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A semiclassical Birkhoff normal form for symplectic magnetic wells

Léo Morin

Abstract. In this paper we construct a Birkhoff normal form for a semiclassical magnetic Schrödinger operator with non-degenerate magnetic field, and discrete magnetic well, defined on an even-dimensional Riemannian manifold M. We use this normal form to get an expansion of the first eigenvalues in powers of $\hbar^{1/2}$, and semiclassical Weyl asymptotics for this operator.

1. Introduction

The analysis of the magnetic Schrödinger operator, or magnetic Laplacian, on a Riemannian manifold

$$\mathcal{L}_\hbar = (i\hbar\,\mathrm{d} + A)^*(i\hbar\,\mathrm{d} + A)$$

in the semiclassical limit $\hbar \to 0$ has given rise to many investigations in the last twenty years. Asymptotic expansions of the lowest eigenvalues have been studied in many cases involving the geometry of the possible boundary of M and the variations of the magnetic field. For discussions about the subject, the reader is referred to the books and review [9, 14, 25]. The classical picture associated with the Hamiltonian

$$|p - A(q)|^2$$

has started being investigated to describe the semiclassical bound states (the eigenfunctions of low energy) of $\mathcal{L}_\hbar$, in [26] (on $\mathbf{R}^2$) and [12] (on $\mathbf{R}^3$). In these two papers, semiclassical Birkhoff normal forms were used to describe the first eigenvalues. In [27], Sjöstrand introduced the semiclassical Birkhoff normal form to study the spectrum of an electric Schrödinger operator, and some resonance phenomenons appeared. In [5], the resonant case for the same electric Schrödinger operator

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was tackled (see also [28, 29]). In this paper, we adapt this method to $\mathscr{L}_\hbar$, generalizing the results of [26] to higher dimensions and manifolds. Our normal forms give a great geometric interpretation of the semiclassical spectral asymptotics of $\mathscr{L}_\hbar$. Indeed, it enlightens the contributions of the cyclotron motion (the first oscillator) and the variations of the magnetic field near its well (the second oscillator) in the Weyl asymptotics and the eigenvalue asymptotics. In [18], there is a discussion on how normal forms can yield Weyl laws for an electro-magnetic Schrödinger operator, and our approach clearly follows the same kind of ideas (See Remark 3).

In this paper, we get an expansion of the first eigenvalues of $\mathscr{L}_\hbar$ in powers of $\hbar^{1/2}$, and semiclassical Weyl asymptotics. It would be interesting to have a precise description of the eigenfunctions too, as was done in the 2D case by Bonthonneau and Raymond [3] (Euclidian case) and Nguyen Duc Tho [11] (general Riemannian metric). Moreover, we only have investigated the spectral theory of the stationary Schrödinger equation with a pure magnetic field; it would be interesting to describe the long-time dynamics of the full Schrödinger evolution, as was done in the Euclidian 2D case by Boil and Vũ Ngọc [2]. Finally, on a Riemannian manifold M, the magnetic Laplacian is related to the Bochner Laplacian (see the recent papers [20–22], where bounds and asymptotic expansions of the first eigenvalues of Bochner Laplacians are given). We explain in [24] how the results of this paper apply to the Bochner Laplacian.

1.1. Definition of the magnetic Schrödinger operator

Let (M, g) be a smooth d-dimensional oriented Riemannian manifold. We assume that M is compact with boundary, or that $M = \mathbf{R}^d$ with the Euclidean metric. For $q \in M$, g_q is a scalar product on T_qM. Since M is oriented, there is a canonical volume form, denoted either by $\mathrm{d}x_g$ or by $\mathrm{d}q_g$. If $f \in \mathrm{L}^2(M)$, we denote its norm by

$$\|f\| = \left(\int_M |f(q)|^2 \,\mathrm{d}q_g\right)^{1/2}.$$

If $p \in T_qM^*$, we denote by $|p|_{g_q^\star}$ or $|p|$ the norm of p, defined by

$$|Q|_{g_q}^2 = |g_q(Q,\cdot)|_{g_q^*}^2, \quad \text{for all } Q \in T_qM.$$

We denote by g_q^* the associated scalar product. The norm of a 1-form α on M is

$$\|\alpha\| = \left(\int_M |\alpha(q)|^2 \,\mathrm{d}q_g\right)^{1/2}.$$

It is associated with a scalar product, denoted by brackets $\langle\cdot,\cdot\rangle$.

We denote by d the exterior derivative, associating to any p-form α a $(p+1)$-form $\mathrm{d}\,\alpha$. Using the scalar products induced by the metric, we can define its adjoint d^*, associating to any p-form α a $(p-1)$-form $\mathrm{d}^*\,\alpha$.

We take a 1-form A on M called the magnetic potential, and we denote by $B = \mathrm{d}\,A$ its exterior derivative. B is called the magnetic 2-form. The associated classical Hamiltonian is defined on T^*M by

$$H(q,p) = |p - A(q)|^2_{g_q^*}, \quad p \in T_qM^*.$$

Using the isomorphism $T_qM \simeq T_qM^*$ given by the metric, we define the magnetic operator $\mathbf{B}(q)\colon T_qM \to T_qM$ by

$$B_q(Q_1, Q_2) = g_q\big(\mathbf{B}(q)Q_1, Q_2\big), \quad \text{for all } Q_1, Q_2 \in T_qM. \tag{1.1}$$

The norm of $\mathbf{B}(q)$ is

$$|\mathbf{B}(q)| = [\mathrm{Tr}(\mathbf{B}^*(q)\mathbf{B}(q))]^{1/2}.$$

On the quantum side, for $\hbar > 0$, we define the magnetic quadratic form $q_\hbar$ on

$$D(q_\hbar) = \{u \in \mathrm{L}^2(M), (i\hbar d + A)u \in \mathrm{L}^2\,\Omega^1(M), u_{\partial M} = 0\},$$

by

$$q_\hbar(u) = \int_M |(i\hbar d + A)u|^2\,\mathrm{d}\,q_g,$$

where $\mathrm{L}^2\,\Omega^1(M)$ denotes the space of square-integrable 1-forms on M. By the Lax–Milgram theorem, this quadratic form defines a self-adjoint operator $\mathcal{L}_\hbar$ on

$$D(\mathcal{L}_\hbar) = \{u \in \mathrm{L}^2(M), (i\hbar\,\mathrm{d} + A)^*(i\hbar\,\mathrm{d} + A)u \in \mathrm{L}^2(M), u_{\partial M} = 0\},$$

by the formula

$$\langle \mathcal{L}_\hbar u, v\rangle = q_\hbar[u,v], \quad \text{for all } u, v \in \mathcal{C}_0^\infty(M),$$

where $q_\hbar[\cdot,\cdot]$ is the inner product associated with the quadratic form $q_\hbar(.)$. $\mathcal{L}_\hbar$ is the magnetic Schrödinger operator with Dirichlet boundary conditions.

1.2. Local coordinates

If we choose local coordinates $q = (q_1, \ldots, q_d)$ on M, we get the corresponding vector fields basis $(\partial_{q_1}, \ldots, \partial_{q_d})$ on T_qM, and the dual basis $(\mathrm{d}\,q_1, \ldots, \mathrm{d}\,q_d)$ on T_qM^*. In these basis, g_q can be identified with a symmetric matrix $(g_{ij}(q))$ with determinant $|g|$, and g_q^* is associated with the inverse matrix $(g^{ij}(q))$. We can write the 1-form A in the coordinates:

$$A \equiv A_1 \,\mathrm{d}q_1 + \cdots + A_d \,\mathrm{d}q_d,$$

with $\mathbf{A} = (A_j)_{1\le j\le d} \in \mathcal{C}^\infty(\mathbf{R}^d, \mathbf{R}^d)$. We denote by

$$T_q A\colon T_q M \to T_q M^*$$

the linear operator whose matrix is the Jacobian of $\mathbf{A}$:

$$(\nabla \mathbf{A}(q))_{ij} = \partial_j A_i(q).$$

In the coordinates, the 2-form B is

$$B = \sum_{i<j} B_{ij} \,\mathrm{d}q_i \wedge \mathrm{d}q_j,$$

with

$$B_{ij} = \partial_i A_j - \partial_j A_i = ({}^t\nabla\mathbf{A} - \nabla\mathbf{A})_{ij}.$$

Let us denote by $(\mathbf{B}_{ij}(q))_{1\le i,j\le d}$ the matrix of the operator $\mathbf{B}(q)\colon T_qM \to T_qM$ in the basis $(\partial q_1, \ldots, \partial q_d)$. With this notation, equation (1.1) relating $\mathbf{B}$ to B can be rewritten

$$\sum_{ijk} g_{kj}\mathbf{B}_{ki} Q_i \tilde{Q}_j = \sum_{ij} B_{ij} Q_i \tilde{Q}_j \quad \text{for all } Q, \tilde{Q} \in \mathbf{R}^d,$$

which means that

$$B_{ij} = \sum_k g_{kj}\mathbf{B}_{ki} \quad \text{for all } i, j. \tag{1.2}$$

Also note for later use that

$$\begin{aligned} \iota_Q B &= \sum_{i<j} B_{ij}(Q_i \,\mathrm{d}q_j - Q_j \,\mathrm{d}q_i) = \sum_j \Big(\sum_i B_{ij} Q_i\Big) \mathrm{d}q_j \\ &= \sum_j [({}^t\nabla\mathbf{A} - \nabla\mathbf{A})Q]_j \,\mathrm{d}q_j = ({}^tT_qA - T_qA)Q, \end{aligned} \tag{1.3}$$

Finally, in the coordinates H is given by

$$H(q,p) = \sum_{i,j} g^{ij}(q)(p_i - A_i(q))(p_j - A_j(q)), \tag{1.4}$$

and $\mathcal{L}_\hbar$ acts as the differential operator

$$\mathcal{L}_\hbar^{\mathrm{coord}} = \sum_{k,l=1}^{d} |g|^{-1/2}(i\hbar\partial_k + A_k) g^{kl} |g|^{1/2} (i\hbar\partial_l + A_l). \tag{1.5}$$

1.3. Pseudodifferential operators

We refer to [23,31] for the general theory of $\hbar$-pseudodifferential operators. If $m \in \mathbf{Z}$, we denote by

$$S^m(\mathbf{R}^{2n}) = \{a \in \mathscr{C}^\infty(\mathbf{R}^{2n}), |\partial_x^\alpha \partial_\xi^\beta a| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|} \text{ for all } \alpha, \beta \in \mathbf{N}^d\}$$

the class of Kohn–Nirenberg symbols. If a depends on the semiclassical parameter $\hbar$, we require that the coefficients $C_{\alpha\beta}$ are uniform with respect to $\hbar \in (0, \hbar_0]$. For $a_\hbar \in S^m(\mathbf{R}^{2n})$, we define its associated Weyl quantization $\mathrm{Op}_\hbar^w(a_\hbar)$ by the oscillatory integral

$$\mathcal{A}_\hbar u(x) = \mathrm{Op}_\hbar^w(a_\hbar) u(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbf{R}^{2n}} e^{\frac{i}{\hbar}\langle x-y,\xi\rangle} a_\hbar\Big(\frac{x+y}{2}, \xi\Big) u(y)\, \mathrm{d}y\, \mathrm{d}\xi,$$

and we denote

$$a_\hbar = \sigma_\hbar(\mathcal{A}_\hbar).$$

A pseudodifferential operator $\mathcal{A}_\hbar$ on $\mathrm{L}^2(M)$ is an operator acting as a pseudodifferential operator in coordinates. Then the principal symbol of $\mathcal{A}_\hbar$ does not depend on the coordinates, and we denote it by $\sigma_0(\mathcal{A}_\hbar)$. The subprincipal symbol $\sigma_1(\mathcal{A}_\hbar)$ is also well defined, up to imposing the charts to be volume-preserving (in other words, if we see $\mathcal{A}_\hbar$ as acting on half-densities, its subprincipal symbol is well defined).

If M is compact, in any local coordinates, the coefficients A_j of A (as a function of $q \in \mathbf{R}^d$) are in $S^0(\mathbf{R}^{2d}_{(q,p)})$. Hence, we see from (1.5) that $\mathcal{L}_\hbar$ is a pseudodifferential operator on $\mathrm{L}^2(M)$. Its principal and subprincipal Weyl symbols are

$$\sigma_0(\mathcal{L}_\hbar) = H, \quad \sigma_1(\mathcal{L}_\hbar) = 0.$$

This is well known, but we detail the computation of the subprincipal symbol in Appendix A (Lemma A.1).

If $M = \mathbf{R}^d$, we assume that $A_j \in S^0(\mathbf{R}^{2d})$ for $1 \leq j \leq d$. We could also assume that A_j belongs to some standard class of symbol defined by a general order function on $\mathbf{R}^{2d}$.

1.4. Assumptions

Since $\mathbf{B}(q)$, defined in (1.1), is a skew-symmetric operator for the scalar product g_q, its eigenvalues are in $i\mathbf{R}$. We define the magnetic intensity, which is equivalent to the trace-norm, by

$$b(q) = \mathrm{Tr}^+ \mathbf{B}(q) = \frac{1}{2} \mathrm{Tr}([\mathbf{B}^*(q)\mathbf{B}(q)]^{1/2}) = \sum_{\substack{i\beta_j \in \mathrm{sp}(\mathbf{B}(q)) \\ \beta_j > 0}} \beta_j.$$

It is a continuous function of q, but not smooth in general. We also denote

$$b_0 = \inf_{q \in M} b(q),$$

and in the non-compact case $M = \mathbf{R}^d$,

$$b_\infty = \liminf_{|q| \to +\infty} b(q).$$

We first assume that the magnetic field satisfies the following inequality.

Assumption 1. We assume that there exist $\hbar_0 > 0$ and $C_0 > 0$ such that, for $\hbar \in (0, \hbar_0]$,

$$(1 + \hbar^{1/4} C_0) q_\hbar(u) \geq \int_M \hbar (b(q) - \hbar^{1/4} C_0) |u(q)|^2 \, \mathrm{d} q_g \quad \text{for all } u \in D(q_\hbar).$$

In [15], Helffer and Morame proved such an inequality in the case M compact. If $M = \mathbf{R}^d$, they prove that it is sufficient to assume

$$\|\nabla \mathbf{B}_{ij}(q)\| \leq C(1 + |\mathbf{B}(q)|), \quad 1 \leq i, j \leq d$$

for some $C > 0$ to deduce the inequality.

We consider the case of a unique discrete magnetic well:

Assumption 2. We assume that the magnetic intensity b admits a unique and non-degenerate minimum b_0 at $q_0 \in M \setminus \partial M$, such that $0 < b_0 < b_\infty$.

Finally, we make a non-degeneracy assumption.

Assumption 3. We assume that d is even and $\mathbf{B}(q_0)$ is invertible.

In particular, $\mathbf{B}(q)$ is invertible for q in a neighborhood of q_0, which means that the 2-form B is symplectic near q_0. Under this Assumption, the eigenvalues of $\mathbf{B}(q_0)$ can be written

$$\pm i \beta_1(q_0), \quad \dots, \quad \pm i \beta_{d/2}(q_0),$$

with $\beta_j(q_0) > 0$. We define the resonance order $r_0 \in \mathbf{N}^* \cup \{\infty\}$ of the eigenvalues by

$$r_0 := \min\{|\alpha| : \alpha \in \mathbf{Z}^{d/2}, \alpha \neq 0, \langle \alpha, \beta(q_0) \rangle = 0\}, \tag{1.6}$$

with the notations

$$|\alpha| = \sum_{j=1}^{d/2} |\alpha_j|, \quad \langle \alpha, \beta(q_0) \rangle := \sum_{j=1}^{d/2} \alpha_j \beta_j(q_0).$$

We make a non-resonance assumption.

Assumption 4. We assume that the eigenvalues of $\mathbf{B}(q_0)$ are simple (which is equivalent to assuming that $r_0 \geq 3$).

In particular, there is a neighborhood $\Omega \subset\subset M \setminus \partial M$ of q_0 on which the eigenvalues of $\mathbf{B}(q)$ are simple, and defined by smooth positive functions

$$\beta_j : \Omega \to \mathbf{R}_+^*.$$

We can choose Ω such that every β_j is bounded from below by a positive constant on Ω. We can also find smooth orthonormal vectors on Ω,

$$u_1(q), v_1(q), \dots, u_{d/2}(q), v_{d/2}(q) \in T_q M,$$

such that

$$\mathbf{B}(q)u_j(q) = -\beta_j(q)v_j(q), \quad \mathbf{B}(q)v_j(q) = \beta_j(q)u_j(q). \tag{1.7}$$

If r_0 is finite, we take $r = r_0$. Otherwise, we take any integer $r \geq 3$ as large as we want. Then, up to reducing Ω (depending on r), we also have (since r is finite), for $0 < |\alpha| < r$,

$$\langle \alpha, \beta(q) \rangle \neq 0, \quad \text{for all } q \in \Omega. \tag{1.8}$$

Under Assumption 2, we can find $b_0 < \tilde{b}_1 < b_\infty$ such that

$$K := \{b(q) \leq \tilde{b}_1\} \subset \Omega. \tag{1.9}$$

Note that the larger is r, the smaller are K and $\tilde{b}_1$. In the case $M = \mathbf{R}^d$, using the inequality in Assumption 1, it is proved in [15] that there exist $\hbar_0$ and $c > 0$ such that, for $\hbar \in (0, \hbar_0]$,

$$\mathrm{sp}_{\mathrm{ess}}(\mathcal{L}_\hbar) \subset [\hbar(\tilde{b}_1 - c\hbar^{1/4}), +\infty),$$

and so, for $\hbar$ small enough, the spectrum of $\mathcal{L}_\hbar$ below $\hbar b_1$ (for a given $b_1 < \tilde{b}_1$) is discrete. When M is compact, $\mathcal{L}_\hbar$ has compact resolvent, and its full spectrum is discrete.

1.5. Main results

On the classical part, we first prove the following reduction of the Hamiltonian. For $z = (x, \xi) \in \mathbf{R}^d$ and $w = (y, \eta) \in \mathbf{R}^d$, we denote $z_j = (x_j, \xi_j)$, $w_j = (y_j, \eta_j)$, and $B_z(\varepsilon) = \{|z| \leq \varepsilon\}$. We use the notation $\mathbf{R}_z^d$ (or $\mathbf{R}_w^d$) to emphasize that an element of $\mathbf{R}^d$ is denoted by z (or w).

Theorem 1.1. *Under Assumptions 1–4, for Ω and $\varepsilon > 0$ small enough, there exist a neighborhood V of $0 \in \mathbf{R}^d_w$, a neighborhood U of $(q_0, A_{q_0}) \in T^*M$, and symplectomorphisms*

$$\varphi\colon (\Omega, B) \to (V, \mathrm{d}\eta \wedge \mathrm{d}y),$$

and

$$\Phi\colon (V \times B_z(\varepsilon), \mathrm{d}\eta \wedge \mathrm{d}y + \mathrm{d}\xi \wedge \mathrm{d}x) \to (U \subset T^*M, \omega),$$

with $\Phi(\varphi(q), 0) = (q, A(q))$, under which the Hamiltonian H becomes

$$\widehat{H}(w, z) = H \circ \Phi(w, z) = \sum_{j=1}^{d/2} \hat{\beta}_j(w)|z_j|^2 + \mathcal{O}(|z|^3),$$

locally uniformly in w, with the notation $\hat{\beta}_j(w) = \beta_j \circ \varphi^{-1}(w)$.

Our next aim is to construct a semiclassical Birkhoff normal form for $\mathcal{L}_\hbar$, that is to say a pseudodifferential operator $\mathcal{N}_\hbar$ on $\mathrm{L}^2(\mathbf{R}^d)$, commuting with suitable harmonic oscillators such that

$$U_\hbar \mathcal{L}_\hbar U_\hbar^* = \mathcal{N}_\hbar + R_\hbar,$$

with $U_\hbar\colon \mathrm{L}^2(M) \to \mathrm{L}^2(\mathbf{R}^d)$ a microlocally unitary Fourier integral operator and $R_\hbar$ a remainder. We will construct the remainder so that the first eigenvalues of $\mathcal{L}_\hbar$ coincide with the first eigenvalues of $\mathcal{N}_\hbar$, up to a small error of order $\mathcal{O}(\hbar^{r/2-\varepsilon})$, where r is defined in (1.6). More precisely, we prove the following theorem.

Theorem 1.2 (semiclassical Birkhoff normal form). *We denote by $z=(x,\xi)\in T^*\mathbf{R}^{d/2}_x$ and $w = (y, \eta) \in T^*\mathbf{R}^{d/2}_y$ the canonical variables. For $\zeta > 0$ and $\hbar \in (0, \hbar_0]$ small enough, there exist a Fourier integral operator*

$$U_\hbar\colon \mathrm{L}^2(\mathbf{R}^d_{(x,y)}) \to \mathrm{L}^2(M),$$

a pseudodifferential operator $R_\hbar$ on $\mathbf{R}^d$, and a smooth function $f^\star(w, I_1, \dots, I_{d/2}, \hbar)$, such that

i. $U_\hbar^* \mathcal{L}_\hbar U_\hbar = \mathcal{N}_\hbar + R_\hbar$,
ii. $\mathcal{N}_\hbar = \mathrm{Op}_\hbar^w(H_0 + f^\star(w, \mathcal{I}_\hbar^{(1)}, \dots, \mathcal{I}_\hbar^{(d/2)}, \hbar))$,
iii. $|f^\star(w, I, \hbar)| \leq C(\hbar + |I|)^2$,
iv. $\sigma_\hbar^w(R_\hbar) \in \mathcal{O}((|z| + \hbar^{1/2})^r)$ *on a neighborhood of* $w = 0$,
v. $U_\hbar^* U_\hbar = I$ *microlocally near* $(z, w) = 0$,
vi. $U_\hbar U_\hbar^* = I$ *microlocally near* $(q, p) = (q_0, A_{q_0})$,
vii. $(1 - \zeta)\langle \mathrm{Op}_\hbar^w H_0\psi, \psi\rangle \leq \langle \mathcal{N}_\hbar\psi, \psi\rangle \leq (1 + \zeta)\langle \mathrm{Op}_\hbar^w H_0\psi, \psi\rangle$, *for all* $\psi \in \mathcal{S}(\mathbf{R}^d)$,

with

$$\mathcal{I}_\hbar^{(j)} = \mathrm{Op}_\hbar^w(|z_j|^2) = -\hbar^2 \frac{\partial^2}{\partial x_j^2} + x_j^2, \quad H_0 = \sum_{j=1}^{d/2} \hat{\beta}_j(w)|z_j|^2. \tag{1.10}$$

We call $\mathcal{N}_\hbar$ the *normal form,* and $R_\hbar$ the *remainder.*

Remark 1. The notation $\mathrm{Op}_\hbar^w\, f^\star(w, \mathcal{I}_\hbar^{(1)}, \dots, \mathcal{I}_\hbar^{(d/2)}, \hbar)$ should be understood as the quantization of an operator-valued symbol. The operator-symbol is

$$f^\star(w, \mathcal{I}_\hbar^{(1)}, \dots, \mathcal{I}_\hbar^{(d/2)}, \hbar)$$

and we consider its quantization with respect to $w = (y, \eta)$. We could also construct a normal form such that

$$\widetilde{\mathcal{N}}_\hbar = \mathrm{Op}_\hbar^w\big(H_0 + f(w, |z_1|^2, \dots, |z_{d/2}|^2, \hbar)\big),$$

with f having the same general properties as $f^\star$. Such a normal form is a function of the *classical* harmonic oscillators $|z_j|^2$, while our $\mathcal{N}_\hbar$ is a function of the *quantized* oscillators. The advantage of this version is that, being a function of $\mathcal{I}_\hbar^{(j)}$, we can easily describe the spectrum of $\mathcal{N}_\hbar$.

Theorem 1.3. *For $k \geq 0$, let us denote by h_k the Hermite function, satisfying*

$$\mathcal{I}_\hbar^{(j)} h_k(x_j) = \hbar(2k+1) h_k(x_j).$$

For $n = (n_1, \dots, n_{d/2}) \in \mathbf{N}^{d/2}$, there exists a pseudodifferential operator $\mathcal{N}_\hbar^{(n)}$ acting on $\mathrm{L}^2(\mathbf{R}_y^{d/2})$ such that

$$\mathcal{N}_\hbar(u \otimes h_{n_1} \otimes \dots \otimes h_{n_{d/2}}) = \mathcal{N}_\hbar^{(n)}(u) \otimes h_{n_1} \otimes \dots \otimes h_{n_{d/2}}, \quad u \in \mathcal{S}(\mathbf{R}_y^{d/2}).$$

Its symbol is

$$F^{(n)}(w) = \hbar \sum_{j=1}^{d/2} \hat{\beta}_j(w)(2n_j + 1) + f^\star(w, \hbar(2n+1), \hbar),$$

and we have

$$\mathrm{sp}(\mathcal{N}_\hbar) = \bigcup_n \mathrm{sp}(\mathcal{N}_\hbar^{(n)}).$$

Moreover, the multiplicity of λ as eigenvalue of $\mathcal{N}_\hbar$ is the sum over n of the multiplicities of λ as eigenvalue of $\mathcal{N}_\hbar^{(n)}$.

Using microlocalization properties of the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$, we prove that they have the same spectra in the following sense. We recall that $\tilde{b}_1$, defined in (1.9), is chosen such that

$$\{b(q) \leq \tilde{b}_1\} \subset \Omega.$$

Theorem 1.4. *Let $\varepsilon > 0$ and $b_1 \in (0, \tilde{b}_1)$. We denote by*

$$\lambda_1(\hbar) \le \lambda_2(\hbar) \le \cdots \quad \text{and} \quad \nu_1(\hbar) \le \nu_2(\hbar) \le \cdots$$

the first eigenvalues of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$, respectively. Then

$$\lambda_n(\hbar) = \nu_n(\hbar) + \mathcal{O}(\hbar^{r/2-\varepsilon}),$$

uniformly in n such that $\lambda_n(\hbar) \le \hbar b_1$ and $\nu_n(\hbar) \le \hbar b_1$.

Finally, we deduce an expansion of the $N > 0$ first eigenvalues of $\mathcal{L}_\hbar$ in powers of $\hbar^{1/2}$.

Theorem 1.5 (expansion of the first eigenvalues). *Let $\varepsilon > 0$ and $N \ge 1$. There exist $\hbar_0 > 0$ and $c_0 \in \mathbf{R}$ such that, for $\hbar \in (0, \hbar_0]$, the N first eigenvalues of $\mathcal{L}_\hbar$, denoted $(\lambda_j(\hbar))_{1 \le j \le N}$, admit an expansion in powers of $\hbar^{1/2}$ of the form*

$$\lambda_j(\hbar) = \hbar b_0 + \hbar^2(E_j + c_0) + \hbar^{5/2} c_{j,5} + \cdots + \hbar^{(r-1)/2} c_{j,r-1} + \mathcal{O}(\hbar^{r/2-\varepsilon}),$$

where $\hbar E_j$ is the j-th eigenvalue of the $d/2$-dimensional harmonic oscillator

$$\mathrm{Op}_\hbar^w(\mathrm{Hess}_0(b \circ \varphi^{-1})).$$

Remarks 2. 1. In dimension $d = 2$, the two-term asymptotics for $\lambda_1(\hbar)$ were first proved by Helffer and Morame in [16]. They have an explicit formula for c_0, depending on the Hessian of b at q_0. A full expansion of $\lambda_j(\hbar)$ in powers of $\hbar^{1/2}$ was proved in [13], always in dimension $d = 2$. The result of [26] proved that no odd powers of $\hbar^{1/2}$ could appear in the expansion. Theorem 1.5 extends this result to higher dimensions: our proof shows that the odd powers of $\hbar^{1/2}$ may only appear if they do appear in the corresponding eigenvalue asymptotics of a $d/2$-dimensional operator with symbol

$$\mathrm{Hess}_0(b \circ \varphi^{-1}) + \mathcal{O}(\hbar).$$

If $d = 2$, this never happens, as proved in [27] for instance, using a Birkhoff normal form.

2. Unfortunately, our method is too implicit to be able to compute c_0. Indeed, the perceptive reader will notice that c_0 comes from the $\hbar^2$ order term of $f^\star(w, \hbar, \ldots, \hbar)$. This term is computed from the $\hbar^2$ term of the symbol $\sigma_\hbar$ in (4.4) via a Birkhoff algorithm. However, the $\hbar^2$ term of $\sigma_\hbar$ is not computable since it comes from the conjugation by a Fourier integral operator associated to the canonical transformation Φ, which is itself implicitly defined.

3. When M is compact but the 2-form B is not exact, one can not define the magnetic Laplacian $\mathcal{L}_\hbar$, but one can consider the Bochner Laplacian on a complex line

bundle L endowed with a connection of curvature iB. Then, the semiclassical limit corresponds to the high tensor product limit of L. Using quasimodes, Kordyukov [21] proved asymptotic expansions of the first eigenvalues of the Bochner Laplacian, in the case of non-degenerate magnetic wells. Thus, his result is closely related to Theorem 1.5. When M is compact, our case corresponds to the Bochner Laplacian of a trivial line bundle. However, our normal form gives a geometrical interpretation of the coefficients, and also describes higher eigenvalues (semi-excited states).

Note that, from Theorems 1.4 and 1.3, we deduce Weyl estimates for $\mathcal{L}_\hbar$. Here $N(\mathcal{L}_\hbar, b_1\hbar)$ denotes the number of eigenvalues λ of $\mathcal{L}_\hbar$ such that $\lambda \leq b_1\hbar$, counted with multiplicities.

Corollary 1.1 (Weyl estimates). *For any $b_1 \in (b_0, \tilde{b}_1)$,*

$$N(\mathcal{L}_\hbar, b_1\hbar) = \frac{1}{(2\pi\hbar)^{d/2}} \sum_{n\in\mathbf{N}^{d/2}} \int_{b^{[n]}(q)\leq b_1} \frac{B^{d/2}}{(d/2)!} + o(\hbar^{-d/2})$$

in the limit $\hbar \to 0$, where

$$b^{[n]}(q) = \sum_{j=1}^{d/2} (2n_j + 1)\beta_j(q).$$

The sum is finite because the β_j are bounded from below by a positive constant on Ω. In particular, if $M = \mathbf{R}^d$, we get

$$N(\mathcal{L}_\hbar, b_1\hbar) = \frac{1}{(2\pi\hbar)^{d/2}} \sum_{n\in\mathbf{N}^{d/2}} \int_{b^{[n]}(q)\leq b_1} \beta_1(q)\dots\beta_{d/2}(q)\,\mathrm{d}q + o(\hbar^{-d/2}).$$

Remark 3. In their works, Demailly [6, 7] and Bouche [4] proved similar Weyl asymptotics for Bochner Laplacians on a compact complex manifold. They used an expansion of the associated heat kernel and an local approximation of the magnetic field by a constant. In Ivrii's book ([18, Chapter 19, p. 1988]) there is a discussion on how normal forms can yield Weyl asymptotics for an electro-magnetic Schrödinger operator. In this book, Weyl laws are stated in various regimes according to $\hbar$ and the intensity of the magnetic field. Our Corollary 1.1 is in the setting of [18, Theorem 19.6.25] (with $\mu = 1/h$ and $h = \hbar^{1/2}$) giving a remainder $\mathcal{O}(\hbar^{1-d/2})$ for this Weyl law, but it seems not clear that the mentioned microhyperbolicity assumption corresponds to our localization assumptions on b. The approach is very close to ours, since it relies on microlocal normal forms.

1.6. Organization and strategy

In Section 2, we construct a symplectomorphism which simplify H near its zero set $\Sigma = H^{-1}(0)$ (Theorem 1.1). In the new coordinates, H becomes

$$\widehat{H}(q,z) = \sum_{j=1}^{d/2} \beta_j(q)|z_j|^2 + \mathcal{O}(|z|^3).$$

In Section 3, we construct a formal Birkhoff normal form: in the space of formal series in variables $(x, \xi, \hbar)$, we change $\widehat{H}$ into $H^0 + \kappa + \rho$, with $H^0 = \sum_{j=1}^{d/2} \beta_j |z_j|^2$, κ a series in $|z_j|^2$ ($1 \leq j \leq d/2$), and ρ a remainder of order r (Theorem 3.1). In Section 4, we quantize the changes of coordinates constructed in Sections 2 and 3, and we get the semiclassical Birkhoff normal form (Theorem 1.2). In Section 5, we reduce $\mathcal{N}_\hbar$ (Theorem 1.3) and we deduce an expansion of its first eigenvalues. It remains to prove that the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ below $b_1\hbar$ coincide. Before doing it, we need microlocalization results stated in Section 6 (the same as in the 2D case [26]). We show that the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ are microlocalized near the zero set of H, where our formal construction is valid. In Section 7, we use the results of Section 6, to prove that $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ have the same spectrum below $b_1\hbar$ (Theorem 1.4). This theorem, together with the results of Section 5, finishes the proof of Theorem 1.5. We also prove the Weyl estimates (Corollary 1.1) here.

2. Reduction of the classical Hamiltonian

2.1. A symplectic reduction of T^*M

The zero set of H:

$$\Sigma = H^{-1}(0) = \{(q, A(q)) \in T^*M : q \in \Omega\},$$

is a d-dimensional smooth submanifold of the cotangent bundle T^*M. We denote by $j : \Omega \to T^*M$ the embedding

$$j(q) = (q, A(q)).$$

The symplectic structure on T^*M is defined by the form

$$\omega = \mathrm{d}p \wedge \mathrm{d}q = \mathrm{d}\alpha, \quad \alpha = p\,\mathrm{d}q.$$

In other words, for $p \in T_qM^*$ and $\mathcal{V} \in T_{(q,p)}(T^*M)$,

$$\alpha_{(q,p)}(\mathcal{V}) = p(\pi_*\mathcal{V}), \tag{2.1}$$

Where the map $\pi_*\colon T_{(q,p)}(T^*M) \to T_qM$ is the differential of the canonical projection

$$\pi\colon T^*M \to M, \quad \pi(q,p) = q.$$

Using local coordinates with the notations of Section 1.2, at any point $(q,p) \in T^*M$ with

$$p = p_1\,\mathrm{d}q_1 + \cdots + p_d\,\mathrm{d}q_d,$$

the tangent vectors $\mathcal{V} \in T_{(q,p)}(T^*M)$ are identified with $(Q,P) \in T_qM \times T_qM^*$, with

$$Q = Q_1\partial q_1 + \cdots + Q_d\partial q_d, \quad P = P_1\,\mathrm{d}q_1 + \cdots + P_d\,\mathrm{d}q_d.$$

With this notation,

$$\pi_*(Q,P) = Q,$$
$$\alpha_{(q,p)}(Q,P) = p(Q),$$
$$\omega_{(q,p)}((Q,P),(Q',P')) = \langle P',Q\rangle - \langle P,Q'\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between T_qM and T_qM^*.

Lemma 2.1. *Σ is a symplectic submanifold of (T^*M,ω), and*

$$j^*\omega = B.$$

In particular, at each point $j(q) \in \Sigma$,

$$T_{j(q)}(T^*M) = T_{j(q)}\Sigma \oplus T_{j(q)}\Sigma^\perp, \tag{2.2}$$

where $\perp$ denotes the symplectic orthogonal for ω.

Proof. To say that Σ is a symplectic submanifold of T^*M means that the restriction of ω to Σ is non-degenerate. Written with the embedding j, this restriction is $j^*\omega$. Actually, using the definition (2.1) of α with $p = A_q$ and $\mathcal{V} = d_qj(Q)$, we get

$$(j^*\alpha)_q(Q) = A_q(\pi_*d_qj(Q)) = A_q(Q) \quad \text{for all } Q \in T_qM.$$

Hence,

$$j^*\alpha = A,$$

so

$$j^*(\mathrm{d}\alpha) = \mathrm{d}A = B. \qquad \blacksquare$$

Since any $j(q)$ is a critical point of H, the Hessian of H at $j(q)$ is well defined and independent of any choice of coordinates. We now compute this Hessian according to the decomposition (2.2):

Lemma 2.2. *The Hessian $T^2_{j(q)}H$, as a bilinear form on $T_{j(q)}(T^*M)$, satisfies*

$$T^2_{j(q)}H(\mathcal{V},\mathcal{V}) = 0 \qquad \text{if } \mathcal{V} \in T_{j(q)}\Sigma,$$
$$T^2_{j(q)}H(\mathcal{V},\mathcal{V}) = 2|\mathbf{B}(q)\pi_*\mathcal{V}|^2_{g_q} \quad \text{if } \mathcal{V} \in T_{j(q)}\Sigma^\perp.$$

Proof. Using local coordinates on M, we will denote every $\mathcal{V} \in T_{(q,p)}(T^*M)$, as $(Q,P) \in T_qM \times T_qM^*$. In these coordinates, with the notations introduced in Section 1.2,

$$\Sigma \equiv \{(q,\mathbf{A}(q)), q \in \mathbf{R}^d\}$$

so that

$$T_{j(q)}\Sigma = \{(Q,P) \in T_qM \times T_qM^*, P = T_qA \cdot Q\}. \tag{2.3}$$

We can also describe $T_{j(q)}\Sigma^\perp$ using these coordinates. Indeed,

$$\begin{aligned}(Q,P) \in T_{j(q)}\Sigma^\perp &\iff \omega((Q,P),(Q_0,T_qA\cdot Q_0)) = 0 \quad \text{for all } Q_0 \in T_qM\\ &\iff \langle P, Q_0\rangle = \langle T_qA\cdot Q_0, Q\rangle \quad \text{for all } Q_0 \in T_qM\\ &\iff P = {}^tT_qA\cdot Q.\end{aligned}$$

Hence

$$T_{j(q)}\Sigma^\perp = \{(Q,P), P = {}^tT_qA\cdot Q\}. \tag{2.4}$$

From the expression (1.4) of H in coordinates, we deduce that

$$\begin{aligned}T_{(q,p)}H(Q,P) = {}&2\sum_{ij} g^{ij}(q)(p_i - A_i(q))(P_j - \nabla_q A_j\cdot Q)\\ &+ \sum_{ijk}\partial_k g^{ij}(q)Q_k(p_i - A_i(q))(p_j - A_j(q)),\end{aligned}$$

so that the Hessian of H in coordinates is

$$\begin{aligned}T^2_{j(q)}H((Q,P),(Q,P)) &= 2\sum_{ij} g^{ij}(q)(P_i - \nabla_q A_i\cdot Q)(P_j - \nabla_q A_j\cdot Q)\\ &= 2|P - T_qA\cdot Q|^2_{g_q^*}.\end{aligned}$$

It follows from (2.3) that

$$T^2_{j(q)}H((Q,P),(Q,P)) = 0 \quad \text{for all } (Q,P) \in T_{j(q)}\Sigma,$$

and from (2.4) and (1.3) that

$$T^2_{j(q)}H((Q,P),(Q,P)) = 2|({}^tT_qA - T_qA)Q|^2_{g_q^*} = |\iota_Q B|^2_{g_q^*}$$

for all $(Q,P)\in T_{j(q)}\Sigma^{\perp}$. Let us rewrite this using $\mathbf{B}$. Note that

$$|\iota_Q B|^2_{g_q^*} = \sum_{ij} g^{ij}(q)\Big(\sum_{ki} B_{ki}Q_k\Big)\Big(\sum_{\ell j} B_{\ell j}Q_\ell\Big) = \sum_{k\ell}\Big(\sum_{ij} g^{ij}B_{ki}B_{\ell j}\Big)Q_kQ_\ell,$$

and keeping in mind that (g^{ij}) is the inverse matrix of (g_{ij}) together with the relation (1.2) between B and $\mathbf{B}$, we have

$$\sum_{ij} g^{ij}B_{ki}B_{\ell j} = \sum_{ijk'\ell'} g^{ij}g_{k'i}g_{\ell' j}\mathbf{B}_{k'k}\mathbf{B}_{\ell'\ell} = \sum_{k'\ell'} g_{k'\ell'}\mathbf{B}_{k'k}\mathbf{B}_{\ell'\ell},$$

and so

$$|\iota_Q B|^2_{g_q^*} = \sum_{k'\ell'} g_{k'\ell'}\Big(\sum_k \mathbf{B}_{k'k}Q_k\Big)\Big(\sum_\ell \mathbf{B}_{\ell'\ell}Q_\ell\Big) = |\mathbf{B}(q)Q|^2_{g_q}. \qquad \blacksquare$$

We endow $\Omega\times\mathbf{R}^d_z$ with the symplectic form

$$\omega_0(q,z) = B\oplus\sum_{j=1}^{d/2} \mathrm{d}\xi_j\wedge\mathrm{d}x_j,$$

with the notation $z=(x,\xi)$. (Σ,B) is a d-dimensional symplectic submanifold of (T^*M,ω). The following Darboux–Weinstein lemma claims that this situation is modelled on the submanifold $\Sigma_0=\Omega\times\{0\}$ of $(\Omega\times\mathbf{R}^d_z,\omega_0)$.

Lemma 2.3. *There exists a local diffeomorphism*

$$\Phi_0\colon \Omega\times\mathbf{R}^d_z\to T^*M$$

such that

$$\Phi_0^*\omega=\omega_0, \quad \textit{and} \quad \Phi_0(\Sigma_0)=\Sigma.$$

In order to keep track on the construction of Φ_0, we will give the proof of this result.

Proof. Again, we use local coordinates on M to denote every $\mathcal{V}\in T_{(q,p)}(T^*M)$ as $(Q,P)\in T_qM\times T_q^*M$. For $q\in\Omega$, using the vectors $u_j(q), v_j(q)\in T_qM$ defined in (1.7), we define the vectors

$$e_j(q)=\frac{1}{\sqrt{\beta_j(q)}}\big(u_j(q),{}^tT_qAu_j(q)\big), \quad f_j(q)=\frac{1}{\sqrt{\beta_j(q)}}\big(v_j(q),{}^tT_qAv_j(q)\big),$$

which are in $T_{j(q)}\Sigma^{\perp}$ by (2.4). These vectors satisfy

$$\omega_{j(q)}(e_i(q), f_j(q)) = \delta_{ij}, \tag{2.5a}$$

$$\omega_{j(q)}(e_i(q), e_j(q)) = 0, \tag{2.5b}$$

$$\omega_{j(q)}(f_i(q), f_j(q)) = 0. \tag{2.5c}$$

Indeed, the first equality follows from

$$\begin{aligned}
\omega_{j(q)}(e_i, f_j) &= -\frac{1}{\sqrt{\beta_i \beta_j}} \langle ({}^t T_q A - T_q A) u_j, v_j \rangle \\
&= -\frac{1}{\sqrt{\beta_i \beta_j}} B(u_i, v_j) \\
&= -\frac{1}{\sqrt{\beta_i \beta_j}} g_q(\mathbf{B}(q) u_i, v_j) \\
&= \frac{\beta_i}{\sqrt{\beta_i \beta_j}} g_q(v_i, v_j) \\
&= \delta_{ij},
\end{aligned}$$

and the two others from similar calculations.

Let us construct a $\widetilde{\Phi}_0 \colon \Omega \times \mathbf{R}^d_z \to T^*M$ such that

$$\widetilde{\Phi}_0(q, 0) = j(q), \tag{2.6}$$

$$\partial_z \widetilde{\Phi}_0(q, 0) = L_q, \tag{2.7}$$

where $L_q \colon \mathbf{R}^d \to T_{j(q)}\Sigma^{\perp}$ is the linear map sending the canonical basis onto

$$(e_1(q), f_1(q), \ldots, e_{d/2}(q), f_{d/2}(q)).$$

For this, we take local vector fields $\hat{e}_j(q, p), \hat{f}_j(q, p) \in T_{(q,p)}(T^*M)$ defined in a neighborhood of Σ, such that

$$\hat{e}_j(j(q)) = e_j(q), \quad \hat{f}_j(j(q)) = f_j(q).$$

In other words, if we see e_j and f_j as vector fields on Σ using $j(q)$, we extend them to a neighborhood of Σ. Then we consider the associated flows, defined on a neighborhood of Σ by

$$\begin{aligned}
\frac{\partial \phi_j^{x_j}}{\partial x_j}(q, p) &= \hat{e}_j(\phi_j^{x_j}(q, p)), \quad x_j \in \mathbf{R}, \\
\frac{\partial \psi_j^{\xi_j}}{\partial \xi_j}(q, p) &= \hat{f}_j(\psi_j^{\xi_j}(q, p)), \quad \xi_j \in \mathbf{R}, \\
\phi_j^0(q, p) &= \psi_j^0(q, p) = (q, p).
\end{aligned}$$

Then

$$\widetilde{\Phi}_0(q,z) := \phi_1^{x_1} \circ \psi_1^{\xi_1} \circ \cdots \circ \phi_{d/2}^{x_{d/2}} \circ \psi_{d/2}^{\xi_{d/2}}(j(q))$$

satisfies (2.6) and (2.7). Hence, if $q \in \Omega$, the linear tangent map

$$T_{(q,0)}\widetilde{\Phi}_0 \colon T_q M \oplus \mathbf{R}^d \to T_{j(q)}\Sigma \oplus T_{j(q)}\Sigma^{\perp}$$

acts as

$$\begin{pmatrix} T_q j & 0 \\ 0 & L_q \end{pmatrix}.$$

In particular, $\widetilde{\Phi}_0^* \omega = \omega_0$ on $\{z = 0\}$ by (2.5) and lemma 2.1. By the Weinstein lemma (Lemma A.2), for $\varepsilon > 0$ small enough there exists a diffeomorphism $S \colon \Omega \times B_z(\varepsilon) \to \Omega \times B_z(\varepsilon)$ such that $S(q,z) = (q,z) + \mathcal{O}(|z|^2)$ and $S^* \widetilde{\Phi}_0^* \omega = \omega_0$. Then $\Phi_0 = \widetilde{\Phi}_0 \circ S$ is the desired symplectomorphism. ■

2.2. Proof of Theorem 1.1

Now we can prove the normal form for the classical Hamiltonian. Up to reducing Ω, we can take symplectic coordinates $w = (y,\eta) \in \mathbf{R}^d$ to describe Ω, thanks to the Darboux lemma:

$$\varphi \colon \Omega \to V \subset \mathbf{R}^d_w.$$

We get a new symplectomorphism

$$\Phi \colon V \times B_z(\varepsilon) \to U \subset T^*M,$$

defined by

$$\Phi(w,z) = \Phi_0(\varphi^{-1}(w), z).$$

It remains to compute a Taylor expansion of H in these coordinates. Using the Taylor Formula for $\widehat{H} = H \circ \Phi$, we get

$$\widehat{H}(w,z) = \widehat{H}(w,0) + \partial_z \widehat{H}|_{z=0}(z) + \frac{1}{2}\partial_z^2 \widehat{H}|_{z=0}(z,z) + \mathcal{O}(|z|^3). \tag{2.8}$$

By the chain rule, we have (with $q = \varphi^{-1}(w)$)

$$\partial_z \widehat{H}|_{z=0}(z) = T_{j(q)} H(\partial_z \Phi|_{z=0}(z)) = 0,$$

because $T_{j(q)} H = 0$, and

$$\partial_z^2 \widehat{H}|_{z=0}(z,z) = T^2_{j(q)} H(\partial_z \Phi|_{z=0}(z), \partial_z \Phi|_{z=0}(z)).$$

But $\partial_z \Phi|_{z=0}$ sends the canonical basis onto $(e_1(q), f_1(q), \ldots, e_{d/2}(q), f_{d/2}(q))$, so we get from Lemma 2.2:

$$\frac{1}{2}\partial_z^2 \widehat{H}|_{z=0}(z,z) = \sum_{j=1}^{d/2} \beta_j(q)|z_j|^2.$$

Hence, (2.8) gives

$$\widehat{H}(w,z) = H \circ \Phi(w,z) = \sum_{j=1}^{d/2} \hat{\beta}_j(w)|z_j|^2 + \mathcal{O}(|z|^3).$$

3. The formal Birkhoff normal form

3.1. The Hamiltonian $\widehat{H}$

In the new coordinates given by Theorem 1.1, we have a Hamiltonian $\widehat{H}(w,z)$ of the form

$$\widehat{H}(w,z) = H^0(w,z) + \mathcal{O}(|z|^3), \quad \text{where } H^0(w,z) = \sum_{j=1}^{d/2} \hat{\beta}_j(w)|z_j|^2.$$

H^0 is defined for $w \in V$, but we extend the functions $\hat{\beta}_j$ to $\mathbf{R}^d_w$ such that

$$\sum_{j=1}^{d/2} \hat{\beta}_j(w) \geq \tilde{b}_1 \quad \text{for } w \in V^c.$$

This is just technical, since we will prove microlocalization results on V in Section 6. Then we can construct a Birkhoff normal form, in the spirit of [27] and [26], with w as a parameter.

3.2. The space of formal series

We will work in the space of formal series

$$\mathcal{E} = \mathcal{C}^\infty(\mathbf{R}^d_w)[[x, \xi, \hbar]].$$

We endow $\mathcal{E}$ with the Moyal product $\star$, compatible with the Weyl quantization (with respect to all the variables z and w). Given a pseudodifferential operator $\mathcal{A} = \mathrm{Op}_\hbar^w(a)$ we will denote by $\sigma_\hbar^{w,\mathrm{T}}(\mathcal{A})$ or by $[a]$ the formal Taylor series of a at zero, in the variables $x, \xi, \hbar$. With this notation, the compatibility of $\star$ with the Weyl quantization means

$$\sigma_\hbar^{w,\mathrm{T}}(\mathcal{A}\mathcal{B}) = \sigma_\hbar^{w,\mathrm{T}}(\mathcal{A}) \star \sigma_\hbar^{w,\mathrm{T}}(\mathcal{B}).$$

The reader can find the main results on $\hbar$-pseudodifferential operators in [23, 31].

We define the degree of $x^\alpha \xi^\gamma \hbar^\ell$ to be $|\alpha| + |\gamma| + 2\ell$. Hence, we can define the degree and valuation of a series κ, which depends on the point $w \in \mathbf{R}^d$. We denote by $\mathcal{O}_N$ the space of formal series with valuation at least N on V, and $\mathcal{D}_N$ the space spanned by monomials of degree N on V ($V \subset \mathbf{R}^d_w$ is given by Theorem 1.1). We denote by z_j the formal series $x_j + i\xi_j$. Thus, every $\kappa \in \mathcal{E}$ can by written

$$\kappa = \sum_{\alpha\gamma\ell} c_{\alpha\gamma\ell}(w) z^\alpha \bar{z}^\gamma \hbar^\ell,$$

with the notation

$$z^\alpha = z_1^{\alpha_1} \dots z_{d/2}^{\alpha_{d/2}}.$$

For $\kappa_1, \kappa_2 \in \mathcal{E}$, we denote $\mathrm{ad}_{\kappa_1} \kappa_2 = [\kappa_1, \kappa_2] = \kappa_1 \star \kappa_2 - \kappa_2 \star \kappa_1$. It is well known that $[\kappa_1, \kappa_2]$ is of order $\hbar$, so for $N_1 + N_2 \geq 2$, we have

$$\frac{1}{\hbar}[\mathcal{O}_{N_1}, \mathcal{O}_{N_2}] \subset \mathcal{O}_{N_1+N_2-2}. \tag{3.1}$$

Explicitly,

$$[\kappa_1, \kappa_2](z, w, \hbar) = 2\sinh\Big(\frac{\hbar}{2i}\Box\Big)(f(z', w', \hbar)g(z'', w'', \hbar))|_{z'=z''=z, w'=w''=w}, \tag{3.2}$$

where $[f] = \kappa_1$, $[g] = \kappa_2$, and

$$\Box = \sum_{j=1}^{d/2} (\partial_{\xi'_j}\partial_{x''_j} - \partial_{x'_j}\partial_{\xi''_j} + \partial_{\eta'_j}\partial_{y''_j} - \partial_{y'_j}\partial_{\eta''_j}).$$

From formula (3.2), a simple computation gives

$$\frac{i}{\hbar}\,\mathrm{ad}\,|_{z_j|^2}(z^\alpha \bar{z}^\beta \hbar^\ell) = \{|z_j|^2, z^\alpha \bar{z}^\gamma \hbar^\ell\} = (\alpha_j - \gamma_j) z^\alpha \bar{z}^\gamma \hbar^\ell. \tag{3.3}$$

3.3. The formal normal form

In order to prove Theorem 1.2, we look for a pseudodifferential operator $\mathcal{Q}_\hbar$ such that

$$e^{\frac{i}{\hbar}\mathcal{Q}_\hbar}\, \mathrm{Op}_\hbar^w\, \widehat{H} e^{-\frac{i}{\hbar}\mathcal{Q}_\hbar} \tag{3.4}$$

commutes with the harmonic oscillators $\mathcal{I}_\hbar^{(j)}$, $(1 \leq j \leq d/2)$ introduced in (1.10). At the formal level, expression (3.4) becomes

$$e^{\frac{i}{\hbar}\mathrm{ad}_\tau}(H^0 + \gamma), \tag{3.5}$$

where $H_0 + \gamma$ is the Taylor expansion of $\widehat{H}$, and $\tau = \sigma_\hbar^{w,\mathrm{T}}(\mathcal{Q}_\hbar)$. Moreover,

$$\sigma_\hbar^{w,\mathrm{T}}(\mathcal{I}_\hbar^{(j)}) = |z_j|^2,$$

so we want (3.5) to be equal to $H^0 + \kappa$, where $[\kappa, |z_j|^2] = 0$, which is equivalent to say that κ is a series in $(|z_1|^2, \dots, |z_{d/2}|^2, \hbar)$. This is possible modulo $\mathcal{O}_r$, as stated in the following theorem. We recall that r is related to the resonance order r_0 by $r = r_0$ if $r_0 < \infty$, otherwise r is any fixed integer, as large as we want (but related to Ω by (1.8)).

Theorem 3.1. *If $\gamma \in \mathcal{O}_3$, there exist $\tau, \kappa, \rho \in \mathcal{O}_3$ such that*

- $e^{\frac{i}{\hbar}\operatorname{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa + \rho$,
- $[\kappa, |z_j|^2] = 0$ *for* $1 \leq j \leq d/2$,
- $\rho \in \mathcal{O}_r$.

Proof. Let $3 \leq N \leq r - 1$. Assume that, for a $\tau_N \in \mathcal{O}_3$

$$e^{\frac{i}{\hbar}\operatorname{ad}_{\tau_N}}(H^0 + \gamma) = H^0 + K_3 + \dots + K_{N-1} + R_N + \mathcal{O}_{N+1},$$

where $K_i \in \mathcal{D}_i$ commutes with $|z_j|^2$ $(1 \leq j \leq d/2)$ and where $R_N \in \mathcal{D}_N$. Using (3.1), for any $\tau' \in \mathcal{D}_N$,

$$\begin{aligned} e^{\frac{i}{\hbar}\operatorname{ad}_{\tau_N + \tau'}}(H^0 + \gamma) &= e^{\frac{i}{\hbar}\operatorname{ad}_{\tau'}}(H^0 + K_3 + \dots + K_{N-1} + R_N + \mathcal{O}_{N+1}) \\ &= H^0 + K_3 + \dots + K_{N-1} + R_N + \frac{i}{\hbar}\operatorname{ad}_{\tau'} H^0 + \mathcal{O}_{N+1}. \end{aligned}$$

Thus, we look for τ' and $K_N \in \mathcal{D}_N$ such that

$$R_N = K_N + \frac{i}{\hbar}\operatorname{ad}_{H^0}\tau' \mod \mathcal{O}_{N+1}. \tag{3.6}$$

To solve this equation, we need to study ad_{H^0}. Since $H^0 = \sum_j \hat{\beta}_j(w)|z_j|^2$,

$$\frac{i}{\hbar}\operatorname{ad}_{H^0}\tau' = \sum_{j=1}^{d/2}\Big(\hat{\beta}_j(w)\frac{i}{\hbar}\operatorname{ad}_{|z_j|^2}(\tau') + \frac{i}{\hbar}\operatorname{ad}_{\hat{\beta}_j}(\tau')|z_j|^2\Big).$$

Since $\hat{\beta}_j$ only depends on w,

$$\frac{i}{\hbar}\operatorname{ad}_{\hat{\beta}_j}(\tau') \in \mathcal{O}_{N-1},$$

(see formula (3.2)). Hence

$$\frac{i}{\hbar}\operatorname{ad}_{H^0}\tau' = \sum_{j=1}^{d/2}\hat{\beta}_j(w)\frac{i}{\hbar}\operatorname{ad}_{|z_j|^2}(\tau') + \mathcal{O}_{N+1}.$$

Thus, equation (3.6) can be rewritten

$$R_N = K_N + T(\tau') + \mathcal{O}_{N+1}, \tag{3.7}$$

with the notation

$$T = \sum_{j=1}^{d/2} \hat{\beta}_j(w) \frac{i}{\hbar} \operatorname{ad}_{|z_j|^2}.$$

From formula (3.3) we see that T acts on monomials as

$$T(c(w)z^\alpha \bar{z}^\gamma) = \langle \alpha - \gamma, \hat{\beta}(w) \rangle c(w) z^\alpha \bar{z}^\gamma. \tag{3.8}$$

Thus, if we write

$$R_N = \sum |_{|\alpha|+|\gamma|+2\ell=N} r_{\alpha\gamma\ell}(w) z^\alpha \bar{z}^\gamma \hbar^\ell,$$

we choose

$$K_N = \sum_{\alpha=\gamma} r_{\alpha\gamma\ell} |z|^{2\alpha} \hbar^\ell,$$

which commutes with $|z_j|^2$ ($1 \leq j \leq d/2$). The rest $R_N - K_N$ is a sum of monomials of the form $r_{\alpha\gamma\ell} z^\alpha \bar{z}^\gamma \hbar^\ell$ with $\alpha \neq \gamma$. As soon as $0 < |\alpha - \gamma| < r$, we have $\langle \alpha - \gamma, \hat{\beta}(w) \rangle \neq 0$ by (1.8) (because $r \leq r_0$), so we can define the smooth coefficient

$$c_{\alpha\gamma\ell}(w) = \frac{r_{\alpha\gamma\ell}(w)}{\langle \alpha - \gamma, \hat{\beta}(w) \rangle}.$$

Thus, (3.8) yields

$$T(c_{\alpha\gamma\ell} z^\alpha \bar{z}^\gamma \hbar^\ell) = r_{\alpha\gamma\ell}(w) z^\alpha \bar{z}^\gamma \hbar^\ell,$$

so $R_N - K_N$ is in the range of T modulo $\mathcal{O}_{N+1}$ because $N \leq r - 1$. Hence, we solved equation (3.7), and thus we can iterate until $N = r - 1$. The series ρ is the $\mathcal{O}_r$ that remains:

$$e^{i\hbar^{-1} \operatorname{ad}_{\tau_N}} (H^0 + \gamma) = H^0 + K_3 + \cdots + K_{r-1} + \rho. \qquad \blacksquare$$

4. The semiclassical Birkhoff normal form

The next step is to quantize Theorems 1.1 and 3.1.

4.1. Quantization of Theorem 1.1

Theorem 1.1 gives a symplectomorphism Φ reducing H to $\hat{H} = H \circ \Phi$. We can quantize this result in the following way. The Egorov theorem ([23, Theorem 5.5.9]) implies the existence of a Fourier integral operator

$$V_\hbar \colon \mathrm{L}^2(\mathbf{R}^d_{(x,y)}) \to \mathrm{L}^2(M),$$

associated to the symplectomorphism Φ, and a pseudo-differential operator $\hat{\mathcal{L}}_\hbar$ with principal symbol $\hat{H}$ on $V \times B_z(\varepsilon)$ and subprincipal symbol 0, such that

$$V_\hbar^* \mathcal{L}_\hbar V_\hbar = \hat{\mathcal{L}}_\hbar, \tag{4.1}$$

$$\begin{aligned} V_\hbar^* V_\hbar &= I \quad \text{microlocally on } V \times B_z(\varepsilon), \\ V_\hbar V_\hbar^* &= I \quad \text{microlocally on } U. \end{aligned} \tag{4.2}$$

4.2. Proof of Theorem 1.2

By (4.1), we are reduced to the pseudodifferential operator $\hat{\mathcal{L}}_\hbar$, which has a total symbol of the form

$$\sigma_\hbar = \hat{H} + \hbar^2 \tilde{r}_\hbar \quad \text{on } V \times B_z(\varepsilon).$$

In particular, $\sigma_\hbar^{w,\mathrm{T}}(\hat{\mathcal{L}}_\hbar) = H_0 + \gamma$ for some $\gamma \in \mathcal{O}_3$, with the notation of Section 3.2. We want to construct a normal form using a bounded pseudodifferential operator $\mathcal{Q}_\hbar$:

$$e^{\frac{i}{\hbar}\mathcal{Q}_\hbar} \hat{\mathcal{L}}_\hbar e^{-\frac{i}{\hbar}\mathcal{Q}_\hbar} = \mathcal{N}_\hbar + R_\hbar. \tag{4.3}$$

In Theorem 3.1, applied to γ, we have constructed formal series τ, κ, and ρ such that

$$e^{\frac{i}{\hbar}\mathrm{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa + \rho.$$

The idea is to choose pseudodifferential operators $\mathcal{Q}_\hbar$ and $\mathcal{N}_\hbar$ such that $\sigma_\hbar^{w,\mathrm{T}}(\mathcal{Q}_\hbar) = \tau$ and $\sigma_\hbar^{w,\mathrm{T}}(\mathcal{N}_\hbar) = H^0 + \kappa$, and to check that they satisfy (4.3). Following this idea, we prove the following theorem.

Theorem 4.1. *For $\hbar \in (0, \hbar_0]$ small enough, there exist a unitary operator*

$$U_\hbar \colon \mathrm{L}^2(\mathbf{R}^d) \to \mathrm{L}^2(\mathbf{R}^d),$$

a smooth function $f^\star(w, I_1, \ldots, I_{d/2}, \hbar)$, and a pseudodifferential operator $R_\hbar$ such that

i. $U_\hbar^* \hat{\mathcal{L}}_\hbar U_\hbar = \mathcal{L}_\hbar^0 + \mathrm{Op}_\hbar^w f^\star(w, \mathcal{I}_\hbar^{(1)}, \ldots, \mathcal{I}_\hbar^{(d/2)}, \hbar) + R_\hbar$,
ii. *$f^\star$ has an arbitrarily small compact $(I_1, \ldots, I_{d/2}, \hbar)$-support (containing 0),*
iii. $\sigma_\hbar^{w,\mathrm{T}}(R_\hbar) \in \mathcal{O}_r$ *and* $\sigma_\hbar^{w,\mathrm{T}}(U_\hbar R_\hbar U_\hbar^*) \in \mathcal{O}_r$.

with $\mathcal{I}_\hbar^{(j)} = \mathrm{Op}_\hbar^w(|z_j|^2)$ *and* $\mathcal{L}_\hbar^0 = \mathrm{Op}_\hbar^w(H^0)$. *We call*

$$\mathcal{N}_\hbar = \mathcal{L}_\hbar^0 + \mathrm{Op}_\hbar^w f^\star(w, \mathcal{I}_\hbar^{(1)}, \dots, \mathcal{I}_\hbar^{(d/2)}, \hbar) \tag{4.4}$$

the normal form, and $R_\hbar$ *the remainder.*

Proof. The pseudodifferential operator $\widehat{\mathcal{L}}_\hbar$ defined by (4.1) has a symbol of the form

$$\sigma_\hbar = \widehat{H} + \hbar^2 \tilde{r}_\hbar \quad \text{on } V \times B_z(\varepsilon),$$

so $\sigma_\hbar = H^0 + r_\hbar$ with $\gamma := [r_\hbar] \in \mathcal{O}_3$. We apply Theorem 3.1 with this $\gamma \in \mathcal{O}_3$. The formal series $\kappa \in \mathcal{O}_3$ that we get commutes with $|z_j|^2$ $(1 \le j \le d/2)$, so by formula (3.3) we can write it

$$\kappa = \sum_{k \ge 2} \sum_{l+|m|=k} c_{l,m}(w)|z_1|^{2m_1} \dots |z_{d/2}|^{2m_{d/2}} \hbar^l.$$

We take a function $f(w, I_1, \dots, I_{d/2}, \hbar)$ with Taylor series

$$\sum_{k \ge 2} \sum_{l+|m|=k} c_{l,m}(w) I_1^{m_1} \dots I_{d/2}^{m_{d/2}} \hbar^l$$

and arbitrarily small compact support in $(I_1, \dots, I_{d/2}, \hbar)$ (containing 0).

Let $c(w, z, \hbar)$ be a smooth function with compact support with Taylor series τ, given by Theorem 3.1. Then, by the Taylor formula, we have

$$\begin{aligned}
&e^{\frac{i}{\hbar}\mathrm{Op}_\hbar^w(c)} \mathrm{Op}_\hbar^w(H^0 + r_\hbar) e^{-\frac{i}{\hbar}\mathrm{Op}_\hbar^w(c)} \\
&= \sum_{n=0}^{r-1} \frac{1}{n!} \mathrm{ad}^n_{i\hbar^{-1}\mathrm{Op}_\hbar^w(c)} \mathrm{Op}_\hbar^w(H^0 + r_\hbar) \\
&\quad + \int_0^1 \frac{1}{(r-1)!}(1-t)^{r-1} e^{it\hbar^{-1}\mathrm{Op}_\hbar^w(c)} \\
&\qquad\qquad \mathrm{ad}^r_{i\hbar^{-1}\mathrm{Op}_\hbar^w(c)} \mathrm{Op}_\hbar^w(H^0 + r_\hbar) e^{-it\hbar^{-1}\mathrm{Op}_\hbar^w(c)} \, \mathrm{d}t.
\end{aligned}$$

By the Egorov theorem and the fact that $\mathrm{ad}^r_{i\hbar^{-1}\mathrm{Op}_\hbar^w(c)} : \mathcal{E} \to \mathcal{O}_r$ (see (3.1)), the integral remainder has a symbol with Taylor series in $\mathcal{O}_r$. Moreover,

$$\begin{aligned}
\sigma_\hbar^{w,\mathrm{T}}\Big(\sum_{n=0}^{r-1} \frac{1}{n!} \mathrm{ad}^n_{i\hbar^{-1}\mathrm{Op}_\hbar^w(c)} \mathrm{Op}_\hbar^w(H^0 + r_\hbar)\Big) &= \sum_{n=0}^{r-1} \frac{1}{n!} \mathrm{ad}^n_{i\hbar^{-1}\tau}(H^0 + \gamma) \\
&= e^{\frac{i}{\hbar}\mathrm{ad}_\tau}(H^0 + \gamma) + \mathcal{O}_r \\
&= H^0 + \kappa + \mathcal{O}_r.
\end{aligned}$$

Thus, by the definition of f, there exists $s(w,z,\hbar)$ such that $[s]\in\mathcal{O}_r$ and

$$\begin{aligned} &e^{\frac{i}{\hbar}\operatorname{Op}_\hbar^w(c)}\operatorname{Op}_\hbar^w(H^0+r_\hbar)e^{-\frac{i}{\hbar}\operatorname{Op}_\hbar^w(c)} \\ &\quad=\operatorname{Op}_\hbar^w(H^0)+\operatorname{Op}_\hbar^w(f(w,|z_1|^2,\dots,|z_{d/2}|^2,\hbar))+\operatorname{Op}_\hbar^w(s). \end{aligned}$$

Finally, we want to change the function $f(w,|z_1|^2,\dots,|z_{d/2}|^2,\hbar)$ of *classical* harmonic oscillators into a function $f^\star(w,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(d/2)},\hbar)$ of the *quantized* oscillators. To do so, note that the m-th power

$$(|z_j|^2)^{\star m}:=|z_j|^2\star\dots\star|z_j|^2$$

is a polynomial function of $(|z_j|^2,\hbar)$, and that we can rewrite the series κ as a function of $(|z_j|^2)^{\star m}$. We get new coefficients $c^\star_{l,m}$ such that

$$\kappa=\sum_{k\geq 2}\sum_{l+|m|=k}c^\star_{l,m}(w)(|\hat{z}_1|^2)^{\star m_1}\dots(|z_{d/2}|^2)^{\star m_{d/2}}\hbar^l.$$

We take a function $f^\star$ with Taylor series

$$\sum_{k\geq 2}\sum_{l+|m|=k}c^\star_{l,m}(w)I_1^{m_1}\dots I_{d/2}^{m_{d/2}}\hbar^l,$$

and using the compatibility of the quantization with the Moyal product, we deduce that

$$\sigma_\hbar^{w,\mathrm{T}}(f^\star(w,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(d/2)},\hbar))=[f(w,|z_1|^2,\dots,|z_{d/2}|^2,\hbar)],$$

so we get

$$\begin{aligned} &e^{\frac{i}{\hbar}\operatorname{Op}_\hbar^w(c)}\operatorname{Op}_\hbar^w(H^0+r_\hbar)e^{-\frac{i}{\hbar}\operatorname{Op}_\hbar^w(c)} \\ &\quad=\operatorname{Op}_\hbar^w(H^0)+\operatorname{Op}_\hbar^w(f^\star(w,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(d/2)},\hbar))+\operatorname{Op}_\hbar^w(\tilde{s}), \end{aligned}$$

for a new symbol $\tilde{s}(w,z,\hbar)$ with $[\tilde{s}]\in\mathcal{O}_r$. Hence, we get

$$U_\hbar^*\hat{\mathcal{L}}_\hbar U_\hbar=\operatorname{Op}_\hbar^w(H^0)+\operatorname{Op}_\hbar^w(f^\star(w,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(d/2)},\hbar))+\operatorname{Op}_\hbar^w(\tilde{s}),$$

with $U_\hbar=e^{-\frac{i}{\hbar}\operatorname{Op}_\hbar^w(c)}$. To prove (iii) with $R_\hbar=\operatorname{Op}_\hbar^w(\tilde{s})$, note that

$$\sigma_\hbar^{w,\mathrm{T}}(R_\hbar)=[\tilde{s}]\in\mathcal{O}_r$$

and

$$\sigma_\hbar^{w,\mathrm{T}}(U_\hbar R_\hbar U_\hbar^*)=e^{\frac{i}{\hbar}\operatorname{ad}_\tau}([\tilde{s}])\in\mathcal{O}_r. \qquad\blacksquare$$

Theorem 1.2 follows with the new operator $\widetilde{U}_\hbar=V_\hbar U_\hbar$ given by (4.1) and Theorem 4.1. Point Theorem 1.2 (ii) is remaining. We prove it here, using that the function $f^\star$ can be chosen with arbitrarily small compact support.

Proposition 4.1. *For any $\zeta \in (0,1)$, up to reducing the support of $f^\star$, the normal form $\mathcal{N}_\hbar$ of Theorem 4.1 satisfies for $\hbar \in (0,\hbar_0]$ small enough:*

$$(1-\zeta)\langle \mathcal{L}_\hbar^0 \psi, \psi\rangle \le \langle \mathcal{N}_\hbar \psi, \psi\rangle \le (1+\zeta)\langle \mathcal{L}_\hbar^0 \psi, \psi\rangle, \quad \textit{for all } \psi \in \mathcal{S}(\mathbf{R}^d).$$

Proof. For a given $K > 0$, we can take a cutoff function χ supported in $\{\lambda \in \mathbf{R}^{d/2} : \|\lambda\| \le K\}$, and change $f^\star$ into $\chi f^\star$. Thus, for $\lambda_j \in \mathrm{sp}(\mathcal{I}_\hbar^{(j)})$,

$$\begin{aligned}|\chi f^\star(w,\lambda_1,\dots,\lambda_{d/2},\hbar)| &\le CK\|\lambda\| \\ &\le CK\sum_j \frac{1}{\min \hat\beta_j}\hat\beta_j(w)\lambda_j \\ &\le \tilde C K \sum_j \hat\beta_j(w)\lambda_j .\end{aligned}$$

Hence, using functional calculus and the Gårding inequality, we deduce that

$$\begin{aligned}|\langle \mathrm{Op}_\hbar^w\, f^*(w,\mathcal{I}_\hbar^{(1)},\dots,\mathcal{I}_\hbar^{(d/2)},\hbar)\psi,\psi\rangle| &\le \tilde C K \langle \mathcal{L}_\hbar^0\psi,\psi\rangle + c\hbar\|\psi\|^2 \\ &\le \zeta\langle \mathcal{L}_\hbar^0\psi,\psi\rangle,\end{aligned}$$

for K and $\hbar$ small enough. ∎

5. Spectral reduction of $\mathcal{N}_\hbar$

In this section, we prove an expansion of the first eigenvalues of $\mathcal{N}_\hbar$ in powers of $\hbar^{1/2}$. In order to prove Theorem 1.5, it will only remain to compare the spectra of $\mathcal{N}_\hbar$ and $\mathcal{L}_\hbar$. This will be done in the next sections.

Let $1 \le j \le d/2$. For $n_j \ge 0$, we denote by $h_{n_j} : \mathbf{R} \to \mathbf{R}$ the n_j-th Hermite function of the variable x_j. In particular, for every $1 \le j \le d/2$ we have

$$\mathcal{I}_\hbar^{(j)} h_{n_j}(x_j) = \hbar(2n_j+1)h_{n_j}(x_j). \tag{5.1}$$

Moreover, $(h_{n_j})_{n_j\ge 0}$ is a Hilbertian basis of $\mathrm{L}^2(\mathbf{R}_{x_j})$:

$$\mathrm{L}^2(\mathbf{R}_{x_j}) = \bigoplus_{n_j\ge 0}\langle h_{n_j}\rangle.$$

On $\mathbf{R}_x^{d/2}$, we define the functions $\mathbf{h}_n$ for any $n = (n_1,\dots,n_{d/2}) \in \mathbf{N}^{d/2}$ by

$$\mathbf{h}_n(x) = h_{n_1}\otimes\cdots\otimes h_{n_{d/2}}(x) = h_{n_1}(x_1)\dots h_{n_{d/2}}(x_{d/2}).$$

We have the following space decomposition:

$$\mathrm{L}^2(\mathbf{R}_x^{d/2}) = \bigoplus_{n\in\mathbf{N}^{d/2}}\langle \mathbf{h}_n\rangle.$$

In particular,

$$\mathrm{L}^2(\mathbf{R}^d_{x,y}) = \bigoplus_{n\in\mathbf{N}^{d/2}} \big(\mathrm{L}^2(\mathbf{R}^{d/2}_y) \otimes \langle \mathbf{h}_n\rangle\big). \tag{5.2}$$

Since $\mathcal{N}_\hbar$ commutes with the harmonic oscillators $\mathcal{I}_\hbar^{(j)}$ $(1 \le j \le d/2)$, it is reduced in the decomposition (5.2). More precisely,

Lemma 5.1. *For $n = (n_1, \dots, n_{d/2}) \in \mathbf{N}^{d/2}$, there exists a classical pseudodifferential operator $\mathcal{N}_\hbar^{(n)}$ acting on $\mathrm{L}^2(\mathbf{R}^{d/2}_y)$ such that*

$$\mathcal{N}_\hbar(u \otimes h_{n_1} \otimes \cdots \otimes h_{n_{d/2}}) = \mathcal{N}_\hbar^{(n)}(u) \otimes h_{n_1} \otimes \cdots \otimes h_{n_{d/2}} \quad \textit{for all } u \in \mathcal{S}(\mathbf{R}^{d/2}_y).$$

Its symbol is

$$F^{(n)}(w) = \hbar \sum_{j=1}^{d/2} \hat{\beta}_j(w)(2n_j+1) + f^\star(w, \hbar(2n+1), \hbar),$$

and we have

$$\mathrm{sp}(\mathcal{N}_\hbar) = \bigcup_n \mathrm{sp}(\mathcal{N}_\hbar^{(n)}).$$

Moreover, the multiplicity of λ as eigenvalue of $\mathcal{N}_\hbar$ is the sum over n of the multiplicities of λ as eigenvalue of $\mathcal{N}_\hbar^{(n)}$.

This follows directly from (5.1) and (4.4). Moreover, we can prove the following more precise inclusions of the spectra.

Lemma 5.2. *Let $b_1 \in (b_0, \tilde{b}_1)$. There exist $\hbar_0, n_{\max}, c > 0$ such that, for any $\hbar \in (0, \hbar_0)$,*

$$\mathrm{sp}(\mathcal{N}_\hbar) \cap (-\infty, b_1\hbar] \subset \bigcup_{0\le|n|\le n_{\max}} \mathrm{sp}(\mathcal{N}_\hbar^{(n)}), \tag{5.3}$$

and, for any $n \in \mathbf{N}^{d/2}$ with $1 \le |n| \le n_{\max}$,

$$\mathrm{sp}(\mathcal{N}_\hbar^{(n)}) \subset [\hbar(b_0 + c|n|), +\infty). \tag{5.4}$$

Proof. Remember that the functions $\hat{\beta}_j$ are bounded from below by a positive constant. Thus, the Gårding inequality implies that there are $\hbar_0, c > 0$ such that, for every $\hbar \in (0, \hbar_0)$,

$$\langle \mathrm{Op}_\hbar^w(\hat{\beta}_j)u, u\rangle \ge c\|u\|^2 \quad \text{for all } u \in \mathrm{L}^2(\mathbf{R}^{d/2}_y). \tag{5.5}$$

For any $n \in \mathbf{N}^{d/2}$, we have

$$\begin{aligned}\langle \mathcal{N}_\hbar^{(n)} u, u\rangle &= \langle \mathcal{N}_\hbar(u \otimes \mathbf{h}_n), u \otimes \mathbf{h}_n\rangle \\ &\geq (1-\zeta)\langle \mathcal{L}_\hbar^0(u \otimes \mathbf{h}_n), u \otimes \mathbf{h}_n\rangle \qquad \text{(by Proposition 4.1)} \\ &= (1-\zeta)\sum_{j=1}^{d/2} \hbar(2n_j+1)\langle \mathrm{Op}_\hbar^w(\hat{\beta}_j)u, u\rangle\end{aligned}$$

because $\mathcal{L}_\hbar^0 = \sum_j \mathrm{Op}_\hbar^w(\hat{\beta}_j)\mathcal{I}_\hbar^{(j)}$. Thus, using (5.5) and the Gårding inequality,

$$\begin{aligned}\langle \mathcal{N}_\hbar^{(n)} u, u\rangle &\geq \hbar(1-\zeta)(2c|n|\|u\|^2 + \langle \mathrm{Op}_\hbar^w(\hat{b})u, u\rangle) \\ &\geq \hbar(1-\zeta)(2c|n| + b_0 - \tilde{c}\hbar)\|u\|^2.\end{aligned}$$

This proves (5.4) for a new $c > 0$. Moreover, if you take any eigenpair (λ, ψ) of $\mathcal{N}_\hbar$ with $\lambda \leq b_1\hbar$, it is an eigenpair of some $\mathcal{N}_\hbar^{(n)}$, with $\psi = u \otimes \mathbf{h}_n$, and

$$\hbar(1-\zeta)(2c|n| + b_0 - \tilde{c}\hbar)\|u\|^2 \leq \langle \mathcal{N}_\hbar^{(n)} u, u\rangle = \langle \mathcal{N}_\hbar \psi, \psi\rangle \leq b_1\hbar\|\psi\|^2.$$

Thus, there is a $n_{\max} > 0$ independent of $\hbar, \lambda, \psi$ such that

$$|n| \leq n_{\max}.$$

We deduce (5.3). ∎

Using the previous Lemma and the well-known expansion of the first eigenvalues of $\mathrm{Op}_\hbar^w(\hat{b})$, we deduce an expansion of the first eigenvalues of $\mathcal{N}_\hbar$.

Theorem 5.1. *Let $\varepsilon > 0$ and $N \geq 1$. There exist $\hbar_0 > 0$ and $c_0 > 0$ such that, for $\hbar \in (0, \hbar_0]$, the N first eigenvalues of $\mathcal{N}_\hbar$: $(\lambda_j(\hbar))_{1 \leq j \leq N}$ admit an expansion in powers of $\hbar^{1/2}$ of the form*

$$\lambda_j(\hbar) = \hbar b_0 + \hbar^2(E_j + c_0) + \hbar^{5/2}c_{j,5} + \hbar^3 c_{j,6} + \cdots,$$

where $\hbar E_j$ is the j-th eigenvalue of the $d/2$-dimensional harmonic oscillator associated to the Hessian of $\hat{b}$ at 0, counted with multiplicity.

Proof. The smallest eigenvalues of $\mathcal{N}_\hbar$ are those of $\mathcal{N}_\hbar^{(0)}$, which has the symbol

$$\hbar\hat{b}(w) + f^\star(w, \hbar, \ldots, \hbar) = \hbar(\hat{b}(w) + \hbar c_0 + \mathcal{O}(\hbar^2)).$$

The first eigenvalues of a semiclassical pseudodifferential operator with principal symbol $\hat{b}$ (which admits a unique and non-degenerate minimum) have an expansion of the form

$$\mu_j(\hbar) = b_0 + \hbar E_j + \hbar^{3/2}\sum_{m \geq 0} a_{j,m}\hbar^{m/2},$$

where $\hbar E_j$ is the j-th eigenvalue of the $d/2$-dimensional harmonic oscillator associated to the Hessian of $\hat{b}$ at the minimum. Let us recall the idea of the proof of this result. Since the minimum of $\hat{b}$ is non-degenerate, we can write

$$\hat{b}(w) = b_0 + \frac{1}{2}\operatorname{Hess}_0 \hat{b}(w, w) + \mathcal{O}(|w|^3).$$

A linear symplectic change of coordinates changes $\operatorname{Hess}_0 \hat{b}$ into

$$\sum_{j=1}^{d/2} \nu_j (y_j^2 + \eta_j^2),$$

for some positive numbers $(\nu_j)_{1 \leq j \leq d/2}$. In these coordinates the symbol becomes

$$\hat{b}(y, \eta) = b_0 + \sum_{j=1}^{d/2} \nu_j (y_j^2 + \eta_j^2) + \mathcal{O}(|w|^3) + \mathcal{O}(\hbar),$$

and Helffer and Sjöstrand proved in [17] that the first eigenvalues of a pseudo-differential operator with such a symbol admits an expansion in powers of $\hbar^{1/2}$. Sjöstrand [27] recovered this result using a Birkhoff normal form in the case where the coefficients $(\nu_j)_j$ are non-resonant. Charles and Vũ Ngọc also tackled the resonant case in [5]. ∎

6. Microlocalization results

In Section 4, we proved Theorem 1.2. We constructed a normal form, which is only valid on a neighborhood U of $\Sigma = H^{-1}(0)$ since the rest $R_\hbar$ can be large outside this neighborhood. Hence, we now show that the eigenfunctions of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$ are microlocalized on a neighborhood of Σ. These results, and their proofs, are the same as in the case $d = 2$, so we refer the reader to [26].

6.1. Microlocalization of the eigenfunctions

Recall that

$$K = \{b(q) \leq \tilde{b}_1\} \subset \Omega.$$

For $\varepsilon > 0$, we denote

$$K_\varepsilon = \{q \colon \mathrm{d}(q, K) \leq \varepsilon\}. \tag{6.1}$$

For $\varepsilon > 0$ small enough, $K_\varepsilon \subset \Omega$. The following theorem states the well-known Agmon estimates (see Agmon's paper [1] for the electric Schrödinger case), which gives exponential decay of the eigenfunctions of the magnetic Laplacian $\mathcal{L}_\hbar$ outside the minimum q_0 of the magnetic intensity b. In particular, these eigenfunctions are localized in Ω.

Lemma 6.1 (Agmon estimates). *Let $\alpha \in (0, 1/2)$ and $b_0 < b_1 < \tilde{b}_1$. There exist $C, \hbar_0 > 0$ such that for all $\hbar \in (0, \hbar_0]$ and for all eigenpair (λ, ψ) of $\mathcal{L}_\hbar$ with $\lambda \le \hbar b_1$, we have*

$$\int_M |e^{d(q,K)\hbar^{-\alpha}} \psi|^2 \, \mathrm{d}q \le C \|\psi\|^2.$$

In particular, if $\chi_0 \colon M \to [0, 1]$ is a smooth function being 1 on K_ε,

$$\psi = \chi_0 \psi + \mathcal{O}(\hbar^\infty) \quad \textit{in } \mathrm{L}^2(M).$$

Proof. This result is now classical, and the proof in the 2-dimensional case ([26, Proposition 4.1]) is still valid, using Assumption 1. ∎

Now, we state the microlocalization of the eigenfunctions of $\mathcal{L}_\hbar$ near Σ.

Lemma 6.2. *Let $\varepsilon > 0$, $\delta \in (0, \frac{1}{2})$, and $0 < b_1 < \tilde{b}_1$. Let $\chi_0 \colon M \to [0, 1]$ be a smooth function being 1 on K_ε. Let $\chi_1 \colon \mathbf{R} \to [0, 1]$ be a smooth compactly supported cutoff function being 1 near 0. Then for any normalized eigenpair (λ, ψ) of $\mathcal{L}_\hbar$ such that $\lambda \le \hbar b_1$ we have*

$$\psi = \chi_1(\hbar^{-2\delta} \mathcal{L}_\hbar) \chi_0(q) \psi + \mathcal{O}(\hbar^\infty) \quad \textit{in } \mathrm{L}^2(M).$$

Proof. Again, the same proof as in the 2-dimensional case holds (see [26, Proposition 4.3]). ∎

The next two theorems state the microlocalization of the eigenfunctions of the normal form. We recall that if φ is defined by Theorem 1.1, we have

$$\varphi(K) = \{w \in V \colon \hat{b}(w) \le \tilde{b}_1\},$$

with $\hat{b}(w) = b \circ \varphi^{-1}(w)$. We also recall the definition (6.1) of K_ε. This first lemma gives a microlocalization result on the w variable.

Lemma 6.3. *Let $\hbar \in (0, \hbar_0]$ and $b_1 \in (0, \tilde{b}_1)$. Let χ_0 be a smooth cutoff function on $\mathbf{R}^d_w$ supported on V such that $\chi_0 = 1$ on $\varphi(K_\varepsilon)$. Then for any normalized eigenpair (λ, ψ) of $\mathcal{N}_\hbar$ such that $\lambda \le \hbar b_1$, we have*

$$\psi = \mathrm{Op}^w_\hbar(\chi_0)\psi + \mathcal{O}(\hbar^\infty) \quad \textit{in } \mathrm{L}^2(\mathbf{R}^d_{x,y}).$$

Proof. This is standard symbolic calculus. See the 2-dimensional case ([26, Proposition 4.4]). ∎

Now, we prove the microlocalization of the eigenfunctions of $\mathcal{N}_\hbar$ on a neighborhood of $\varphi(\Sigma) = \{(z,w) : z = 0\}$.

Lemma 6.4. *Let $\hbar \in (0,\hbar_0]$, $b_1 \in (0,\tilde{b}_1)$, and $\delta \in (0,1/2)$. Let χ_0 be a smooth cutoff function on $\mathbf{R}^{d/2}_w$ supported on V such that $\chi_0 = 1$ on $\varphi(K_\varepsilon)$ and χ_1 a real cutoff function being 1 near 0. Then for any normalized eigenpair (λ,ψ) of $\mathcal{N}_\hbar$ such that $\lambda \le \hbar b_1$, we have*

$$\psi = \chi_1(\hbar^{-2\delta}\mathcal{I}_\hbar^{(1)})\dots\chi_1(\hbar^{-2\delta}\mathcal{I}_\hbar^{(d/2)})\,\mathrm{Op}_\hbar^w(\chi_0(w))\psi + \mathcal{O}(\hbar^\infty) \quad \textit{in } \mathrm{L}^2(\mathbf{R}^d).$$

Proof. This is the analogue of [26, Proposition 4.5] in the 2-dimensional case. ∎

6.2. Rank of the spectral projections

We want the microlocalization Theorems 6.2 and 6.4 to be uniform with respect to $\lambda \in (-\infty, b_1\hbar]$. That is why we need the rank of the spectral projections to be bounded by some finite power of $\hbar^{-1}$. If $\mathcal{A}$ is a bounded from below self-adjoint operator, and $\alpha \in \mathbf{R}$, we denote by $N(\mathcal{A},\alpha)$ the number of eigenvalues of $\mathcal{A}$ smaller than α, counted with multiplicities. It is the rank of the spectral projection $\mathbf{1}_{]-\infty,\alpha]}(\mathcal{A})$.

The proof of the following estimate is based on the inequality of Assumption 1, together with a magnetic Lieb–Thirring inequality, which can be found in [10] for instance. The proof is the same as in the 2-dimensional case ([26, Lemma 4.6]).

Lemma 6.5. *Let $b_0 < b_1 < \tilde{b}_1$. There exists $C > 0$ and $\hbar_0 > 0$ such that for all $\hbar \in (0,\hbar_0]$, we have*

$$N(\mathcal{L}_\hbar, \hbar b_1) \le C\hbar^{-d/2}.$$

The same result holds for $\mathcal{N}_\hbar$:

Lemma 6.6. *Let $b_1 \in (0,\tilde{b}_1)$. There exists $C > 0$ and $\hbar_0 > 0$ such that*

$$N(\mathcal{N}_\hbar, \hbar b_1) \le C\hbar^{-d/2} \quad \textit{for all } \hbar \in (0,\hbar_0).$$

Proof. By Lemma 4.1, we have

$$\langle \mathcal{N}_\hbar\psi,\psi\rangle \ge (1-\zeta)\langle \mathcal{L}_\hbar^0\psi,\psi\rangle \ge (1-\zeta)\hbar\langle \mathcal{B}_\hbar\psi,\psi\rangle,$$

with $\mathcal{B}_\hbar = \mathrm{Op}_\hbar^w(\hat{b})$. Using the min-max principle, it follows that

$$N(\mathcal{N}_\hbar, \hbar b_1) \le N(\mathcal{B}_\hbar, (1-\zeta)^{-1}b_1),$$

and using Weyl estimates ([8, Chapter 9] or [19]), we get

$$N(\mathcal{B}_\hbar, (1-\zeta)^{-1}b_1) = \mathcal{O}(\hbar^{-d/2}).$$ ∎

7. Comparison of the spectra of $\mathcal{L}_\hbar$ and $\mathcal{N}_\hbar$

7.1. Proof of Theorem 1.4

We denote by

$$\lambda_1(\hbar) \leq \lambda_2(\hbar) \leq \cdots$$

the smallest eigenvalues of $\mathcal{L}_\hbar$ and

$$\nu_1(\hbar) \leq \nu_2(\hbar) \leq \cdots$$

the smallest eigenvalues of $\mathcal{N}_\hbar$. The goal of this section is to prove the following theorem, using the results of Section 6.

Theorem 7.1. *If $b_1 < \tilde{b}_1$ and $\delta \in (0, 1/2)$, then*

$$\lambda_n(\hbar) = \nu_n(\hbar) + \mathcal{O}(\hbar^{\delta r}),$$

uniformly in n such that $\lambda_n(\hbar) \leq \hbar b_1$ and $\nu_n(\hbar) \leq \hbar b_1$.

Together with Theorem 5.1, this theorem concludes the proofs of Theorems 1.4 and 1.5.

Proof. We will prove that $\nu_n(\hbar) \leq \lambda_n(\hbar) + \mathcal{O}(\hbar^{\delta r})$, the other inequality being similar. Let $1 \leq n \leq N(\mathcal{L}_\hbar, \hbar b_1)$, and let us denote by $\psi_{1,\hbar}, \ldots, \psi_{n,\hbar}$ the normalized eigenfunctions associated to the first eigenvalues of $\mathcal{L}_\hbar$. We also denote

$$V_{n,\hbar} = \operatorname{span}\{\chi_1(\hbar^{-2\delta}\mathcal{L}_\hbar)\chi_0(q)\psi_{j,\hbar}\colon 1 \leq j \leq n\},$$

where χ_0 and χ_1 are defined in Lemma 6.2. We have the normal form

$$U_\hbar^* \mathcal{L}_\hbar U_\hbar = \mathcal{N}_\hbar + R_\hbar, \tag{7.1}$$

and we will use the min-max principle. For $\psi \in \operatorname{span}_{1 \leq j \leq n} \psi_{j,\hbar}$, we denote

$$\tilde{\psi} = \chi_1(\hbar^{-2\delta}\mathcal{L}_\hbar)\chi_0(q)\psi \in V_{n,\hbar}$$

Such a $\tilde{\psi}$ is microlocalized on $\Omega_\hbar \subset U \subset T^*M$, where

$$\Omega_\hbar = \{(q, p) \in T^*M\colon |p - A(q)|^2 < c\hbar^{2\delta}, q \in \Omega\}.$$

(Indeed, the symbol of $\chi_1(\hbar^{-2\delta}\mathcal{L}_\hbar)$ is $\mathcal{O}(\hbar^\infty)$ where $\chi_1(\hbar^{-2\delta}|p - A(q)|^2) \equiv 0$). Thus, since $U_\hbar U_\hbar^* = I$ microlocally on U (4.2) we deduce from (7.1) that

$$\langle \mathcal{N}_\hbar U_\hbar^* \tilde{\psi}, U_\hbar^* \tilde{\psi}\rangle = \langle \mathcal{L}_\hbar \tilde{\psi}, \tilde{\psi}\rangle - \langle U_\hbar R_\hbar U_\hbar^* \tilde{\psi}, \tilde{\psi}\rangle + \mathcal{O}(\hbar^\infty)\|\tilde{\psi}\|^2, \tag{7.2}$$

On the first hand, by Theorem 6.2, we can change $\tilde{\psi}$ into ψ up to an error of order $\hbar^\infty$. Indeed, by Lemma 6.5, the estimates of Theorem 6.2 remain true for ψ. We get

$$\langle \mathcal{L}_\hbar \tilde{\psi}, \tilde{\psi}\rangle = \langle \mathcal{L}_\hbar \psi, \psi\rangle + \mathcal{O}(\hbar^\infty)\|\psi\|^2 \leq (\lambda_n(\hbar) + \mathcal{O}(\hbar^\infty))\|\psi\|^2.$$

On the other hand, the remainder is $\langle U_\hbar R_\hbar U_\hbar^* \tilde{\psi}, \tilde{\psi}\rangle$, where the function $U_\hbar^*\tilde{\psi}$ is microlocalized in

$$\mathcal{V}_\hbar = \{(w,z) \colon w \in V, |z|^2 \leq c\hbar^{2\delta}\},$$

because $U_\hbar$ is a Fourier integral operator with phase function associated to the canonical transformation Φ, which is sending $\Omega_\hbar$ (where $\tilde{\psi}$ is microlocalized) on $\mathcal{V}_\hbar$. Moreover, the symbol of the pseudo-differential operator $R_\hbar$ on V is $\mathcal{O}((\hbar + |z|^2)^{r/2})$ (Theorem 4.1), so we get

$$U_\hbar R_\hbar U_\hbar^* \tilde{\psi} = \mathcal{O}(\hbar^{\delta r}).$$

Thus, equation (7.2) yields

$$\langle \mathcal{N}_\hbar U_\hbar^* \tilde{\psi}, U_\hbar^* \tilde{\psi}\rangle \leq (\lambda_n(\hbar) + \mathcal{O}(\hbar^{\delta r}))\|U_\hbar^* \tilde{\psi}\|^2,$$

for all $\tilde{\psi} \in V_{n,\hbar}$. Since $V_{n,\hbar}$ is n-dimensional, the min-max principle gives

$$\nu_n(\hbar) \leq \lambda_n(\hbar) + \mathcal{O}(\hbar^{\delta r}).$$

The same arguments give the opposite inequality, replacing Lemma 6.2 and 6.5 by Lemma 6.4 and 6.6. ∎

7.2. Proof of Corollary 1.1

Let us prove the Weyl estimates stated in Corollary 1.1. The proof relies on the classical Weyl asymptotics for pseudo-differential operators with elliptic principal symbol ([8, Chapter 9], [19, Appendix]). Let us first prove the Weyl estimates for the normal form. For any $n \in \mathbf{N}^{d/2}$, $\mathcal{N}_\hbar^{(n)}$ is a pseudo-differential operator with principal symbol

$$\hbar \hat{b}^{[n]}(w) = \hbar \sum_{j=1}^{d/2} (2n_j + 1)\hat{\beta}_j(w).$$

Note that

$$V_n := \{\hat{b}^{[n]}(w) \leq b_1\}$$

is empty for all but finitely many n. For these n, the Gårding inequality gives

$$\langle \mathcal{N}_\hbar^{(n)} \psi, \psi\rangle \geq \hbar(b_1 - c\hbar)\|\psi\|, \quad \text{for all } \psi \in \mathcal{S}(\mathbf{R}^{d/2}),$$

so that

$$N(\mathcal{N}_\hbar^{(n)}, b_1\hbar) = N\Big(\frac{1}{\hbar}\mathcal{N}_\hbar^{(n)}, [b_1 - c\hbar, b_1]\Big)$$

which is $o(\hbar^{-d/2})$ by the classical Weyl asymptotics. For the other finitely many n,

$$V_n \subset \{\hat{b}(w) \leq b_1\}$$

is a compact set with positive volume and thus the classical Weyl asymptotics gives

$$N(\mathcal{N}_\hbar^{(n)}, b_1\hbar) = N\Big(\frac{1}{\hbar}\mathcal{N}_\hbar^{(n)}, b_1\Big) \sim \frac{1}{(2\pi\hbar)^{d/2}} \operatorname{Vol}(V_n).$$

Using

$$\operatorname{sp}(\mathcal{N}_\hbar) = \bigcup_n \operatorname{sp}(\mathcal{N}_\hbar^{(n)}),$$

we deduce that

$$N(\mathcal{N}_\hbar, b_1\hbar) = \frac{1}{(2\pi\hbar)^{d/2}} \sum_n \operatorname{Vol}(V_n) + o(\hbar^{-d/2}).$$

Moreover,

$$\operatorname{Vol}(V_n) = \int_{V_n} \mathrm{d}\,y\,\mathrm{d}\,\eta = \int_{\varphi^{-1}(V_n)} \varphi^*(\mathrm{d}\,y\,\mathrm{d}\,\eta),$$

where φ is defined in Theorem 1.1. Since φ is a symplectomorphism, we have

$$B = \varphi^*(\mathrm{d}\,\eta \wedge \mathrm{d}\,y)$$

and thus

$$\frac{B^{d/2}}{(d/2)!} = \frac{1}{(d/2)!}\varphi^*((\mathrm{d}\,\eta \wedge \mathrm{d}\,y)^{d/2}) = \varphi^*(\mathrm{d}\,y\,\mathrm{d}\,\eta).$$

Hence

$$\operatorname{Vol}(V_n) = \int_{b^{[n]}(q)\leq b_1} \frac{B^{d/2}}{(d/2)!},$$

so that

$$N(\mathcal{N}_\hbar, b_1\hbar) = \frac{1}{(2\pi\hbar)^{d/2}} \sum_{n\in\mathbf{N}^{d/2}} \int_{b^{[n]}(q)\leq b_1} \frac{B^{d/2}}{(d/2)!} + o(\hbar^{-d/2}),$$

where the sum is finite. It remains to compare

$$N_1 := N(\mathcal{N}_\hbar, b_1\hbar) \quad \text{and} \quad N_2 := N(\mathcal{L}_\hbar, b_1\hbar).$$

If we apply Theorem 1.4 with some $b_1 + \delta > b_1$, we get a $c > 0$ such that for $\hbar$ small enough,

$$N(\mathcal{N}_\hbar, \hbar b_1 - c\hbar^{r/2-\varepsilon}) \leq N_2 \leq N(\mathcal{N}_\hbar, \hbar b_1 + c\hbar^{r/2-\varepsilon}),$$

so

$$|N_1 - N_2| \leq N(\mathcal{N}_\hbar, [\hbar b_1 - c\hbar^{r/2-\varepsilon}, \hbar b_1 + c\hbar^{r/2-\varepsilon}]).$$

Classical Weyl asymptotics gives

$$N(\mathcal{N}_\hbar^{(n)}, [\hbar b_1 - c\hbar^{r/2-\varepsilon}, \hbar b_1 + c\hbar^{r/2-\varepsilon}]) = o(\hbar^{-d/2}),$$

for any $n \in \mathbf{N}^{d/2}$, so $|N_1 - N_2| = o(\hbar^{-d/2})$, and the proof is complete.

A. Appendix

Lemma A.1. *The principal and subprincipal symbols of the operator*

$$\mathcal{L}_\hbar = (i\hbar\,\mathrm{d} + A)^*(i\hbar\,\mathrm{d} + A)$$

are

$$\sigma_0(\mathcal{L}_\hbar) = |p - A(q)|^2_{g^*(q)}, \quad \textit{and} \quad \sigma_1(\mathcal{L}_\hbar) = 0.$$

Proof. We will compute these symbols in coordinates, in which $\mathcal{L}_\hbar$ acts as

$$\mathcal{L}_\hbar^{\text{coord}} = \sum_{k\ell} |g|^{-1/2}(i\hbar\partial_k + A_k)g^{k\ell}|g|^{1/2}(i\hbar\partial_\ell + A_\ell).$$

The principal symbol is always well defined. The subprincipal symbol is well defined if we restrict the changes of coordinates to be volume-preserving. This amounts to conjugating $\mathcal{L}_\hbar^{\text{coord}}$ by $|g|^{1/4}$. Thus, the subprincipal symbol is defined in coordinates by

$$\sigma_1(\mathcal{L}_\hbar) = \sigma_1(|g|^{1/4}\mathcal{L}_\hbar^{\text{coord}}|g|^{-1/4}).$$

The total symbol of $-i\hbar\partial_k - A_k$ is

$$\sigma(-i\hbar\partial_k - A_k) = p_k - A_k,$$

so we can use the star product $\star$ on symbols to compute the symbol of $\mathcal{L}_\hbar$:

$$\sigma(|g|^{1/4}\mathcal{L}_\hbar^{\text{coord}}|g|^{-1/4}) = \sum_{k\ell} |g|^{1/4} \star |g|^{-1/2} \star (p_k - A_k) \star g^{k\ell}|g|^{1/2} \star (p_\ell - A_\ell) \star |g|^{-1/4}.$$

Now, we will use the formula

$$\sigma(f \star g) = fg + \frac{\hbar}{2i}\{f, g\} + \mathcal{O}(\hbar^2)$$

several times to compute the symbol, where $\{f, g\}$ denotes the Poisson brackets. Of course, we directly deduce the principal symbol

$$\sigma_0(|g|^{1/4}\mathcal{L}_\hbar^{\text{coord}}|g|^{-1/4}) = \sum_{k\ell} g^{k\ell}(p_k - A_k)(p_\ell - A_\ell)$$

so that

$$\sigma_0(\mathcal{L}_\hbar) = |p - A(q)|^2_{g^*(q)}.$$

To compute the subprincipal symbol, we will use

$$\sigma(|g|^{1/4}\mathcal{L}_\hbar^{\text{coord}}|g|^{-1/4}) = \sum_{k\ell}[|g|^{-1/4} \star (p_k - A_k) \star |g|^{1/4}] \star g^{k\ell} \star [|g|^{1/4} \star (p_\ell - A_\ell) \star |g|^{-1/4}].$$

Let us compute $a_k = |g|^{-1/4} \star (p_k - A_k) \star |g|^{1/4}$:

$$\begin{aligned} a_k &= (p_k - A_k) + \frac{\hbar}{2i}[\{|g|^{-1/4}(p_k - A_k), |g|^{1/4}\} + \{|g|^{-1/4}, p_k - A_k\}|g|^{1/4}] \\ &\quad + \mathcal{O}(\hbar^2) \\ &= (p_k - A_k) + \frac{\hbar}{2i}\left[|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_k} - \frac{\partial |g|^{-1/4}}{\partial q_k}|g|^{1/4}\right] + \mathcal{O}(\hbar^2) \\ &= (p_k - A_k) + \frac{\hbar}{i}|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_k} + \mathcal{O}(\hbar^2). \end{aligned}$$

We also get the similar result for $b_\ell = |g|^{1/4} \star (p_\ell - A_\ell) \star |g|^{-1/4}$:

$$b_\ell = (p_\ell - A_\ell) - \frac{\hbar}{i}|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_\ell} + \mathcal{O}(\hbar^2)$$

Thus, we can compute

$$\begin{aligned} a_k \star g^{k\ell} &= g^{k\ell}(p_k - A_k) + \frac{\hbar}{2i}\{p_k - A_k, g^{k\ell}\} + \frac{\hbar}{i}|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_k}g^{k\ell} + \mathcal{O}(\hbar^2) \\ &= g^{k\ell}(p_k - A_k) + \frac{\hbar}{2i}\frac{\partial g^{k\ell}}{\partial q_k} + \frac{\hbar}{i}|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_k}g^{k\ell} + \mathcal{O}(\hbar^2), \end{aligned}$$

and

$$\begin{aligned} a_k \star g^{k\ell} \star b_\ell = g^{k\ell}(p_k - A_k)(p_l - A_l) &+ \frac{\hbar}{2i}\{g^{k\ell}(p_k - A_k), p_\ell - A_\ell\} \\ &- \frac{\hbar}{i}g^{k\ell}(p_k - A_k)|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_\ell} + \frac{\hbar}{2i}\frac{\partial g^{k\ell}}{\partial q_k}(p_\ell - A_\ell) \\ &+ \frac{\hbar}{i}|g|^{-1/4}\frac{\partial |g|^{1/4}}{\partial q_k}(p_\ell - A_\ell) + \mathcal{O}(\hbar^2). \end{aligned}$$

Summing over k, ℓ, we get

$$\begin{aligned}\sum_{k\ell} a_k \star g^{k\ell} \star b_\ell &= \sum_{k\ell} g^{k\ell}(p_k - A_k)(p_l - A_l) + \frac{\hbar}{2i}\{g^{k\ell}(p_k - A_k), p_\ell - A_\ell\} \\ &\quad + \frac{\hbar}{2i}\frac{\partial g^{k\ell}}{\partial q_k}(p_\ell - A_\ell) + \mathcal{O}(\hbar^2) \\ &= \sum_{k\ell} g^{k\ell}(p_k - A_k)(p_\ell - A_\ell) \\ &\qquad + \frac{\hbar}{2i} g^{k\ell}\frac{\partial(p_\ell - A_\ell)}{\partial q_k} - \frac{\hbar}{2i}\frac{\partial g^{k\ell}(p_k - A_k)}{\partial q_\ell} \\ &\qquad + \frac{\hbar}{2i}\frac{\partial g^{k\ell}}{\partial q_k}(p_\ell - A_\ell) + \mathcal{O}(\hbar^2) \\ &= \sum_{k\ell} g^{k\ell}(p_k - A_k)(p_\ell - A_\ell) + \mathcal{O}(\hbar^2).\end{aligned}$$

Since

$$\sigma(|g|^{1/4}\mathcal{L}_\hbar^{\mathrm{coord}}|g|^{-1/4}) = \sum_{k\ell} a_k \star g^{k\ell} \star b_\ell,$$

we deduce that:

$$\sigma_1(|g|^{1/4}\mathcal{L}_\hbar^{\mathrm{coord}}|g|^{-1/4}) = 0,$$

and

$$\sigma_1(\mathcal{L}_\hbar) = 0. \qquad \blacksquare$$

The following Lemma due to Weinstein [30] tells that, if two 2-forms coincide on a submanifold, they are equal up to a transformation tangent to the identity. The proof can be found in [26, Lemma 2.4] as well.

Lemma A.2 (relative Darboux lemma). *Let ω_0 and ω_1 be two 2-forms on $\Omega \times \mathbf{R}_z^d$ which are closed and non-degenerate. Assume that $\omega_{0|z=0} = \omega_{1|z=0}$. Then there exists a change of coordinates S on a neighborhood of $\Omega \times \{0\}$ such that*

$$S^*\omega_1 = \omega_0 \quad \textit{and} \quad S = I + \mathcal{O}(|z|^2).$$

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Léo Morin
Département Mathématiques, Université de Rennes, CNRS, IRMAR – UMR 6625,
35000 Rennes, France; leo.morin@ens-rennes.fr