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Carleman estimate for complex second order elliptic operators with discontinuous Lipschitz coefficients

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Abstract. In this paper, we derive a local Carleman estimate for the complex second order elliptic operator with Lipschitz coefficients having jump discontinuities. Combing the result by M. Bellassoued and J. Le Rousseau (2018) and the arguments by M. Di Cristo, E. Francini, C.-L. Lin, S. Vessella, and J.-N. Wang (2017), we present an elementary method to derive the Carleman estimate under the optimal regularity assumption on the coefficients.

1. Introduction

Carleman estimates are important tools for proving the unique continuation property for partial differential equations. Additionally, Carleman estimates have been successfully applied to study inverse problems and controllability of partial differential equations. Most of Carleman estimates are proved under the assumption that the leading coefficients possess certain regularity. For example, for general second order elliptic operators, Carleman estimates were proved when the leading coefficients are at least Lipschitz [9]. In general, the Lipschitz regularity assumption is the optimal condition for the unique continuation property to hold in $\mathbb{R}^n$ with $n \geq 3$ (see counterexamples constructed by Pliś [17] and Miller [16]). Therefore, Carleman estimates for second order elliptic operators with general discontinuous coefficients are most likely not valid. Nonetheless, recently, in the case of coefficients having jump discontinuities at an interface with homogeneous or non-homogeneous transmission conditions, one can still prove useful Carleman estimates, see, for example, Le Rousseau and Robbiano [14, 15], Le Rousseau and Lerner [13], and [5].

Above mentioned results are proved for real coefficients. In many real world problems, the case of complex-valued coefficients arises naturally. The modeling of the current flows in biological tissues or the propagation of the electromagnetic waves in conductive media are typical examples. In the former case, the electric current is due

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to the ions in intracellular and extracellular fluids. The intracellular ions are separated by cell membranes which act like capacitors. When a constant voltage (DC) is applied, a conductance current I_c flows through the extracellular fluid. At the same time, a constant amount of charges will be stored in the cell membrane. If now an alternating voltage (AC) is applied, the charges stored in the cell membranes will change according to the frequency, resulting in a displacement current I_d through the intracellular fluid. From Ohm's law, I_c depends on the material's conductivity σ while I_d depends on its permittivity ε. Due to the 90 degrees phase difference, we see that the total current $I = I_c + I_d$ is proportional to the product of the complex-valued admittivity $\sigma + i\omega\varepsilon$ and the applied voltage, where $\omega = 2\pi f$ and f is the frequency of the applied signal. On the other hand, in some situations, the conductivities are not continuous functions. For instance, in the human body, different organs have different conductivities. Therefore, to model the current flow in the human body, it is more reasonable to consider an anisotropic complex-valued conductivity with jump-type discontinuities [11].

The electromagnetic waves propagating in conductive media is modeled by the time-harmonic Maxwell equations:

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\epsilon E + \sigma E \quad \text{in } \Omega, \tag{1.1}$$

where μ is the magnetic permeability, ϵ is the electric permittivity, and σ is the conductivity. Assume that μ is small. Then the first equation of (1.1) can be approximated by $\nabla \times E = 0$ in Ω. From this, there exists a potential function u such that $E = \nabla u$ provided Ω is simply connected. Taking the divergence of the second equation of (1.1) and using $E = \nabla u$ implies

$$\operatorname{div}((\sigma - i\omega\epsilon)\nabla u) = 0 \quad \text{in } \Omega,$$

which is an elliptic equation with complex-valued coefficients. Furthermore, if the properties of the medium (ϵ, σ) vary sharply across an interface in Ω, then it is legitimate to consider discontinuous coefficients $\sigma - i\omega\epsilon$.

With potential applications in mind, our goal in this paper is to derive a Carleman estimate for the second order elliptic equations with complex-valued leading coefficients having jump-type discontinuities. Although such a Carleman estimate has been derived in [2], we want to remark that the method used in [2, 13–15] are based on the technique of pseudodifferential operators and hence requires C^∞ coefficients and interface; while the method in [5] (and its parabolic counterpart, [7]) relies on the Fourier transform and a version of partition of unity which requires only Lipschitz coefficients and $C^{1,1}$ interface. Hence, the main purpose of the paper is to extend the method in [5, 7] to second order elliptic operators with complex-valued coefficients. It is important to point out that even though second order elliptic operators with complex-valued coefficients can be written as a coupled second order elliptic system with real coefficients, neither the method in [13–15] nor that in [5] can be applied to coupled elliptic systems. Therefore, we need to work on operators with complex-valued coefficients directly.

Our strategy to derive the Carleman estimate consists of two major steps. In the first step, we treat second order elliptic operators with constant complex coefficients. Based on [2], by checking the strong pseudoconvexity and the transmission conditions in a neighborhood of a fixed point at the interface, we can derive a Carleman estimate for second order elliptic operators with constant complex coefficients from [2, Theorem 1.6]. We will pay more attention to the verification of the transmission conditions at the interface. Such transmission conditions at the interface are the Lopatinskii-type conditions for the conjugate operator and the transmission operators at the interface, which an extension of the Lopatinskii-type conditions corresponding to the conjugate operator and the boundary operators [1, 18]. Note that the result in [2] is stated for quite general complex coefficients and the corresponding weight function such that the definition of the transmission condition [2, Definition 1.4] holds. However, no explicit form of weight function is given. In this work, one of the motivations in proving such a Carleman estimate is to derive the propagation of smallness across the interface. Having established the propagation of smallness, we can study some interesting applied questions such as inverse problems. Therefore, we need an explicit form of weight function and check the transmission condition for such weight function. To ensure the transmission condition, we are led to bound the size of the imaginary parts of the complex coefficients. In the second step, we extend the Carleman estimate to the operator with non-constant complex coefficients with small imaginary parts. This method in this step is taken from the argument in [5, Section 4]. The key tool is a version of partition of unity.

Furthermore, in the second step, we need an interior Carleman estimate for second order elliptic operators having Lipschitz leading coefficients and with the weight function ψ_ε. An interior Carleman estimate was proved in [8, Theorem 8.3.1], but for operators with C^1 leading coefficients. Another interior estimate was established in [9, Proposition 17.2.3] for operators with Lipschitz leading coefficients, but with a different weight function. Hörmander remarked in [10, p. 703, lines 7–8] that "inspection of proof of [8, Theorem 8.3.1] shows that only Lipschitz continuity was actually used in the proof." But, as far as we can check, there is no formal proof of this statement in literature. To make the paper self contained, we would like give a detailed proof of interior Carleman estimate for second order elliptic operator with Lipschitz leading coefficients and with a rather general weight function, see Proposition 4.1. This interior Carleman estimate may be useful on other occasions. We also refer to [12] for other related results on the interior Carleman estimates with nearly optimal assumptions on coefficients.

In this paper, we present a detailed and elementary derivation of the Carleman estimate for the second order elliptic equations with complex-valued coefficients having jump-type discontinuities following our method in [5]. Having established the Carleman estimate, we then can apply the ideas in [6] to prove a three-region inequality and those in [4] to prove a three-ball inequality across the interface. With the help of the three-ball inequality, we can study the size estimate problem for the complex conductivity equation following the ideas in [3]. We will present these quantitative uniqueness results and the application to the size estimate in the forthcoming paper.

The paper is organized as follows. In Section 2, we introduce notations that will be used in the paper and the statement of the theorem. In Section 3, we derive a Carleman estimate for the operator having discontinuous piecewise constant coefficients. This Carleman estimate is a special case of [2, Theorem 1.6]. Therefore, the main task of Section 3 is to check the transmission condition and the strong pseudoconvexity condition. Finally, the main Carleman estimate is proved in Section 4. The key ingredient is a partition of unity introduced in [5].

2. Notations and statement of the main theorem

We will state and prove the Carleman estimate for the case where the interface is flat. Since our Carleman estimate is local near any point at the interface, for general $C^{1,1}$ interface, it can be flatten by a suitable change of coordinates. Moreover, the transformed coefficients away from the interface remain Lipschitz. Define $H_\pm = \chi_{\mathbb{R}^n_\pm}$ where $\mathbb{R}^n_\pm = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} | x_n \gtrless 0\}$ and $\chi_{\mathbb{R}^n_\pm}$ is the characteristic function of $\mathbb{R}^n_\pm$. In places we will use equivalently the symbols ∂, ∇ and $D = -i\nabla$ to denote the gradient of a function and we will add the index x' or x_n to denote gradient in $\mathbb{R}^{n-1}$ and the derivative with respect to x_n respectively. We further denote $\partial_\ell = \partial/\partial x_\ell$, $D_\ell = -i\partial_\ell$, and $\partial_{\xi_\ell} = \partial/\partial\xi_\ell$.

Let $u_\pm \in C^\infty(\mathbb{R}^n)$. We define

$$u = H_+u_+ + H_-u_- = \sum_\pm H_\pm u_\pm,$$

hereafter, we denote $\sum_\pm a_\pm = a_+ + a_-$, and

$$\mathcal{L}(x, D)u := \sum_\pm H_\pm \operatorname{div}(A_\pm(x)\nabla u_\pm), \tag{2.1}$$

where

$$A_\pm(x) = \{a^\pm_{\ell j}(x)\}^n_{\ell,j=1} = \{a^\pm_{\ell j}(x', x_n)\}^n_{\ell,j=1}, \quad x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R} \tag{2.2}$$

is a Lipschitz symmetric matrix-valued function. Assume that

$$a^\pm_{\ell j}(x) = a^\pm_{j\ell}(x), \quad \text{for all } \ell, j = 1, \dots, n, \tag{2.3}$$

and furthermore

$$a_{\ell j}^{\pm}(x) = M_{\ell j}^{\pm}(x) + i\gamma N_{\ell j}^{\pm}(x), \tag{2.4}$$

where $(M_{\ell j}^{\pm})$ and $(N_{\ell j}^{\pm})$ are real-valued matrices and $\gamma > 0$. We further assume that there exist $\lambda_0, \Lambda_0 > 0$ such that for all $\xi \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we have

$$\lambda_0|\xi|^2 \le M^{\pm}(x)\xi \cdot \xi \le \Lambda_0|\xi|^2 \tag{2.5}$$

and

$$\lambda_0|\xi|^2 \le N^{\pm}(x)\xi \cdot \xi \le \Lambda_0|\xi|^2. \tag{2.6}$$

In the paper, we consider Lipschitz coefficients $A_{\pm}$, i.e., there exists a constant $M_0 > 0$ such that

$$|A_{\pm}(x) - A_{\pm}(y)| \le M_0|x - y|. \tag{2.7}$$

To treat the transmission conditions, for all $x' \in \mathbb{R}^{n-1}$, we write

$$h_0(x') := u_+(x', 0) - u_-(x', 0), \tag{2.8}$$

$$h_1(x') := A_+(x', 0)\nabla u_+(x', 0) \cdot \nu - A_-(x', 0)\nabla u_-(x', 0) \cdot \nu, \tag{2.9}$$

where $\nu = e_n$.

Let us now introduce the weight function. Let φ be

$$\varphi(x_n) = \begin{cases} \varphi_+(x_n) := \alpha_+ x_n + \beta x_n^2/2, & x_n \ge 0, \\ \varphi_-(x_n) := \alpha_- x_n + \beta x_n^2/2, & x_n < 0, \end{cases} \tag{2.10}$$

where α_+, α_- and β are positive numbers which will be determined later. In what follows we denote by φ_+ and φ_- the restriction of the weight function φ to $[0, +\infty)$ and to $(-\infty, 0)$ respectively. We use similar notation for any other weight functions. For any $\varepsilon > 0$ let

$$\psi_\varepsilon(x) := \varphi(x_n) - \frac{\varepsilon}{2}|x'|^2, \tag{2.11}$$

and let

$$\phi_\delta(x) := \psi_\delta(\delta^{-1}x), \quad \delta > 0. \tag{2.12}$$

For a function $h \in L^2(\mathbb{R}^n)$, we define

$$\hat{h}(\xi', x_n) = \int_{\mathbb{R}^{n-1}} h(x', x_n) e^{-ix'\cdot\xi}\, dx', \quad \xi' \in \mathbb{R}^{n-1}.$$

As usual we denote by $H^{1/2}(\mathbb{R}^{n-1})$ the space of the functions $f \in L^2(\mathbb{R}^{n-1})$ satisfying

$$\int_{\mathbb{R}^{n-1}} |\xi'||\hat{f}(\xi')|^2\, d\xi' < \infty,$$

with the norm

$$\|f\|^2_{H^{1/2}(\mathbb{R}^{n-1})} = \int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{1/2}|\hat{f}(\xi')|^2\,d\xi'. \tag{2.13}$$

Moreover, we define

$$[f]_{1/2,\mathbb{R}^{n-1}} = \left[\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|f(x)-f(y)|^2}{|x-y|^n}\,dy\,dx\right]^{1/2},$$

and recall that there is a positive constant C, depending only on n, such that

$$C^{-1}\int_{\mathbb{R}^{n-1}} |\xi'||\hat{f}(\xi')|^2\,d\xi' \le [f]^2_{1/2,\mathbb{R}^{n-1}} \le C\int_{\mathbb{R}^{n-1}} |\xi'||\hat{f}(\xi')|^2\,d\xi',$$

so that the norm (2.13) is equivalent to the norm $\|f\|_{L^2(\mathbb{R}^{n-1})} + [f]_{1/2,\mathbb{R}^{n-1}}$. We use the letters $C, C_0, C_1, \ldots$ to denote constants. The value of the constants may change from line to line, but it is always greater than 1.

We will denote by $B'_r(x')$ the $(n-1)$-ball centered at $x' \in \mathbb{R}^{n-1}$ with radius $r > 0$. Whenever $x' = 0$ we denote $B'_r = B'_r(0)$. Likewise, we denote $B_r(x)$ be the n-ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ and $B_r = B_r(0)$.

Theorem 2.1. *Let u and $A_\pm(x)$ satisfy (2.1)-(2.9). There exist $\alpha_+, \alpha_-, \beta, \delta_0, r_0, \gamma_0$ and C depending on $\lambda_0, \Lambda_0, M_0$ such that if $\gamma < \gamma_0$, $\delta \le \delta_0$ and $\tau \ge C$, then*

$$\begin{aligned}
&\sum_\pm \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^n_\pm} |D^k u_\pm|^2 e^{2\tau\phi_{\delta,\pm}(x',x_n)}\,dx'\,dx_n \\
&\quad + \sum_\pm \sum_{k=0}^{1} \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k u_\pm(x',0)|^2 e^{2\phi_\delta(x',0)}\,dx' \\
&\quad + \sum_\pm \tau^2 [e^{\tau\phi_\delta(\cdot,0)} u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_\pm [D(e^{\tau\phi_{\delta,\pm}} u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\
&\quad \le C\Bigg(\sum_\pm \int_{\mathbb{R}^n_\pm} |\mathcal{L}(x,D)(u_\pm)|^2 e^{2\tau\phi_{\delta,\pm}(x',x_n)}\,dx'\,dx_n \\
&\qquad + [e^{\tau\phi_\delta(\cdot,0)} h_1]^2_{1/2,\mathbb{R}^{n-1}} + [D_{x'}(e^{\tau\phi_\delta} h_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\
&\qquad + \tau^3 \int_{\mathbb{R}^{n-1}} |h_0|^2 e^{2\tau\phi_\delta(x',0)}\,dx' + \tau \int_{\mathbb{R}^{n-1}} |h_1|^2 e^{2\tau\phi_\delta(x',0)}\,dx'\Bigg).
\end{aligned} \tag{2.14}$$

where $u = H_+u_+ + H_-u_-$, $u_\pm \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} u \subset B'_{\delta r_0} \times [-\delta r_0, \delta r_0]$*, and ϕ_δ is given by (2.12).*

Remark 2.2. Estimate (2.14) is a local Carleman estimate near $x_n = 0$. As mentioned above, by flattening the interface, we can derive a local Carleman estimate near a $C^{1,1}$ interface from (2.14). Nonetheless, an estimate like (2.14) is sufficient for some applications such as the inverse problem of estimating the size of an inclusion by one pair of boundary measurement (see, for example, [6]).

3. Carleman estimate for operators with constant coefficients

The purpose of this section is to derive (2.14) for $\mathcal{L}(x, D)$ with discontinuous piecewise constant coefficients. More precisely, we derive (2.14) for $\mathcal{L}_0(D)$, where $\mathcal{L}_0(D)$ is obtained from $\mathcal{L}(x, D)$ by freezing the variable x at $(x_0', 0)$. Without loss of generality, we take $(x_0', 0) = (0, 0) = 0$ and thus

$$\mathcal{L}_0(D)u = \mathcal{L}(0, D)u = \sum_{\pm} H_\pm \operatorname{div}(A_\pm(0)\nabla u_\pm).$$

Since $\mathcal{L}_0$ has piecewise constant coefficients, to prove (2.14), we will apply [2, Theorem 1.6]. So, the task here is to verify the strong pseudoconvexity and transmission conditions for operator $\mathcal{L}_0$ with the weight function given in (2.11).

To streamline the presentation, we define $\Omega_1 := \{x_n < 0\}$, $\Omega_2 := \{x_n > 0\}$. On each side of the interface, we have complex second order elliptic operators. We denote

$$P_k = \sum_{1\le j,\ell\le n} a_{\ell j}^{(k)} D_\ell D_j, \quad k = 1, 2,$$

where $a_{\ell j}^{(1)} = a_{\ell j}^-$ and $a_{\ell j}^{(2)} = a_{\ell j}^+$. Here we denote $a_{\ell j}^{(k)} = a_{\ell j}^{(k)}(0)$. The principal symbol of P_k is denoted by

$$p_k(\xi) = \sum_{1\le j,\ell\le n} a_{\ell j}^{(k)} \xi_\ell \xi_j.$$

Corresponding to (2.3)–(2.6), we have

$$a_{\ell j}^{(k)} = a_{j\ell}^{(k)}, \tag{3.1}$$

$$a_{\ell j}^{(k)} = M_{\ell j}^{(k)} + i\gamma N_{\ell j}^{(k)}, \tag{3.2}$$

$$\lambda_0|\xi|^2 \le M^{(k)}\xi\cdot\xi \le \Lambda_0|\xi|^2, \tag{3.3}$$

$$\lambda_0|\xi|^2 \le N^{(k)}\xi\cdot\xi \le \Lambda_0|\xi|^2. \tag{3.4}$$

Since some computations in the verification of the transmission conditions are useful in proving the strong pseudoconvexity condition, we will begin with the discussion of the transmission conditions at the interface $\{x_n = 0\}$.

3.1. Transmission conditions

We consider the natural transmission conditions that use the interface operators

$$T_k^1 = (-1)^k, \quad T_k^2 = (-1)^k \sum_{1\le j\le n} a_{nj}^{(k)} D_j$$

that correspond to the continuity of the solution and of the normal flux, respectively. We now write the weight function

$$\psi_\varepsilon(x) = \varphi(x_n) - \frac{\varepsilon}{2}|x'|^2, \tag{3.5}$$

where

$$\varphi(x_n) = \begin{cases} \varphi_1(x_n), & x_n < 0, \\ \varphi_2(x_n), & x_n \ge 0, \end{cases}$$

and

$$\varphi_k(x_n) = \alpha_k x_n + \frac{1}{2}\beta x_n^2$$

with $\alpha_1, \alpha_2 > 0$ (corresponding to α_- and α_+ in (2.10), respectively) and $\beta > 0$. Notice that φ is smooth in Ω_1, Ω_2 and is continuous across the interface. Then we have

$$\nabla\psi_\varepsilon(0) = \begin{cases} (0,\dots,0,\alpha_1), & x_n < 0 \\ (0,\dots,0,\alpha_2), & x_n \ge 0. \end{cases}$$

As mentioned in the introduction, the transmission conditions considered here are Lopatinkii-type conditions at the interface associated with the conjugate operators $e^{\tau\psi_\varepsilon} P_k(e^{-\tau\psi_\varepsilon}\cdot)$ and the interface operators T_k^1, T_k^2. Roughly speaking, these conditions guarantee the coercivity of the system $\{e^{\tau\psi_\varepsilon} P_k(e^{-\tau\psi_\varepsilon}\cdot), T_k^1, T_k^2\}$. Before stating the formal definition of the transmission conditions [2, Definition 1.4], we will follow the notations and the calculations used in [2, Section 1.7.1]. Let us first denote $\omega := (0, \xi', \nu, \tau)$ with $\xi' = (\xi_1, \dots, \xi_{n-1}) \ne 0$, $\nu = e_n$ and $\lambda \in \mathbb{C}$,

$$\tilde{t}^1_{k,\psi_\varepsilon}(\omega,\lambda) = (-1)^k$$

and

$$\begin{aligned}\tilde{t}^2_{k,\psi_\varepsilon}(\omega,\lambda) &= (-1)^k a_{nn}^{(k)}((-1)^k\lambda + i\tau\partial_{x_n}\psi_\varepsilon(0)) \\ &\quad + (-1)^k \sum_{1\le j\le n-1} a_{nj}^{(k)}(\xi_j + i\tau\partial_{x_j}\psi_\varepsilon(0)) \\ &= (-1)^k a_{nn}^{(k)}((-1)^k\lambda + i\tau\alpha_k) + (-1)^k \sum_{1\le j\le n-1} a_{nj}^{(k)}\xi_j.\end{aligned}$$

The principal symbols of P_k, $k = 1, 2$, can be written as

$$p_k(\xi) = a_{nn}^{(k)}\Big(\Big(\xi_n + \sum_{1\le j\le n-1} \frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_j\Big)^2 + b_k(\xi')\Big), \tag{3.6}$$

where

$$b_k(\xi') = (a_{nn}^{(k)})^{-2}\sum_{1\le \ell, j\le n-1}(a_{\ell j}^{(k)}a_{nn}^{(k)} - a_{n\ell}^{(k)}a_{nj}^{(k)})\xi_\ell\xi_j. \tag{3.7}$$

We also need to introduce the principal symbol of the conjugate operators

$$\begin{aligned}\tilde{p}_{k,\psi_\varepsilon}(\omega,\lambda) &= a_{nn}^{(k)}\Big[\Big((-1)^k\lambda + i\tau\partial_{x_n}\psi_\varepsilon(0) + \sum_{1\le j\le n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}(\xi_j + i\tau\partial_{x_j}\psi_\varepsilon(0))\Big)^2 \\ &\qquad + b_k(\xi' + i\tau\partial_{x'}\psi_\varepsilon(0))\Big] \\ &= a_{nn}^{(k)}\Big[\Big((-1)^k\lambda + i\tau\alpha_k + \sum_{1\le j\le n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_j\Big)^2 + b_k(\xi')\Big].\end{aligned} \tag{3.8}$$

Let us introduce $A^{(k)}, B^{(k)} \in \mathbb{R}$ for $k = 1, 2$ such that

$$b_k(\xi') = (a_{nn}^{(k)})^{-2}\sum_{1\le \ell, j\le n-1}(a_{\ell j}^{(k)}a_{nn}^{(k)} - a_{n\ell}^{(k)}a_{nj}^{(k)})\xi_\ell\xi_j = (A^{(k)} - iB^{(k)})^2, \tag{3.9}$$

where $A^{(k)} \ge 0$. We also denote

$$\sum_{1\le j\le n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_j = E^{(k)} + iF^{(k)}, \tag{3.10}$$

where $E^{(k)}, F^{(k)} \in \mathbb{R}$. Using (3.8), (3.9), and (3.10), we can write

$$\begin{aligned}\tilde{p}_{2,\psi_\varepsilon} &= a_{nn}^{(2)}[(\lambda + i\tau\alpha_2 + E^{(2)} + iF^{(2)})^2 + (A^{(2)} - iB^{(2)})^2] \\ &= a_{nn}^{(2)}[(\lambda + i\tau\alpha_2 + E^{(2)} + iF^{(2)} + i(A^{(2)} - iB^{(2)})) \\ &\quad \cdot (\lambda + i\tau\alpha_2 + E^{(2)} + iF^{(2)} - i(A^{(2)} - iB^{(2)}))] \\ &= a_{nn}^{(2)}(\lambda - \sigma_1^{(2)})(\lambda - \sigma_2^{(2)}),\end{aligned}$$

where

$$\begin{aligned}\sigma_1^{(2)} &= -E^{(2)} - B^{(2)} - i(\tau\alpha_2 + F^{(2)} + A^{(2)}), \\ \sigma_2^{(2)} &= -E^{(2)} + B^{(2)} - i(\tau\alpha_2 + F^{(2)} - A^{(2)}).\end{aligned}$$

On the other hand, we can write

$$\begin{aligned}\tilde{p}_{1,\psi_\varepsilon} &= a_{nn}^{(1)}[(-\lambda + i\tau\alpha_1 + E^{(1)} + iF^{(1)})^2 + (A^{(1)} - iB^{(1)})^2] \\ &= a_{nn}^{(1)}[(\lambda - i\tau\alpha_1 - E^{(1)} - iF^{(1)} + i(A^{(1)} - iB^{(1)})) \\ &\quad \cdot (\lambda - i\tau\alpha_1 - E^{(1)} - iF^{(1)} - i(A^{(1)} - iB^{(1)}))] \\ &= a_{nn}^{(1)}(\lambda - \sigma_1^{(1)})(\lambda - \sigma_2^{(1)}),\end{aligned}$$

where

$$\begin{aligned}\sigma_1^{(1)} &= E^{(1)} + B^{(1)} + i(\tau\alpha_1 + F^{(1)} + A^{(1)}), \\ \sigma_2^{(1)} &= E^{(1)} - B^{(1)} + i(\tau\alpha_1 + F^{(1)} - A^{(1)}).\end{aligned}$$

Let us introduce the polynomial

$$K_{k,\psi_\varepsilon}(\omega,\lambda) := \prod_{\operatorname{Im}\sigma_j^{(k)} \geq 0} (\lambda - \sigma_j^{(k)}).$$

Now, we state the definition of transmission conditions given in [2, Definition 1.4].

Definition 3.1. The pair $\{P_k, \psi_\varepsilon, T_k^j,\ k = 1,2,\ j = 1,2\}$ satisfies the transmission conditions at ω if for any polynomials $q_1(\lambda), q_2(\lambda)$, there exist polynomials $U_1(\lambda), U_2(\lambda)$ and constant c_1, c_2 such that

$$\begin{cases} q_1(\lambda) = c_1\tilde{t}^1_{1,\psi_\varepsilon}(\omega,\lambda) + c_2\tilde{t}^2_{1,\psi_\varepsilon}(\omega,\lambda) + U_1(\lambda)K_{1,\psi_\varepsilon}(\omega,\lambda), \\ q_2(\lambda) = c_1\tilde{t}^1_{2,\psi_\varepsilon}(\omega,\lambda) + c_2\tilde{t}^2_{2,\psi_\varepsilon}(\omega,\lambda) + U_2(\lambda)K_{2,\psi_\varepsilon}(\omega,\lambda). \end{cases}$$

In this section, we will show that if we choose α_1, α_2 appropriately and restrict the size of the imaginary parts of the complex coefficients, the transmission conditions described in Definition 3.1 are satisfied.

Theorem 3.1. *Assume that $a_{\ell j}^{(k)}$ have properties (3.1)–(3.4). Moreover, the number γ in (3.2) satisfies $\gamma < \gamma_0$, where γ_0 is explicitly given by*

$$\gamma_0 = \frac{\sqrt{2}\lambda_0^5}{\Lambda_0^3\sqrt{n\lambda_0^4 + n^2\Lambda_0^4}}. \tag{3.11}$$

Let ψ_ε be given by (3.5) with α_1, α_2 satisfying (3.25). Then $\{P_k, \psi_\varepsilon, T_k^j, . = 1,2, j = 1,2\}$ satisfies the transmission conditions at 0.

Before going through the lengthy computation of checking the transmission conditions, we give an overview of what we have to do. In order to check the transmission conditions, we need to study the polynomial $K_{k,\psi_\varepsilon}(\omega,\lambda)$. More precisely, being able to satisfy the conditions depends on the degree of K_{1,ψ_ε} and K_{2,ψ_ε}, that is, on the number of roots with negative imaginary parts. For this reason, we need to determine the signs of the imaginary parts of the roots $\sigma_j^{(k)}$ defined above. Under the ellipticity assumptions (3.3) and (3.4), we can show that $\operatorname{Im}\sigma_1^{(2)} < 0$ and $\operatorname{Im}\sigma_1^{(1)} > 0$. Thus, we only need to treat three cases:

$$\begin{cases} \text{Case 1.} & K_{2,\psi_\varepsilon} = 1, & \deg K_{1,\psi_\varepsilon} = 1 \text{ or } 2, \\ \text{Case 2.} & \deg K_{1,\psi_\varepsilon} = 2, & \deg K_{2,\psi_\varepsilon} = 1, \\ \text{Case 3.} & \deg K_{1,\psi_\varepsilon} = 1, & \deg K_{2,\psi_\varepsilon} = 1, \end{cases} \tag{3.12}$$

where $\deg K_{k,\psi_\varepsilon}$ denotes the degree of K_{k,ψ_ε}. For Case 1, it is not hard to show that the transmission conditions hold. In the second case, the ratio α_2/α_1 (see (3.25)) will come into play. Case 3 is the most complicate one. It turns out satisfying a non-degenerate condition (see (3.28)) will lead to the transmission conditions. This non-degenerate condition holds trivially when coefficients of the equation are real. In view of continuity, one can expect that the non-degenerate condition still holds true if the imaginary parts of the coefficients are not too large. From the viewpoint of applications, restricting the size of imaginary parts is reasonable. For instance, the portion of electric currents in the biological tissues due to cell membranes is small. In other words, the conductivities of biological tissue have small imaginary parts.

We now prepare to prove Theorem 3.1. We begin with some preliminary computations about the signs of the imaginary parts of $\sigma_j^{(k)}$. Note that we can write

$$b_k(\xi') = \frac{1}{a_{nn}^{(k)}} \sum_{1\le \ell,j\le n-1} a_{\ell j}^{(k)}\xi_\ell\xi_j - (E^{(k)} + iF^{(k)})^2. \tag{3.13}$$

Since b_k plays an essential role, we begin by working some calculations on the matrix $\frac{1}{a_{nn}^{(k)}}\mathcal{A}^{(k)}$, where $\mathcal{A}^{(k)}$ is the matrix $(a_{\ell j}^{(k)})$. Let $a_{nn}^{(k)} = |a_{nn}^{(k)}|e^{i\theta}$. Choosing $\xi = e_n$, we have that

$$a_{nn}^{(k)} = \sum_{1\le \ell,j\le n} a_{\ell j}^{(k)}\xi_\ell\xi_j = \sum_{1\le \ell,j\le n} M_{\ell j}^{(k)}\xi_\ell\xi_j + i\gamma \sum_{1\le \ell,j\le n} N_{\ell j}^{(k)}\xi_\ell\xi_j.$$

Hence, from (3.3) and (3.4), we have that

$$\lambda_0 \le \operatorname{Re}(a_{nn}^{(k)}) \le \Lambda_0 \quad \text{and} \quad \lambda_0 \le \frac{\operatorname{Im}(a_{nn}^{(k)})}{\gamma} \le \Lambda_0$$

and so that $\theta \in [0, \pi/2)$. Let us evaluate

$$\begin{aligned} (a_{nn}^{(k)})^{-1}\mathcal{A}^{(k)} &= |a_{nn}^{(k)}|^{-1}(M^{(k)} + i\gamma N^{(k)})(\cos\theta - i\sin\theta) \\ &= |a_{nn}^{(k)}|^{-1}[\cos\theta M^{(k)} + \gamma\sin\theta N^{(k)} \\ &\qquad + i(-\sin\theta M^{(k)} + \gamma\cos\theta N^{(k)})]. \end{aligned} \tag{3.14}$$

Using (3.3) and (3.4) again, we see that for $\xi \in \mathbb{R}^n$

$$\begin{aligned} \mathrm{Re}((a_{nn}^{(k)})^{-1}\mathcal{A}^{(k)}\xi\cdot\xi) &= |a_{nn}^{(k)}|^{-1}[\cos\theta M^{(k)}\xi\cdot\xi + \gamma\sin\theta N^{(k)}\xi\cdot\xi] \\ &\geq |a_{nn}^{(k)}|^{-1}\lambda_0(\cos\theta+\gamma\sin\theta)|\xi|^2. \end{aligned} \tag{3.15}$$

In fact, since $\cos\theta = M_{nn}^{(k)}|a_{nn}^{(k)}|^{-1}$ and $\sin\theta = \gamma N_{nn}^{(k)}|a_{nn}^{(k)}|^{-1}$, while

$$|a_{nn}^{(k)}|^2 = (M_{nn}^{(k)})^2 + \gamma^2(N_{nn}^{(k)})^2,$$

we have

$$|a_{nn}^{(k)}|^{-1}(\cos\theta+\gamma\sin\theta) = \frac{M_{nn}^{(k)}+\gamma^2 N_{nn}^{(k)}}{(M_{nn}^{(k)})^2+\gamma^2(N_{nn}^{(k)})^2} \geq \frac{\lambda_0(1+\gamma^2)}{\Lambda_0^2(1+\gamma^2)} = \frac{\lambda_0}{\Lambda_0^2}. \tag{3.16}$$

Combining (3.15) and (3.16) implies

$$\mathrm{Re}((a_{nn}^{(k)})^{-1}\mathcal{A}^{(k)}\xi\cdot\xi) \geq \frac{\lambda_0^2}{\Lambda_0^2}|\xi|^2 := \tilde{\lambda}_1|\xi|^2. \tag{3.17}$$

Now, let us write

$$\begin{aligned} \tilde{\lambda}_1|\xi|^2 &\leq \mathrm{Re}((a_{nn}^{(k)})^{-1}\mathcal{A}^{(k)}\xi\cdot\xi) \\ &= \mathrm{Re}\Big[\sum_{1\leq\ell,j\leq n-1}\frac{a_{\ell j}^{(k)}}{a_{nn}^{(k)}}\xi_\ell\xi_j + 2\sum_{1\leq j\leq n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_n\xi_j + \xi_n^2\Big] \\ &= \xi_n^2 + 2b_0^{(k)}(\xi')\xi_n + b_1^{(k)}(\xi'), \end{aligned} \tag{3.18}$$

where

$$b_0^{(k)}(\xi') = \mathrm{Re}\Big(\sum_{1\leq j\leq n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_j\Big) = \mathrm{Re}(E^{(k)}+iF^{(k)}) = E^{(k)}$$

and

$$b_1^{(k)}(\xi') = \mathrm{Re}\Big(\sum_{1\leq\ell,j\leq n-1}\frac{a_{\ell j}^{(k)}}{a_{nn}^{(k)}}\xi_\ell\xi_j\Big).$$

Substituting $\tilde{\xi}_n = \xi_n = -b_0^{(k)}(\xi')$ into (3.18) gives

$$\tilde{\lambda}_1(|\xi'|^2+|\tilde{\xi}_n|^2) \leq \tilde{\xi}_n^2 - 2b_0^{(k)}(\xi')\tilde{\xi}_n + b_1^{(k)}(\xi') = -(b_0^{(k)}(\xi')^2 + b_1^{(k)}(\xi'),$$

which implies

$$\tilde{\lambda}_1|\xi'|^2 \leq \mathrm{Re}\Big(\sum_{1\leq\ell,j\leq n-1}\frac{a_{\ell j}^{(k)}}{a_{nn}^{(k)}}\xi_\ell\xi_j\Big) - E_k^2. \tag{3.19}$$

Putting (3.13) and (3.19) together gives

$$\begin{aligned}\operatorname{Re}(b_k(x_0,\xi')) &= \operatorname{Re}\Big(\sum_{1\le \ell,j\le n-1}\frac{a_{\ell j}^{(k)}}{a_{nn}^{(k)}}\xi_\ell\xi_j\Big) - (E^{(k)})^2 + (F^{(k)})^2\\ &\ge \tilde\lambda_1|\xi'|^2 + (F^{(k)})^2 > 0. \end{aligned}\tag{3.20}$$

The following lemma guarantees the positivity of $A^{(k)}$.

Lemma 3.2. *Assume that* (3.3) *and* (3.4) *hold. Then*

$$A^{(k)} \ge \sqrt{\tilde\lambda_1|\xi'|^2 + |F^{(k)}|^2} > |F^{(k)}|.\tag{3.21}$$

Proof. From (3.9), it is easy to see that

$$A^{(k)} = \operatorname{Re}\sqrt{b_k} = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}},$$

where $a = \operatorname{Re} b_k$ and $b = \operatorname{Im} b_k$. We have from (3.20) that $a > 0$ and thus

$$A^{(k)} \ge \sqrt{a} \ge \sqrt{\tilde\lambda_1|\xi'|^2 + (F^{(k)})^2} > |F^{(k)}|. \qquad \blacksquare$$

Lemma 3.2 implies

$$\begin{aligned}\operatorname{Im}\sigma_1^{(2)} &= -(\tau\alpha_2 + F^{(2)} + A^{(2)}) = -\tau\alpha_2 - F^{(2)} - A^{(2)}\\ &\le -\tau\alpha_2 - |F^{(2)}| - F^{(2)} \le -\tau\alpha_2 < 0\end{aligned}\tag{3.22}$$

and

$$\operatorname{Im}\sigma_1^{(1)} = \tau\alpha_1 + F^{(1)} + A^{(1)} > \tau\alpha_1 + F^{(1)} + |F^{(1)}| \ge \tau\alpha_1 > 0.\tag{3.23}$$

In view of (3.22) and (3.23), to verify the transmission conditions, it suffices to consider three cases listed in (3.12).

Proof of Theorem 3.1. We discuss three cases separately.

Case 1. $\tilde p_{2,\psi_\varepsilon}$ has two roots in $\{\operatorname{Im} z < 0\}$, i.e., $-\tau\alpha_2 - F^{(2)} + A^{(2)} < 0$ in view of (3.22). In this case, we have that

$$K_{2,\psi_\varepsilon} = 1, \quad \text{while } K_{1,\psi_\varepsilon} \text{ has degree 1 or 2 (note (3.23))}.$$

Since $\tilde t^1_{2,\psi_\varepsilon}(\omega,\lambda) = 1$ and

$$\tilde t^2_{2,\psi_\varepsilon}(\omega,\lambda) = a_{nn}^{(2)}\Big(\lambda + i\tau\alpha_2 + \sum_{1\le j\le n-1}\frac{a_{nj}^{(2)}}{a_{nn}^{(2)}}\xi_j\Big),$$

for any $q_2(\lambda)$, we simply choose

$$U_2(\lambda) = q_2(\lambda) - c_1\tilde{t}^1_{2,\psi_\varepsilon} - c_2\tilde{t}^2_{2,\psi_\varepsilon}.$$

On the other hand, we have $\tilde{t}^1_{1,\psi_\varepsilon}(\omega,\lambda) = -1$ and

$$\tilde{t}^2_{1,\psi_\varepsilon}(\omega,\lambda) = a^{(1)}_{nn}(\lambda - i\tau\alpha_1 - \sum_{1\le j\le n-1}\frac{a^{(1)}_{nj}}{a^{(1)}_{nn}}\xi_j).$$

Then, for any polynomial $q_1(\lambda)$, we choose $U_1(\lambda)$ to be the quotient of the division between q_1 and K_{1,ψ_ε}. The remainder term is equal to $c_1\tilde{t}_{1,\psi_\varepsilon} + c_2\tilde{t}_{2,\psi_\varepsilon}$ with suitable c_1, c_2.

Case 2. Assume that $\operatorname{Im}\sigma^{(2)}_2 \ge 0$ and $\operatorname{Im}\sigma^{(1)}_2 \ge 0$, i.e.,

$$-\tau\alpha_2 - F^{(2)} + A^{(2)} \ge 0, \quad \tau\alpha_1 + F^{(1)} - A^{(1)} \ge 0.$$

Then K_{1,ψ_ε} has degree 2 and K_{2,ψ_ε} has degree 1. In order to avoid this case, we need to be sure that if $-\tau\alpha_2 - F^{(2)} + A^{(2)} \ge 0$, then $\tau\alpha_1 + F^{(1)} - A^{(1)} < 0$, that is,

$$\tau\alpha_2 + F^{(2)} - A^{(2)} \le 0 \implies \tau\alpha_1 + F^{(1)} - A^{(1)} < 0.$$

This can be achieved by assuming that

$$\frac{\alpha_2}{\alpha_1} > \frac{A^{(2)} - F^{(2)}}{A^{(1)} - F^{(1)}}, \quad \text{for all } \xi' \ne 0. \tag{3.24}$$

Recall that $A^{(k)} - F^{(k)} > 0$, $k = 1,2$. We remark that all $A^{(k)}$ and $F^{(k)}$ are homogeneous of degree 1 in ξ'. Hence, (3.24) holds provided

$$\frac{\alpha_2}{\alpha_1} = \max_{|\xi'|=1}\left\{\frac{A^{(2)} - F^{(2)}}{A^{(1)} - F^{(1)}}\right\} + 1. \tag{3.25}$$

Hence, if we assume (3.25), then the transmission conditions are satisfied.

Case 3. Each symbol has exactly one root in $\{\operatorname{Im} z < 0\}$, i.e.,

$$\tau\alpha_1 + F^{(1)} - A^{(1)} < 0, \quad -\tau\alpha_2 - F^{(2)} + A^{(2)} > 0.$$

In this case, we have

$$K_{1,\psi_\varepsilon} = (\lambda - \sigma^{(1)}_1), \quad K_{2,\psi_\varepsilon} = (\lambda - \sigma^{(2)}_2).$$

Given polynomials $q_1(\lambda), q_2(\lambda)$, there exist $U_1(\lambda), U_2(\lambda)$ such that

$$q_1(\lambda) = U_1(\lambda)K_{1,\psi_\varepsilon} + \tilde{q}_1,$$
$$q_2(\lambda) = U_2(\lambda)K_{2,\psi_\varepsilon} + \tilde{q}_2,$$

where $\tilde{q}_1, \tilde{q}_2$ are constants in λ. The transmission conditions are satisfied if there exists constants μ_1, μ_2, c_1, c_2 so that

$$\begin{cases} \tilde{q}_1 = \mu_1 K_{1,\psi_\varepsilon} + c_1 \tilde{t}^1_{1,\psi_\varepsilon} + c_2 \tilde{t}^2_{1,\psi_\varepsilon}, \\ \tilde{q}_2 = \mu_2 K_{2,\psi_\varepsilon} + c_1 \tilde{t}^1_{2,\psi_\varepsilon} + c_2 \tilde{t}^2_{2,\psi_\varepsilon}, \end{cases}$$

namely,

$$\begin{cases} \tilde{q}_1 = \mu_1(\lambda - \sigma_1^{(1)}) - c_1 + c_2 a_{nn}^{(1)}(\lambda - i\tau\alpha_1 - E^{(1)} - iF^{(1)}), \\ \tilde{q}_2 = \mu_2(\lambda - \sigma_2^{(2)}) + c_1 + c_2 a_{nn}^{(2)}(\lambda + i\tau\alpha_2 + E^{(2)} + iF^{(2)}). \end{cases} \tag{3.26}$$

System (3.26) is equivalent to

$$\begin{cases} \mu_1 + c_2 a_{nn}^{(1)} = 0, \\ \mu_2 + c_2 a_{nn}^{(2)} = 0, \\ \mu_1 \sigma_1^{(1)} + c_1 + c_2 a_{nn}^{(1)}(i\tau\alpha_1 + E^{(1)} + iF^{(1)}) = -\tilde{q}_1, \\ -\mu_2 \sigma_2^{(2)} + c_1 + c_2 a_{nn}^{(2)}(i\tau\alpha_2 + E^{(2)} + iF^{(2)}) = \tilde{q}_2. \end{cases} \tag{3.27}$$

System (3.27) has a unique solution if and only if the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & a_{nn}^{(1)} \\ 0 & 1 & 0 & a_{nn}^{(2)} \\ \sigma_1^{(1)} & 0 & 1 & \zeta_1 \\ 0 & -\sigma_2^{(2)} & 1 & \zeta_2 \end{pmatrix}$$

with $\zeta_1 = a_{nn}^{(1)}(i\tau\alpha_1 + E^{(1)} + iF^{(1)})$, $\zeta_2 = a_{nn}^{(2)}(i\tau\alpha_2 + E^{(2)} + iF^{(2)})$, is nonsingular. We compute

$$\begin{aligned} \det T &= \det \begin{pmatrix} 1 & 0 & a_{nn}^{(1)} \\ 0 & 1 & a_{nn}^{(2)} \\ 0 & -\sigma_2^{(2)} & \zeta_2 \end{pmatrix} - \det \begin{pmatrix} 1 & 0 & a_{nn}^{(1)} \\ 0 & 1 & a_{nn}^{(2)} \\ \sigma_1^{(1)} & 0 & \zeta_1 \end{pmatrix} \\ &= \zeta_2 + \sigma_2^{(2)} a_{nn}^{(2)} - \zeta_1 + \sigma_1^{(1)} a_{nn}^{(1)} \\ &= a_{nn}^{(2)}(i\tau\alpha_2 + E^{(2)} + iF^{(2)} - E^{(2)} + B^{(2)} - i\tau\alpha_2 - iF^{(2)} + iA^{(2)}) \\ &\quad + a_{nn}^{(1)}(-i\tau\alpha_1 - E^{(1)} - iF^{(1)} + E^{(1)} + B^{(1)} + i\tau\alpha_1 + iF^{(1)} + iA^{(1)}) \\ &= a_{nn}^{(2)}(B^{(2)} + iA^{(2)}) + a_{nn}^{(1)}(B^{(1)} + iA^{(1)}). \end{aligned}$$

Therefore, if

$$a_{nn}^{(2)}(B^{(2)} + iA^{(2)}) + a_{nn}^{(1)}(B^{(1)} + iA^{(1)}) \neq 0, \tag{3.28}$$

then the transmission conditions hold.

We now verify (3.28). In the real case where $a_{nn}^{(2)}, a_{nn}^{(1)}$ are positive real numbers, it is easy to see that

$$a_{nn}^{(2)}A^{(2)} + a_{nn}^{(1)}A^{(1)} > 0$$

and thus (3.28) holds.

For the complex case, we want to show that there exists $\gamma_0 > 0$ such that if $\gamma < \gamma_0$, then (3.28) is satisfied. Let $u_k = A^{(k)} + iB^{(k)}$ and $v_k = iu_k = -B^{(k)} + iA^{(k)}$. We will consider u_k and v_k as vectors in $\mathbb{R}^2$, i.e., $u_k = (A^{(k)}, B^{(k)})$, $v_k = u_k^{\perp} = (-B^{(k)}, A^{(k)})$. Let $a_{nn}^{(k)} = \eta^{(k)} + i\gamma\delta^{(k)}$ for $\eta^{(k)}, \delta^{(k)} \in \mathbb{R}$. By the ellipticity conditions (3.3) and (3.4), we have

$$\lambda_0 \leq \eta^{(k)} \leq \Lambda_0, \quad \lambda_0 \leq \delta^{(k)} \leq \Lambda_0.$$

Notice that $\det T = 0$ if and only if

$$(\eta^{(2)} + i\gamma\delta^{(2)})(B^{(2)} + iA^{(2)}) + (\eta^{(1)} + i\gamma\delta^{(1)})(B^{(1)} + iA^{(1)}) = 0,$$

i.e.,

$$\begin{aligned}(\eta^{(2)}B^{(2)} - \gamma\delta^{(2)}A^{(2)} + \eta^{(1)}B^{(1)} - \gamma\delta^{(1)}A^{(1)})\\ + i(\eta^{(2)}A^{(2)} + \gamma\delta^{(2)}B^{(2)} + \eta^{(1)}A^{(1)} + \gamma\delta^{(1)}B^{(1)}) = 0,\end{aligned}$$

which is equivalent to

$$\eta^{(2)}\begin{pmatrix} A^{(2)} \\ B^{(2)} \end{pmatrix} + \eta^{(1)}\begin{pmatrix} A^{(1)} \\ B^{(1)} \end{pmatrix} = \gamma\delta^{(2)}\begin{pmatrix} -B^{(2)} \\ A^{(2)} \end{pmatrix} + \gamma\delta^{(1)}\begin{pmatrix} -B^{(1)} \\ A^{(1)} \end{pmatrix} \tag{3.29}$$

or simply

$$\eta^{(2)}u_2 + \eta^{(1)}u_1 = \gamma\delta^{(2)}v_2 + \gamma\delta^{(1)}v_1. \tag{3.30}$$

Recall that $A^{(k)} \geq |F^{(k)}| > 0$. Therefore, in the real case $\gamma\delta^{(k)} = 0$, then (3.29) will never be satisfied. If $B^{(1)}$ and $B^{(2)}$ have the same sign, that is, either $B^{(k)} \geq 0$ or $B^{(k)} \leq 0$ for $k = 1, 2$, (3.30) can not hold. To see this, let us consider $B^{(k)} \geq 0$, $k = 1, 2$. Then u_1, u_2 are in the first quadrant of the plane and v_1, v_2 are in the second quadrant of the plane. The sets

$$C_u = \{\eta^{(2)}u_2 + \eta^{(1)}u_1 \colon \eta^{(k)} \geq 0\}, \quad C_u = \{\gamma\delta^{(2)}v_2 + \gamma\delta^{(1)}v_1 \colon \gamma\delta^{(k)} \geq 0\}$$

can only intersect at the original. Same thing happens if $B^{(k)} \leq 0$ for $k = 1, 2$.

The only case we need to investigate is when $B^{(1)}$ and $B^{(2)}$ have different signs. For example, let us assume

$$B^{(1)} > 0, \quad B^{(2)} < 0.$$

Even in this case, the intersection between C_u and C_v is non-trivial if the angle ϕ between u_1 and u_2 is less than $\pi/2$. Note that u_1 is the first quadrant and u_2 is in the fourth quadrant. So, the angle between u_1 and u_2 is less than π. We would like to show that (3.30) cannot hold for $\phi \in [\pi/2, \pi)$ if we choose γ_0 small enough.

Note that in this case $\cos\phi \le 0$. To do so, we estimate $\|\eta^{(2)}u_2 + \eta^{(1)}u_1\|$ from below and $\|\delta^{(2)}v_2 + \delta^{(1)}v_1\|$ from above. We now discuss the estimate of $\|\delta^{(2)}v_2 + \delta^{(1)}v_1\|$ from above. Compute

$$\begin{aligned} &\|\delta^{(2)}v_2 + \delta^{(1)}v_1\|^2 \\ &\quad= (\delta^{(2)})^2[(A^{(2)})^2 + (B^{(2)})^2] + (\delta^{(1)})^2[(A^{(1)})^2 + (B^{(1)})^2] \\ &\qquad + 2\delta^{(1)}\delta^{(2)}(-B^{(2)}, A^{(2)}) \cdot (-B^{(1)}, A^{(1)}) \\ &\quad= (\delta^{(2)})^2[(A^{(2)})^2 + (B^{(2)})^2] + (\delta^{(1)})^2[(A^{(1)})^2 + (B^{(1)})^2] \\ &\qquad + 2\delta^{(1)}\delta^{(2)}[(A^{(2)})^2 + (B^{(2)})^2]^{1/2}[(A^{(1)})^2 + (B^{(1)})^2]^{1/2}\cos\phi \\ &\quad\le (\delta^{(2)})^2[(A^{(2)})^2 + (B^{(2)})^2] + (\delta^{(1)})^2[(A^{(1)})^2 + (B^{(1)})^2]. \end{aligned} \tag{3.31}$$

In view of (3.9) and (3.13), we have

$$\begin{aligned} (A^{(k)})^2 + (B^{(k)})^2 = |b_k| &= \Big|\sum_{1\le \ell, j\le n-1} \frac{a^{(k)}_{\ell j}}{a^{(k)}_{nn}}\xi_\ell\xi_j - (E^{(k)} + iF^{(k)})^2\Big| \\ &\le \Big|\sum_{1\le \ell, j\le n-1} \frac{a^{(k)}_{\ell j}}{a^{(k)}_{nn}}\xi_\ell\xi_j\Big| + |(E^{(k)} + iF^{(k)})^2\Big|. \end{aligned} \tag{3.32}$$

By (3.3), (3.4), and (3.14), we obtain

$$\begin{aligned} \Big|\sum_{1\le \ell, j\le n-1} \frac{a^{(k)}_{\ell j}}{a^{(k)}_{nn}}\xi_\ell\xi_j\Big|^2 &= \Big|\frac{1}{a^{(k)}_{nn}}\mathcal{A}^{(k)}\xi\cdot\xi\Big|^2 \qquad\qquad (\text{with } \xi = (\xi', 0)) \\ &= |a^{(k)}_{nn}|^{-2}|\cos\theta M^{(k)}\xi\cdot\xi + \gamma\sin\theta N^{(k)}\xi\cdot\xi \\ &\quad + i(-\sin\theta M^{(k)}\xi\cdot\xi + \gamma\cos\theta N^{(k)}\xi\cdot\xi)|^2 \\ &= |a^{(k)}_{nn}|^{-2}[(M^{(k)}\xi\cdot\xi)^2 + \gamma^2(N^{(k)}\xi\cdot\xi)^2] \\ &\le \frac{\Lambda^2(1+\gamma^2)|\xi|^4}{\lambda_0^2(1+\gamma^2)} = \tilde\lambda_1^{-1}|\xi|^4, \end{aligned}$$

where we have used the estimate

$$\lambda_0(1+\gamma^2)^{1/2} \le |a^{(k)}_{nn}| \le \Lambda_0(1+\gamma^2)^{1/2} \tag{3.33}$$

in deriving the inequality above. We thus obtain

$$\Big|\sum_{1\le \ell, j\le n-1} \frac{a^{(k)}_{\ell j}}{a^{(k)}_{nn}}\xi_\ell\xi_j\Big| \le \tilde\lambda_1^{-1/2}|\xi'|^2. \tag{3.34}$$

Furthermore, we can estimate

$$
\begin{aligned}
|(E^{(k)}+iF^{(k)})^2| &= \Big|\Big(\sum_{1\le j\le n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_j\Big)^2\Big| = \Big|\sum_{1\le j\le n-1}\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\xi_j\Big|^2 \\
&\le \Big(\sum_{1\le j\le n-1}\Big|\frac{a_{nj}^{(k)}}{a_{nn}^{(k)}}\Big|^2\Big)|\xi'|^2 \le \frac{(n-1)\Lambda_0^2(1+\gamma^2)}{\lambda_0^2(1+\gamma^2)}|\xi'|^2 \\
&= (n-1)\tilde{\lambda}_1^{-1}|\xi'|^2.
\end{aligned}
\tag{3.35}
$$

Substituting (3.34) and (3.35) into (3.32) gives

$$
(A^{(k)})^2+(B^{(k)})^2 \le (\tilde{\lambda}_1^{-1/2}+(n-1)\tilde{\lambda}_1^{-1})|\xi'|^2 \le n\frac{\Lambda_0^2}{\lambda_0^2}|\xi'|^2. \tag{3.36}
$$

It follows from (3.31) and (3.36) that

$$
\|\delta^{(2)}v_2+\delta^{(1)}v_1\|^2 \le 2\Lambda_0^2 n\frac{\Lambda_0^2}{\lambda_0^2}|\xi'|^2. \tag{3.37}
$$

Next, we want to estimate $\|\eta^{(2)}u_2+\eta^{(1)}u_1\|$ from below. As above, we have

$$
\begin{aligned}
&\|\eta^{(2)}u_2+\eta^{(1)}u_1\|^2 \\
&\quad = (\eta^{(2)})^2[(A^{(2)})^2+(B^{(2)})^2]+(\eta^{(1)})^2[(A^{(1)})^2+(B^{(1)})^2] \\
&\qquad + 2\eta^{(1)}\eta^{(2)}[(A^{(2)})^2+(B^{(2)})^2]^{1/2}[(A^{(1)})^2+(B^{(1)})^2]^{1/2}\cos\phi.
\end{aligned}
\tag{3.38}
$$

Recall that $B_1>0$, $B_2<0$. Thus,

$$
\begin{aligned}
\cos\phi &= \frac{A^{(1)}A^{(2)}+B^{(1)}B^{(2)}}{[(A^{(2)})^2+(B^{(2)})^2]^{1/2}[(A^{(1)})^2+(B^{(1)})^2]^{1/2}} \\
&= \frac{A^{(1)}A^{(2)}-|B^{(1)}||B^{(2)}|}{[(A^{(2)})^2+(B^{(2)})^2]^{1/2}[(A^{(1)})^2+(B^{(1)})^2]^{1/2}} \\
&= \frac{1-\frac{|B^{(1)}|}{A^{(1)}}\frac{|B^{(2)}|}{A^{(2)}}}{(1+(\frac{B^{(2)}}{A^{(2)}})^2)^{1/2}(1+(\frac{B^{(1)}}{A^{(1)}})^2)^{1/2}}.
\end{aligned}
$$

Notice that by (3.21) and (3.37)

$$
0\le\frac{|B^{(k)}|}{A^{(k)}}\le\frac{\sqrt{(A^{(k)})^2+(B^{(k)})^2}}{A^{(k)}}\le\frac{\sqrt{n}\frac{\Lambda_0}{\lambda_0}|\xi'|}{\sqrt{\tilde{\lambda}_1}|\xi'|}=\frac{\sqrt{n}\Lambda_0^2}{\lambda_0^2}:=\tilde{\lambda}_2\ge 1.
$$

It is readily seen that the function

$$
f(x,y)=\frac{1-xy}{\sqrt{1+x^2}\sqrt{1+y^2}}
$$

defined on $(x, y) \in [0, \tilde{\lambda}_2] \times [0, \tilde{\lambda}_2]$ attains its minimum at $x = y = \tilde{\lambda}_2$. Hence, we have

$$\cos\phi \geq \frac{1-\tilde{\lambda}_2^2}{1+\tilde{\lambda}_2^2} = -1 + \frac{2}{1+\tilde{\lambda}_2^2}.$$

Now, (3.38) gives

$$\begin{aligned}
&\|\eta^{(2)}u_2 + \eta^{(1)}u_1\|^2 \\
&\quad \geq (\eta^{(2)})^2[(A^{(2)})^2 + (B^{(2)})^2] + (\eta^{(1)})^2[(A^{(1)})^2 + (B^{(1)})^2] \\
&\qquad + 2\eta^{(1)}\eta^{(2)}[(A^{(2)})^2 + (B^{(2)})^2]^{1/2}[(A^{(1)})^2 + (B^{(1)})^2]^{1/2}\Big(-1 + \frac{2}{1+\tilde{\lambda}_2^2}\Big) \\
&\quad = \big((\eta^{(2)})^2[(A^{(2)})^2 + (B^{(2)})^2]^{1/2} - (\eta^{(1)})^2[(A^{(1)})^2 + (B^{(1)})^2]^{1/2}\big)^2 \\
&\qquad + \frac{4}{1+\tilde{\lambda}_2^2}\eta^{(1)}\eta^{(2)}[(A^{(2)})^2 + (B^{(2)})^2]^{1/2}[(A^{(1)})^2 + (B^{(1)})^2]^{1/2} \\
&\quad \geq \frac{4}{1+\tilde{\lambda}_2^2}A^{(1)}A^{(2)}\lambda_0^2 \geq \frac{4}{1+\tilde{\lambda}_2^2}\tilde{\lambda}_1|\xi'|^2\lambda_0^2 = \frac{4}{1+\tilde{\lambda}_2^2}\frac{\lambda_0^4}{\Lambda_0^2}|\xi'|^2.
\end{aligned} \tag{3.39}$$

Hence, in view of (3.37), (3.39), if we choose γ_0 given in (3.11), i.e.,

$$\gamma_0 = \frac{\sqrt{2}\lambda_0^5}{\Lambda_0^3\sqrt{n\lambda_0^4 + n^2\Lambda_0^4}},$$

then for $\gamma < \gamma_0$ we have

$$\|\eta^{(2)}u_2 + \eta^{(1)}u_1\|^2 > \gamma^2\|\delta^{(2)}v_2 + \delta^{(1)}v_1\|^2.$$

In other words, (3.30) cannot hold (i.e., $\det T \neq 0$), and equivalently, (3.28) is satisfied. The proof of Theorem 3.1 is now completed. ∎

3.2. Strong pseudoconvexity

Here we want to check the strong pseudoconvexity condition for the operator $\mathcal{L}_0$ and the weight function $\psi_\varepsilon(x)$ in $B_{\delta'} \cap \Omega_1$ and $B_{\delta'} \cap \Omega_2$ for some small $\delta' > 0$. Even though $\mathcal{L}_0$ is represented by P_k in Ω_k, $k = 1, 2$, it is not necessary to discuss the strong pseudoconvexity condition for P_1 and P_2 separately. We suppress the index k in notations and denote the symbol

$$p(\xi) = \sum_{1\leq j,\ell\leq n} a_{\ell j}\xi_\ell\xi_j$$

with $a_{\ell j} = M_{\ell j} + i\gamma N_{\ell j}$ and consider the weight function

$$\psi_\varepsilon(x) = \alpha x_n + \frac{\beta}{2}x_n^2 - \frac{\varepsilon}{2}|x'|^2.$$

In view of the definition of $\psi_\varepsilon(x)$ in (2.10), α here represents either $\alpha_2 = \alpha_+$ or $\alpha_1 = \alpha_-$. Hence, we have that

$$(\partial_j \psi_\varepsilon(x))_{j=1}^n = \nabla\psi_\varepsilon(x) = (-\varepsilon x', \alpha + \beta x_n)$$

and

$$(\partial^2_{\ell j}\psi_\varepsilon(x))_{\ell,j=1}^n = \nabla^2\psi_\varepsilon(x) = \begin{pmatrix} -\varepsilon I_{n-1} & 0 \\ 0 & \beta \end{pmatrix}. \tag{3.40}$$

The strong pseudoconvexity condition reads that in $B_{\delta'}$, if

$$\begin{cases} p(\xi + i\tau\nabla\psi_\varepsilon(x)) = 0, \\ (\xi,\tau) \neq 0, \quad \nabla\psi_\varepsilon(x) \neq 0, \quad x \in \overline{B_{\delta'}}, \end{cases}$$

then

$$\begin{aligned} Q(x,\xi,\tau) &:= \sum_{\ell,j=1}^n \partial^2_{\ell j}\psi_\varepsilon(x)\partial_{\xi_j} p(\xi + i\tau\nabla\psi_\varepsilon(x))\overline{\partial_{\xi_\ell} p(\xi + i\tau\nabla\psi_\varepsilon(x))} \\ &\quad + \frac{1}{\tau}\operatorname{Im}\sum_{j=1}^n \partial_j p(\xi + i\tau\nabla\psi_\varepsilon(x))\overline{\partial_{\xi_j} p(\xi + i\tau\nabla\psi_\varepsilon(x))} \\ &= \sum_{\ell,j=1}^n \partial^2_{\ell j}\psi_\varepsilon(x)\partial_{\xi_j} p(\xi + i\tau\nabla\psi_\varepsilon(x))\overline{\partial_{\xi_\ell} p(\xi + i\tau\nabla\psi_\varepsilon(x))} > 0 \end{aligned} \tag{3.41}$$

(see [8, (8.3.2)]).

We now write

$$\begin{aligned} &p(\xi + i\tau\nabla\psi_\varepsilon) \\ &\quad = \sum_{1\le \ell,j\le n} a_{\ell j}(\xi_\ell + i\tau\partial_\ell\psi_\varepsilon)(\xi_\ell + i\tau\partial_\ell\psi_\varepsilon) \\ &\quad = \sum_{1\le \ell,j\le n} a_{\ell j}\xi_\ell\xi_j + 2i\sum_{1\le \ell,j\le n} a_{\ell j}\xi_l(\tau\partial_j\psi_\varepsilon) - \sum_{1\le \ell,j\le n} a_{\ell j}(\tau\partial_\ell\psi_\varepsilon)(\tau\partial_j\psi_\varepsilon). \end{aligned}$$

Hence, $p(\xi + i\tau\nabla\psi_\varepsilon) = 0$ implies

$$\sum_{1\le \ell,j\le n} a_{\ell j}(\tau\partial_\ell\psi_\varepsilon)(\tau\partial_j\psi_\varepsilon) = \sum_{1\le \ell,j\le n} a_{\ell j}\xi_\ell\xi_j + 2i\sum_{1\le \ell,j\le n} a_{\ell j}\xi_l(\tau\partial_j\psi_\varepsilon) \tag{3.42}$$

By (3.2)–(3.4), we have

$$\left|\sum_{1\le \ell,j\le n} a_{\ell j}(\tau\partial_\ell\psi_\varepsilon)(\tau\partial_j\psi_\varepsilon)\right| \le \sqrt{1+\gamma^2}\Lambda_0|\tau\nabla\psi_\varepsilon|^2.$$

From this estimate, we obtain from (3.42) that

$$\begin{aligned}
&\sqrt{1+\gamma^2}\Lambda_0|\tau\nabla\psi_\varepsilon|^2 \\
&\quad \ge \Big|\sum_{1\le \ell,j\le n} a_{\ell j}\xi_\ell\xi_j + 2i\sum_{1\le \ell,j\le n} a_{\ell j}\xi_l(\tau\partial_j\psi_\varepsilon)\Big| \\
&\quad \ge \sqrt{1+\gamma^2}\lambda_0|\xi|^2 - 2\sqrt{1+\gamma^2}\Lambda_0|\xi||\tau\nabla\psi_\varepsilon| \\
&\quad \ge \sqrt{1+\gamma^2}\lambda_0|\xi|^2 - \frac{\sqrt{1+\gamma^2}\lambda_0}{2}|\xi|^2 - 2\frac{\sqrt{1+\gamma^2}\Lambda_0^2}{\lambda_0}|\tau\nabla\psi_\varepsilon|^2, \qquad (3.43)
\end{aligned}$$

which leads to

$$\frac{\lambda_0}{2}|\xi|^2 \le \Big(\Lambda_0 + \frac{2\Lambda_0^2}{\lambda_0}\Big)|\tau\nabla\psi_\varepsilon|^2. \qquad (3.44)$$

By (3.44) and exchanging the roles of ξ and $\tau\nabla\psi_\varepsilon$ in (3.43), we thus conclude that there exist positive constants C_1, C_2, depending on λ_0, Λ_0 such that

$$C_1|\xi| \le |\tau\nabla\psi_\varepsilon| \le C_2|\xi| \qquad (3.45)$$

whenever $p(\xi + i\tau\nabla\psi_\varepsilon) = 0$.

As in (3.6) and (3.7), we can write

$$p(\xi) = a_{nn}\Big[\Big(\xi_n + \sum_{1\le j\le n-1}\frac{a_{nj}}{a_{nn}}\xi_j\Big)^2 + b(\xi')\Big],$$

where

$$b(\xi') = \frac{1}{a_{nn}^2}\sum_{1\le \ell,j\le n-1}(a_{\ell j}a_{nn} - a_{n\ell}a_{nj})\xi_\ell\xi_j.$$

Similar to (3.9) and (3.10), we further express

$$b(\xi') = (A(\xi') - iB(\xi'))^2, \qquad (3.46)$$

with $A(\xi') \ge 0$ and

$$\sum_{1\le j\le n-1}\frac{a_{nj}}{a_{nn}}\xi_j = E(\xi') + iF(\xi'), \qquad (3.47)$$

where $E(\xi'), F(\xi') \in \mathbb{R}$.

To verify that (3.41) for x near 0, we first derive an estimate of $Q(0,\xi,\tau)$. At $x = 0$, we have $\partial_j\psi_\varepsilon(0) = 0$, $1 \le j \le n-1$ and $\partial_n\psi_\varepsilon(0) = \alpha$, i.e.,

$$\xi + i\tau\nabla\psi_\varepsilon(0) = (\xi', \xi_n + i\tau\alpha).$$

Thus, we can rewrite

$$p(\xi + i\tau\nabla\psi_\varepsilon(0)) = p(\xi + i\tau\alpha e_n) = a_{nn}(\xi_n - \sigma_1)(\xi_n - \sigma_2), \qquad (3.48)$$

where

$$\begin{cases} \sigma_1 = -E - B - i(\tau\alpha + F + A), \\ \sigma_2 = -E + B - i(\tau\alpha + F - A). \end{cases} \tag{3.49}$$

From now on, we suppress the dependence of coefficients at 0 if there is no danger of causing confusion.

By (3.40), we have that

$$Q(0,\xi,\tau) = -\varepsilon \sum_{1\le j\le n-1} |\partial_{\xi_j} p(\xi + i\tau\alpha e_n)|^2 + \beta|\partial_{\xi_n} p(\xi + i\tau\alpha e_n)|^2,$$

where for $1 \le j \le n-1$

$$\partial_{\xi_j} p(\xi + i\tau\alpha e_n) = 2\sum_{1\le\ell\le n-1} a_{\ell j}\xi_\ell + a_{nj}(\xi_n + i\tau\alpha)$$

and

$$\partial_{\xi_n} p(\xi + i\tau\alpha e_n) = 2\sum_{1\le\ell\le n-1} a_{\ell n}\xi_\ell + a_{nn}(\xi_n + i\tau\alpha).$$

Therefore, we can write

$$\begin{aligned} Q(0,\xi,\tau) = & -4\varepsilon \sum_{1\le j\le n-1} \Big| \sum_{1\le\ell\le n-1} a_{\ell j}\xi_\ell + a_{nj}(\xi_n + i\tau\alpha)\Big|^2 \\ & + 4\beta\Big| \sum_{1\le\ell\le n-1} a_{\ell n}\xi_\ell + a_{nn}(\xi_n + i\tau\alpha)\Big|^2. \end{aligned} \tag{3.50}$$

It follows from (3.48) and (3.49) that $p(\xi + i\tau\alpha e_n) = 0$ if and only if

$$\xi_n + i\tau\alpha = -E - B - i(F + A) \tag{3.51}$$

or

$$\xi_n + i\tau\alpha = -E + B - i(F - A). \tag{3.52}$$

Therefore, if $p(\xi + i\tau\alpha e_n) = 0$, then the second term in (3.50) can be further simplified as

$$\begin{aligned} \Big| \sum_{1\le\ell\le n-1} a_{\ell n}\xi_\ell + a_{nn}(\xi_n + i\tau\alpha)\Big|^2 &= |a_{nn}|^2 \Big| \sum_{1\le\ell\le n-1} \frac{a_{\ell n}}{a_{nn}}\xi_\ell + (\xi_n + i\tau\alpha)\Big|^2 \\ &= |a_{nn}|^2 |E + iF + (\xi_n + i\tau\alpha)|^2 = |a_{nn}|^2(A^2 + B^2), \end{aligned} \tag{3.53}$$

where we have used (3.47), (3.51) or (3.52). Combining (3.46), (3.47), (3.51) or (3.52), we have that

$$|\xi_n + i\tau\alpha| \le |E| + |B| + |F| + |A| \le C\Lambda_0|\xi'|, \tag{3.54}$$

which implies

$$\sum_{1\leq j\leq n-1}\Big|\sum_{1\leq \ell\leq n-1} a_{\ell j}\xi_\ell + a_{nj}(\xi_n + i\tau\alpha)\Big|^2 \leq C\Lambda_0^2|\xi'|^2. \tag{3.55}$$

Putting (3.50), (3.53), and (3.55) together gives

$$Q(0,\xi,\tau) \geq 4\beta|a_{nn}|^2(A^2+B^2) - 4\varepsilon C\Lambda_0|\xi'|^2. \tag{3.56}$$

Recall the estimate (3.21) in Lemma 3.2

$$A^2 \geq \tilde{\lambda}_1|\xi'|^2 + |F|^2 \geq \tilde{\lambda}_1|\xi'|^2.$$

Using this estimate in (3.56) and choosing ε sufficiently small leads to

$$Q(0,\xi,\tau) \geq 4(\beta\tilde{\lambda}_1\lambda_0^2 - \varepsilon C\Lambda_0)|\xi'|^2 \geq 2\beta\tilde{\lambda}_1\lambda_0^2|\xi'|^2,$$

whenever $p(\xi + i\tau\alpha e_n) = 0$. Furthermore, (3.54) implies

$$|\xi + i\tau\alpha e_n|^2 \leq (1 + C^2\Lambda_0^2)|\xi'|^2,$$

and it follows that if $p(\xi + i\tau\alpha e_n) = 0$ then

$$Q(0,\xi,\tau) \geq C\beta|\xi + i\tau\alpha e_n|^2. \tag{3.57}$$

In conclusion, we have shown that

$$(\xi,\tau) \in \{(\xi,\tau)\in S\colon p(\xi + i\tau\alpha e_n) = 0\} \implies Q(0,\xi,\tau) > 0, \tag{3.58}$$

where $S := \{(\xi,\tau)\in\mathbb{R}^{n+1}\colon |\xi|^2+\tau^2 = 1\}$.

Now, we recall the following elementary theorem. Let X be a compact subset of $\mathbb{R}^N$ and $F, G\colon X\to\mathbb{R}$ be two continuous functions, then the following two statements are equivalent:

i. $F(x) = 0$ for all $x\in X \implies G(x) > 0$;

ii. there exist positive constants C_1, C_2 such that $C_1G(x) + |F(x)| \geq C_2$, for all $x \in X$.

With the help of this theorem, (3.58) is equivalent to

$$C_1Q(0,\xi,\tau) + |p(\xi + i\tau\alpha e_n)| \geq C_2 \tag{3.59}$$

for all $(\xi,\tau)\in S$. Thanks to (3.59), we can estimate

$$\begin{aligned} &C_1Q(x,\xi,\tau) + |p(\xi + i\tau\nabla\psi_\varepsilon(x))| \\ &\quad = C_1Q(0,\xi,\tau) + |p(\xi + i\tau\alpha e_n)| + R(x,\xi,\tau) \geq C_2 + R(x,\xi,\tau), \end{aligned}$$

where

$$R(x,\xi,\tau) = C_1[Q(x,\xi,\tau) - Q(0,\xi,\tau)] + |p(\xi + i\tau\nabla\psi_\varepsilon(x))| - |p(\xi + i\tau\alpha e_n)|.$$

Observe that $R(0,\xi,\tau) = 0$ for $(\xi,\tau) \in S$. Since R is continuous, there exists a small number $\delta' > 0$ such that

$$|R(x,\xi,\tau)| \le \frac{C_2}{2}$$

for all x with $|x| \le \delta' < \frac{\alpha}{2\beta}$ and $(\xi,\tau) \in S$. In other words, we have that

$$C_1 Q(x,\xi,\tau) + |p(\xi + i\tau\nabla\psi_\varepsilon(x))| \ge \frac{C_2}{2} \tag{3.60}$$

in $\{|x| \le \delta'\} \times S$. By the elementary theorem stated above, (3.60) is equivalent to

$$\begin{aligned} p(\xi + i\tau\nabla\psi_\varepsilon(x)) = 0, &\quad \text{for all } x \in \overline{B_{\delta'}}, .\xi,\tau) \in S \\ \Longrightarrow \; Q(x,\xi,\tau) > 0, &\quad \text{for all } x \in \overline{B_{\delta'}}, .\xi,\tau) \in S, \end{aligned}$$

which immediately implies the strong pseudoconvexity condition near 0 in view of the homogeneity of p and Q in (ξ,τ).

Having verified the strong pseudoconvexity in a neighborhood of 0 and the transmission conditions at 0, we can derive a Carleman estimate with weight $\psi_\varepsilon(x)$ for the operator $\mathcal{L}_0$.

Theorem 3.3 ([2, Theorem 1.6]). *Assume that coefficients $A_\pm(0)$ satisfy conditions (3.1)–(3.4). There exist $\alpha_+, \alpha_-, \beta, \varepsilon_0, \gamma_0, r_0$ and C, depending on λ_0, Λ_0, such that if $\varepsilon \le \varepsilon_0$, $\gamma \le \gamma_0$, $\tau \ge C$, then*

$$\begin{aligned} &\sum_\pm \sum_{k=0}^{2} \tau^{3-2k} \int_{\mathbb{R}^n_\pm} |D^k u_\pm|^2 e^{2\tau\psi_{\varepsilon,\pm}(x)} dx \\ &\quad + \sum_\pm \sum_{k=0}^{1} \tau^{3-2k} \int_{\mathbb{R}^{n-1}} |D^k u_\pm(x',0)|^2 e^{2\psi_\varepsilon(x',0)} dx' \\ &\quad + \sum_\pm \tau^2 [e^{\tau\psi_\varepsilon(\cdot,0)} u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_\pm [D(e^{\tau\psi_{\varepsilon,\pm}} u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\ &\le C\Bigg(\int_{\mathbb{R}^n_\pm} |\mathcal{L}_0(D)(u_\pm)|^2 e^{2\tau\psi_{\varepsilon,\pm}(x)}\, dx + [e^{\tau\psi_\varepsilon(\cdot,0)} h_1^{(0)}]^2_{1/2,\mathbb{R}^{n-1}} \\ &\quad + [D_{x'}(e^{\tau\psi_\varepsilon} h_0^{(0)})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \tau^3 \int_{\mathbb{R}^{n-1}} |h_0^{(0)}|^2 e^{2\tau\psi_\varepsilon(x,0)}\, dx \\ &\quad + \tau \int_{\mathbb{R}^{n-1}} |h_1^{(0)}|^2 e^{2\tau\psi_\varepsilon(x,0)}\, dx \Bigg). \end{aligned} \tag{3.61}$$

for $u = H_+u_+ + H_-u_-$, $u_\pm \in C^\infty(\mathbb{R}^n)$ *and* $\operatorname{supp} u \subset B'_{r_0} \times [-r_0, r_0]$, *and, for all* $x' \in \mathbb{R}^{n-1}$,

$$h_0^{(0)}(x') := u_+(x', 0) - u_-(x', 0),$$
$$h_1^{(0)}(x') := A_+(0)\nabla u_+(x', 0) \cdot e_n - A_-(0)\nabla u_-(x', 0) \cdot e_n,$$

4. Derivation of the Carleman estimate

This section is devoted to the derivation of the Carleman estimate (2.14) following the ideas used in [5]. We first introduce the partition of unity given in [5]. For any $r > 0$ and $x' \in \mathbb{R}^{n-1}$, denote the $(n-1)$-cube

$$Q_r(x') = \{y' \in \mathbb{R}^{n-1}: |y'_j - x'_j| \le r, . = 1, 2, \dots, n-1\}.$$

Let $\vartheta_0 \in C_0^\infty(\mathbb{R})$ such that

$$0 \le \vartheta_0 \le 1, \quad \operatorname{supp} \vartheta_0 \subset (-3/2, 3/2), \quad \vartheta_0(t) = 1 \text{ for } t \in [-1, 1]. \tag{4.1}$$

Let $\vartheta(x') = \vartheta_0(x_1) \dots \vartheta_0(x_{n-1})$, so that

$$\operatorname{supp} \vartheta \subset \overset{\circ}{Q}_{3/2}(0) \quad \text{and} \quad \vartheta(x') = 1 \text{ for } x' \in Q_1(0),$$

where $\overset{\circ}{Q}$ denotes the interior of the set Q. Given $\mu \ge 1$ and $g \in \mathbb{Z}^{n-1}$, we define

$$x'_g = \frac{g}{\mu}$$

and

$$\vartheta_{g,\mu}(x') = \vartheta(\mu(x' - x'_g)).$$

Thus, we can see that

$$\operatorname{supp} \vartheta_{g,\mu} \subset \overset{\circ}{Q}_{3/2\mu}(x'_g) \subset Q_{2/\mu}(x'_g)$$

and

$$|D^k \vartheta_{g,\mu}| \le C_1 \mu^k (\chi_{Q_{3/2\mu}(x'_g)} - \chi_{Q_{1/\mu}(x'_g)}), \quad k = 0, 1, 2, \tag{4.2}$$

where $C_1 \ge 1$ depends only on n.

Notice that, for any $g \in \mathbb{Z}^{n-1}$,

$$\operatorname{card}(\{g' \in \mathbb{Z}^{n-1} \colon \operatorname{supp} \vartheta_{g',\mu} \cap \operatorname{supp} \vartheta_{g,\mu} \neq \emptyset\}) = 5^{n-1}. \tag{4.3}$$

Thus, we can define

$$\bar{\vartheta}_\mu(x') := \sum_{g \in \mathbb{Z}^{n-1}} \vartheta_{g,\mu} \geq 1, \quad x' \in \mathbb{R}^{n-1}. \tag{4.4}$$

By (4.2), we get that

$$|D^k \bar{\vartheta}_\mu| \leq C_2 \mu^k, \tag{4.5}$$

where $C_2 \geq 1$ depends on n. Define

$$\eta_{g,\mu}(x') = \vartheta_{g,\mu}(x')/\bar{\vartheta}_\mu(x'), \quad x' \in \mathbb{R}^{n-1}, \tag{4.6}$$

then we have that

$$\begin{cases} \sum_{g \in \mathbb{Z}^{n-1}} \eta_{g,\mu} = 1, & x' \in \mathbb{R}^{n-1}, \\ \operatorname{supp} \eta_{g,\mu} \subset Q_{3/2\mu}(x'_g) \subset Q_{2/\mu}(x'_g), & \\ |D^k \eta_{g,\mu}| \leq C_3 \mu^k \chi_{Q_{3/2\mu}(x'_g)}, & k = 0, 1, 2, \end{cases} \tag{4.7}$$

where $C_3 \geq 1$ depends on n.

We will first extend (3.61) to operators with leading coefficients depending on the vertical variable x_n. To do so, we need to derive an interior Carleman estimate for second order elliptic operators having Lipschitz leading coefficients and with the weight function ψ_ε. To derive such Carleman estimate, we define the n-cube $K_R = \{x = (x_1, \dots, x_n) \colon |x_j| \leq R, 1 \leq j \leq n\}$ for $R > 0$. Let us denote

$$P(x, D) = \sum_{1 \leq j, \ell \leq n} a_{j\ell}(x) D^2_{j\ell}$$

and its symbol $p(x, \xi) = \sum_{1 \leq j, \ell \leq n} a_{j\ell}(x) \xi_j \xi_\ell$. Assume that for all $1 \leq j, \ell \leq n$ and $x, y \in K_1$,

$$\begin{cases} a_{j\ell}(x) = a_{\ell j}(x), & \\ |a_{j\ell}(x)| \leq \Lambda, & \\ |a_{j\ell}(x) - a_{j\ell}(y)| \leq M_0 |x - y|, & \\ |p(x, \xi)| \geq \lambda |\xi|^2, & \text{for all } \xi \in \mathbb{R}^n, \end{cases} \tag{4.8}$$

where $\Lambda, \lambda > 0$. Let $\varphi(x) \in C^2(K_1)$ be real-valued and satisfy $|\nabla \varphi(x)| \neq 0$ for all $x \in K_1$. We denote

$$S(x, y; \xi, \tau) = \sum_{\ell, j = 1}^{n} \partial^2_{\ell j} \varphi(x) \partial_{\xi_j} p(y, \xi + i\tau \nabla \varphi(x)) \overline{\partial_{\xi_\ell} p(y, \xi + i\tau \nabla \varphi(x))}$$

for $x, y \in K_1$, $\xi \in \mathbb{R}^n$, $\tau > 0$.

Proposition 4.1. *Assume that the following condition holds:*

$$\left.\begin{array}{l} p(0,\xi+i\tau\varphi(0))=0 \\ (\xi,\tau)\neq(0,0) \end{array}\right\} \implies S(0,0;\xi,\tau)>0. \tag{4.9}$$

Then there exist $\overline{R}\in(0,1]$, $\delta_0\in(0,1]$, $C_0\geq 1$, $\tau_0\geq 1$, *depending on* λ,Λ,M_0, $\|\varphi\|_{C^2(Q_1)}$, *such that*

$$\sum_{|\alpha|\leq 2}\tau^{3-2|\alpha|}\int|D^\alpha u|^2e^{2\tau\varphi(x)}dx\leq C_0\int|P(\delta x,D)u|^2e^{2\tau\varphi(x)}dx, \tag{4.10}$$

for all $u\in C_0^\infty(\overset{\circ}{K}_{\overline{R}})$, $\tau\geq\tau_0$, $0<\delta\leq\delta_0$.

Proof. In view of the homogeneity in (ξ,τ), (4.9) is equivalent to that there exist $C_1>0, C_2>0$ such that for all $(\xi,\tau)\in\mathbb{R}^{n+1}$,

$$C_2|p(0,\xi+i\tau\nabla\varphi(0)|^2+(|\xi|^2+\tau^2)S(0,0;\xi,\tau)\geq C_1(|\xi|^2+\tau^2)^2.$$

From (4.8), we can see that there exists $\overline{R}\in(0,1]$ such that, for all $x,y\in K_{\overline{R}},.\xi,\tau)\in\mathbb{R}^{n+1}$,

$$\widetilde{C}_2|p(y,\xi+i\tau\nabla\varphi(x)|^2+(|\xi|^2+\tau^2)S(x,y;\xi,\tau)\geq\widetilde{C}_1(|\xi|^2+\tau^2)^2, \tag{4.11}$$

where $\widetilde{C}_1>0,\widetilde{C}_2>0$ are independent of x,y. Thanks to (4.11), the Carleman derived in [8, Theorem 8.3.1] holds for

$$P(\delta y,D_x)u=\sum_{1\leq\ell,j\leq n}a_{j\ell}(\delta y)D^2_{x_jx_\ell}u(x),$$

that is,

$$\sum_{|\alpha|\leq 2}\tau^{3-2|\alpha|}\int|D_x^\alpha u|^2e^{2\tau\varphi(x)}dx\leq C_3\int|P(\delta y,D_x)u|^2e^{2\tau\varphi(x)}dx \tag{4.12}$$

for all $u\in C_0^\infty(\overset{\circ}{K}_{\overline{R}})$, $0\leq\delta\leq 1$, and $\tau\geq\tau_1$, where C_3 and τ_1 do not depend on δ and y. Note that for fixed δ, y, $P(\delta y,D_x)$ is an operator having constant coefficients.

Now, we use the partition of unity introduced above, but with $n-1$ being replaced by n. In particular, for $h\in\mathbb{Z}^n$, we define

$$x_h=\frac{h}{\mu},\quad \mu=\sqrt{\varepsilon\tau},\quad \text{with } \tau\geq\frac{1}{\varepsilon},$$

where $\varepsilon\in(0,1]$ will be chosen later. Let $u\in C_0^\infty(\overset{\circ}{K}_{\overline{R}})$, in view of the first relation in (4.7), we have

$$u(x)=\sum_{h\in\mathbb{Z}^n}u(x)\eta_{h,\mu}(x),$$

where $\eta_{h,\mu}(x)$ is defined similarly as in (4.6) with $n-1, g$ being replaced by n, h, respectively. Applying (4.12) with $y = x_h$ implies

$$\begin{aligned}\sum_{|\alpha|\le 2}\tau^{3-2|\alpha|}\int |D^\alpha u|^2 e^{2\tau\varphi(x)}\,dx\\ &\le c\sum_{h\in\mathbb{Z}^n}\sum_{|\alpha|\le 2}\tau^{3-2|\alpha|}\int |D^\alpha(u\eta_{h,\mu})|^2 e^{2\tau\varphi(x)}dx\\ &\le cC_3\int |P(\delta x_h, D)(u\eta_{h,\mu})|^2 e^{2\tau\varphi(x)}dx,\end{aligned} \tag{4.13}$$

for all $\tau \ge \tau_2 = \min\{\tau_1, \frac{1}{\varepsilon}\}$, where $c = c(n)$.

Now, we write

$$|P(\delta x_h, D)(u\eta_{h,\mu})| \le |P(\delta x, D)(u\eta_{h,\mu})| + |(P(\delta x_h, D) - P(\delta x, D))(u\eta_{h,\mu})| \tag{4.14}$$

and use (4.5), the second inequality of (4.8), to estimate

$$|P(\delta x, D)(u\eta_{h,\mu})| \le |P(\delta x, D)u|\eta_{h,\mu} + C_4\Lambda(\sqrt{\varepsilon\tau}|Du| + \varepsilon\tau|u|)\chi_{K_{2/\mu}(x_h)} \tag{4.15}$$

and

$$\begin{aligned}&|(P(\delta x_h, D) - P(\delta x, D))(u\eta_{h,\mu})|\\ &= \Big|\sum_{1\le j,\ell\le n}(a_{j\ell}(\delta x_h) - a_{j\ell}(\delta x))D^2_{j\ell}(u\eta_{h,\mu})\Big|\\ &\le \eta_{h,\mu}\sum_{1\le j\ell,\le n}|a_{j\ell}(\delta x_h) - a_{j\ell}(\delta x)||D^2_{j\ell}u| + 2C_4\Lambda(\sqrt{\varepsilon\tau}|Du| + \varepsilon\tau|u|)\chi_{K_{2/\mu}(x_h)}\\ &\le c\eta_{h,\mu}\frac{\delta M_0}{\mu}|D^2u| + 2C_4\Lambda(\sqrt{\varepsilon\tau}|Du| + \varepsilon\tau|u|)\chi_{K_{2/\mu}(x_h)}\end{aligned} \tag{4.16}$$

with $c = c(n)$. Here $K_{2/\mu}(x_h)$ denotes the n-cube centered at x_h with length $4/\mu$ and $\chi_{K_{2/\mu}(x_h)}$ is the characteristic function of $K_{2/\mu}(x_h)$. Substituting (4.14)–(4.16) into (4.13) gives

$$\begin{aligned}&\sum_{|\alpha|\le 2}\tau^{3-2|\alpha|}\int |D^\alpha u|^2 e^{2\tau\varphi(x)}dx\\ &\le C_5\int |P(\delta x, D)u|^2 e^{2\tau\varphi(x)}dx\\ &\quad + C_5\Big\{\frac{\delta^2 M_0^2}{\varepsilon\tau}\int |D^2u|^2 e^{2\tau\varphi(x)}\,dx + \varepsilon\tau\int |Du|^2 e^{2\tau\varphi(x)}\,dx\\ &\qquad + (\varepsilon\tau)^2\int |u|^2 e^{2\tau\varphi(x)}\,dx\Big\}\end{aligned} \tag{4.17}$$

for all $\tau \geq \tau_2$, where $C_5 \geq 1$. Finally, by choosing $\varepsilon = 1/(2C_5)$ and $\delta_0 = \varepsilon$, all terms inside of the curved brace on the right-hand side of (4.17) can be absorbed by its left-hand side and (4.10) follows immediately. ∎

4.1. Carleman estimate for operators depending on the vertical variable

Here we would like to prove a Carleman estimate for the operator that satisfies conditions (3.1)–(3.4) but depending only on the x_n variable. That is, we consider

$$\mathcal{L}(x_n, D)u := \sum_{\pm} H_\pm \operatorname{div}(A_\pm(x_n)\nabla u_\pm),$$

where $u_\pm \in C^\infty(\mathbb{R}^n)$ and $\operatorname{supp} u \subset B'_{r_0} \times [-r_0, r_0]$, where r_0 is the number obtained in Theorem 3.3. Introduce $\delta \in (0,1)$ that will be chosen later, define

$$\phi_\delta(x) := \psi_\delta(\delta^{-1}x) = \psi_\delta(\delta^{-1}x', \delta^{-1}x_n),$$

and consider the scaled operator

$$\mathcal{L}(\delta x_n, D)u := \sum_{\pm} H_\pm \operatorname{div}(A_\pm(\delta x_n)\nabla u_\pm).$$

Notice that $A_\pm(\delta x_n)$ satisfies assumptions (3.3), (3.4) and also the Lipschitz condition

$$|A_\pm(\delta\tilde{x}_n) - A_\pm(\delta x_n)| \leq M_0\delta|\tilde{x}_n - x_n|. \tag{4.18}$$

Let $\vartheta_0 \in C_0^\infty(\mathbb{R})$ be given as in (4.1). For $\mu \geq 1$ satisfying $2/\mu < r_0$, we define

$$\eta_\mu(x_n) = \vartheta_0(\mu x_n), \tag{4.19}$$

$$v_\mu(x', x_n) = \eta_\mu(x_n)u(x', x_n) \quad \text{and} \quad z_\mu(x', x_n) = (1 - \eta_\mu(x_n))u(x', x_n). \tag{4.20}$$

Since $v_{\mu,\pm}(x', 0) = u_\pm(x', 0)$ and $\nabla v_{\mu,\pm}(x', 0) = \nabla u_\pm(x', 0)$, we have trivially, for all $x' \in \mathbb{R}^{n-1}$,

$$v_{\mu,+}(x', 0) - v_{\mu,-}(x', 0) = u_+(x', 0) - u_-(x', 0) = h_0^{(0)}(x'), \tag{4.21}$$

and

$$\begin{aligned} &A_+(0)\nabla v_{\mu,+}(x', 0) \cdot e_n - A_-(0)\nabla v_{\mu,-}(x', 0) \cdot e_n \\ &\quad = A_+(0)\nabla u_+(x', 0) \cdot e_n - A_-(0)\nabla u_-(x', 0) \cdot e_n = h_1^{(0)}(x'). \end{aligned} \tag{4.22}$$

The aim of this section is to prove a simple version of (2.14):

$$
\begin{aligned}
&\sum_{\pm}\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n_{\pm}}|D^k u_{\pm}|^2 e^{2\tau\psi_{\varepsilon,\pm}(x)}\,dx\\
&\qquad+\sum_{\pm}\sum_{k=0}^{1}\tau^{3-2k}\int_{\mathbb{R}^{n-1}}|D^k u_{\pm}(x',0)|^2 e^{2\psi_{\varepsilon}(x',0)}\,dx'\\
&\qquad+\sum_{\pm}\tau^2[e^{\tau\psi_\varepsilon(\cdot,0)}u_{\pm}(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\sum_{\pm}[D(e^{\tau\psi_{\varepsilon,\pm}}u_{\pm})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}\\
&\le C\Bigg(\int_{\mathbb{R}^n_{\pm}}|\mathcal{L}(\delta x_n,D)(u_{\pm})|^2 e^{2\tau\psi_{\varepsilon,\pm}(x)}\,dx+[e^{\tau\psi_\varepsilon(\cdot,0)}h_1^{(0)}]^2_{1/2,\mathbb{R}^{n-1}}\\
&\qquad+[D_{x'}(e^{\tau\psi_\varepsilon}h_0^{(0)})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\tau^3\int_{\mathbb{R}^{n-1}}|h_0^{(0)}|^2 e^{2\tau\psi_\varepsilon(x,0)}\,dx\\
&\qquad+\tau\int_{\mathbb{R}^{n-1}}|h_1^{(0)}|^2 e^{2\tau\psi_\varepsilon(x,0)}\,dx\Bigg).
\end{aligned}
\tag{4.23}
$$

To proceed the proof of (4.23), we first note that $\operatorname{supp} z_\mu\subset B'_{r_0}\times[-r_0,r_0]$ and vanishes in the strip $\mathbb{R}^{n-1}\times[-\frac{1}{\mu},\frac{1}{\mu}]$. It is clear that $A_{\pm}(\delta x_n)$ satisfies (3.3), (3.4), and (2.7). Estimate (3.59) implies that (4.9) holds for $\sum_{1\le j,\ell\le n}a^{\pm}_{j\ell}(x)D^2_{j\ell}$ with $\varphi=\psi_\varepsilon$. Observe that z_μ is supported away from $x_n=0$. It follows from (4.10) in Proposition 4.1 that there exist $\delta_0\in(0,1]$, $\tau_0>0$, and choose a small r_0 if necessary, such that

$$
\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n}|D^k z_\mu|^2 e^{2\tau\psi_\varepsilon(x)}dx\le C\int_{\mathbb{R}^n}|\mathcal{L}(\delta x_n,D)z_\mu|^2\,.^{2\tau\psi_\varepsilon(x)}dx
\tag{4.24}
$$

for all $\tau\ge\tau_0$, $0<\delta\le\delta_0$, where C depends on Λ_0, λ_0, and M_0.

Let us denote by $\mathrm{LHS}(u)$ the left-hand side of inequality (4.23). We have

$$
\begin{aligned}
\mathrm{LHS}(u)&\le 2(\mathrm{LHS}(v_\mu)+\mathrm{LHS}(z_\mu))\\
&=2\Bigg(\mathrm{LHS}(v_\mu)+\sum_{k=0}^{2}\tau^{3-2k}\int_{\mathbb{R}^n}|D^k z_\mu|^2 e^{2\tau\psi_\varepsilon(x)}dx\Bigg).
\end{aligned}
\tag{4.25}
$$

Then applying (3.61) to v_μ and using (4.24) leads to

$$
\begin{aligned}
\mathrm{LHS}(u)\le C\Bigg(&\int_{\mathbb{R}^n}|\mathcal{L}_0(D)v_\mu|^2\,.^{2\tau\psi_\varepsilon(x)}dx+[e^{\tau\psi_\varepsilon(\cdot,0)}h_1^{(0)}]^2_{1/2,\mathbb{R}^{n-1}}\\
&+[D_{x'}(e^{\tau\psi_\varepsilon}h_0^{(0)})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}+\tau^3\int_{\mathbb{R}^{n-1}}|h_0^{(0)}|^2 e^{2\tau\psi_\varepsilon(x',0)}dx'\\
&+\tau\int_{\mathbb{R}^{n-1}}|h_1^{(0)}|^2 e^{2\tau\psi_\varepsilon(x',0)}dx'+\int_{\mathbb{R}^n}|\mathcal{L}(\delta x_n,D)z_\mu|^2\,.^{2\tau\psi_\varepsilon(x)}dx\Bigg).
\end{aligned}
\tag{4.26}
$$

By (3.3), (3.4), (4.18), and (4.19) and since $\mu > 1$, we can estimate

$$
\begin{aligned}
&|\mathcal{L}_0(D)v_\mu| \\
&\quad \le |\mathcal{L}(\delta x_n, D)v_\mu| + |\mathcal{L}(\delta x_n, D)v_\mu - \mathcal{L}_0(D)v_\mu| \\
&\quad \le |\mathcal{L}(\delta x_n, D)u|\eta_\mu + \frac{2\delta M_0}{\mu}\sum_{\pm}|D^2 u_\pm|\eta_\mu \\
&\qquad + C(\delta M_0 + \Lambda_0)\sum_{\pm}(\mu|Du_\pm| + \mu^2|u_\pm|)\chi_{\mathbb{R}^{n-1}\times([-\frac{2}{\mu},\frac{2}{\mu}]\setminus[-\frac{1}{\mu},\frac{1}{\mu}])}.
\end{aligned} \tag{4.27}
$$

On the other hand, we have

$$
\begin{aligned}
&|\mathcal{L}(\delta x_n, D)z_\mu| \\
&\quad \le |\mathcal{L}(\delta x_n, D)u|(1-\eta_\mu) \\
&\qquad + C(\delta M_0 + \Lambda_0)\sum_{\pm}(\mu|Du_\pm| + \mu^2|u_\pm|\chi_{\mathbb{R}^{n-1}\times([-\frac{2}{\mu},\frac{2}{\mu}]\setminus[-\frac{1}{\mu},\frac{1}{\mu}])}.
\end{aligned} \tag{4.28}
$$

Putting (4.27), (4.28), and (4.26) together implies

$$
\mathrm{LHS}(u) \le C_1\Big(\int_{\mathbb{R}^n} |\mathcal{L}(\delta x_n, D)u|^2 e^{2\tau\psi_\varepsilon(x)}\,dx + \mathcal{T}_R\Big) + C_2\mathcal{R} \tag{4.29}
$$

where

$$
\begin{aligned}
\mathcal{T}_R &= [e^{\tau\psi_\varepsilon(\cdot,0)}h_1^{(0)}]^2_{1/2,\mathbb{R}^{n-1}} + [D_{x'}(e^{\tau\psi_\varepsilon}h_0^{(0)})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} \\
&\quad + \tau^3\int_{\mathbb{R}^{n-1}} |h_0^{(0)}|^2 e^{2\tau\psi_\varepsilon(x',0)}\,dx' + \tau\int_{\mathbb{R}^{n-1}} |h_1^{(0)}|^2 e^{2\tau\psi_\varepsilon(x',0)}\,dx', \\
\mathcal{R} &= \frac{\delta^2}{\mu^2}\sum_{\pm}\int_{\mathbb{R}^n_\pm} |D^2u_\pm|^2\,dx + \mu^2\sum_{\pm}\int_{\mathbb{R}^n_\pm} |Du_\pm|^2\,dx + \mu^4\int_{\mathbb{R}^n} |u|^2\,dx,
\end{aligned}
$$

C_1 depends only on Λ_0 and λ_0, and C_2 depends only on Λ_0, λ_0 and M_0.

Now, we choose $\mu = \sqrt{\varepsilon\tau}$ and calculate

$$
\begin{aligned}
\mathrm{LHS}(u) - C_2\mathcal{R} &= \frac{1}{\tau}\Big(1 - \frac{C_2\delta^2}{\varepsilon}\Big)\sum_{\pm}\int_{\mathbb{R}^n_\pm} |D^2u_\pm|^2 e^{2\tau\psi_\varepsilon}\,dx \\
&\quad + \tau(1 - C_2\varepsilon)\sum_{\pm}\int_{\mathbb{R}^n_\pm} |Du_\pm|^2 e^{2\tau\psi_\varepsilon} - : dx \\
&\quad + \tau^3\Big(1 - \frac{C_2\varepsilon}{\tau}\Big)\int_{\mathbb{R}^n} |u|^2 e^{2\tau\psi_\varepsilon}\,dx + \mathcal{T}_L,
\end{aligned} \tag{4.30}
$$

where

$$\mathcal{T}_L = \sum_{\pm}\sum_{k=0}^{1} \tau^{3-2k} \int\limits_{\mathbb{R}^{n-1}} |D^k u_\pm(x',0)|^2 e^{2\psi_\varepsilon(x',0)}\, dx' \\ + \sum_{\pm} \tau^2 [e^{\tau\psi_\varepsilon(\cdot,0)} u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm} [D(e^{\tau\psi_{\varepsilon,\pm}} u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}.$$

By choosing ε and δ satisfying

$$\delta^2 \le \frac{\varepsilon}{2C_2} \quad \text{and} \quad \varepsilon \le \frac{1}{2C_2}, \tag{4.31}$$

estimate (4.23) follows easily from (4.29) and (4.30).

4.2. Carleman estimate for operators depending on all variables

We now want to extend the estimate (4.23) to operators with coefficients depending also on the variables x'. To treat this case we proceed exactly as in [5, Section 4.2, pp. 198–200], that is, we approximate with coefficients depending only on x_n. We use the partition of unity introduced at the beginning of Section 4 and show that

$$\mathrm{LHS}(u) \le C \sum_{g\in\mathbb{Z}^{n-1}} \mathrm{LHS}(u\eta_{g,\mu}) + CR_1, \tag{4.32}$$

where we define

$$\mathrm{LHS}(u) = \sum_{\pm}\sum_{k=0}^{2} \tau^{3-2k} \int\limits_{\mathbb{R}^n_\pm} |D^k u_\pm|^2 e^{2\tau\psi_{\varepsilon,\pm}(x',x_n)}\, dx'\, dx_n \\ + \sum_{\pm}\sum_{k=0}^{1} \tau^{3-2k} \int\limits_{\mathbb{R}^{n-1}} |D^k u_\pm(x',0)|^2 e^{2\psi_\varepsilon(x',0)}\, dx' \\ + \sum_{\pm} \tau^2 [e^{\tau\psi_\varepsilon(\cdot,0)} u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \sum_{\pm} [D(e^{\tau\psi_{\varepsilon,\pm}} u_\pm)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}$$

and

$$R_1 := (\varepsilon\tau)^{1/2} \sum_{\pm} \int\limits_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x',0)} (|D_{x_n} u_\pm(x',0)|^2 + |D_{x'} u_\pm(x',0)|^2 \\ + \tau^2 |u_\pm(x',0)|^2)\, dx'.$$

Remind that $\eta_{g,\mu}$ is defined in (4.6). Notice that Ξ in [5, (4.25)] corresponds to LHS here.

As in [5, Section 4.3], we introduce some local differential operators that only depend on x_n, in such a way that we can apply estimate (4.23). Let us define

$$A^{\delta}_{\pm}(x', x_n) := A_{\pm}(\delta x', \delta x_n), \tag{4.33}$$

$$\mathcal{L}_{\delta}(x', x_n, D)u := \sum_{\pm} H_{\pm} \operatorname{div}(A^{\delta}_{\pm}(x', x_n)\nabla u_{\pm}), \tag{4.34}$$

and the transmission conditions

$$\begin{cases} \theta_0(x') = u_+(x', 0) - u_-(x', 0), \\ \theta_1(x') = A^{\delta}_{+}(x', 0)\nabla u_+(x', 0) \cdot e_n - A^{\delta}_{-}(x', 0)\nabla u_-(x', 0) \cdot e_n. \end{cases}$$

Next, recalling that $x'_g = g/\mu$ and $g \in \mathbb{Z}^{n-1}$, we define

$$A^{\delta,g}_{\pm}(x_n) := A^{\delta}_{\pm}(x'_g, x_n) = A_{\pm}(\delta x'_g, \delta x_n),$$

$$\mathcal{L}_{\delta,g}(x_n, D)u := \sum_{\pm} H_{\pm} \operatorname{div}(A^{\delta,g}_{\pm}(x_n)\nabla u_{\pm}).$$

We notice that $A^{\delta,g}_{\pm}(x_n)$ satisfies assumptions (3.3), (3.4) and also the Lipschitz condition

$$|A^{\delta,g}_{\pm}(\tilde{x}_n) - A^{\delta,g}_{\pm}(x_n)| \le M_0\delta|\tilde{x}_n - x_n|.$$

We now apply (4.23) to each summand and add up with respect to $g \in \mathbb{Z}^{n-1}$ to obtain that

$$\sum_{g\in\mathbb{Z}^{n-1}} \mathrm{LHS}(u\eta_{g,\mu}) \le C \sum_{g\in\mathbb{Z}^{n-1}} (d^{(1)}_{g,\mu} + d^{(2)}_{g,\mu} + d^{(3)}_{g,\mu}), \tag{4.35}$$

where

$$d^{(1)}_{g,\mu} = \int_{\mathbb{R}^n} |\mathcal{L}_{\delta,g}(x_n, D)(u\eta_{g,\mu})|^2 e^{2\tau\psi_\varepsilon(x)}\, dx,$$

$$d^{(2)}_{g,\mu} = \tau^3 \int_{\mathbb{R}^{n-1}} |e^{\tau\psi_\varepsilon(x',0)}\theta_{0;g,\mu}(x')|^2\, dx' + [D_{x'}(e^{\tau\psi_\varepsilon}\theta_{0;g,\mu})(\cdot, 0)]^2_{1/2,\mathbb{R}^{n-1}},$$

$$d^{(3)}_{g,\mu} = \tau \int_{\mathbb{R}^{n-1}} |e^{\tau\psi_\varepsilon(x',0)}\theta_{1;g,\mu}(x')|^2\, dx' + [e^{\tau\psi_\varepsilon(\cdot,0)}\theta_{1;g,\mu}(\cdot)]^2_{1/2,\mathbb{R}^{n-1}},$$

where we set

$$\theta_{0;g,\mu}(x') := u_+(x', 0)\eta_{g,\mu}(x') - u_-(x', 0)\eta_{g,\mu}(x') = \theta_0(x')\eta_{g,\mu},$$

$$\theta_{1;g,\mu}(x') := A^{\delta,g}_{+}(0)\nabla(u_+\eta_{g,\mu}) \cdot e_n - A^{\delta,g}_{-}(0)\nabla(u_-\eta_{g,\mu}) \cdot e_n.$$

We now proceed as in [5, Section 4.3, pp. 201–204] for the estimates of the terms $d^{(j)}_{g,\mu}$, $j=1,2,3$ in (4.35). For the sake of clarity, we show here the estimate of the term $d^{(1)}_{g,\mu}$. By (3.3), (3.4), (2.7), (4.7), and (4.33) we obtain that

$$
\begin{aligned}
&|\mathcal{L}_{\delta,g}(x_n,D)(u\eta_{g,\mu})| \\
&\quad\le |\mathcal{L}_{\delta}(x',x_n,D)(u\eta_{g,\mu})| + |\mathcal{L}_{\delta}(x',x_n,D)(u\eta_{g,\mu}) - \mathcal{L}_{\delta,g}(x_n,D)(u\eta_{g,\mu})| \\
&\quad\le \eta_{g,\mu}|\mathcal{L}_{\delta}(x',x_n,D)u| + C\eta_{g,\mu}\sum_{\pm}|A^{\delta}_{\pm}(x',x_n) - A^{\delta}_{\pm}(x'_g,x_n)||D^2u_{\pm}| \\
&\qquad + C\chi_{Q_{\frac{2}{\mu}}(x'_g)}\sum_{\pm}(\mu|Du_{\pm}| + \mu^2|u_{\pm}|) \\
&\quad\le \eta_{g,\mu}|\mathcal{L}_{\delta}(x',x_n,D)u| + C\chi_{Q_{\frac{2}{\mu}}(x'_g)}\sum_{\pm}(\delta\mu^{-1}|D^2u_{\pm}| + \mu|Du_{\pm}|.\mu^2|u_{\pm}|),
\end{aligned}
$$

which, together with (4.3) and since $\mu=(\varepsilon\tau)^{1/2}>1$, implies

$$
\sum_{g\in\mathbb{Z}^{n-1}} d^{(1)}_{g,\mu} \le C\int_{\mathbb{R}^n}|\mathcal{L}_{\delta}(x',x_n,D)u|^2 .^{2\tau\psi_\varepsilon}\,dx + CR_2, \tag{4.36}
$$

where

$$
\begin{aligned}
R_2 = {}& \frac{\delta^2}{\mu^2}\sum_{\pm}\int_{\mathbb{R}^n_{\pm}}|D^2u_{\pm}|^2 .^{2\tau\psi_{\varepsilon,\pm}}\,dx + \mu^2\sum_{\pm}\int_{\mathbb{R}^n_{\pm}}|Du_{\pm}|^2 .^{2\tau\psi_{\varepsilon,\pm}}\,dx \\
&+ \mu^4\int_{\mathbb{R}^n}|u|^2 .^{2\tau\psi_\varepsilon}\,dx.
\end{aligned}
$$

With similar calculations, which are explicitly written in the above mentioned pages of [5], we can estimate $d^{(2)}_{g,\mu}$, $d^{(3)}_{g,\mu}$ and get

$$
\begin{aligned}
\mathrm{LHS}(u) \le C\Bigg(&\int_{\mathbb{R}^n}|\mathcal{L}_{\delta}(x',x_n,D)u|^2 e^{2\tau\psi_\varepsilon}\,dx + [e^{\tau\psi_\varepsilon(\cdot,0)}\theta_1]^2_{1/2,\mathbb{R}^{n-1}} \\
&+ [D_{x'}(e^{\tau\psi_\varepsilon}\theta_0)(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}} + \tau^3\int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x',0)}|\theta_0(x')|^2\,dx' \\
&+ \tau\int_{\mathbb{R}^{n-1}} e^{2\tau\psi_\varepsilon(x',0)}|\theta_1(x')|^2\,dx' + R_3\Bigg).
\end{aligned}
\tag{4.37}
$$

where

$$
R_3 = \frac{\delta^2}{\mu^2}\sum_{\pm}\int_{\mathbb{R}^n_{\pm}}|D^2u_{\pm}|^2 e^{2\tau\psi_{\varepsilon,\pm}}\,dx + \mu^2\sum_{\pm}\int_{\mathbb{R}^n_{\pm}}|Du_{\pm}|^2 e^{2\tau\psi_{\varepsilon,\pm}}\,dx
$$

$$+ \mu^4 \int_{\mathbb{R}^n} |u|^2 e^{2\tau\psi_\varepsilon}\,dx + (\mu + \delta^2\varepsilon^{-1}) \sum_{\pm} \int_{\mathbb{R}^{n-1}} |Du_\pm(x',0)|^2 e^{2\tau\psi_\varepsilon(x,0)}\,dx$$
$$+ \mu\tau^2 \sum_{\pm} \int_{\mathbb{R}^{n-1}} |u_\pm(x',0)|^2 e^{2\tau\psi_\varepsilon(x',0)}\,dx'$$
$$+ (\mu^4 + \delta^2\mu^{-2}\tau^2) \sum_{\pm} [e^{\tau\psi_\varepsilon(\cdot,0)} u_\pm(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}$$
$$+ \delta^2\mu^{-2} \sum_{\pm} [D(u_\pm e^{\tau\psi_{\varepsilon,\pm}})(\cdot,0)]^2_{1/2,\mathbb{R}^{n-1}}.$$

We now set $\varepsilon = \delta$ and choose a sufficiently small δ_0 and a sufficiently large τ_0, both depending on λ_0, Λ_0, M_0, and n such that if $\varepsilon = \delta \le \delta_0$ inequalities (4.31) are satisfied and if $\tau \ge \tau_0$, then R_3 on the right-hand side of (4.37) can be absorbed by $\mathrm{LHS}(u)$. We finally get the estimate (4.23) by the standard change of variable $u(\delta x', \delta x_n)$.

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