quad \text{and} \quad \lambda^{-m}a - a_m = \mathcal{O}(\lambda^{-1+0}).$$

This implies that we must have the leading asymptotic decomposition as in the statement of lemma. ■

For later reference, we record a sharpening of the symbol computation above. In particular, note that if $\square u \in \mathcal{C}^\infty$ then we can compute the symbol of $(R - i\alpha)u$ explicitly by substituting the full transport equation (9) into (8) to obtain

$$(R - i\alpha)u = \int e^{i(t \pm r)\lambda} b(r, \theta, \lambda) d\lambda$$

where

$$b = \left(-\lambda D_\lambda - im + \frac{r}{2\lambda} \Delta\right) a. \tag{10}$$

Therefore, $(R - i\alpha)$ acting on u can be characterized by $(-\lambda D_\lambda - im + \frac{r}{2\lambda} \Delta)$ acting on its symbol a . We now generalize Lemma 3.7 to get a characterization of full polyhomogeneity by induction. The result given in the following lemma is similar to the characterization given by Joshi [12] for polyhomogeneous Lagrangian distributions on smooth manifolds, though the Hamilton vector field of our operator R is not a multiple of the radial vector field of fiber variables as in [12].

Lemma 3.8 (characterization of the complete polyhomogeneity). *Assume that $\square u \in \mathcal{C}^\infty(\mathbf{R} \times X^\circ)$, u is conormal to $\{t \pm r = 0\}$ and takes the oscillatory integral form (7) for $t \leq 0$ with $a \in S^m(X; \mathbf{R}_\lambda)$. Set*

$$\alpha_j = m + \frac{n + 1}{2} - j + 1.$$

Then

1. if a is polyhomogeneous of order m , then

$$\prod_{j=1}^k (R - i\alpha_j)u \in I\mathcal{D}_{-m-1/2+k-0}$$

is conormal to $\{t \pm r = 0\}$ for $t \leq 0$;

2. conversely, suppose that for the above $\alpha_1, \dots, \alpha_k$,

$$\prod_{l=1}^j (R - i\alpha_l) u \in L^2_{\text{loc}}(\mathbf{R}_{\mp}; \mathcal{D}_{-m-1/2+j-0});$$

for $1 \leq j \leq k$. Then in the oscillatory integral expression (7)

$$a = a_m + a_{m-1} + \dots + a_{m-k+1} + r_{m-k}$$

where $a_m, a_{m-1}, \dots, a_{m-k+1}$ are homogeneous symbols with the degrees same as their indices and $r_{m-k} \in S^{m-k+0}(X; \mathbf{R}_\lambda)$.

Proof. First assume $a \in S^m(X; \mathbf{R}_\lambda)$ is polyhomogeneous. By an integration by parts argument and the fact that $\square u \in \mathcal{C}^\infty$, we see that

$$\prod_{j=1}^k (R - i\alpha_j) u \in I \mathcal{D}_{-m-1/2+k-0}$$

is implied by

$$\prod_{j=1}^k \left(-\lambda D_\lambda + i \frac{n+1}{2} - i\alpha_j + \frac{r}{2\lambda} \Delta \right) a \in S^{m-k}(X; \mathbf{R}_\lambda) \tag{11}$$

in the Kohn–Nirenberg class. This is aided by the commutator relation (6) which makes $\prod_{j=1}^k (R - i\alpha_j) u$ for all $k \in \mathbb{N}$ again a solution to the wave equation modulo smooth terms, so we can apply the transport equation substitution iteratively as we did in Lemma 3.7. At each step, $(-\lambda D_\lambda + i \frac{n+1}{2} - i\alpha_j + \frac{r}{2\lambda} \Delta)$ acting on polyhomogeneous symbols still gives polyhomogeneous ones. Also note that $\frac{r}{2\lambda} \Delta$ always lowers the symbol order by one, and the coefficient α_j , when combining with R to annihilate the principal part of each step, only depends on the order of the symbol it acts on.

By Lemma 3.7, we know for $\alpha_1 = \frac{n+1}{2} + m$,

$$\left(-\lambda D_\lambda + i \frac{n+1}{2} - i\alpha_1 + \frac{r}{2\lambda} \Delta \right) a \in S^{m-1}(X; \mathbf{R}_\lambda) \text{ polyhomogeneous.}$$

Taking this as the new symbol b of the conormal distribution, then applying $(R - i\alpha_2)$ to the new distribution gives

$$\left(-\lambda D_\lambda + i \frac{n+1}{2} - i\alpha_2 + \frac{r}{2\lambda} \Delta \right) b \in S^{m-2}(X; \mathbf{R}_\lambda) \text{ polyhomogeneous.}$$

Applying this argument repeatedly with

$$\alpha_j = \frac{n+1}{2} + (m - j + 1) \quad \text{up to } j = k,$$

we proved that the first statement is true.

For the second part, we can work by induction. In the previous lemma, we have showed that $(R - i\alpha_1)$ raises regularity almost by one implies the leading one-step polyhomogeneity. Assume the conclusion in the second part of this lemma is true for up to k -terms one-step polyhomogeneity, i.e., assume

$$\prod_{l=1}^j (R - i\alpha_l)u \in L^2_{\text{loc}}(\mathbf{R}_{\mp}; \mathcal{D}_{-m-1/2+j-0})$$

for $1 \leq j \leq k$ imply

$$a = a_m + a_{m-1} + \dots + a_{m-k+1} + r_{m-k} \tag{12}$$

with $a_m, a_{m-1}, \dots, a_{m-k+1}$ homogeneous symbols and $r_{m-k} \in S^{m-k+0}(X; \mathbf{R}_\lambda)$. Thus

$$b_k := \prod_{j=1}^k \left(-\lambda D_\lambda + i \frac{n+1}{2} - i\alpha_j + \frac{r}{2\lambda} \Delta \right) a \in S^{m-k+0}(X; \mathbf{R}_\lambda)$$

as in the proof of the first part. We consider

$$\prod_{j=1}^{k+1} (R - i\alpha_j)u \in L^2_{\text{loc}}(\mathbf{R}_{\mp}; \mathcal{D}_{-m-1/2+k+1-0})$$

This has the oscillatory integral form

$$(R - i\alpha_{k+1}) \int e^{i(t \pm r)\lambda} b_k(r, \theta, \lambda) d\lambda.$$

Now, we apply Lemma 3.6, since by the assumption

$$\prod_{j=1}^{k+1} (R - i\alpha_j)u \in L^2_{\text{loc}}(\mathbf{R}_{\mp}; \mathcal{D}_{-m-1/2+k+1-0})$$

and

$$\prod_{j=1}^k (R - i\alpha_j)u \in I\mathcal{D}_{-m-1/2+k-0}$$

conormal to $\{t \pm r = 0\}$. Therefore, by Lemma 3.6,

$$\prod_{j=1}^{k+1} (R - i\alpha_j)u \in I\mathcal{D}_{-m-1/2+k+1-0}$$

conormal to $\{t \pm r = 0\}$ and thus

$$\left(-\lambda D_\lambda + i \frac{n+1}{2} - i\alpha_{k+1} + \frac{r}{2\lambda} \Delta \right) b_k \in S^{m-k-1+0}(X; \mathbf{R}_\lambda).$$

Since $\frac{r}{2\lambda}\Delta$ lowers the symbol order by one, we have

$$\left(-\lambda D_\lambda + i\frac{n+1}{2} - i\alpha_{k+1}\right)b_k \in S^{m-k-1+0}(X; \mathbf{R}_\lambda).$$

Plugging in α_{k+1} defined above, and using the same argument as in the proof of Lemma 3.7, this forces b_k to take the form $b_k = \tilde{b}_{m-k} + \tilde{r}_{m-k-1}$, where \tilde{b}_{m-k} is homogeneous of order $m-k$ and $\tilde{r}_{m-k-1} \in S^{m-k-1+0}(X; \mathbf{R}_\lambda)$. Considering that the action of $(-\lambda D_\lambda + i\frac{n+1}{2} - i\alpha_j + \frac{r}{2\lambda}\Delta)$ for $1 \leq j \leq k$ on the symbol a gives b_k , together with a being k -term one-step polyhomogeneous (12), a must take the form

$$a = a_m + a_{m-1} + \dots + a_{m-k} + r_{m-k-1}$$

where $a_m, a_{m-1}, \dots, a_{m-k}$ are homogeneous with degrees given by their indices and $r_{m-k-1} \in S^{m-k-1+0}(X; \mathbf{R}_\lambda)$ (cf. [12, Proposition 2.1]). We thus finished our induction. ■

Remark 3.9. It is worth to point out that this lemma cannot be generalized directly to give a similar characterization of one-step polyhomogeneity on general non-product cones $(\mathbf{R} \times Y, dr^2 + r^2h(r, \theta, d\theta))$. This is due to the fact that the commutator equation is now $[\square, R] = -2i\square + E$ with E an error term. The existence of this error term makes even $(R - i\alpha_1)u$ no longer a solution to the wave equation, for which being a solution is an essential property for us to build our characterization.

Recall that our ultimate goal in this section is to develop the propagation of polyhomogeneity. Prior to this, we use the foregoing results to obtain a propagation of leading order polyhomogeneity which will be used later to give a diffraction symbol estimate. Then we can show the propagation of full polyhomogeneity as a corollary.

First, we recall from [14, Theorem 4.8] the more basic results on propagation of conormality. These follow easily in the situation at hand by commutation of R^k and Y_s through the equation, together with the observation that the symbols of R, Y_1 , and \square form a set of defining functions to the conormal bundle of $\{t \pm x = 0\}$. The continuity of the evolution map asserted in the following proposition follows from the proof of [14, Theorem 4.8] or, as usual, from the inverse mapping theorem; the essence of the direct proof is that norms of powers of the test operators are conserved relative to domains of powers of the Laplacian which agree with Sobolev spaces away from $r = 0$; converting these estimates to estimates in symbol spaces requires a Sobolev embedding step, which loses at most a fixed number of derivatives (which can then be interpolated away up to an ε). For brevity, we abbreviate the restriction to a time interval by

$$a_{(c,d)} \equiv a|_{t \in (c,d)}.$$

Proposition 3.10 (continuity of symbol evolution [2]). *Suppose that $\square u = 0$ and that for $t < 0$, $u \in I^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t + r = 0\})$.*

1. *For $t > 0$, $u \in I^{m-(n-1)/4+0}(\mathbf{R} \times X; N^*\{t - r = 0\})$, thus the conormality is conserved.*
2. *The map from negative time data to positive-time data is continuous in the following sense: for any $a < b < 0 < c < d$, any $\varepsilon > 0$, and any M , there exist M' and C such that if we write u in the form (7) with symbol a then*

$$\|a_{(c,d)}\|_{S_M^{m+\varepsilon}} \leq C \|a_{(a,b)}\|_{S_{M'}^m}.$$

Here the symbol norm S_M^m is given by

$$\sum_{|\alpha|+|\beta|\leq M} \sup |\langle \lambda \rangle^{-m} (\partial_{r,\theta})^\alpha (\lambda \partial_\lambda)^\beta a|.$$

Remark 3.11. We will abbreviate the existence of symbol estimates of this type as “ a satisfies effective estimates” in what follows. The effective estimates here is crucial to prove the propagation of polyhomogeneity and to compute the diffraction coefficient using the mode-by-mode solutions.

Proposition 3.12 (propagation of leading order polyhomogeneity). *Suppose that $\square u = 0$ and that for $t < 0$, $u \in I_{\text{phg}}^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t + r = 0\})$, i.e., conormal distributions with polyhomogeneous symbols, Then*

1. *for $t > 0$, $u \in I^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t - r = 0\})$ and has an oscillatory integral representation of the form (7) where its symbol a is of the form $a_m + r_{m-1}$ with $r_{m-1} \in S^{m-1+0}(X; \mathbf{R}_\lambda)$;*
2. *for any fixed $\varepsilon > 0$, each symbol seminorm of r_{m-1} in $S^{m-1+\varepsilon}(X; \mathbf{R}_\lambda)$ is bounded in terms of finitely many seminorms of the symbol of u for $t < 0$.*

Proof. Take α as above. Then for $t < 0$,

$$(R - i\alpha)u \in I_{\text{phg}}^{m-(n-1)/4-1}(\mathbf{R} \times X; N^*\{t + r = 0\}).$$

We let $b \in S^{m-1}(X; \mathbf{R}_\lambda)$ denote the total symbol of $(R - i\alpha)u$, i.e.,

$$(R - i\alpha)u = \int e^{i(t+r)\lambda} b(r, \theta, \lambda) d\lambda \quad \text{for } t < 0,$$

while

$$u = \int e^{i(t+r)\lambda} a(r, \theta, \lambda) d\lambda$$

for $a \in S_{\text{phg}}^m(X; \mathbf{R}_\lambda)$.

Then for $t > 0$, since by commutator relations (6) in the beginning of this section $\square(R - i\alpha)u = 0$, Proposition 3.10 implies that for $t > 0$ and for all $\varepsilon > 0$,

$$(R - i\alpha)u \in I^{m-(n-1)/4-1+\varepsilon}(\mathbf{R} \times X; N^*\{t - r = 0\}),$$

with symbol seminorms depending on those of b . By definition of conormal distribution, this implies

$$(R - i\alpha)u \in L^2_{\text{loc}}(\mathbf{R}_+; \mathcal{D}_{-m+1/2-0})$$

for $t > 0$. Thus, by Lemma 3.7 we have the corresponding symbol decomposition of a and we therefore proved the first part of our proposition.

To prove the quantitative estimates of the second part, we note that by (10) and the quantitative propagation of conormality from Proposition 3.10 we have for $t > 0$

$$b \equiv \left(-\lambda D_\lambda - im + \frac{r}{2\lambda} \Delta\right)a \in S^{m-1+\varepsilon}(X; \mathbf{R}_\lambda)$$

for all $\varepsilon > 0$, together with symbol estimates: whenever $\alpha_- < \beta_- < 0 < \alpha_+ < \beta_+$, for all M ,

$$\begin{aligned} \|b_{(\alpha_+, \beta_+)}\|_{S^{m-1+\varepsilon}_{M'}} &\leq C \|b_{(\alpha_-, \beta_-)}\|_{S^{m-1}_{M'}} \\ &= C \|(-\lambda D_\lambda - im + \frac{r}{2\lambda} \Delta)a_{(\alpha_-, \beta_-)}\|_{S^{m-1}_{M'}} \\ &\leq C \|a_{(\alpha_-, \beta_-)}\|_{S^m_{M'}}, \end{aligned} \tag{13}$$

for M' .

Now, without loss of generality we can assume $m = 0$, otherwise we use

$$\tilde{a} = \lambda^{-m}a, \quad \tilde{b} = \lambda^{-m}b$$

to make the order of a zero. This yields

$$b = \left(-\lambda D_\lambda + \frac{r}{2\lambda} \Delta\right)a \in S^{-1+\varepsilon}(X; \mathbf{R}_\lambda) \quad \text{for } t \in (\alpha_+, \beta_+)$$

again enjoying the same type of effective estimates as b . Thus

$$D_\lambda a = -\lambda^{-1}b + \frac{r}{2\lambda^2} \Delta a \in S^{-2-\varepsilon}(X; \mathbf{R}_\lambda) \quad \text{for } t \in (\alpha_+, \beta_+), \tag{14}$$

again the right hand side enjoys effective estimates since $r/(2\lambda^2)\Delta$ is a continuous map from symbols of order s to symbols of order $s - 2$, and multiplication by powers of λ is a continuous symbol map. In particular, then

$$\|\lambda^{-1}b_{(\alpha_+, \beta_+)}\|_{S^{m-2+\varepsilon}_M} \leq C \|\lambda^{-1}a_{(\alpha_-, \beta_-)}\|_{S^{m-1}_M} \tag{15}$$

and

$$\left\| \frac{r}{2\lambda^2} a_{(\alpha_+, \beta_+)} \right\|_{S_M^{m-2+\varepsilon}} \leq C \left\| \lambda^{-2} a_{(\alpha_-, \beta_-)} \right\|_{S_M^{m-2}} \tag{16}$$

Integrating (14) from $\lambda = \pm\infty$ with $C(r, \theta, \operatorname{sgn} \lambda)$ as constant of integration (cf. [12, Proposition 2.1] for this strategy) yields

$$\begin{aligned} a &= C(r, \theta, \operatorname{sgn}(\lambda)) + \int -\lambda^{-1} b + \frac{r}{2\lambda^2} \Delta \tilde{a} \, d\lambda \\ &\equiv C(r, \theta, \operatorname{sgn}(\lambda)) + e(r, \theta, \operatorname{sgn} \lambda). \end{aligned}$$

Here the term $C(r, \theta, \operatorname{sgn}(\lambda))$ corresponds to the homogeneous term in a , while the integral term $e(r, \theta, \lambda)$ is a remainder that lies in $S^{-1+\varepsilon}$, which corresponds to r_{m-1} , and satisfies effective estimates directly following from the above two estimates (15) and (16). ■

We finally state the propagation of one-step polyhomogeneity as a corollary of our previous results, which we summarized as Theorem 1.2 in the introduction:

Corollary 3.13 (propagation of one-step polyhomogeneity). *Suppose that $\square u = 0$ and that for $t < 0$, $u \in I_{\text{phg}}^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t + r = 0\})$. Then for $t > 0$, $u \in I_{\text{phg}}^{m-(n-1)/4}(\mathbf{R} \times X; N^*\{t - r = 0\})$ and has an oscillatory integral representation of the form (7) where the symbol a is polyhomogeneous.*

Proof. We know

$$\prod_{j=1}^k (R - i\alpha_j) u \in I^{m-k-(n-1)/4}(\mathbf{R} \times X; N^*\{t + r = 0\})$$

with $\alpha_j = \frac{n+1}{2} + (m - j + 1)$ as in Lemma 3.8 for $t < 0$ and all $k \in \mathbf{N}$. By Proposition 3.10 and the commutator relations (6),

$$\prod_{j=1}^k (R - i\alpha_j) u \in I^{m-k-(n-1)/4+0}(\mathbf{R} \times X; N^*\{t - r = 0\})$$

for $t > 0$ and all $k \in \mathbf{N}$. Thus, by Lemma 3.8 and the fact that conormal distribution $I^{m-k-(n-1)/4+0}(\mathbf{R} \times X; N^*\{t - r = 0\})$ corresponds to iterative regularity $I\mathcal{D}_{-m-1/2+k-0}$, we conclude that u is a polyhomogeneous distribution. ■

This corollary in particular shows that the diffractive singularities enjoy a one-step polyhomogeneity (cf. Theorem 1.2), which improves the result of half-step polyhomogeneity given implicitly in Cheeger and Taylor [4] and [5, Theorem 5.1, 5.3]. This half-step polyhomogeneity is later explicitly pointed out by Ford and Wunsch in [7, Proof of Proposition 2.1]. Now, we prove Theorem 1.2 given in Section 1:

Proof of Theorem 1.2. By Corollary 3.13, for any $\varphi \in \mathcal{C}_c^\infty(N)$, if $\square u_\varphi = 0$ near $\{r + r' = t\}$ satisfies

$$u_\varphi = \int e^{i\lambda(r+r'-t)} K(r, r', \theta, \theta'; \lambda) \cdot \varphi(\theta') d\theta' d\lambda,$$

$$u_\varphi|_{t=0} = \varphi(\theta') \delta(r - r'),$$

then $\langle K, \varphi \rangle \in S_{\text{phg}}^0(X; \mathbf{R})$. We need to show K is also a polyhomogeneous symbol of the same order near the diffractive front and away from the geometric front, i.e., polyhomogeneity in the weak sense implies polyhomogeneity in the strong sense for $d_N(\theta, \theta') \neq \pi$. This can be done by interpolation.

First for any $\theta \in N$ we take $\theta'_0 \in N$ such that $d_h(\theta, \theta'_0) \neq \pi$. Then we take a compact neighborhood S of θ'_0 such that $d_h(\theta, \theta') \neq \pi$ for any $\theta' \in S$.

By the characterization of polyhomogeneous symbols, for any $n \in \mathbb{N}$,

$$\int \left[\prod_{k=1}^n (\lambda D_\lambda - ik) K \right] \cdot \varphi d\theta' \in S^{-n}(X; \mathbf{R}).$$

We define $c_n := \prod_{k=1}^n (\lambda D_\lambda - ik) K$, so that $\langle c_n, \varphi \rangle \in S^{-n}(X; \mathbf{R})$ for any $\varphi \in \mathcal{C}_c^\infty(S)$. Note that in particular $c_n \in S^0(X; \mathbf{R})$ for $d_h(\theta, \theta') \neq \pi$. It remains to show that for $d_h(\theta, \theta') \neq \pi$ we can obtain $c_n \in S^{-n}(X; \mathbf{R})$ indeed, i.e., it satisfies classical symbol estimates of the order $-n$.

We only show the sup-norm of the 0-th order derivative of the symbol satisfies the estimates here; the estimates for higher derivatives of the symbol hold similarly. Note that $\langle c_n, \varphi \rangle \in S^{-n}(X; \mathbf{R})$ for any $\varphi \in \mathcal{C}_c^\infty(S)$ implies that $\lambda^n c_n$ is weakly bounded in θ' -variable as $\lambda \rightarrow \infty$, i.e.,

$$|\lambda^n \langle c_n, \varphi \rangle| \leq C_\varphi$$

uniformly in λ for all $\varphi \in \mathcal{C}_c^\infty(S)$. By the principle of uniform boundedness, there exist $k \in \mathbb{N}$ and $C > 0$ independent of λ such that

$$\sup |\lambda^n \langle c_n, \varphi \rangle| \leq C \|\varphi\|_k$$

where $\|\cdot\|_k$ denotes the semi-norm in $\mathcal{C}_c^\infty(S)$. Therefore, there exists some $k' > 0$ such that $\lambda^n c_n$ is bounded in $H^{-k'}$ by the Closed Graph Theorem and Sobolev embedding. Moreover, for $d_h(\theta, \theta') \neq \pi$, we note that c_n is bounded in H^s uniformly in λ for any $s > 0$ since $c_n \in S^0(X; \mathbf{R})$ ³. Therefore, by interpolation, for any $\varepsilon > 0$, $\lambda^{n-\varepsilon} c_n$

³We know *a priori* $c_n \in S^0$ for $d_N(\theta, \theta') \neq \pi$. This would not be true at the intersection of the geometric and diffractive front so we are only able to obtain the desired symbol estimates for purely diffractive singularities.

is bounded in $H^{s'}$ for any $s' > 0$, which yields the desired estimate for the sup-norm of the 0-th order derivative of the symbol:

$$c_n = \mathcal{O}_{H^{s'}}(\lambda^{n-\varepsilon})$$

for any $\varepsilon > 0$ with $d_h(\theta, \theta') \neq \pi$. ■

4. Diffraction coefficient on product cones

Assume $t > r'$. Away from the intersection of the geometric and diffractive front, i.e., for $d_h(\theta, \theta') \neq \pi$, the half wave propagator can be decomposed as

$$U(t) \equiv U_D(t) + U_G(t)$$

modulo smooth terms. In the right hand side of the above equation, the first part is the diffractive wave propagator with

$$\text{WF } U_D(t) \subset N^*\{t = r + r'\}$$

and the second part is the geometric wave propagator with

$$\text{WF } U(t) \setminus \text{WF } U_D(t) \subset N^*\{t = d_g(z, z')\}$$

where $z = (r, \theta)$, $z' = (r', \theta')$. The wavefront set WF is used to denote the singularities of the distribution in the cotangent bundle $T^*(\mathbf{R} \times X^0 \times X^0)$. We refer to [19, Appendix 1] for a detailed discussion on this topic. On the diffractive front \mathcal{D} and away from the geometric front \mathcal{G} (see Figure 2), we can write the kernel of the diffractive half wave propagator as

$$U_D(t) = (2\pi)^{-\frac{n+1}{2}} \int e^{i(r+r'-t)\lambda} K(r, \theta; r', \theta'; \lambda) d\lambda |r^{n-1} dr d\theta r'^{n-1} dr' d\theta'|^{\frac{1}{2}} \tag{17}$$

where $K(r, \theta; r', \theta'; \lambda)$ is a polyhomogeneous symbol of order 0, and $U_D(t)$ is a conormal distribution to $\{r + r' = t\}$. Here we confuse the propagator with its Schwartz kernel and the assumption $t > r'$ is to ensure the existence of the diffractive front. This expression is due to Cheeger and Taylor [4, 5] and Melrose and Wunsch [14], and it can be seen as a consequence of conormality of the diffractive wave away from the intersection region. We show later in this section that the symbol of the diffractive half wave kernel has the form

$$K(r, \theta; r', \theta'; \lambda) \equiv \sum_j \sigma_j^T(t, r, r'; \lambda) \varphi_j(\theta) \varphi_j(\theta') \pmod{S^{-\infty}} \tag{18}$$

where φ_j is the j -th Fourier mode of Δ_θ on N and $\sigma_j^T(t, r, r'; \lambda)$ is the total symbol of j -th mode of diffractive fundamental solution $E_j(t, r, r')$. We define $\sigma_j^P(t, r, r'; \lambda)$ to be the principal symbol of σ_j^T . And the *diffraction coefficient* is defined to be

$$K_0(r, \theta; r', \theta') := \sum_j \sigma_j^P(t, r, r'; \lambda) \varphi_j(\theta) \varphi_j(\theta').$$

The idea of this section is the following. We first consider diffractions of spherical waves. By [14, Theorem 4.8], the spherical diffractive wave is conormal to $\{r + r' = t\}$ for $t > r'$. Then we consider a mode-by-mode decomposition of the diffractive fundamental solution, and shows that the principal symbol of the diffractive half wave kernel is given by the sum of the principal symbol of the diffractive wave of each Fourier mode. This reduces the computation of the diffraction coefficient to each mode.

The construction is based on the functional calculus [20] on product cones. We first consider the exact solution of a single eigenmode to the half wave equation. Recall the Laplacian on a product cone is

$$\Delta = D_r^2 - i \frac{n-1}{r} D_r + \frac{1}{r^2} \Delta_\theta$$

and we define $\nu_j := \sqrt{\mu_j^2 + \alpha^2}$ with $\alpha = -\frac{n-2}{2}$, where μ_j^2, φ_j denote eigenvalues and eigenfunctions of Δ_θ . If we take

$$g_j(r) = r^{-\frac{n-2}{2}} J_{\nu_j}(\lambda r),$$

then

$$\Delta(g_j \varphi_j) = \lambda^2 (g_j \varphi_j).$$

This can be seen by reducing $(\Delta - \lambda^2)(g_j \varphi_j) = 0$ to a Bessel equation by change of variables. For a detailed treatment on this solution on a single Fourier mode, we refer to [3]. By the functional calculus on product cones,

$$f(\Delta)g(r, \theta) = r^\alpha \sum_j \left(\int_0^\infty f(\lambda^2) J_{\nu_j}(\lambda r) \lambda \left(\int_0^\infty s^{1-\alpha} J_{\nu_j}(\lambda s) g_j(s) ds \right) d\lambda \right) \varphi_j(\theta) \tag{19}$$

for $g(r, \theta) = \sum_j g_j(r) \varphi_j(\theta) \in L^2(C(N); \mathbb{C})$. We define the operator ν on N by

$$\nu = (\Delta_\theta + \alpha^2)^{\frac{1}{2}};$$

the kernel of $f(\Delta)$ is thus a function on $\mathbf{R}_+ \times \mathbf{R}_+$ taking values in operators on N , by the formula

$$f(\Delta) = (rr')^\alpha \int_0^\infty f(\lambda^2) J_\nu(\lambda r) J_\nu(\lambda r') \lambda d\lambda. \tag{20}$$

Using (19), we take $g(r, x)$ to be one single mode of spherical wave $\delta(r - r')\varphi_j(\theta)$ and the operator as half wave operator $U(t) = e^{-it\sqrt{\Delta}}$. Then

$$U(t)(\delta(r - r')\varphi_j) = r^\alpha \left(\int e^{-it\lambda} J_{\nu_j}(\lambda r) H(\lambda) \lambda (H(r')r'^{1-\alpha} J_{\nu_j}(\lambda r')) d\lambda \right) \varphi_j(\theta)$$

for fixed $r' > 0$, where H is the Heaviside function.

We define $\tilde{\chi}_{\nu_j}(\lambda) := \rho(\lambda)\lambda(\tilde{\rho}(r')r'^{1-\alpha} J_{\nu_j}(\lambda r'))$ with $\rho(\lambda)$ and $\tilde{\rho}(\lambda)$ being smooth cutoffs away from 0 and equal to 1 for $\lambda > 1$. Thus, the solution should have the form

$$u_{\nu_j}(t, r, r', \theta) \equiv r^\alpha \left(\int e^{-it\lambda} J_{\nu_j}(\lambda r) \tilde{\chi}_{\nu_j}(\lambda) d\lambda \right) \varphi_j(\theta) \pmod{\mathcal{C}^\infty}$$

for fixed $r' > 0$. Note that the solution also has singularities at $\{r = r' + t\}$ apart from the diffractive singularities.

From the previous discussion, we have

$$U(t)(\delta(r - r')\varphi_j(\theta)) \equiv \int E_D(t, r, r', \theta, \theta')\varphi_j(\theta')d\theta' = E_j(t, r, r')\varphi_j(\theta),$$

modulo the singularities at $\{r = r' + t\}$ for the first equation, which correspond to geometric singularities. Now, we compute the diffractive fundamental solution $E_D(t, r, r', \theta, \theta')$. We first regularize it by averaging it angularly to instead study

$$u_\varphi(t, r, r', \theta) = \int E_D(t, r, \theta, r', \theta')\varphi(\theta') d\theta'$$

for an arbitrary $\varphi \in \mathcal{C}_c^\infty(N)$ supported close to a single point $\theta_0 \in N$. Using the Plancherel theorem, Fourier expanding φ in N , i.e., taking the eigenfunction expansion gives

$$u_\varphi = \sum_j c_{\nu_j} E_j(t, r, r')\varphi_j(\theta)$$

with $c_{\nu_j} = \langle \varphi, \varphi_j \rangle_{L^2}$ the corresponding Fourier coefficient.

Now, what we can compute by the asymptotic expansion of Bessel functions is the principal symbol of the conormal solution u_{ν_j} at $N^*\{r + r' = t\}$ (as we will do in the later part of this section); for now we write these principal symbols $\sigma_j^P(t, r, r', \lambda)\varphi_j(\theta)$. Formally, the principal symbol of u_φ is the sum of the principal symbols of u_{ν_j} , though we have to be careful to show that the subprincipal symbols of u_{ν_j} will not add up and contribute to the principal symbol of u_φ . This leads to the following theorem due to Baskin and Wunsch [2]:

Theorem 4.1 (principal symbols of diffraction [2]). *The principal symbol σ^P of u_φ is equal to the sum of principal symbols of mode-by-mode solutions:*

$$\sigma^P(u_\varphi) = \sum_j c_{\nu_j} \sigma_j^P(t, r, r', \lambda)\varphi_j(\theta), \tag{21}$$

i.e., the subprincipal symbols of u_{v_j} will not add up and contribute to principal symbol of u_φ .

Proof. The convergence of this sum is due to the fact that c_{v_j} are rapidly decaying with respect to v_j as $j \rightarrow \infty$, which comes from the fact that φ is \mathcal{C}^∞ so the Fourier coefficient rapidly decays; we can check that the series of principal symbols converges. However, we need information about the growth rate of the symbol remainders: the equality of conormal distributions tells us that really

$$\sigma^T(u_\varphi) = \sum_j c_{v_j} \sigma_j^T(t, r, r', \lambda) \varphi_j(\theta),$$

where we use σ^T to denote the total symbol, i.e., the full amplitude of the conormal distribution with the canonical choice of phase function $\phi(t, r, \lambda) = (t \pm r)\lambda$. Thus, it remains to check that the sum of remainder terms

$$\sum_j c_{v_j} (\sigma_j^T - \sigma_j^P) \varphi_j(\theta) \tag{22}$$

converges in the topology of symbols of order $m - 1 + \varepsilon$. By Proposition 3.12 we find that any desired symbol semi-norm of $\sigma_j^T - \sigma_{m,j}$ is bounded by some symbol semi-norm of the symbol of solution u_{v_j} for $t \ll 0$. Examination of the initial data shows that each of these norms grows at most *polynomially* in v_j (with the growth arising from θ derivatives). Thus, since c_{v_j} decays rapidly, the series (22) does indeed converge in every symbol semi-norm with respect to the $S^{m-1+\varepsilon}$ topology, and the subprincipal terms cannot affect the principal symbol of the sum.⁴ ■

Following Theorem 4.1, we now construct the principal symbol of the diffractive fundamental solution. We fix $\chi \in \mathcal{C}^\infty(N)$ equal to 1 near θ_0 and use the above results for all φ supported on $\{\chi = 1\}$. We have established that

$$\sigma^P(u_\varphi) = \sum_j c_{v_j} \sigma_j^P(t, r, r', \lambda) \varphi_j(\theta).$$

Now, let φ approach $\delta(\theta')$ in the sense of distribution (with θ' not geometrically related⁵ to θ), so that its Fourier coefficients c_{v_j} approach $\varphi_j(\theta')$. We then obtain

⁴It certainly *can* happen that lower-order terms in a sum affect the principal symbol of the result: consider $\delta = \sum_k e^{ik\theta}$ on the circle, and regard the right hand side as a sum of conormal distributions with symbols of order 0, which happen to have vanishing principal symbol. Of course, the problem is that the symbol remainder terms grow nastily in k .

⁵We know, *a priori*, for $d_h(\theta, \theta') \neq \pi$ the diffractive singularities are conormal from [4, 5, 14].

in the limit, in a neighborhood of any pair θ and θ' that are related by strictly diffractively geodesics, i.e., for θ, θ' with $d_h(\theta, \theta') \neq \pi$,

$$\sigma^P(E_D(t, r, r', \theta, \theta')) = \sum \sigma_j^P(t, r, r', \lambda) \varphi_j(\theta) \varphi_j(\theta'),$$

as desired.

In order to get the diffraction coefficient, now it remains to compute the principal symbol of the diffractive fundamental solution for each mode: $\sigma_j^P(t, r, r', \lambda)$. We employ the functional calculus on product cones and the conormality of diffractive waves.

We consider the kernel of the half wave propagator given by (20). Again, here we use the Hankel asymptotics to get the diffractive coefficient. Consider (20) acting on a single mode $\varphi_j(\theta)$:

$$E_j(t, r, r') \varphi_j(\theta) \equiv (rr')^\alpha \left(\int_0^\infty e^{-it\lambda} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \lambda d\lambda \right) \varphi_j(\theta) \tag{23}$$

modulo the singularities of the right-hand side at $N^*\{r = r' + t\}$.

Note that for positive ν and z , the Bessel function $J_\nu(z)$ is the real part of Hankel function $H_\nu^{(1)}(z)$. Thus, using the asymptotic formulas of Hankel functions from [17, 10.17.5], we can extract the leading part of $J_{\nu_j}(\lambda r)$ as the principal symbol with phase variable λ :

$$J_{\nu_j}(\lambda r) \equiv \left(\frac{1}{2\pi\lambda r} \right)^{\frac{1}{2}} \left(e^{i(\lambda r - \frac{\nu_j\pi}{2} - \frac{\pi}{4})} + e^{-i(\lambda r - \frac{\nu_j\pi}{2} - \frac{\pi}{4})} \right) \text{ mod } S^{-\frac{3}{2}+0}. \tag{24}$$

Note that here we are not requiring the uniformity in ν in the asymptotic expansion, since the remainder terms have been taken care of by Theorem 4.1. We now combine this Bessel asymptotics together with (23) to get the diffractive principal symbol. Thus,

$$\begin{aligned} E_j(t, r, r') \varphi_j(\theta) &\equiv (rr')^\alpha \left(\int \rho(\lambda) e^{-it\lambda} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \lambda d\lambda \right) \varphi_j(\theta) \text{ mod } \mathcal{C}^\infty \\ &\equiv \frac{1}{2\pi} (rr')^{\alpha-\frac{1}{2}} \left(\int e^{i\lambda(r+r'-t)} e^{-i(\nu_j\pi + \frac{\pi}{2})} d\lambda \right) \varphi_j(\theta) \end{aligned}$$

modulo singularities at the conormal bundle $N^*\{r = r' + t\}$ and the lower order singularities at the conormal bundle $N^*\{r + r' = t\}$. The second equality is due to the fact that diffractive wave is conormal to $\{r + r' = t\}$ [14, Theorem 4.8], so the only part in $J_{\nu_j}(\lambda r)$ and $J_{\nu_j}(\lambda r')$ that contributes to the diffractive principal symbol is each of their first terms in the asymptotic expansion (24), and the remaining terms are smooth near $N^*\{r + r' = t\}$, hence will not contribute to the diffractive singularities.

Now, comparing the above equation with the general formula for the diffractive half wave kernel (17), we have the diffraction coefficient:

$$K_0(r, \theta; r', \theta') = -\frac{i}{2\pi} (rr')^{-\frac{n-1}{2}} e^{-i\nu\pi} \tag{25}$$

where $\nu = \sqrt{\Delta_\theta + (\frac{n-2}{2})^2}$.

5. Scattering matrix on product cones

Consider the leading order behaviors of the solutions of

$$(\Delta - \lambda^2) u = 0$$

under the asymptotic condition:

$$u \sim a_+(\theta)r^{-\frac{n-1}{2}} e^{i\lambda r} + a_-(\theta)r^{-\frac{n-1}{2}} e^{-i\lambda r} + \mathcal{O}(r^{-\frac{n+1}{2}}) \text{ as } r \rightarrow \infty, \tag{26}$$

where a_-/a_+ is called the incoming/outgoing coefficient. Then $a_+(\theta)$ is uniquely determined by $a_-(\theta)$ and the scattering matrix $S(\lambda)$ is the unitary map from $a_-(\theta)$ to $a_+(\theta)$ for $\lambda \in \mathbf{R} \setminus \{0\}$. This property is known for smooth asymptotically Euclidean manifolds [16], and we show below in Proposition 5.1 it is also true on product cones. Meanwhile, we show that the scattering matrix on a product cone is

$$S(\lambda) = -i e^{-i\pi\nu}$$

where $\nu = \sqrt{\Delta_\theta + (\frac{n-2}{2})^2}$, which is related to the diffraction coefficient (25) as

$$S(\lambda) = 2\pi (rr')^{\frac{n-1}{2}} K_0(r, \theta; r', \theta').$$

Here we should note that we only consider the smooth part of the scattering matrix. By [15], on smooth asymptotically Euclidean manifolds (smooth manifolds with large conical ends), the scattering matrix is a Fourier integral operator with the canonical relation given by geodesic flow at time π . In the previous section, we found the diffraction coefficient of the points on product cones which are strictly diffractively related. This corresponds to the smooth part of the scattering matrix, i.e., $S(\lambda, \theta, \theta')$ for $d_h(\theta, \theta') \neq \pi$, where $S(\lambda, \theta, \theta')$ is the kernel of the scattering matrix. From now on, we use the name *scattering matrix* without saying that it means the smooth part.

We define the scattering matrix through the following proposition:

Proposition 5.1 (the scattering matrix on product cones). *The scattering matrix $S(\lambda)$ on the product cone $C(N)$ for $\lambda \in \mathbf{R} \setminus \{0\}$ is a unitary operator:*

$$S(\lambda): \mathcal{C}^\infty(N) \rightarrow \mathcal{C}^\infty(N), \quad a_-(\theta) \mapsto a_+(\theta), \tag{27}$$

where $a_+(\theta)$ is the outgoing coefficient in the asymptotic expansion (26) and it is uniquely determined by the incoming coefficient $a_-(\theta)$. Moreover, the scattering matrix on a product cone takes the form

$$S(\lambda) = -ie^{-i\pi\nu}$$

where $\nu = \sqrt{\Delta_\theta + (\frac{n-2}{2})^2}$.

Proof. Consider the homogeneous equation on $C(N)$

$$(\Delta - \lambda^2)u = 0. \tag{28}$$

By Section 2, this is

$$\left(\partial_r^2 + \frac{n-1}{r}\partial_r - \frac{1}{r^2}\Delta_\theta + \lambda^2\right)u = 0.$$

Now, we consider an eigenfunction decomposition of $u \in L^2(C(N))$ by eigenfunctions on N of Δ_θ . i.e.,

$$L^2(C(N); \mathbb{C}) = \bigoplus_{j=0}^\infty L^2(\mathbf{R}_+; E_j), \quad u(r, \theta) = \sum_{j=0}^\infty v_j(r)\varphi_j(\theta),$$

where the first space is defined via the volume form induced by the conic metric, and the latter spaces can be identified with the space $L^2(\mathbf{R}_+; r^{n-1}dr)$. E_j denotes the j -th eigenspace.

The equation (28) thus becomes

$$\sum_j \varphi_j(\theta) \cdot \left(\partial_r^2 + \frac{n-1}{r}\partial_r + \lambda^2 - \frac{\mu_j^2}{r^2}\right)v_j(r) = 0,$$

where μ_j^2 is the eigenvalue to Δ_θ with the eigenfunction φ_j . We thus reduce the equation (28) to

$$\left(\partial_r^2 + \frac{n-1}{r}\partial_r + \left(\lambda^2 - \frac{\mu_j^2}{r^2}\right)\right)v_j(r) = 0 \tag{29}$$

for all j . By changing variable ρ with $\rho = \lambda r$, we have

$$\left(\partial_\rho^2 + \frac{n-1}{\rho}\partial_\rho + \left(1 - \frac{\mu_j^2}{\rho^2}\right)\right)v_j(r) = 0.$$

Writing $v_j(r) = \rho^\sigma \omega_j(\rho)$, we can replace the previous equation by

$$\begin{aligned} &\sigma(\sigma - 1)\rho^{\sigma-2}\omega_j + 2^\sigma \rho^{\sigma-1}\partial_\rho\omega_j + \rho^\sigma\partial_\rho^2\omega_j \\ &+ \frac{n-1}{\rho}(\sigma\rho^{\sigma-1}\omega_j + \rho^\sigma\partial_\rho\omega_j) + \left(1 - \frac{\mu_j^2}{\rho^2}\right)\rho^\sigma\omega_j = 0. \end{aligned}$$

Rewrite it into the following form

$$\left(\partial_\rho^2 + \frac{2\rho + (n-1)}{\rho}\partial_\rho + \frac{\sigma(\sigma-1) + (n-1)\sigma}{\rho^2} + \left(1 - \frac{\mu_j^2}{\rho^2}\right)\right)\omega_j = 0.$$

Setting $2\sigma + (n-1) = 1$, i.e., $\sigma = 1 - \frac{n}{2}$, the equation above then becomes

$$\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \left(1 - \frac{\mu_j^2 + (1-n/2)^2}{\rho^2}\right)\right)\omega_j = 0.$$

This is a homogeneous Bessel equation

$$\omega_j'' + \frac{1}{\rho}\omega_j' + \left(1 - \frac{v_j^2}{\rho^2}\right)\omega_j = 0 \tag{30}$$

with $v_j = \sqrt{\mu_j^2 + (1 - \frac{n}{2})^2}$. Its general solutions in terms of Bessel/Hankel functions are the linear combinations of $H_v^{(1)}(z)$ and $H_v^{(2)}(z)$. We can then construct solutions to (28) with the prescribed boundary condition using Bessel functions asymptotics as following. Consider the general solutions to (30)

$$\omega_j = C_1 H_{v_j}^{(1)}(\rho) + C_2 H_{v_j}^{(2)}(\rho).$$

Noticing that from the above change of variables, $v_j = \rho^{1-\frac{n}{2}}\omega_j(\rho)$ and $\rho = \lambda r$, we can obtain general solutions to (29):

$$v_j = C_1 \cdot (\lambda r)^{1-n/2} H_{v_j}^{(1)}(\lambda r) + C_2 \cdot (\lambda r)^{1-n/2} H_{v_j}^{(2)}(\lambda r)$$

and thus they have general solutions to (28):

$$u = \sum_j (C_{1,j} \cdot (\lambda r)^{1-n/2} H_{v_j}^{(1)}(\lambda r) + C_{2,j} \cdot (\lambda r)^{1-n/2} H_{v_j}^{(2)}(\lambda r))\varphi_j(\theta). \tag{31}$$

By the asymptotic behaviors of Hankel functions, we notice that the solutions thus have the asymptotic expansion

$$u \sim a_+(\theta)r^{-\frac{n-1}{2}}e^{i\lambda r} + a_-(\theta)r^{-\frac{n-1}{2}}e^{-i\lambda r} + \mathcal{O}(r^{-\frac{n+1}{2}}) \text{ as } r \rightarrow \infty, \tag{32}$$

where the first term is called outgoing and the second incoming.

As in the construction of the resolvent on the product cone [3, Theorem 2.1], we consider the boundary behaviors of Bessel functions. The choice of the Friedrichs extension requires that both v_j and v'_j lie in the weighted L^2 space near 0. By the asymptotic formula of Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ from [17, 10.7.3 and 10.7.4], the existence of $Y_\nu(z)$ fails the Friedrichs extension condition. And since

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z) \text{ and } H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)$$

for $\nu > 0$, the facts above and in particular the Friedrichs extension imply that in (31) the coefficients $C_{1,j} = C_{2,j} = C_j$. Thus, v_j must be a multiple of $r^{-(n-2)/2} J_{\nu_j}(\lambda r)$ near $r = 0$. And the solution then becomes

$$u = \sum_j 2C_j \cdot (\lambda r)^{1-n/2} J_{\nu_j}(\lambda r) \varphi_j(\theta). \tag{33}$$

Again, from [17, 10.17.3], by asymptotics of $J_\nu(z)$,

$$J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\cos \omega \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k}(\nu)}{z^{2k}} - \sin \omega \sum_{k=0}^{\infty} (-1)^k \frac{a_{2k+1}(\nu)}{z^{2k+1}}\right), \text{ as } z \rightarrow 0$$

where $\omega = z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi$ and $a_k(\nu)$ constants of ν , we can extract the leading part of u as

$$u \sim \sum_j 2C_j \cdot (\lambda r)^{1-n/2} \left(\frac{1}{2\pi\lambda r}\right)^{\frac{1}{2}} (e^{-\frac{i\pi}{4} - \frac{1}{2}\pi\nu_j} e^{i\lambda r} + e^{\frac{i\pi}{4} + \frac{1}{2}\pi\nu_j} e^{-i\lambda r}) \varphi_j(\theta).$$

Note that no uniformity in ν is needed here since we only focus on the structure of the scattering matrix based on the functional calculus; the existence is known *a priori*. For each mode $\varphi_j(\theta)$, the scattering matrix $S(\lambda)$ acts like

$$a_{-,j}(\theta) := e^{\frac{i\pi}{4} + \frac{1}{2}\pi\nu_j} \varphi_j \mapsto a_{+,j}(\theta) := e^{-\frac{i\pi}{4} - \frac{1}{2}\pi\nu_j} \varphi_j.$$

Thus, the scattering matrix acts diagonally on modes and takes the form

$$S(\lambda) = -ie^{-i\pi\nu},$$

where $\nu = \sqrt{\Delta_\theta + (\frac{n-2}{2})^2}$ is defined by the functional calculus for the self-adjoint operator Δ_θ on N . The unitarity follows directly from this expression.

Now, we show the uniqueness of the scattering matrix. It suffices to show the uniqueness for each mode. For any $g(\theta) \in \mathcal{C}^\infty(N)$, if there are two solutions u_1 and u_2 to

$$(\Delta - \lambda^2)u = 0 \tag{34}$$

with $g(\theta)$ the incoming boundary condition, then $u = u_1 - u_2$ is an outgoing solution to the homogeneous stationary wave equation (34). Thus, each mode component of u needs to satisfy the outgoing condition and it takes the form

$$H_{\nu_j}^{(1)}(\lambda r) = J_{\nu_j}(\lambda r) + iY_{\nu_j}(\lambda r),$$

since $H_{\nu_j}^{(1)}(\lambda r)$ gives the outgoing part in the asymptotic expansions (26). Then each mode component of u has to be zero since $Y_{\nu_j}(\lambda r)$ does not lie in L^2 at $r = 0$, otherwise it contradicts to the requirement of the Friedrichs extension. Hence, $a_+(\theta)$ must be uniquely determined by $g = a_-(\theta)$. ■

Combining the expression of the scattering matrix with the conclusion of the diffraction coefficient in the previous section, we have proved Theorem 1.1 which we restate more precisely as follows:

Theorem. *Away from the intersection of geometric wave front and diffractive front, i.e., for $d_h(\theta, \theta') \neq \pi$, the diffraction coefficient, which is the principal symbol of the diffractive half wave kernel, $K_0(r, \theta; r', \theta')$ and the kernel of the scattering matrix $S(\lambda)$ satisfy the following relation:*

$$K_0(r, \theta, r', \theta') = (2\pi)^{-1} (rr')^{-\frac{n-1}{2}} S(\lambda, \theta, \theta'),$$

where $S(\lambda, \theta, \theta')$ is the kernel of the scattering matrix.

6. From the radiation field to the scattering matrix

We now proceed to study the diffraction part of the radiation field of the wave equation on product cones. By using the notion of the radiation field, we can give an alternative proof of the Theorem 1.1.

We use some ideas developed by Friedlander [9] and Sá Barreto [18] on the radiation field of asymptotically Euclidean manifolds. The radiation field on product cones is also studied by Baskin and Marzuola [1].⁶ We define the radiation field through a theorem proved by Friedlander in [9] for smooth manifolds, and in the setting of product cones $X = C(N)$ by Baskin and Marzuola [1]:

⁶In [1], they studied the radiation field of the solution to the wave equation away from the cone point, while we only use the radiation field corresponding to the diffractive fundamental solution.

Theorem 6.1 (radiation field). *Let $f_0, f_1 \in \mathcal{C}_0^\infty(X)$ be smooth functions with compact support in X . If $u(t, r, \theta) \in \mathcal{C}^\infty(\mathbf{R}_+ \times X)$ solves the wave equation with Cauchy data:*

$$\begin{cases} \square u(t, r, \theta) = 0 & \text{on } \mathbf{R} \times X, \\ u(0, r, \theta) = f_0, \quad D_t u(0, r, \theta) = f_1, \end{cases}$$

where $(r, \theta) \in \mathbf{R}_+ \times N$, then there exist $w_k \in \mathcal{C}^\infty(\mathbf{R} \times N)$, such that

$$r^{\frac{n-1}{2}} (Hu)(s+r, r, \theta) \sim \sum_{k=0}^\infty r^{-k} w_k(s, \theta), \quad \text{as } r \rightarrow \infty$$

where $H(t)$ denotes the Heaviside function. In particular,

$$r^{\frac{n-1}{2}} (Hu)(s+r, r, \theta)|_{r \rightarrow \infty} = w_0(s, \theta)$$

is well defined, and it is called the radiation field of u as in [9, Proposition 2].

Remark 6.2. Friedlander showed the existence of the radiation field in the context of the smooth scattering manifold. In our case there is a singularity at the cone point. This is not an issue here, since we are considering the radiation field away from the cone point.

For solutions to the wave equation, we have the energy norm $\|\cdot\|_E$:

$$\|u\|_E = \frac{1}{2} \int_X (|du|_g^2 + |u_t|^2) dg$$

and define the wave group

$$W(t): \mathcal{C}_0^\infty(X^\circ) \times \mathcal{C}_0^\infty(X^\circ) \rightarrow \mathcal{C}_0^\infty(X^\circ) \times \mathcal{C}_0^\infty(X^\circ),$$

by

$$W(t)(f_0, f_1) = (u, u_t), \quad t \in \mathbf{R}.$$

We know by conservation of energy that $W(t)$ is a strongly continuous group of unitary operators.

We now define a map

$$\mathcal{R}_+: \mathcal{C}_0^\infty(X^\circ) \times \mathcal{C}_0^\infty(X^\circ) \longrightarrow \mathcal{C}^\infty(\mathbf{R} \times N)$$

by

$$\begin{aligned} \mathcal{R}_+(f_0, f_1)(s, \theta) &= r^{\frac{n-1}{2}} (D_t Hu)(s+r, r, \theta)|_{r \rightarrow \infty} \\ &=: D_s w_0^+(s, \theta) (= D_s w_0(s, \theta)), \end{aligned}$$

which is called the *forward radiation field*. Its existence follows from Theorem 6.1. Similarly, we can define the *backward radiation field*

$$\mathcal{R}_-: \mathcal{C}_0^\infty(X^o) \times \mathcal{C}_0^\infty(X^o) \rightarrow \mathcal{C}^\infty(\mathbf{R} \times N)$$

as

$$\mathcal{R}_-(f_0, f_1)(s, \theta) = r^{\frac{n-1}{2}} (D_t H_- u)(s - r, r, \theta)|_{r \rightarrow \infty} = D_s w_0^-(s, \theta),$$

where $H_-(t) = H(-t)$.

Sá Barreto also proved in [18] that the forward and backward radiation fields are in fact unitary on smooth asymptotically Euclidean manifolds under the energy norm of the Cauchy data. It leads to the definition of the scattering operator which is essentially the Fourier conjugation of the scattering matrix.

Theorem 6.3. *The maps \mathcal{R}_\pm extend to isometries*

$$\mathcal{R}_\pm: H_E(X) \rightarrow L^2(\mathbf{R} \times N).$$

The scattering operator defined by

$$S = \mathcal{R}_+ \circ \mathcal{R}_-^{-1} \tag{35}$$

is unitary on $L^2(\mathbf{R} \times N)$; the scattering matrix S is given by conjugating the scattering operator with the partial Fourier transform in the s -variable:

$$S = \mathcal{F} S \mathcal{F}^{-1}. \tag{36}$$

Here we note that the proof of this theorem in [18] can be extended to product cones from smooth asymptotically Euclidean manifolds. This is because the proof relies on the fact that the Laplacian on product cones has purely absolute continuous spectrum so that we could apply the proposition that the Poisson operators give isometries from the absolute continuous spectral subspace of Δ to $L^2(\mathbf{R} \times N)$ [10, Proposition 9.1] to give an isometry between the energy norm of the initial data (f_0, f_1) and the Fourier transform of the forward/backward radiation field as in [18]. Otherwise, although there are still isometries between the absolute continuous spectral subspace of Δ and $L^2(\mathbf{R} \times N)$, the energy norm would need to include the contribution from the discrete eigenmodes.

In the rest of this section, we construct the scattering matrix from the diffractive coefficient using the radiation field, and we shall see that the scattering matrix agrees with the diffraction coefficient up to scaling. In other words, using the radiation field, we give a second proof to Theorem 1.1.

Assume $z = (r, \theta)$ and $z' = (r', \theta')$ are points on the cone $C(N)$. We first consider the forward fundamental solution $E(z, z', t)$ solving

$$\square_{(r, \theta, t)} E = \delta(z - z') \delta(t), \quad \text{supp } E \subset \{t \geq d_{C(N)}(z, z')\}.$$

For fixed $z' \in C(N)$, the fundamental solution has a radiation field

$$E_\infty(s, \theta, z') = \lim_{r \rightarrow \infty} r^{\frac{n-1}{2}} E(s+r, r, \theta, z').$$

Friedlander in [9] also points out that we can get the inverse of the radiation field through the following formula:

$$u(r, \theta, t) = -2 \int E_\infty(s' - t, \theta', r, \theta) \partial_s w(s', \theta') |h(0, \theta')|^{1/2} |ds' d\theta'|.$$

Then from (35), we get the kernel of the scattering operator \mathcal{S} in terms of the fundamental solution:

$$\begin{aligned} K_{\mathcal{S}} &= \lim_{r \rightarrow \infty} r^{\frac{n-1}{2}} \partial_t E_\infty(s + s' + r, \theta', r, \theta) \\ &= \lim_{\substack{r \rightarrow \infty \\ r' \rightarrow \infty}} 2(rr')^{\frac{n-1}{2}} \partial_t E(s - s' + r + r', r, \theta, r', \theta'). \end{aligned} \tag{37}$$

In terms of the wave propagator, we know the fundamental solution is given by

$$E(z, z', t) = \frac{\sin t \sqrt{\Delta}}{\sqrt{\Delta}} \delta(z - z').$$

Now, we compute the scattering operator using (37) and the kernel of the diffractive wave propagator. Since we showed the diffractive wave enjoys one-step polyhomogeneity in Section 3, by the formula of the kernel of the diffractive wave propagator, we can make a WKB style ansatz for the diffractive part of $E(z, z', t)$ as the following:

$$E(z, z', t) \equiv (rr')^{-\frac{n-1}{2}} \int e^{i(r+r'-t)\lambda} a(r, r', \theta, \theta', \lambda) d\lambda$$

modulo singularities other than $N^*\{r + r' = t\}$ with symbol

$$a = \sum_{k=0}^{\infty} \bar{a}_k(r) \lambda^{-k},$$

where \bar{a}_0 is the diffraction coefficient. Here the symbol a and each term \bar{a}_k in its expansion depend on (r, r', θ, θ') , though we only emphasis the r dependence for our purpose. Consider the equation

$$\square u \in \mathcal{C}^\infty$$

near $\{r + r' = t\}$. In terms of the symbol of u , a straightforward computation yields that

$$\left(2i\lambda \partial_r + \partial_r^2 + i\lambda \frac{n-1}{r} + \frac{n-1}{r} \partial_r - \frac{\Delta_\theta}{r^2} \right) (r^{-\frac{n-1}{2}} a) \in \mathcal{O}(\lambda^{-\infty}).$$

A further computation (cf. Proof of Lemma 3.7) yields a series of transport equations for the symbol asymptotic expansion coefficients \bar{a}_k :

$$r \partial_r \bar{a}_0 = 0 \text{ and } r \partial_r \bar{a}_k = \frac{1}{r} \left(\frac{n^2 - 1}{4} - \Delta_\theta \right) \bar{a}_{k-1}.$$

Therefore, solving the transport equations on coefficients \bar{a}_k 's leads to the result that the symbol of diffractive singularities enjoys polyhomogeneity in λr :

$$(r r')^{-\frac{n-1}{2}} \int e^{i(r+r'-t)\lambda} \left(\sum_{k=0}^{\infty} a_k \cdot (\lambda r)^{-k} \right) d\lambda$$

where a_k does not depend on r or λ . Now, the kernel of the scattering operator of the diffractive wave can be computed as the following using equation (37):

$$K_S = \lim_{\substack{r \rightarrow \infty \\ r' \rightarrow \infty}} 2(r r')^{\frac{n-1}{2}} \int_{\mathbf{R}_\lambda} e^{i(s'-s)\lambda} (-i) \frac{\lambda}{2|\lambda|} \rho(\lambda) \frac{(r r')^{-\frac{n-1}{2}}}{2\pi} \times (H(\lambda)e^{-i\pi\nu} + H(-\lambda)e^{i\pi\nu}) d\lambda$$

as the Schwartz kernel, where $\rho(\lambda) \in \mathcal{C}^\infty(\mathbf{R})$ satisfies $\rho \equiv 1$ for $|\lambda| > 2$ and $\rho \equiv 0$ for $|\lambda| < 1$. The equation holds with only the leading part on r, r' left, because the remainder terms have lower order in r, r' , and they become zero in the limit. In other words, the diffraction coefficient is the only part that contributes to this piece of the scattering operator.

Taking the Fourier conjugation of the scattering operator computed above, by (36) the scattering matrix is

$$\begin{aligned} S(\lambda) &= -i \frac{\lambda}{|\lambda|} \rho(\lambda) (H(\lambda)e^{-i\pi\nu} + H(-\lambda)e^{i\pi\nu}) \\ &= -iH(\lambda)e^{-i\pi\nu} + iH(-\lambda)e^{i\pi\nu}. \end{aligned}$$

For $\lambda \neq 0$, the scattering matrix takes the form

$$S(\lambda) = -i e^{-i\pi\nu}$$

where $\nu = \sqrt{\Delta_\theta + (\frac{n-2}{2})^2}$, which agrees with what we obtained from the direct computation in Section 5.

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