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# Bounds on orthogonal polynomials and separation of their zeros

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**Abstract.** Let $\{p_n\}$ denote the orthonormal polynomials associated with a measure $\mu$ with compact support on the real line. Let $\mu$ be regular in the sense of Stahl, Totik, and Ullmann, and $I$ be a subinterval of the support in which $\mu$ is absolutely continuous, while $\mu'$ is positive and continuous there. We show that boundedness of the $\{p_n\}$ in that subinterval is closely related to the spacing of zeros of $p_n$ and $p_{n-1}$ in that interval. One ingredient is proving that "local limits" imply universality limits.

## 1. Results

Let $\mu$ be a finite positive Borel measure with compact support, which we denote by $\mathrm{supp}[\mu]$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

The zeros of $p_n$ are real and simple. We list them in decreasing order:

$$x_{1n} > x_{2n} > \cdots > x_{n-1,n} > x_{nn}.$$

They interlace the zeros $y_{jn}$ of $p_n'$ :

$$p_n(y_{jn}) = 0 \quad \text{and} \quad y_{jn} \in (x_{j+1,n}, x_{jn}), \quad 1 \le j \le n-1.$$

It is a classic result that the zeros of $p_n$ and $p_{n-1}$ also interlace. The three term recurrence relation has the form

$$(x - b_n)p_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x),$$

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where for $n \geq 1$,

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} = \int x p_{n-1}(x) p_n(x) d\mu(x); \; b_n = \int x p_n^2(x) d\mu(x).$$

Uniform boundedness of orthonormal polynomials is a long studied topic. For example, given an interval $I$, one asks whether

$$\sup_{n \geq 1} \|p_n\|_{L_\infty(I)} < \infty.$$

There is an extensive literature on this fundamental question - see for example [1–4, 12]. In this paper, we establish a connection to the distance between zeros of $p_n$ and $p_{n-1}$.

The results require more terminology: we let $\mathrm{dist}(a, \mathbb{Z})$ denote the distance from a real number $a$ to the integers. We say that $\mu$ is *regular* (in the sense of Stahl, Totik, and Ullmann) if for every sequence of non-zero polynomials $\{P_n\}$ with degree $P_n$ at most $n$,

$$\limsup_{n \to \infty} \left( \frac{|P_n(x)|}{(\int |P_n|^2 d\mu)^{1/2}} \right)^{1/n} \leq 1$$

for quasi-every $x \in \mathrm{supp}[\mu]$ (that is except in a set of logarithmic capacity 0). If the support consists of finitely many intervals, and $\mu' > 0$ a.e. in each subinterval, then $\mu$ is regular, though much less is required [16]. An equivalent formulation involves the leading coefficients $\{\gamma_n\}$ of the orthonormal polynomials for $\mu$:

$$\lim_{n \to \infty} \gamma_n^{1/n} = \frac{1}{\mathrm{cap}(\mathrm{supp}[\mu])},$$

where cap denotes logarithmic capacity.

Recall that the equilibrium measure for the compact set $\mathrm{supp}[\mu]$ is the probability measure that minimizes the energy integral

$$\iint \log \frac{1}{|x-y|} d\nu(x)\, d\nu(y)$$

amongst all probability measures $\nu$ supported on $\mathrm{supp}[\mu]$. If $I$ is an interval contained in $\mathrm{supp}[\mu]$, then the equilibrium measure is absolutely continuous in $I$, and moreover its density, which we denote throughout by $\omega$, is positive and continuous in the interior $\mathring{I}$ of $I$ [13, Theorem IV.2.5, p. 216]. Given sequences $\{x_n\}, \{y_n\}$ of non-0 real numbers, we write

$$x_n \sim y_n$$

if there exists $C > 1$ such that for $n \geq 1$,

$$C^{-1} \leq x_n / y_n < C.$$

Similar notation is used for functions and sequences of functions.

Our main result is the following.

**Theorem 1.1.** *Let $\mu$ be a regular measure on $\mathbb{R}$ with compact support. Let $I$ be a closed subinterval of the support, and assume that in some open interval containing $I$, $\mu$ is absolutely continuous, while $\mu'$ is positive and continuous. Let $\omega$ be the density of the equilibrium measure for the support of $\mu$. Let $A > 0$. The following statements are equivalent:*

a. *there exists $C > 0$ such that, for $n \geq 1$ and $x_{jn} \in I$,*

$$\operatorname{dist}\big(n\omega(x_{jn})(x_{jn} - x_{j,n-1}), \mathbb{Z}\big) \geq C; \tag{1.1}$$

b. *there exists $C > 0$ such that, for $n \geq 1$ and $y_{jn} \in I$,*

$$\operatorname{dist}\big(n\omega(y_{jn})(y_{jn} - y_{j,n-1}), \mathbb{Z}\big) \geq C; \tag{1.2}$$

c. *uniformly for $n \geq 1$ and $x \in I$,*

$$\|p_{n-1}\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]}\|p_n\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \sim 1; \tag{1.3}$$

d. *there exists $C > 0$ such that, for $n \geq 1$ and $x \in I$,*

$$\|p_{n-1}\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]}\|p_n\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \leq C. \tag{1.4}$$

*Moreover, under any of* (a)–(d), *we have*

$$\sup_{n\geq 1}\sup_{x\in I}\big||x - b_n|^{1/2} p_n(x)\big| < \infty. \tag{1.5}$$

**Remarks.** (a) The main idea behind the proof is that universality limits and "local" limits give

$$|p_{n-1}(y_{j,n-1})p_n(y_{jn})||\sin[\pi n\omega(y_{jn})(y_{jn} - y_{j,n-1})] + o(1)| \sim 1,$$

uniformly in $j, n$, while $p_n$ has a local extremum at $y_{jn}$.

(b) We could replace $x_{j,n-1} - x_{jn}$ in (1.1) by $x_{j,n-1} - x_{j,n+k}$, for any fixed integer $k$ (see Lemma 4.1).

(d) Under additional assumptions, involving the spacing of zeros of $p_n$ and $p_{n-2}$, we can remove the factor $|x - b_n|^{1/2}$ in (1.5).

**Theorem 1.2.** *Let $\mu$ be a regular measure on $\mathbb{R}$ with compact support. Let $I$ be a closed subinterval of the support and assume that in some open interval containing $I$, $\mu$ is absolutely continuous, while $\mu'$ is positive and continuous. Let $\omega$ be the density of the equilibrium measure for the support of $\mu$. Let $A > 0$. Assume that* (1.1) *holds in $I$. The following statements are equivalent:*

a. *there exist* $C_1 > 0$ *such that, for* $n \geq 1$ *and* $x_{jn} \in I$,

$$|n(x_{jn} - x_{j-1,n-2})| \geq C_1 |x_{jn} - b_{n-1}|; \tag{1.6}$$

b. *uniformly for* $x \in I$ *and* $n \geq 1$,

$$\|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \sim 1; \tag{1.7}$$

c. *one has*

$$\sup_{n \geq 1} \|p_n\|_{L_\infty(I)} < \infty. \tag{1.8}$$

**Remark.** We note that because of the interlacing, both $x_{jn}$ and $x_{j-1,n-2}$ belong to the interval $(x_{j,n-1}, x_{j-1,n-1})$.

Two important ingredients in our proofs are universality and local limits. The so-called "universality limit" involves the reproducing kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)}{x - y}. \tag{1.9}$$

For $x$ in the interior of supp$[\mu]$ (the "bulk" of the support), at least when $\mu'(x)$ is finite and positive, the universality limit typically takes the form [6, 8, 14, 15, 18]

$$\lim_{n \to \infty} \frac{K_n(x + \frac{a}{\mu'(x) K_n(x,x)}, x + \frac{b}{\mu'(x) K_n(x,x)})}{K_n(x, x)} = \mathbb{S}(a - b), \tag{1.10}$$

uniformly for $a, b$ in compact subsets of $\mathbb{C}$. Here $\mathbb{S}$ is the sinc kernel,

$$\mathbb{S}(a) = \frac{\sin \pi a}{\pi a}.$$

Universality limits holds far more generally than pointwise asymptotics for orthonormal polynomials, that at one stage were used to prove them. In a series of recent papers [7, 9–11], it was shown that one can go in the other direction, namely from universality limits, to "local ratio limits" for orthogonal polynomials.

Under fairly general conditions on $\mu$, the *Christoffel function* $K_n(x, x)$ admits the asymptotic [17]

$$\lim_{n \to \infty} \frac{1}{n} K_n(x, x) \mu'(x) = \omega(x)$$

for $x$ in the interior of the support of $\mu$. This allows us to reformulate the universality limit (1.10) as

$$\lim_{n \to \infty} \frac{K_n(x + \frac{a}{n\omega(x)}, x + \frac{b}{n\omega(x)}) \mu'(x)}{n\omega(x)} = \mathbb{S}(a - b), \tag{1.11}$$

uniformly for $a, b$ in compact subsets of $\mathbb{C}$.

Using this universality limit, we proved in [9]:

**Theorem A.** *Assume that $\mu$ is a regular measure with compact support. Let $I$ be a closed subinterval of the support in which $\mu$ is absolutely continuous, and $\mu'$ is positive and continuous. Let $J$ be a compact subset of the interior $\mathring{I}$ of $I$. Then*

$$\lim_{n\to\infty} \frac{p_n(y_{jn} + \frac{z}{n\omega(y_{jn})})}{p_n(y_{jn})} = \cos \pi z \tag{1.12}$$

*uniformly for $y_{jn} \in J$ and $z$ in compact subsets of $\mathbb{C}$.*

A secondary goal of this paper is to prove a converse of Theorem A, namely to show that local limits such as (1.12) imply a universality limit like (1.11). For measures on the unit circle this was undertaken in [10] – however the results necessarily take a quite different form.

**Theorem 1.3.** *Let $\mu$ be a measure with compact support. Assume that we are given both a bounded sequence of real numbers $\{\xi_n\}$ such that*

$$\sup_{n\geq 1} n|\xi_n - \xi_{n-1}| < \infty, \tag{1.13}$$

*and a sequence $\{\tau_n\}$ of positive numbers with $\tau_n \sim 1$ such that*

$$\lim_{n\to\infty} \frac{\tau_n}{\tau_{n-1}} = 1 \tag{1.14}$$

*and, uniformly for $z$ in compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} \frac{p_n(\xi_n + \frac{\tau_n}{n} z)}{p_n(\xi_n)} = \cos \pi z. \tag{1.15}$$

*Let $A > 0$. Then uniformly for $a, b$ in compact subsets of $\mathbb{C}$, and $x_n$ such that*

$$|x_n - \xi_n| \leq \frac{A}{n} \tag{1.16}$$

*we have*

$$\frac{K_n(x_n + \frac{\tau_n}{n} a, x_n + \frac{\tau_n}{n} b)}{K_n(x_n, x_n)} = \mathbb{S}(a-b) + o\Big(\frac{\frac{\gamma_{n-1}}{\gamma_n} n|p_{n-1}(\xi_{n-1})p_n(\xi_n)|}{K_n(x_n, x_n)}\Big). \tag{1.17}$$

*Moreover,*

$$\frac{K_n(x_n + \frac{\tau_n}{n} a, x_n + \frac{\tau_n}{n} b)}{K_n(x_n, x_n)} = \mathbb{S}(a-b) + o(1), \tag{1.18}$$

*provided either*

$$\liminf_{n\to\infty} \operatorname{dist}\Big(\frac{n}{\tau_n}(\xi_n - \xi_{n-1}), \mathbb{Z}\Big) > 0 \tag{1.19}$$

*or*

$$\sup_{n\geq 1} \frac{\frac{\gamma_{n-1}}{\gamma_n} n |p_{n-1}(\xi_{n-1}) p_n(\xi_n)|}{K_n(x_n, x_n)} < \infty. \tag{1.20}$$

We prove Theorem 1.3 in Section 2 and Theorem 1.1 in Section 3. Theorem 1.2 is proved in Section 4. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of $n, x, \theta$. The same symbol does not necessarily denote the same constant in different occurrences.

## 2. Proof of Theorem 1.3

Throughout this section, we assume the hypotheses of Theorem 1.3. Write for $n \geq 1$ and $m = n-1, n$,

$$x_n = \xi_m + \Delta_{n,m} \frac{\tau_m}{m} \tag{2.1}$$

and

$$\chi_n = \left(\frac{\tau_n}{n}\right) / \left(\frac{\tau_{n-1}}{n-1}\right). \tag{2.2}$$

Recall from (1.14) that $\chi_n \to 1$ as $n \to \infty$. Note too that in view of (1.13), (1.14), and (1.16), $\{\Delta_{n,n}\}$ and $\{\Delta_{n,n-1}\}$ are bounded sequences. We start with:

**Lemma 2.1.** a. *Uniformly for $z$ in compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} \frac{\tau_n}{n} \frac{p_n'(\xi_n + \frac{\tau_n}{n} z)}{p_n(\xi_n)} = -\pi \sin \pi z. \tag{2.3}$$

b. *Uniformly for $a, b$ in compact subsets of $\mathbb{C}$,*

$$\left(p_n\left(x_n + \frac{\tau_n}{n} a\right) - p_n\left(x_n + \frac{\tau_n}{n} b\right)\right) / p_n(\xi_n)$$
$$= -\pi \int_b^a \sin \pi(\Delta_{n,n} + t)\, dt + o(|a-b|).$$

c. *Moreover,*

$$\left(p_{n-1}\left(x_n + \frac{\tau_n}{n} a\right) - p_{n-1}\left(x_n + \frac{\tau_n}{n} b\right)\right) / p_{n-1}(\xi_{n-1})$$
$$= -\pi \int_b^a \sin \pi(\Delta_{n,n-1} + t)\, dt + o(|a-b|).$$

*Proof.* (a) As the asymptotic (1.15) holds uniformly for $z$ in compact subsets of the plane, we can differentiate it to obtain (2.3).

(b) Now

$$\left(p_n\left(x_n + \frac{\tau_n}{n}a\right) - p_n\left(x_n + \frac{\tau_n}{n}b\right)\right)/p_n(\xi_n)$$
$$= \int_b^a p_n'(x_n + \frac{\tau_n}{n}t)\frac{\tau_n}{n}\,dt/p_n(\xi_n).$$

Note that this is meaningful even for complex $a, b$, with the integral being taken over the directed line segment from $b$ to $a$. Using (2.1) and (2.3), we continue this as

$$\int_b^a \frac{p_n'\left(\xi_n + \frac{\tau_n}{n}(\Delta_{n,n} + t)\right)\frac{\tau_n}{n}}{p_n(\xi_n)}\,dt = \int_b^a (-\pi \sin \pi(\Delta_{n,n} + t) + o(1))\,dt$$
$$= -\pi \int_b^a \sin \pi(\Delta_{n,n} + t)\,dt + o(|a - b|).$$

(c) Using (2.2),

$$\left(p_{n-1}\left(x_n + \frac{\tau_n}{n}a\right) - p_{n-1}\left(x_n + \frac{\tau_n}{n}b\right)\right)/p_{n-1}(\xi_{n-1})$$
$$= \int_b^a p_{n-1}'\left(x_n + \frac{\tau_n}{n}t\right)\frac{\tau_n}{n}\,dt/p_{n-1}(\xi_{n-1})$$
$$= \int_b^a \frac{p_{n-1}'\left(\xi_{n-1} + \frac{\tau_{n-1}}{n-1}(\Delta_{n,n-1} + \chi_n t)\right)}{p_{n-1}(\xi_{n-1})}\frac{\tau_{n-1}}{n-1}\chi_n\,dt$$
$$= \int_b^a (-\pi \sin(\pi(\Delta_{n,n-1} + \chi_n t)) + o(1))\,dt$$
$$= -\pi \int_b^a \sin \pi(\Delta_{n,n-1} + t)\,dt + o(|a - b|). \qquad \blacksquare$$

*Proof of Theorem* 1.3. We apply (1.15) and Lemma 2.1(b,c). Now, if $a \neq b$,

$$\frac{\tau_n}{n p_{n-1}(\xi_{n-1}) p_n(\xi_n)} K_n\left(x_n + \frac{\tau_n}{n}a, x_n + \frac{\tau_n}{n}b\right)$$
$$= \frac{\gamma_{n-1}}{\gamma_n}\frac{[p_n(x_n + \frac{\tau_n}{n}a) - p_n(x_n + \frac{\tau_n}{n}b)]p_{n-1}(x_n + \frac{\tau_n}{n}b)}{(a - b)p_n(\xi_n)p_{n-1}(\xi_{n-1})}$$
$$+ \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x_n + \frac{\tau_n}{n}b)[p_{n-1}(x_n + \frac{\tau_n}{n}b) - p_{n-1}(x_n + \frac{\tau_n}{n}a)]}{(a - b)p_n(\xi_n)p_{n-1}(\xi_{n-1})}$$

$$= \frac{\gamma_{n-1}}{\gamma_n}\left[\frac{-\pi}{a-b}\int_b^a \sin\pi(\Delta_{n,n}+t)\,dt + o(1)\right][\cos\pi(\Delta_{n,n-1}+b\chi_n)+o(1)]$$

$$+ \frac{\gamma_{n-1}}{\gamma_n}[\cos\pi(\Delta_{n,n}+b)+o(1)]\left[\frac{\pi}{a-b}\int_b^a \sin\pi(\Delta_{n,n-1}+t)\,dt + o(1)\right]$$

by (1.15) and Lemma 2.1 (b,c). Note that because of the uniformity of the limits, this holds in a confluent form even if $a = b$. We continue this, using $\chi_n = 1 + o(1)$, as

$$= \frac{\gamma_{n-1}}{\gamma_n}\frac{\pi}{a-b}\int_a^b [\sin\pi(\Delta_{n,n}+t)\cos\pi(\Delta_{n,n-1}+b) - \cos\pi(\Delta_{n,n}+b)\sin\pi(\Delta_{n,n-1}+t)]dt + o\left(\frac{\gamma_{n-1}}{\gamma_n}\right). \tag{2.4}$$

Next, we expand the integrand using double angle formulae, in a straightforward but tedious fashion:

$$\begin{aligned}
&\sin\pi(\Delta_{n,n}+t)\cos\pi(\Delta_{n,n-1}+b) - \cos\pi(\Delta_{n,n}+b)\sin\pi(\Delta_{n,n-1}+t)\\
&\quad= [\sin\pi\Delta_{n,n}\cos\pi t + \cos\pi\Delta_{n,n}\sin\pi t]\\
&\qquad\times[\cos\pi\Delta_{n,n-1}\cos\pi b - \sin\pi\Delta_{n,n-1}\sin\pi b]\\
&\qquad- [\cos\pi\Delta_{n,n}\cos\pi b - \sin\pi\Delta_{n,n}\sin\pi b]\\
&\qquad\quad\times[\sin\pi\Delta_{n,n-1}\cos\pi t + \cos\pi\Delta_{n,n-1}\sin\pi t]\\
&\quad= \cos\pi t\cos\pi b\sin\pi(\Delta_{n,n}-\Delta_{n,n-1}) + \sin\pi t\sin\pi b\sin\pi(\Delta_{n,n}-\Delta_{n,n-1})\\
&\quad= \cos(\pi(t-b))\sin\pi(\Delta_{n,n}-\Delta_{n,n-1}).
\end{aligned}$$

We can then continue (2.4) as

$$\begin{aligned}
&\frac{\gamma_{n-1}}{\gamma_n}\frac{\pi}{a-b}\int_a^b [\cos(\pi(t-b))\sin\pi(\Delta_{n,n}-\Delta_{n,n-1})]\,dt + o\left(\frac{\gamma_{n-1}}{\gamma_n}\right)\\
&\quad= \frac{\gamma_{n-1}}{\gamma_n}\sin\pi(\Delta_{n,n}-\Delta_{n,n-1})\frac{1}{a-b}(-\sin\pi(a-b)) + o(\frac{\gamma_{n-1}}{\gamma_n})\\
&\quad= -\pi\frac{\gamma_{n-1}}{\gamma_n}\sin\pi\Big(\Delta_{n,n}-\Delta_{n,n-1}\Big)\mathbb{S}(a-b) + o\left(\frac{\gamma_{n-1}}{\gamma_n}\right).
\end{aligned}$$

In summary, uniformly for $a, b$ in compact subsets of the plane,

$$\begin{aligned}
&\frac{\tau_n}{np_{n-1}(\xi_{n-1})p_n(\xi_n)}K_n\left(x_n+\frac{\tau_n}{n}a, x_n+\frac{\tau_n}{n}b\right)\\
&\quad= -\pi\frac{\gamma_{n-1}}{\gamma_n}\sin\pi(\Delta_{n,n}-\Delta_{n,n-1})\mathbb{S}(a-b) + o\left(\frac{\gamma_{n-1}}{\gamma_n}\right).
\end{aligned} \tag{2.5}$$

Next, observe from (2.1), (1.13), and (1.14), that

$$x_n = \xi_n + \Delta_{n,n}\frac{\tau_n}{n} = \xi_{n-1} + \Delta_{n,n-1}\frac{\tau_n}{n} + o\Big(\frac{1}{n}\Big)$$
$$\implies \frac{\tau_n}{n}[\Delta_{n,n} - \Delta_{n,n-1}] = \xi_{n-1} - \xi_n + o\Big(\frac{1}{n}\Big).$$

As $\tau_n$ is bounded below, this allows us to reformulate (2.5) as

$$\frac{\tau_n}{n}K_n\Big(x_n + \frac{\tau_n}{n}a, x_n + \frac{\tau_n}{n}b\Big)$$
$$= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(\xi_{n-1})p_n(\xi_n)\Big\{\sin\Big[\pi\frac{n}{\tau_n}(\xi_{n-1} - \xi_n)\Big]\mathbb{S}(a-b) + o(1)\Big\}. \quad (2.6)$$

In particular, setting $a = b = 0$,

$$\frac{\tau_n}{n}K_n(x_n, x_n)$$
$$= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(\xi_{n-1})p_n(\xi_n)\Big\{\sin\Big[\pi\frac{n}{\tau_n}(\xi_{n-1} - \xi_n)\Big] + o(1)\Big\}, \quad (2.7)$$

so that (2.6) can be recast as

$$\frac{\tau_n}{n}K_n\Big(x_n + \frac{\tau_n}{n}a, x_n + \frac{\tau_n}{n}b\Big)$$
$$= \frac{\tau_n}{n}K_n(x_n, x_n)\mathbb{S}(a-b) + o\Big(\frac{\gamma_{n-1}}{\gamma_n}|p_{n-1}(\xi_{n-1})p_n(\xi_n)|\Big),$$

giving (1.17). If (1.19) holds, then $\sin[\pi\frac{n}{\tau_n}(\xi_{n-1} - \xi_n)]$ is bounded away from 0, so we can reformulate (2.6) as

$$\frac{\tau_n}{n}K_n\Big(x_n + \frac{\tau_n}{n}a, x_n + \frac{\tau_n}{n}b\Big)$$
$$= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(\xi_{n-1})p_n(\xi_n)\sin\Big[\pi\frac{n}{\tau_n}(\xi_{n-1} - \xi_n)\Big]\{\mathbb{S}(a-b) + o(1)\}$$

and (2.7) as

$$\frac{\tau_n}{n}K_n(x_n, x_n) = -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(\xi_{n-1})p_n(\xi_n)\sin\Big[\pi\frac{n}{\tau_n}(\xi_{n-1} - \xi_n)\Big]\{1 + o(1)\}.$$

Together these give (1.18). Finally, if (1.20) holds, then we see from (2.6) that necessarily $\sin[\pi\frac{n}{\tau_n}(\xi_{n-1} - \xi_n)]$ is bounded away from 0 and again (1.18) follows. ∎

## 3. Proof of Theorem 1.1

Recall that $y_{jn}$ is the zero of $p_n'$ in $(x_{j+1,n}, x_{jn})$. We begin with:

**Lemma 3.1.** *Let $\mu$ be a regular measure on $\mathbb{R}$ with compact support. Let $I$ be a closed subinterval of the support and assume that in some open interval containing $I$, $\mu$ is absolutely continuous, while $\mu'$ is positive and continuous.*

a. *Uniformly for $y_{jn} \in I$,*

$$\lim_{n\to\infty} n(x_{jn} - y_{jn})\omega(x_{jn}) = \frac{1}{2} = \lim_{n\to\infty} n(y_{jn} - x_{j+1,n})\omega(x_{jn}), \tag{3.1}$$

$$\lim_{n\to\infty} n(x_{jn} - x_{j+1,n})\omega(x_{jn}) = 1, \tag{3.2}$$

$$\lim_{n\to\infty} n(y_{jn} - y_{j+1,n})\omega(x_{jn}) = 1. \tag{3.3}$$

b. *Uniformly for $y_{jn} \in I$,*

$$\frac{\gamma_{n-1}}{\gamma_n} |p_{n-1}(y_{j,n-1}) p_n(y_{jn})| \left| \sin[\pi n \omega(y_{jn})(y_{j,n-1} - y_{jn})] + o(1) \right| \sim 1. \tag{3.4}$$

c. *Fix $A > 0$. Uniformly for $n \geq 1$ and $x \in I$,*

$$\|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \sim |p_n(y_{jn})|, \tag{3.5}$$

*where $y_{jn} \in [x - \frac{A}{n}, x + \frac{A}{n}]$ or is the closest zero of $p_n'$ to this interval.*

*Proof.* (a) First note that uniformly for $y_{jn} \in I$ and $z$ in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{p_n(y_{jn} + \frac{z}{n\omega(y_{jn})})}{p_n(y_{jn})} = \cos \pi z. \tag{3.6}$$

This was proved in [9] and is Theorem A above. Next, [18, Theorem 2.1] shows that (3.2) holds uniformly for $x_{jn} \in I$. In particular, $x_{jn} - x_{j+1,n} = O(\frac{1}{n})$ uniformly for $x_{jn} \in I$. Write

$$x_{jn} = y_{jn} + \frac{z_n}{n\omega(y_{jn})},$$

so that $z_n > 0$ and $z_n = O(1)$. From (3.6), we have

$$0 = \frac{p_n(x_{jn})}{p_n(y_{jn})} = \cos \pi z_n + o(1)$$

so necessarily for some non-negative integer $j_n$,

$$z_{jn} = j_n + \frac{1}{2} + o(1).$$

If $j_n \geq 1$ for infinitely many $n$, then Hurwitz' theorem shows that there would be other zeros of $p_n$ between $x_{jn}$ and $y_{jn}$, which contradicts that $y_{jn} \in (x_{j+1,n}, x_{jn})$.

So, $j_n = 0$ for $n$ large enough, which gives the first limit in (3.1). Note too that $\omega(x_{jn})/\omega(y_{jn}) = 1 + o(1)$ by continuity of $\omega$. The second is similar. Both (3.2) and (3.3) follow from (3.1), though as noted, (3.2) appears in [18].

(b) Because of (3.6), we can apply Theorem 1.3 and results from its proof. In that theorem, we set $x_n = y_{jn}$, $\tau_n = \frac{1}{\omega(y_{jn})}$; $\xi_n = y_{jn}$; so that $\xi_{n-1} = y_{j,n-1}$. Note that (1.13), (1.14), and (1.16) are satisfied because of the spacing estimates in Lemma 3.1, and the continuity of $\omega$. From (2.7),

$$\frac{1}{n\omega(y_{jn})}K_n(y_{jn}, y_{jn})$$
$$= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}(y_{j,n-1})p_n(y_{jn})\{\sin[\pi n\omega(y_{jn})(y_{j,n-1} - y_{jn})] + o(1)\}. \quad (3.7)$$

Next, [18, Theorem 2.2] establishes that uniformly for $t \in I$,

$$\lim_{n\to\infty}\frac{1}{n}K_n(t,t)\mu'(t) = \omega(t).$$

Since $\omega$ is positive and continuous in $I$ as is $\mu'$, we then obtain (3.4) from (3.7).

(c) This follows directly from the limit in (3.6) and the fact that $|p_n(y_{jn})|$ is the maximum of $|p_n|$ in $[x_{j+1,n}, x_{jn}]$. ■

*Proof that Theorem 1.1* (a) $\iff$ (b). This follows directly from Lemma 3.1.a). ■

*Proof that Theorem 1.1* (b) $\implies$ (c). First note that as supp[$\mu$] is compact [5, p. 41],

$$\frac{\gamma_{n-1}}{\gamma_n} \leq C. \quad (3.8)$$

Our hypothesis (1.2), as well as (3.4) give that uniformly for $y_{jn} \in I$,

$$\frac{\gamma_{n-1}}{\gamma_n}|p_{n-1}(y_{j,n-1})p_n(y_{jn})| \sim 1. \quad (3.9)$$

Then (3.5) gives uniformly for $x \in I$,

$$\frac{\gamma_{n-1}}{\gamma_n}\|p_{n-1}\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]}\|p_n\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \sim 1. \quad (3.10)$$

Let $I_{jn} = [y_{j+1,n}, y_{jn}]$ for all $j, n$. We similarly obtain from (3.6) and (3.9) and our spacing that

$$\frac{\gamma_{n-1}}{\gamma_n}\left(\int_{I_{j,n-1}} p_{n-1}^2\, d\mu\right)^{1/2}\left(\int_{I_{jn}} p_n^2\, d\mu\right)^{1/2} \geq \frac{C}{n}.$$

Here we are also using that $\mu'$ is positive and continuous in $I$. Adding over $y_{jn} \in I$, and using that there are necessarily $\geq Cn$ such $y_{jn}$, because of the spacing, we obtain

$$\frac{\gamma_{n-1}}{\gamma_n} \sum_{y_{jn} \in I} \left( \int_{I_{j,n-1}} p_{n-1}^2 \, d\mu \right)^{1/2} \left( \int_{I_{jn}} p_n^2 \, d\mu \right)^{1/2} \geq C.$$

Cauchy–Schwarz' inequality gives

$$\frac{\gamma_{n-1}}{\gamma_n} \left( \int p_{n-1}^2 d\mu \int p_n^2 \, d\mu \right)^{1/2} \geq C$$

so that

$$\frac{\gamma_{n-1}}{\gamma_n} \geq C.$$

Together with (3.8), this gives

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} \sim 1. \tag{3.11}$$

So, from (3.10), uniformly in $x \in I$,

$$\|p_{n-1}\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \sim 1. \qquad \blacksquare$$

*Proof that Theorem* 1.1 (c) $\implies$ (d). This is immediate. ■

*Proof that Theorem* 1.1 (d) $\implies$ (b). From (3.4), (3.11), and our bound (1.4),

$$|\sin[\pi n \omega(y_{jn})(y_{j,n-1} - y_{jn})] + o(1)| \geq C.$$

This yields

$$\operatorname{dist}\big(n\omega(y_{jn})(y_{jn} - y_{j,n-1}), \mathbb{Z}\big) \geq C. \qquad \blacksquare$$

*Proof of the bound* (1.5). From the recurrence relation and (3.11),

$$\begin{aligned} &\|(x - b_n) p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \\ &\leq C(\|p_{n+1}\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \\ &\qquad + \|p_{n-1}\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \|p_n\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]}) \leq C, \end{aligned}$$

by (1.4). Then also uniformly in $x \in I$,

$$\|(x - b_n) p_n^2\|_{L_\infty[x-\frac{A}{n}, x+\frac{A}{n}]} \leq C$$

and we obtain (1.5). ■

## 4. Proof of Theorem 1.2

We begin with:

**Lemma 4.1.** *Let $\mu$ be a regular measure on $\mathbb{R}$ with compact support. Let $I$ be a closed subinterval of the support and assume that in some open interval containing $I$, $\mu$ is absolutely continuous, while $\mu'$ is positive and continuous. Assume* (1.1). *Let $A > 0$.*

a. *Let $L \geq 1$. There exists $n_0$ such that, uniformly for $n \geq n_0$, for $x_{jn} \in I$, and $|k - j| \leq L$,*

$$\operatorname{dist}\big(n\omega(x_{jn})(x_{k,n-1} - x_{jn}), \mathbb{Z}\big) \geq C. \tag{4.1}$$

b. *Let*

$$\delta_{jn} := n\omega(x_{jn})(x_{jn} - x_{j-1,n-2}). \tag{4.2}$$

*There exist $n_0, \eta_0 > 0$ such that, uniformly for $n \geq n_0$, and for $x_{jn} \in I$,*

$$|\delta_{jn}| \leq 1 - \eta_0. \tag{4.3}$$

c. *There exist $n_0, C_1 > 0$ such that, uniformly for $n \geq n_0$ and for $x_{jn} \in I$, we have*

$$|x_{jn} - b_{n-1}| \sim \|p_{n-2}\|^2_{L_\infty[x_{jn}-\frac{A}{n}, x_{jn}+\frac{A}{n}]} |\delta_{jn}|. \tag{4.4}$$

*Here if $x_{jn} - b_{n-1} = 0$, both sides are* 0.

*Proof.* (a) Using the spacing (3.2),

$$\operatorname{dist}\big(n\omega(x_{jn})(x_{k,n-1} - x_{jn}), \mathbb{Z}\big) = \operatorname{dist}\big(n\omega(x_{jn})(x_{j,n-1} - x_{jn}), \mathbb{Z}\big) + o(1)$$

so (1.1) gives the result.

(b) The interlacing of the zeros of successive orthogonal polynomials shows that both $x_{jn}$ and $x_{j-1,n-2}$ lie in the interval $(x_{j,n-1}, x_{j-1,n-1})$. Even more, the bounds given in (a) show that for $n$ large enough, both $x_{jn}$ and $x_{j-1,n-2}$ lie in the interval $(x_{j,n-1} + \frac{C_1}{n\omega(x_{jn})}, x_{j-1,n-1} - \frac{C_1}{n\omega(x_{jn})})$ for some $C_1 > 0$. Then

$$\begin{aligned} |\delta_{jn}| &= |n\omega(x_{jn})(x_{jn} - x_{j-1,n-2})| \\ &\leq n\omega(x_{jn})(x_{j,n-1} - x_{j+1,n-1}) - 2C_1 = 1 - 2C_1 + o(1), \end{aligned}$$

by (3.2).

(c) From the recurrence relation,

$$(x_{jn} - b_{n-1})p_{n-1}(x_{jn}) = a_{n-1}p_{n-2}(x_{jn}). \tag{4.5}$$

We now examine the behavior of the left and right-hand side as $n \to \infty$. By (3.1) to (3.3), the local asymptotic (3.6), and the fact that $x_{jn} - y_{j,n-1} = O(\frac{1}{n})$,

$$\begin{aligned}\frac{p_{n-1}(x_{jn})}{p_{n-1}(y_{j,n-1})} &= \cos\pi(n\omega(y_{j,n-1})(x_{jn} - y_{j,n-1})) + o(1)\\ &= \cos\pi(n\omega(y_{j,n-1})(x_{jn} - x_{j,n-1} + x_{j,n-1} - y_{j,n-1})) + o(1)\\ &= \cos\pi(n\omega(y_{j,n-1})(x_{jn} - x_{j,n-1}) + \frac{1}{2}) + o(1)\\ &= -\sin\pi(n\omega(y_{j,n-1})(x_{jn} - x_{j,n-1})) + o(1)\end{aligned}$$

so using our original condition (1.1), we obtain for some threshold $n_0$ that is independent of $j$, and for $n \geq n_0$,

$$|p_{n-1}(x_{jn})| \sim |p_{n-1}(y_{j,n-1})|. \tag{4.6}$$

Next, in analyzing the term on the right in (4.5), we use the differentiated form of (3.6): uniformly for $y_{jn} \in I$ and $z$ in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{p_n'(y_{jn} + \frac{z}{n\omega(y_{jn})})}{n\omega(y_{jn})p_n(y_{jn})} = -\pi \sin \pi z. \tag{4.7}$$

Then noting that we can replace $n$ by $n \pm 2$ in the term involving $z$, we see that

$$\begin{aligned}&\frac{p_{n-2}(x_{jn})}{p_{n-2}(y_{j-1,n-2})}\\ &= \int_{(x_{j-1,n-2}-y_{j-1,n-2})n\omega(y_{j-1,n-2})}^{(x_{jn}-y_{j-1,n-2})n\omega(y_{j-1,n-2})} \frac{p_{n-2}'(y_{j-1,n-2} + \frac{t}{n\omega(y_{j-1,n-2})})}{n\omega(y_{j-1,n-2})p_{n-2}(y_{j-1,n-2})} dt\\ &= \int_{(x_{j-1,n-2}-y_{j-1,n-2})n\omega(y_{j-1,n-2})}^{(x_{jn}-y_{j-1,n-2})n\omega(y_{j-1,n-2})} (-\pi \sin \pi t + o(1)) dt.\end{aligned}$$

Here the lower limit of integration is

$$(x_{j-1,n-2} - y_{j-1,n-2})n\omega(y_{j-1,n-2}) = \frac{1}{2} + o(1),$$

(by (3.1)), so we can continue the above as

$$\begin{aligned}&\frac{p_{n-2}(x_{jn})}{p_{n-2}(y_{j-1,n-2})}\\ &= \int_0^{(x_{jn}-x_{j-1,n-2})n\omega(y_{j-1,n-2})} \left(-\pi \sin\left(\pi\left(t + \frac{1}{2}\right)\right) + o(1)\right) dt + o(\delta_{jn})\end{aligned}$$

$$= \int_0^{(x_{jn}-x_{j-1,n-2})n\omega(y_{j-1,n-2})} \big(-\pi \cos \pi t + o(1)\big)dt + o(\delta_{jn})$$

$$= -\sin \pi\delta_{jn} + o(\delta_{jn}).$$

Here we are also using that $\omega(y_{j-1,n-2})/\omega(x_{jn}) \to 1$ as $n \to \infty$ by continuity of $\omega$ in the interior of $I$. Next, from (b), $|\delta_{jn}| \leq 1-\varepsilon$, so $|\sin \pi\delta_{jn}| \sim |\delta_{jn}|$ and we can continue this as

$$\frac{p_{n-2}(x_{jn})}{p_{n-2}(y_{j-1,n-2})} = -(\sin \pi\delta_{jn})(1+o(1)).$$

It is possible here that $\delta_{jn} = 0$, but in such a case both sides are 0. Combining this with (4.5), (4.6), and (3.11) gives uniformly in $j$ and $n$, for $n \geq n_0$,

$$|x_{jn} - b_{n-1}||p_{n-1}(y_{j,n-1})| \sim |p_{n-2}(y_{j-1,n-2})||\sin \pi\delta_{jn}| \sim |p_{n-2}(y_{j-1,n-2})||\delta_{jn}|.$$

Here by our local limits and (1.3),

$$|p_{n-1}(y_{j,n-1})| = \|p_{n-1}\|_{L_\infty[x_{j+1,n-1},x_{j,n-1}]} \sim \|p_{n-2}\|^{-1}_{L_\infty[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}]}.$$

A related assertion holds for $p_{n-2}(y_{j-1,n-2})$. We deduce that

$$|x_{jn} - b_{n-1}| \sim \|p_{n-2}\|^2_{L_\infty[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}]}|\delta_{jn}|.$$

Again, if $x_{jn} = b_{n-1}$, $\delta_{jn} = 0$. ■

*Proof that Theorem* 1.2 (a) $\iff$ (c). If first (1.6) holds, then $|\delta_{jn}| \geq C|x_{jn} - b_{n-1}|$ and (4.4) gives

$$C|\delta_{jn}| \geq \|p_{n-2}\|^2_{L_\infty[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}]}|\delta_{jn}|,$$

which forces

$$\|p_{n-2}\|^2_{L_\infty[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}]} \leq C_1,$$

uniformly in $x_{jn} \in I$, provided no $\delta_{jn} = 0$. Since $\delta_{jn} = 0$ can occur for at most one $j$, namely when $x_{jn} = b_{n-1}$ (as follows from the recurrence relation), that exceptional interval can be covered by others with $A$ large enough. So, we have (1.8).

Conversely, suppose we have (1.8). Then from (4.4),

$$|x_{jn} - b_{n-1}| \leq C|\delta_{jn}|,$$

so that we have (1.6). ■

*Proof that Theorem 1.2* (b) $\iff$ (c). It is immediate that (b) $\implies$ (c). For the converse we note that if (c) holds, then from Theorem 1.1 (c),

$$\|p_{n-1}\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \geq C$$

uniformly for $x \in I$. This together with (1.8), gives (1.7). ∎

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