J. Spectr. Theory 12 (2022), 707–744
DOI 10.4171/JST/417

The L^p boundedness of the wave operators for matrix Schrödinger equations

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Abstract. We prove that the wave operators for $n \times n$ matrix Schrödinger equations on the half line, with general selfadjoint boundary condition, are bounded in the spaces $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$, for slowly decaying selfadjoint matrix potentials V that satisfy the condition $\int_0^\infty (1+x)|V(x)|\,dx < \infty$. Moreover, assuming that $\int_0^\infty (1+x^\gamma)|V(x)|\,dx < \infty$, $\gamma > \frac{5}{2}$, and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. We also prove that the wave operators for $n \times n$ matrix Schrödinger equations on the line are bounded in the spaces $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$, assuming that the perturbation consists of a point interaction at the origin and of a potential $\mathcal{V}$ that satisfies the condition $\int_{-\infty}^\infty (1+|x|)|\mathcal{V}(x)|\,dx < \infty$. Further, assuming that $\int_{-\infty}^\infty (1+|x|^\gamma)|\mathcal{V}(x)|\,dx < \infty$, $\gamma > \frac{5}{2}$, and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in $L^1(\mathbb{R}, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}, \mathbb{C}^n)$. We obtain our results for $n \times n$ matrix Schrödinger equations on the line from the results for $2n \times 2n$ matrix Schrödinger equations on the half line.

1. Introduction

In this paper we consider the wave operators for the matrix Schrödinger equation on the half line with general selfadjoint boundary condition

$$\begin{cases} i\dfrac{\partial}{\partial t}u(t,x) = \Big(-\dfrac{\partial^2}{\partial x^2} + V(x)\Big)u(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}^+, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^+, \end{cases} \tag{1.1}$$

$$-B^\dagger u(t,0) + A^\dagger \frac{\partial}{\partial x}u(t,0) = 0. \tag{1.2}$$

2020 Mathematics Subject Classification. Primary 34L25; Secondary 34L10, 34L40, 47A40, 81U99.
Keywords. Wave operators, L^p-boundedness, matrix Schrödinger equations, general boundary conditions, half line, line.

Here $\mathbb{R}^+ := (0, +\infty)$, $u(t, x)$ is a function from $\mathbb{R} \times \mathbb{R}^+$ into $\mathbb{C}^n$, A, B are constant $n \times n$ matrices, and the potential V is a $n \times n$ selfadjoint matrix-valued function of x,

$$V(x) = V^\dagger(x), \quad x \in \mathbb{R}^+. \tag{1.3}$$

The dagger designates the matrix adjoint. Let us denote by M_n the set of all $n \times n$ matrices. We assume that V is in the Faddeev class $L^1_1(\mathbb{R}^+, M_n)$, i.e., that it is a Lebesgue measurable $n \times n$ matrix-valued function and

$$\int_{\mathbb{R}^+} (1 + x)|V(x)|\, dx < \infty, \tag{1.4}$$

where by $|V|$ we denote the matrix norm of V. The more general selfadjoint boundary condition at $x = 0$ has been extensively studied. It can be written in many equivalent ways. See [7,9,23,25,26]. For other formulations of the general selfadjoint boundary condition, see [37]. In this paper we use the parametrization of the boundary condition given in [7] and [9, Section 3.4]. We write the boundary condition as in (1.2), with constant matrices A and B satisfying

$$B^\dagger A = A^\dagger B, \tag{1.5}$$

and

$$A^\dagger A + B^\dagger B > 0. \tag{1.6}$$

We prove that the wave operators for the $n \times n$ matrix Schrödinger equation on the half line (1.1) with the general selfadjoint boundary condition (1.2), (1.5), and (1.6), are bounded in the spaces $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$. For this purpose, we suppose that the potential satisfies (1.3) and (1.4). Assuming that $\int_0^\infty (1 + x^\gamma)|V(x)|\, dx < \infty$, $\gamma > \frac{5}{2}$, and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. We also prove that the wave operators for the $n \times n$ Schrödinger equation on the line, with a point interaction at the origin and a potential, are bounded in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$. We assume that the potential, that we denote by $\mathcal{V}$, is selfadjoint, i.e., $\mathcal{V}(x) = \mathcal{V}(x)^\dagger$, and

$$\int_{\mathbb{R}} (1 + |x|)|\mathcal{V}(x)|\, dx < \infty. \tag{1.7}$$

Further, assuming that $\int_{-\infty}^\infty (1 + |x|^\gamma)|\mathcal{V}(x)|\, dx < \infty$, $\gamma > \frac{5}{2}$, and that the scattering matrix is the identity at zero and infinite energy, we prove that the wave operators are bounded in $L^1(\mathbb{R}, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}, \mathbb{C}^n)$. We obtain the boundedness of the wave operators on the line for a $n \times n$ matrix Schrödinger equation from the boundedness of the wave operators for a $2n \times 2n$ matrix Schrödinger equation on the half line.

In the scalar case, there are several results on the boundedness of the wave operators on the line. Recall that in the scalar case the potential is *generic* if the zero energy Jost solutions from the left and from the right are linearly independent, and that it is *exceptional* if the zero energy Jost solutions from the left and from the right are linearly dependent. In the exceptional case, the stationary Schrödinger equation on the line (5.84) with zero energy, $k^2 = 0$, has a bounded solution, that is called a *zero-energy resonance,* or a *half-bound state.* In [41] it was proven that the wave operators are bounded in $L^p(\mathbb{R}), 1 < p < \infty$, under the assumption

$$\int_{\mathbb{R}} (1 + |x|)^{\gamma} |\mathcal{V}(x)|\, dx < \infty, \tag{1.8}$$

with $\gamma > 3/2$ in the generic case and $\gamma > 5/2$ in the exceptional case. Furthermore, in [41] it was proven that in the exceptional case if the Jost solution from the left at zero energy tends to one as $x \to -\infty$, then the wave operators are bounded in $L^1(\mathbb{R})$ and in $L^\infty(\mathbb{R})$. The paper [41] used a constructive proof that allowed to obtain a detailed low-energy expansion, but that was somehow more demanding concerning the decay of the potential. In [10] the boundedness of the wave operators in $L^p(\mathbb{R}), 1 < p < \infty$, was proven assuming that (1.8) holds with $\gamma = 3$ in the generic case and $\gamma = 4$ in the exceptional case and that, moreover, $\frac{d}{dx}\mathcal{V}(x)$ satisfies (1.8) with $\gamma = 2$, both in the generic and the exceptional cases. The boundedness of the wave operators in $L^p(\mathbb{R}), 1 < p < \infty$, was proven in [16] assuming (1.8) with $\gamma = 1$, in the generic case and with $\gamma = 2$ in the exceptional case. Furthermore, in [17] the boundedness of the wave operators in $L^p(\mathbb{R}), 1 < p < \infty$, was proven for a potential that is the sum of a regular potential that satisfies (1.8) with $\gamma > 3/2$ and of a singular potential that is a sum of Dirac delta functions. In [15] the boundedness of the wave operators was proven for the discrete Schrödinger equation on the line. There is a very extensive literature on the L^p-boundedness of the wave operators and on the related problem of dispersive estimates. For surveys, see [20, 39] and [43] for recent results. In these papers also the results in the multidimensional case are discussed.

The matrix Schrödinger equations find their origin at the very beginning of quantum mechanics. They are important in the description of particles with internal structure like spin and isospin, in atoms, molecules and in nuclear physics, and also in the study systems of particles. A well-known example is the Pauli equation, that is the equation for half-spin particles. For further applications and references, see [2–5, 14, 18, 27, 33, 35].

Since a number of years, there is a renew of the interest in matrix Schrödinger equations due to the importance of these equations for quantum graphs. For example, see [11–13, 22, 25, 26, 28–32], as well as the references quoted there. The matrix Schrödinger equation with a diagonal potential corresponds to a star graph. Such a quantum graph describes the dynamics of n connected very thin quantum wires that form a star-graph, that is, a graph with only one vertex and a finite number of edges of infinite length. This situation appears, for example, in the design of elementary gates in quantum computing, in quantum wires, and in nanotubes for microscopic electronic devices. In these cases strings of atoms can form a star-shaped graph. The analysis of the most general boundary condition at the vertex is important in the applications to problems in physics. A relevant example is the Kirchoff boundary condition. A quantum graph is an idealization of wires with a small cross-section that meet at vertices. The graph is obtained in the limit when the cross-section of the wires goes to zero. As it turns out, the boundary conditions on the vertices of the graph depend on how the limit is taken. A priori, all the boundary conditions in (1.2) can appear in this limit procedure. See [12, Section 7.5] for a detailed discussion of the extensive literature on this problem. Hence, it is relevant to study the more general selfadjoint boundary condition.

The boundedness of the wave operators in L^p spaces is an important problem on itself, and it has important applications. Let us elaborate on this point. For any selfadjoint operator H in a Hilbert space we denote by $\mathcal{H}_{\text{ac}}(H)$ the subspace of absolute continuity of H and by $P_{\text{ac}}(H)$ the orthogonal projector onto $\mathcal{H}_{\text{ac}}(H)$. Moreover, for any pair H, H_0 of selfadjoint operators in a Hilbert space, the wave operators are defined as

$$W_{\pm}(H, H_0) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_0} P_{\text{ac}}(H_0),$$

provided that the strong limits exist. The operator H is the perturbed Hamiltonian, and the operator H_0 is the unperturbed Hamiltonian. The wave operators $W_{\pm}(H, H_0)$ are said *complete* if their range is equal to $\mathcal{H}_{\text{ac}}(H)$. In the theory of scattering, the scattering solutions to the interacting Schrödinger equation

$$i\frac{\partial}{\partial t}u(t) = Hu(t), \quad u(0) = \varphi, \tag{1.9}$$

are defined as $e^{-itH}\varphi$, with $\varphi \in \mathcal{H}_{\text{ac}}(H)$. It is a purpose of scattering theory to compare the behavior for large times of the scattering solutions $e^{-itH}\varphi$ with the scattering solutions for free Schrödinger equation

$$i\frac{\partial}{\partial t}v(t) = H_0 v(t), \quad v(0) = \psi, \tag{1.10}$$

with Hamiltonian H_0, that are given by $e^{-itH_0}\psi$, with $\psi \in \mathcal{H}_{\text{ac}}(H_0)$. If the wave operators exist and are complete, all the scattering solutions $e^{-itH}\varphi$ to the interacting Schrödinger equation behave for large positive and negative times as scattering solutions for the free Schödinger equation

$$\lim_{t\to\pm\infty} \|e^{-itH}\varphi - e^{-itH_0} W_{\pm}(H, H_0)^*\varphi\| = 0, \quad \varphi \in \mathcal{H}_{\text{ac}}.$$

Furthermore, the wave operators fulfill the important intertwining relations

$$f(H)P_{\mathrm{ac}}(H) = W_\pm(H, H_0) f(H_0) P_{\mathrm{ac}}(H_0) W_\pm(H, H_0)^*, \tag{1.11}$$

where f is a Borel function. For these results see [36]. The intertwining relations allow us to obtain important properties of $f(H)P_{\mathrm{ac}}(H)$ from those of $f(H_0)P_{\mathrm{ac}}(H_0)$. Let us explain. Assume that the wave operators $W_\pm(H, H_0)$ are bounded in a Banach space Y and the adjoints $W_\pm(H, H_0)^*$ are bounded in a Banach space X. Then, if $f(H_0)P_{\mathrm{ac}}(H_0)$ is bounded from X into Y, it follows from (1.11) that also $f(H)P_{\mathrm{ac}}(H)$ is bounded between the same spaces and, furthermore,

$$\|f(H)P_{\mathrm{ac}}(H)\|_{\mathcal{B}(X,Y)} \leq C\|f(H_0)P_{\mathrm{ac}}(H_0)\|_{\mathcal{B}(X,Y)}, \tag{1.12}$$

for some constant C and where $\mathcal{B}(X, Y)$ denotes the Banach space of bounded operators from X into Y. In the applications, H_0 is often a constant coefficients operator and $f(H_0)$ is a Fourier multiplier. It is usually a simple matter to obtain important dispersive estimates, like the $L^p - L^{p'}$ estimates, $\frac{1}{p} + \frac{1}{p'} = 1, 1 \leq p \leq \infty$, and the Strichartz estimates for the free Schrödinger equation (1.10) with Hamiltonian H_0, and then (1.12) gives us these estimates for the interacting Schrödinger equation (1.9) with Hamiltonian H. These dispersive estimates play a crucial role in the study of initial value problems and in the scattering theory of nonlinear dispersive equations, like the nonlinear Schrödinger equation, and also in other problems, like the stability of soliton solutions. See [20, 39].

The wave operators are singular integral operators in the spectral representation of the unperturbed operator. On this point, see [19] and [42, Section 1 of Chapter 4] where this question is discussed. As singular integral operators are bounded in L^p spaces, the wave operators are bounded in L^p spaces in the spectral representation of the unperturbed operator. I thank D. R. Yafaev for calling this fact to my attention. Note that in this paper we consider the related, but different, problem of the boundedness of the wave operators in L^p spaces in the configuration representation.

The paper is organized as follows. In Section 2 we introduce the notation that we use. In Section 3 we state our results on the boundedness of the wave operators on the half line. In Section 4 we state our results on the boundedness of the wave operators on the line. In Section 5 we mention the results on the scattering theory of matrix Schrödinger equations that we need and we give the proofs of our theorems.

2. Notation

We denote by $\mathbb{R}^+$ the positive half line $(0, \infty)$, and we designate by $\mathbb{C}$ the complex numbers. For a vector $Y \in \mathbb{C}^n$, we denote by Y^T its transpose. By $\langle\cdot,\cdot\rangle$ we designate the scalar product in $\mathbb{C}^n$. We introduce the following convenient notation. For any vector $Y = (y_1, y_2, \ldots, y_{2n})^T \in \mathbb{C}^{2n}$ we denote by $Y_+ := (y_1, y_2, \ldots, y_n)^T \in \mathbb{C}^n$ the vector with the first n components of Y, and $Y_- := (y_{n+1}, y_{n+2}, \ldots, y_{2n})^T \in \mathbb{C}^n$ the vector with the last n components of Y. Further, we use the notation $Y = (Y_+, Y_-)^T$.

We denote the entries of a $n \times m$ matrix M by $\{M\}_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m$. By 0_n and I_n, $n = 1, 2, \ldots$, we designate the $n \times n$ zero and identity matrices, respectively. By $|M|$ we denote the norm of a matrix M. We designate by $L^p(U, \mathbb{C}^n)$, $1 \leq p \leq \infty$, where $U = \mathbb{R}^+$ or $U = \mathbb{R}$, the Lebesgue spaces of $\mathbb{C}^n$ valued functions defined on U. Let us denote by $C_0^\infty(U, \mathbb{C}^n)$ the space of all infinitely differentiable functions defined on U and that have compact support. We designate by $L^p(U, M_n)$, $1 \leq p \leq \infty$, the Lebesgue space of $n \times n$ matrix valued functions defined on U. Further, we designate by $L^1_\gamma(U, M_n)$, $\gamma > 0$, the Lebesgue space of $n \times n$ matrix-valued functions $V(x)$ such that

$$\int_U (1 + |x|)^\gamma |V(x)| \, dx < \infty.$$

For an integer $m \geq 1$, $\mathbf{H}^{(m)}(U, \mathbb{C}^n)$, where $U = \mathbb{R}^+$ or $U = \mathbb{R}$, is the standard Sobolev space of $\mathbb{C}^n$ valued functions (see [1] for the definition and the properties of these spaces). By $\mathbf{H}^{(m,0)}(\mathbb{R}^+, \mathbb{C}^n)$, $m \geq 1$, we denote the closure of $C_0^\infty(\mathbb{R}^+, \mathbb{C}^n)$ in the space $\mathbf{H}^m(\mathbb{R}^+, \mathbb{C}^n)$. Note that the functions in $\mathbf{H}^{(m,0)}(\mathbb{R}^+, \mathbb{C}^n)$, as well as their derivatives of order up to $m - 1$, are zero at $x = 0$.

The Fourier transform, and the inverse Fourier transform are designated by

$$\mathcal{F} f(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} f(x) \, dx, \quad \mathcal{F}^{-1} f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} f(k) \, dk.$$

For any set $O \subset \mathbb{R}$, we denote by χ_O the characteristic function of O.

For any operator G in a Banach space X, we denote by $D[G]$ the domain of G. Further, for a densely defined operator G in a Banach space, we denote by $G^\dagger$ its adjoint. For any selfadjoint operator H in a Hilbert space and for any Borel set O, we designate by $E(O; H)$ the spectral projector of H for O.

We designate by $\mathcal{E}_{\text{even}}$ the extension operator from $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, to even functions in $L^p(\mathbb{R}, \mathbb{C}^n)$ as follows:

$$(\mathcal{E}_{\text{even}} Y)(x) := \begin{cases} Y(x), & x > 0, \\ Y(-x), & x \leq 0. \end{cases}$$

Clearly, $\mathcal{E}_{\text{even}}$ is bounded from $L^p(\mathbb{R}^+, \mathbb{C}^n)$ into $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \leq p \leq \infty$. Moreover, we denote by $\mathcal{E}_{\text{odd}}$ the extension operator from $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, to odd functions in $L^p(\mathbb{R}, \mathbb{C}^n)$ in the following way:

$$(\mathcal{E}_{\text{odd}} Y)(x) := \begin{cases} Y(x), & x > 0, \\ -Y(-x), & x \leq 0. \end{cases}$$

We have that $\mathcal{E}_{\text{odd}}$ is bounded from $L^p(\mathbb{R}^+, \mathbb{C}^n)$ into $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \leq p \leq \infty$.

We denote by $\mathcal{R}$ the restriction operator from $L^p(\mathbb{R}, \mathbb{C}^n)$ into $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, given by

$$(\mathcal{R} Y)(x) := Y(x), \quad x > 0.$$

We have that $\mathcal{R}$ is bounded from $L^p(\mathbb{R}, \mathbb{C}^n)$ into $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$.

For any integrable $n \times n$ matrix valued function $G(x), x \in \mathbb{R}$, we denote by $Q(G)$ the operator of convolution by $G(x)$,

$$(Q(G)Y)(x) := \int_{\mathbb{R}} G(x-y) Y(y)\, dy = \int_{\mathbb{R}} G(y) Y(x-y)\, dy.$$

Since G is integrable, the operator $Q(G)$ is bounded in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \leq p \leq \infty$. For any $n \times n$ matrix valued measurable function $K(x,y)$ defined for $x, y \in \mathbb{R}^+$, we denote by $\mathbf{K}(K)$ the operator

$$\mathbf{K}(K) Y(x) := \int_{\mathbb{R}^+} K(x,y) Y(y)\, dy.$$

The operator $\mathbf{K}(K)$ is bounded in $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \leq p \leq \infty$, provided that the following two conditions are satisfied:

$$\sup_{x \in \mathbb{R}^+} \int_{\mathbb{R}^+} |K(x,y)|\, dy < \infty, \qquad \sup_{y \in \mathbb{R}^+} \int_{\mathbb{R}^+} |K(x,y)|\, dx < \infty. \tag{2.1}$$

The Hilbert transform $\mathcal{H}$ is defined as follows:

$$(\mathcal{H} Y)(x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{Y(y)}{x-y}\, dy,$$

where PV means the principal value of the integral. As is well known [38, 40], the Hilbert transform is a bounded operator in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$.

3. The wave operators on the half line

To define the wave operators we take as unperturbed Hamiltonian H_0 the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal differential operator $-\frac{d^2}{dx^2}$ with the Neumann boundary condition, $\frac{d}{dx}Y(0) = 0$, see Section 5 below and [9, Sections 3.3 and 3.5]. This choice is motivated by the theory of quantum graphs [25,26]. Note that the spectrum of H_0 is absolutely continuous and that it coincides with $[0, \infty)$. The perturbed Hamiltonian H is the selfadjoint realization in $L^2(\mathbb{R}^+, \mathbb{C}^n)$ of the formal differential operator $-\frac{d^2}{dx^2} + V(x)$ with the boundary condition

$$-B^\dagger Y(0) + A^\dagger \frac{d}{dx} Y(0) = 0, \tag{3.1}$$

where the constant matrices A, B satisfy (1.5) and (1.6), and the potential V fulfills (1.3) and (5.2). For the definition of H see Section 5 below and [9, Sections 3.3 and 3.5].

The wave operators $W_\pm(H, H_0)$ are defined as follows:

$$W_\pm(H, H_0) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_0}, \tag{3.2}$$

since $P_{\text{ac}}(H_0) = I$. It is proven in [9, Section 4.4] that the wave operators $W_\pm(H, H_0)$ exist and are complete.

Our result in the case of $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$, is the following theorem.

Theorem 3.1. *Suppose that V fulfills (1.3) and (1.4) and that the constant matrices A, B satisfy (1.5), and (1.6). Then, for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^n)$, we have*

$$W_\pm(H, H_0)Y = \sum_{j=1}^{3} W_\pm^{(j)} Y, \tag{3.3}$$

where

$$W_\pm^{(1)} Y := (I + \mathbf{K}(K))\mathcal{R}\Big(\frac{\pm i}{2}\mathcal{H}\mathcal{E}_{\text{even}} Y + \frac{1}{2}\mathcal{E}_{\text{even}} Y\Big), \tag{3.4}$$

$$W_\pm^{(2)} Y := (I + \mathbf{K}(K))\mathcal{R}\Big(\frac{\mp i}{2}(\mathcal{H} S_\infty \mathcal{E}_{\text{even}} Y) + \frac{1}{2} S_\infty \mathcal{E}_{\text{even}} Y\Big), \tag{3.5}$$

$$W_\pm^{(3)} Y := (I + \mathbf{K}(K))\mathcal{R}\Big(\frac{\mp i}{2}(\mathcal{H} Q(F_s)\mathcal{E}_{\text{even}} Y) + \frac{1}{2}(Q(F_s)\mathcal{E}_{\text{even}} Y)\Big). \tag{3.6}$$

Moreover, the wave operators $W_\pm(H, H_0)$ restricted to $L^2(\mathbb{R}^+,\mathbb{C}^n) \cap L^p(\mathbb{R}^+,\mathbb{C}^n)$, $1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$ and equations (3.3)–(3.6) hold for all $Y \in L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$. Furthermore, the adjoints of the wave operators $W_\pm(H, H_0)^\dagger$, restricted to $L^2(\mathbb{R}^+,\mathbb{C}^n) \cap L^p(\mathbb{R}^+,\mathbb{C}^n)$, $1 < p < \infty$, extend uniquely to bounded operators on $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$. The $n \times n$ matrix valued function $K(x, y)$, $x, y \in \mathbb{R}^+$ is defined in (5.12). Finally, the quantity S_∞ is defined in (5.16), and the $n \times n$ matrix valued function $F_s(x)$, $x \in \mathbb{R}$, is defined in (5.17).

Our result in the case of $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$ is stated in the next theorem. We first prepare a convenient notation, where the scattering matrix $S(k)$ is defined in (5.15):

$$P_+(x) := \frac{1}{\sqrt{2\pi}}\big(\mathcal{F}\chi_{\mathbf{R}^+}(k)(S(-k) - S_\infty)\big)(x), P_-(x) := \frac{1}{\sqrt{2\pi}}\big(\mathcal{F}^{-1}\chi_{\mathbf{R}^+}(k)(S(k) - S_\infty)\big)(x). \tag{3.7}$$

Theorem 3.2. *Suppose that V fulfills (1.3) and (1.4), that $V \in L^1_\gamma(\mathbb{R}^+, M_n), \gamma > \frac{5}{2}$, that the constant matrices A, B satisfy (1.5) and (1.6), and that $S(0) = S_\infty = I_n$. Then, for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^n)$, we have*

$$W_\pm(H, H_0)Y = Y + \mathbf{K}(K)Y + \mathcal{R}Q(P_\pm)\mathcal{E}_{\text{even}}Y + \mathbf{K}(K)\mathcal{R}Q(P_\pm)\mathcal{E}_{\text{even}}Y. \tag{3.8}$$

The wave operators $W_\pm(H, H_0)$ and $W_\pm(H, H_0)^\dagger$ restricted to $L^2(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, respectively to $L^2(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, extend to bounded operators on $L^1(\mathbb{R}^+)$ and to bounded operators on $L^\infty(\mathbb{R})$. The $n \times n$ matrix valued function $K(x, y), x, y \in \mathbb{R}^+$, is defined in (5.12). Moreover, the scattering matrix, $S(k), k \in \mathbb{R}$, is defined in (5.15), the quantity S_∞ is defined in (5.16), and the $n \times n$ matrix valued functions $P_\pm(x)$ are defined in (3.7).

In Remark 5.8 we prove by means of a counter example that the condition $S(0) = S_\infty = I_n$ is necessary for the boundedness of the wave operators on $L^1(\mathbb{R}^+)$ and on $L^\infty(\mathbb{R})$.

Remark 3.3. It follows from [9, (3.10.37)] that $S_\infty = I_n$ if and only if there are no Dirichlet boundary conditions in the diagonal representation of the boundary matrices given in (5.3)–(5.5). Further, by [9, Theorems 3.8.13 and 3.8.14], $S(0) = I_n$ if and only if the geometric multiplicity μ of the eigenvalue zero of the zero energy Jost matrix $J(0)$ (see (5.11)) is equal to n. Moreover, by [9, Remark 3.8.10], the geometric multiplicity of the eigenvalue zero of $J(0)$ is equal to n if and only if there are n linearly independent bounded solutions to the Schrödinger equation (5.1) with zero energy, $k^2 = 0$, that satisfy the boundary condition (3.1). This corresponds to the *purely exceptional case* where there are n linearly independent half-bound states or zero-energy resonances. We provide below a simple example of this situation, with a non-trivial potential. Consider the scalar case, $n = 1$, with the potential

$$V(x) = \begin{cases} 0, & x > 1, \\ 1, & 0 < x < 1. \end{cases}$$

The Jost solution (see (5.10)) is computed in [9, Example 6.4.1]. For $k \neq 1$, it is given by

$$f(k,x) = \begin{cases} e^{ikx}, & x \geq 1, \\ \frac{1}{2}\Big(1+\frac{k}{\gamma}\Big)e^{ik}e^{i\gamma(x-1)} + \frac{1}{2}\Big(1-\frac{k}{\gamma}\Big)e^{ik}e^{-i\gamma(x-1)}, & 0 \leq x \leq 1, \end{cases}$$

where $\gamma := \sqrt{k^2-1}$. We take the boundary matrices $A = -\sin\theta$, $B = \cos\theta$, with $\theta = \operatorname{arctan}\operatorname{coth} 1$. The boundary condition is $\cos\theta Y(0) + \sin\theta Y'(0) = 0$. The Jost function is given by $J(k) = f(k,0)\cos\theta + f'(k,0)\sin\theta$. We have $J(0) = 0$. Then $S(0) = 1$ and, as we have the Robin boundary condition, $S_\infty = 1$. Of course, these results can be obtained by explicit computation.

4. The wave operators on the line

We obtain our results on the line proving that a $2n \times 2n$ matrix Schrödinger equation on the half line is unitarily equivalent to a $n \times n$ matrix Schrödinger equation on the line with a point interaction at $x = 0$. For this purpose, we follow [9, Section 2.4]. Let us denote by $\mathbf{U}$ the unitary operator from $L^2(\mathbb{R}^+, \mathbb{C}^{2n})$ onto $L^2(\mathbb{R}, \mathbb{C}^n)$, defined as follows:

$$Y(x) = \mathbf{U}Z(x) := \begin{cases} Z_+(x), & x \geq 0, \\ Z_-(-x), & x < 0, \end{cases} \tag{4.1}$$

where $Z = (Z_+, Z_-)^T$, with $Z_+, Z_- \in L^2(\mathbb{R}^+, \mathbb{C}^n)$. Let us take as potential the diagonal matrix

$$V(x) := \begin{Bmatrix} V_+(x) & 0_n \\ 0_n & V_-(x) \end{Bmatrix},$$

where both V_+ and V_- are selfadjoint $n \times n$ matrix-valued functions that belong to $L^1_1(\mathbb{R}^+, M_n)$. Under the action of the unitary transformation $\mathbf{U}$ the Hamiltonian in the half line H is unitarily transformed into the Hamiltonian on the line $H_\mathbb{R}$ as follows:

$$H_\mathbb{R} := \mathbf{U}H\mathbf{U}^\dagger, \quad D[H_\mathbb{R}] := \{Y \in L^2(\mathbb{R}, \mathbb{C}^n) \colon \mathbf{U}^\dagger Y \in D[H]\}. \tag{4.2}$$

The operator $H_\mathbb{R}$ is a selfadjoint realization in $L^2(\mathbb{R}, \mathbb{C}^n)$ of the formal differential operator $-\frac{d^2}{dx^2} + \mathcal{V}(x)$, where the selfadjoint $n \times n$ matrix valued potential $\mathcal{V}$ is given by

$$\mathcal{V}(x) = \begin{cases} V_+(x), & x \geq 0, \\ V_-(-x), & x < 0. \end{cases}$$

Note that $\mathcal{V} \in L^1_1(\mathbb{R}, M_n)$. The boundary condition (3.1) satisfied by the functions in the domain of H implies that the functions in the domain of $H_\mathbb{R}$ fulfill a transmission condition at $x = 0$. To compute this transmission condition, it is convenient to write the matrices A and B in (3.1) in the following way:

$$A = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix}, \quad B = \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix}, \tag{4.3}$$

where $A_j, B_j, j = 1, 2$, are $n \times 2n$ matrices. Hence, (3.1) implies that the functions in the domain of $H_{\mathbb{R}}$ satisfy the following transmission condition at $x = 0$:

$$-B_1^\dagger Y(0^+) - B_2^\dagger Y(0^-) + A_1^\dagger \frac{d}{dx} Y(0^+) - A_2^\dagger \frac{d}{dx} Y(0^-) = 0. \tag{4.4}$$

Remark that $u(t, x)$ is a solution of the problem (1.1), (1.2) if and only if $v(t, x) := \mathbf{U}u(t, x)$ is a solution of the following $n \times n$ matrix equation on the line:

$$\begin{cases} i\dfrac{\partial}{\partial t} v(t,x) = \Big(-\dfrac{\partial^2}{\partial x^2} + \mathcal{V}(x)\Big) v(t,x), & t \in \mathbb{R},\ x \in \mathbb{R}, \\ v(0,x) = v_0(x) := \mathbf{U}u_0(x), & x \in \mathbb{R}, \\ -B_1^\dagger v(t,0^+) - B_2^\dagger v(t,0^-) + A_1^\dagger \dfrac{\partial}{\partial x} v(t,0^+) - A_2^\dagger \dfrac{\partial}{\partial x} v(t,0^-) = 0. \end{cases} \tag{4.5}$$

Below we give an example. Let A, B be the following matrices:

$$A = \begin{Bmatrix} 0_n & I_n \\ 0_n & I_n \end{Bmatrix}, \quad B = \begin{Bmatrix} -I_n & \Lambda \\ I_n & 0_n \end{Bmatrix}, \tag{4.6}$$

where Λ is a selfadjoint $n \times n$ matrix. It is easy to check that these matrices satisfy (1.5) and (1.6). The transmission condition in (4.5) is given by

$$v(t,0^+) = v(t,0^-) = v(t,0), \quad \frac{\partial}{\partial x} v(t,0^+) - \frac{\partial}{\partial x} v(t,0^-) = \Lambda v(t,0). \tag{4.7}$$

This transmission condition is a Dirac-delta point interaction at $x = 0$ with coupling matrix Λ. In the particular case $\Lambda = 0$, the functions $v(t, x)$ and $\frac{\partial}{\partial x} v(t, x)$ are continuous at $x = 0$ and we have the matrix Schrödinger equation on the line without a point interaction at $x = 0$.

Let us denote by $H_{0,\mathbb{R}}$ the Hamiltonian (4.2) with the potential $\mathcal{V}$ identically zero and with the boundary condition given by the matrices (4.6) with $\Lambda = 0$. Note that $H_{0,\mathbb{R}}$ is the standard selfadjoint realization of the formal differential operator $-\frac{d^2}{dx^2}$ with domain $D[H_{0,\mathbb{R}}] := \mathbf{H}^{(2)}(\mathbb{R}, \mathbb{C}^n)$. In particular, $H_{0,\mathbb{R}}$ is absolutely continuous and its spectrum consists of $[0, \infty)$. We define the wave operators on the line as follows:

$$W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}}) := \text{s-}\lim_{t \to \pm\infty} e^{itH_{\mathbb{R}}} e^{-itH_{0,\mathbb{R}}}. \tag{4.8}$$

Using Theorem 3.1 and the unitary transformation (4.2) we prove the following theorem, on the boundedness of the wave operators on $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$.

Theorem 4.1. *Let $H_{\mathbb{R}}$ be the Hamiltonian* (4.2)*, with the transmission condition* (4.4)*, and where $\mathcal{V}(x)$, $x \in \mathbb{R}$, is a $n \times n$ selfadjoint matrix-valued function, i.e., $\mathcal{V}(x) = \mathcal{V}^\dagger(x)$ and, moreover, $\mathcal{V}$ satisfies* (1.7)*. Then, the wave operators $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})$ exist and are complete. Moreover, the $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})$, and the adjoint wave operators $W(H_{\mathbb{R}}, H_{0,\mathbb{R}})^\dagger$, restricted to $L^2(\mathbb{R}, \mathbb{C}^n) \cap L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$.*

Theorem 4.1 generalizes the results obtained in [10, 16, 17, 41] to the case of general point interactions at $x = 0$ and to potentials that satisfy (1.8) with $\gamma = 1$.

Below we state our theorem on the boundedness of the wave operators on the line in $L^1(\mathbb{R}, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}, \mathbb{C}^n)$.

Theorem 4.2. *Let $H_{\mathbb{R}}$ be the Hamiltonian* (4.2)*, with the transmission condition* (4.4)*, and where $\mathcal{V}(x)$, $x \in \mathbb{R}$, is a $n \times n$ selfadjoint matrix-valued function, i.e., $\mathcal{V}(x) = \mathcal{V}^\dagger(x)$, and $\mathcal{V} \in L^1_\gamma(\mathbb{R}, M_n)$, $\gamma > \frac{5}{2}$. Let $S_{\mathbb{R}}(0) = S_{\mathbb{R},\infty} = I_{2n}$. Then, the wave operators $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})$ and $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})^\dagger$ restricted to $L^2(\mathbb{R}, \mathbb{C}^n) \cap L^1(\mathbb{R}, \mathbb{C}^n)$, respectively to $L^2(\mathbb{R}, \mathbb{C}^n) \cap L^\infty(\mathbb{R}, \mathbb{C}^n)$, extend to bounded operators on $L^1(\mathbb{R}, \mathbb{C}^n)$ and to bounded operators on $L^\infty(\mathbb{R}, \mathbb{C}^n)$. The scattering matrix on the line $S_{\mathbb{R}}(k)$, $k \in \mathbb{R}$, is defined in* (5.100a) *and the quantity $S_{\mathbb{R},\infty}$ in* (5.102)*.*

Remark 4.3. In the case where the matrices A, B are equal to the matrices in (4.6) and $\Lambda = 0$, there is no point interaction at $x = 0$. The scattering theory in this situation has been studied in [6], and the references quoted there. In this case $S_{\mathbb{R}}(0) = S_{\mathbb{R},\infty} = I_{2n}$ in the *purely exceptional case* where there are n linearly independent bounded solutions to the Schrödinger equation (5.84) with zero energy, $k^2 = 0$ (in this case we say that there are n linearly independent *zero-energy resonances,* or *half-bound states*), and if, moreover, the zero-energy Jost solution from the left $f_l(0, x)$ satisfies $\lim_{x \to -\infty} f_l(0, x) = I_n$ For the definition of $f_l(k, x), k \in \mathbb{R}$, see (5.85).

As we have mentioned above, in [41, Theorem 1.1] we proved that in the scalar case, $n = 1$, and without point interactions, the wave operators $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})$ and $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})^\dagger$ extend to bounded operators on $L^1(\mathbb{R}, \mathbb{C})$ and to bounded operators on $L^\infty(\mathbb{R}, \mathbb{C})$ in the exceptional case, and assuming that $\lim_{x \to -\infty} f_l(0, x) = 1$, for potentials that satisfy (1.8) with $\gamma > \frac{5}{2}$. Theorem 4.2 generalizes this result of [41] to the case where there is a general point interaction.

5. Scattering theory and the L^p-boundedness of the wave operators

5.1. Scattering theory for the matrix Schrödinger equation on the half line

We study the following the stationary matrix Schrödinger equation on the half line

$$-\frac{d^2}{dx^2} Y(x) + V(x) Y(x) = k^2 Y(x), \quad x \in \mathbb{R}^+. \tag{5.1}$$

In this equation, k^2 is the complex-valued spectral parameter, the $n \times n$ matrix valued potential $V(x)$ satisfies (1.3) and, moreover,

$$V \in L^1(\mathbb{R}^+, M_n). \tag{5.2}$$

The solution Y that appears in (5.1) is either a column vector with n components, or a $n \times n$ matrix-valued function. As we already mentioned, the general selfadjoint boundary condition at $x = 0$ can be expressed in terms of two constant $n \times n$ matrices A and B as in (3.1), where the matrices A and B fulfill (1.5), (1.6).

Actually, there is a simpler equivalent form of the boundary condition (3.1). In fact, in [8] and in [9, Section 3.4], it is given the explicit steps to go from any pair of matrices A and B appearing in the selfadjoint boundary condition (3.1), and that satisfy (1.5), (1.6) to a pair $\widetilde{A}$ and $\widetilde{B}$, given by

$$\widetilde{A} = -\operatorname{diag}\{\sin\theta_1, \ldots, \sin\theta_n\}, \quad \widetilde{B} = \operatorname{diag}\{\cos\theta_1, \ldots, \cos\theta_n\}, \tag{5.3}$$

with appropriate real parameters $\theta_j \in (0, \pi]$. The matrices $\widetilde{A}$, $\widetilde{B}$ satisfy (1.5), (1.6). In the case of the matrices $\widetilde{A}$, $\widetilde{B}$, the boundary condition (3.1) is given by

$$\cos\theta_j Y_j(0) + \sin\theta_j \frac{d}{dx} Y_j(0) = 0, \quad j = 1, 2, \ldots, n. \tag{5.4}$$

The case $\theta_j = \pi$ corresponds to the Dirichlet boundary condition and the case $\theta_j = \pi/2$ corresponds to the Neumann boundary condition. In the general case, there are $n_{\mathrm{N}} \le n$ values with $\theta_j = \pi/2$ and $n_{\mathrm{D}} \le n$ values with $\theta_j = \pi$. Further, there are n_{M} remaining values, where $n_{\mathrm{M}} = n - n_{\mathrm{N}} - n_{\mathrm{D}}$ such that those θ_j-values lie in the interval $(0, \pi/2)$ or $(\pi/2, \pi)$. It is proven in [8] and in [9, Section 3.4], that, for any pair of matrices (A, B) that satisfy (1.5), (1.6), there is a pair of matrices $(\widetilde{A}, \widetilde{B})$ as in (5.3), a unitary matrix M, and two invertible matrices T_1, T_2 such

$$A = M\widetilde{A}T_1M^\dagger T_2, \quad B = M\widetilde{B}T_1M^\dagger T_2. \tag{5.5}$$

As we will see, the Hamiltonians with the boundary condition given by matrices A, B and with the matrices $\widetilde{A}$, $\widetilde{B}$, are unitarily equivalent.

We construct a selfadjoint realization of the matrix Schrödinger operator $-\frac{d^2}{dx^2} + V(x)$ by quadratic forms methods. For the following discussion see [9, Sections 3.3 and 3.5]. Let θ_j be as in equations (5.3). We denote

$$\widehat{\mathbf{H}}_j^{(1)}(\mathbb{R}^+, \mathbb{C}) := \mathbf{H}^{(1,0)}(\mathbb{R}^+, \mathbb{C}) \quad \text{if } \theta_j = \pi \tag{5.6a}$$

and

$$\widehat{\mathbf{H}}_j^{(1)}(\mathbb{R}^+, \mathbb{C}) := \mathbf{H}^{(1)}(\mathbb{R}^+, \mathbb{C}) \quad \text{if } \theta_j \neq \pi. \tag{5.6b}$$

We put

$$\widetilde{\mathbf{H}}^{(1)}(\mathbb{R}^+, \mathbb{C}^n) := \bigoplus_{j=1}^{n} \hat{\mathbf{H}}_j^{(1)}(\mathbb{R}^+, \mathbb{C}).$$

We define

$$\Theta := \mathrm{diag}[\widehat{\cot}\,\theta_1, \ldots, \widehat{\cot}\,\theta_n],$$

where $\widehat{\cot}\,\theta_j = 0$, if $\theta_j = \pi/2$, or $\theta_j = \pi$, and $\widehat{\cot}\,\theta_j = \cot\theta_j$, if $\theta_j \neq \pi/2, \pi$. Suppose that the potential V satisfies (1.3) and (5.2). The following quadratic form is closed, symmetric and bounded below:

$$h(Y, Z) := \left(\frac{d}{dx}Y, \frac{d}{dx}Z\right)_{L^2(\mathbb{R}^+,\mathbb{C}^n)} - \langle M\Theta M^\dagger Y(0), Z(0)\rangle + (VY, Z)_{L^2(\mathbb{R}^+,\mathbb{C}^n)}, \tag{5.7a}$$

$$Q(h) := \mathbf{H}^{(A,B)}(\mathbb{R}^+, \mathbb{C}^n), \tag{5.7b}$$

where by $Q(h)$ we denote the domain of h and

$$\mathbf{H}^{(A,B)}(\mathbb{R}^+, \mathbb{C}^n) := M\widetilde{\mathbf{H}}^{(1)}(\mathbb{R}^+, \mathbb{C}^n) \subset \mathbf{H}^{(1)}(\mathbb{R}^+, \mathbb{C}^n). \tag{5.8}$$

We denote by $H_{A,B,V}$ the selfadjoint bounded below operator associated to h, see [24]. The operator $H_{A,B,V}$ is the selfadjoint realization of $-\frac{d^2}{dx^2} + V(x)$ with the selfadjoint boundary condition (3.1). When there will be no possibility of misunderstanding, we will use the notation H, i.e., $H \equiv H_{A,B,V}$. It is proven in [9, Section 3.6] that

$$H_{A,B,V} = MH_{\widetilde{A},\widetilde{B},M^\dagger VM}M^\dagger. \tag{5.9}$$

In the next proposition, we introduce the Jost solution given in [5]. See also [9, Sections 3.1 and 3.2].

Proposition 5.1. *Suppose that the potential V satisfies condition (5.2). For each fixed $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, there exists a unique $n \times n$ matrix-valued Jost solution $f(k, x)$ to equation (5.1) satisfying the asymptotic condition*

$$f(k, x) = e^{ikx}(I + o(1)), \quad x \to +\infty. \tag{5.10}$$

Moreover, for any fixed $x \in [0, \infty)$, $f(k, x)$ is analytic in $k \in \mathbb{C}^+$ and continuous in $k \in \overline{\mathbb{C}^+} \setminus \{0\}$. If V satisfies (1.4), then the Jost solution also exists at $k = 0$, and, for each fixed $x \in [0, \infty)$, $f(k, x)$ is continuous in $k \in \overline{\mathbb{C}^+}$. Furthermore, if (1.4) holds, for $k \in \overline{\mathbb{C}^+} \setminus \{0\}$, the $o(1)$ in (5.10) can be replaced by $o(\frac{1}{x})$.

Using the Jost solution and the boundary matrices A and B satisfying (1.5), (1.6), we construct the Jost matrix $J(k)$,

$$J(k) = f(-k^*, 0)^\dagger B - f'(-k^*, 0)^\dagger A, \quad k \in \overline{\mathbb{C}^+}, \tag{5.11}$$

where the asterisk denotes complex conjugation. For the following result see [5] and also [9, Theorem 3.8.1].

Proposition 5.2. *Suppose that the potential V satisfies (1.3) and (1.4). Then the Jost matrix $J(k)$ is analytic for $k \in \mathbb{C}^+$, continuous for $k \in \overline{\mathbb{C}^+}$, and invertible for $k \in \mathbb{R} \setminus \{0\}$.*

Let $K(x, y)$ be defined as follows:

$$K(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} [f(k, x) - e^{ikx} I] e^{-iky} dk, \quad x, y \geq 0. \tag{5.12}$$

We introduce the quantities

$$\sigma(x) = \int_x^{\infty} |V(y)| \, dy, \quad \sigma_1(x) = \int_x^{\infty} y \, |V(y)| \, dy, \quad x \geq 0.$$

Remark that for potentials satisfying (1.4), both $\sigma(0)$ and $\sigma_1(0)$ are finite, and furthermore, $\int_0^\infty \sigma(x) \, dx = \sigma_1(0) < \infty$.

The following proposition is given in [5]. See also [9, Proposition 3.28].

Proposition 5.3. *Suppose that the potential V satisfies (1.3) and (1.4). Then*

1. *the matrix $K(x, y)$ is continuous in (x, y) in the region $0 \leq x \leq y$ and is related to the potential via*

$$K(x, x^+) = \frac{1}{2} \int_x^{\infty} V(z) dz, \quad x \in [0, +\infty);$$

2. *the matrix $K(x, y)$ satisfies*

$$\begin{cases} K(x, y) = 0, & y < x, \ x, y \in [0, \infty), \\ |K(x, y)| \leq \frac{1}{2} e^{\sigma_1(x)} \sigma\left(\frac{x + y}{2}\right), & x, y, \in \mathbb{R}^+; \end{cases} \tag{5.13}$$

3. *the Jost solution $f(k, x)$ has the representation*

$$f(k, x) = e^{ikx} I + \int_x^{\infty} e^{iky} K(x, y) \, dy. \tag{5.14}$$

The scattering matrix $S(k)$ is a $n \times n$ matrix-valued function of $k \in \mathbb{R}$ that is given by

$$S(k) = -J(-k) J(k)^{-1}, \quad k \in \mathbb{R}. \tag{5.15}$$

In the exceptional case where $J(0)$ is not invertible, the scattering matrix is defined by (5.15) only for $k \neq 0$. However, it is proven in [7] and in [9, Theorem 3.8.14] that, for potentials satisfying (1.3) and (1.4), the limit $S(0) := \lim_{k\to 0} S(k)$ exists in the exceptional case and, moreover, a formula for $S(0)$ is given.

It is proven in [8] and in [9, Theorem 3.10.6] that the following limit exists:

$$S_\infty := \lim_{|k|\to\infty} S(k). \tag{5.16}$$

Let us denote by F_s the following quantity, that up to the factor $1/\sqrt{2\pi}$ is the inverse Fourier transform of $S(k) - S_\infty$:

$$F_s(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - S_\infty] e^{iky}\, dk, \quad y \in \mathbb{R}. \tag{5.17}$$

The following theorem is proven in [34].

Theorem 5.4. *Suppose that the potential V satisfies* (1.3) *and* (1.4). *Then*

$$F_s \in L^1(\mathbb{R}, M_n). \tag{5.18}$$

In terms of the Jost solution $f(k,x)$ and the scattering matrix $S(k)$, we construct the physical solution ([8] and [9, (2.2.29)])

$$\Psi(k,x) = f(-k,x) + f(k,x)S(k), \quad k \in \mathbb{R}. \tag{5.19}$$

The physical solution $\Psi(k,x)$ is the main input to construct the generalized Fourier maps $\mathbf{F}^{\pm}$ for the absolutely continuous subspace of H that are defined in [9, (4.3.44)] (see also [9, Proposition 4.3.4]),

$$(\mathbf{F}^{\pm} Y)(k) = \sqrt{\frac{1}{2\pi}} \int_0^{\infty} (\Psi(\mp k, x))^{\dagger}\, Y(x)\, dx, \tag{5.20}$$

for $Y \in L^1(\mathbb{R}^+, \mathbb{C}^n) \cap L^2(\mathbb{R}^+, \mathbb{C}^n)$.

We have (see [9, (4.3.46)])

$$\|\mathbf{F}^{\pm} Y\|_{L^2(\mathbb{R}^+,\mathbb{C}^n)} = \|E(\mathbb{R}^+; H) Y\|_{L^2(\mathbb{R}^+,\mathbb{C}^n)}. \tag{5.21}$$

Thus, $\mathbf{F}^{\pm}$ extend to bounded operators on $L^2(\mathbb{R}^+, \mathbb{C}^n)$ that we also denote by $\mathbf{F}^{\pm}$.

The following results on the spectral theory of H are proven in [9, Theorems 3.11.1 and 4.3.3 and Proposition 4.3.4].

Theorem 5.5. *Suppose that the potential V satisfies* (1.3) *and* (5.2)*, and that the constant matrices A, B fulfill* (1.5) *and* (1.6)*. Then the Hamiltonian H has no positive eigenvalues and the negative spectrum of H consists of isolated eigenvalues of multiplicity smaller or equal than n that can accumulate only at zero. Furthermore, H has no singular continuous spectrum and its absolutely continuous spectrum is given by* $[0,\infty)$*. The generalized Fourier maps $\mathbf{F}^{\pm}$ are partially isometric with initial subspace $\mathcal{H}_{\mathrm{ac}}(H)$ and final subspace $L^2(\mathbb{R}^+,\mathbb{C}^n)$. Moreover, the adjoint operators are given by*

$$((\mathbf{F}^{\pm})^{\dagger}Z)(x)=\sqrt{\frac{1}{2\pi}}\int_0^{\infty}\Psi(\mp k,x)Z(k)\,dk, \tag{5.22}$$

for $Z\in L^1(\mathbb{R}^+,\mathbb{C}^n)\cap L^2(\mathbb{R}^+,\mathbb{C}^n)$. Furthermore,

$$\mathbf{F}^{\pm}H(\mathbf{F}^{\pm})^{\dagger}=\mathcal{M}, \tag{5.23}$$

where $\mathcal{M}$ is the operator of multiplication by k^2. If, in addition, $V\in L^1_1(\mathbb{R}^+,M_n)$, then 0 *is not an eigenvalue and the number of eigenvalues of H including multiplicities is finite.*

Note that, by (5.21), $(\mathbf{F}^{\pm})^{\dagger}\mathbf{F}^{\pm}$ is the orthogonal projector onto $\mathcal{H}_{\mathrm{ac}}(H)$,

$$(\mathbf{F}^{\pm})^{\dagger}\mathbf{F}^{\pm}=P_{\mathrm{ac}}(H). \tag{5.24}$$

We denote by F_0 the cosine transform,

$$(F_0Y)(k):=\sqrt{\frac{2}{\pi}}\int_0^{\infty}dx\cos(kx)Y(x),\quad Y\in L^2(\mathbf{R}^+). \tag{5.25}$$

F_0 coincides with the generalized Fourier maps for H_0 given by Theorem 5.5.

The following theorem, proven in [9, Theorem 4.4.3], gives the stationary formulae for the wave operators.

Theorem 5.6. *Suppose that V satisfies* (1.3) *and* (5.2)*. Then the wave operators $W_{\pm}(H,H_0)$ exist and are complete. Further, the following the stationary formulae hold:*

$$W_{\pm}(H,H_0)=(\mathbf{F}^{\pm})^{\dagger}F_0. \tag{5.26}$$

5.2. L^p-boundedness of the wave operators in the half-line

We prepare the following proposition.

Proposition 5.7. *Suppose that* $Y \in L^2(\mathbb{R}, \mathbb{C}^n)$. *Then*

$$\mathcal{F}^{-1}\big(\chi_{\mathbb{R}^+}(k)(\mathcal{F}Y)(k)\big)(x) = \frac{i}{2}(\mathcal{H}Y)(x) + \frac{1}{2}Y(x), \quad x \in \mathbb{R}, \tag{5.27}$$

and

$$\mathcal{F}\big(\chi_{\mathbb{R}^+}(k)(\mathcal{F}^{-1}Y)(k)\big)(x) = \frac{-i}{2}(\mathcal{H}Y)(x) + \frac{1}{2}Y(x), \quad x \in \mathbb{R}. \tag{5.28}$$

Proof. The proof is an immediate consequence of [21, equations (3) and (4)]. ∎

Proof of Theorem 3.1. Let us first take $Y \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^n)$. Note that one has $\mathcal{E}_{\text{even}} Y \in C_0^\infty(\mathbb{R}, \mathbb{C}^n)$. By (5.25),

$$(F_0 Y)(k) = (\mathcal{F}\mathcal{E}_{\text{even}} Y)(k), \quad k \in \mathbb{R}^+. \tag{5.29}$$

By (5.14), (5.19), (5.22), (5.26), and (5.29),

$$(W_\pm(H, H_0)Y)(x) := \sum_{j=1}^{6} T_\pm^{(j)}(x), \tag{5.30}$$

where

$$T_\pm^{(1)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ikx} \chi_{\mathbf{R}^+}(k)(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk, \tag{5.31}$$

$$T_\pm^{(2)}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty dz\, K(x,z) \int_{\mathbb{R}} e^{\pm ikz} \chi_{\mathbb{R}^+}(k)(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk, \tag{5.32}$$

$$T_\pm^{(3)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbb{R}^+}(k) S_\infty(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk, \tag{5.33}$$

$$T_\pm^{(4)}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty dz\ K(x,z) \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) S_\infty(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk, \tag{5.34}$$

$$T_\pm^{(5)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbf{R}^+}(k)(S(\mp k) - S_\infty)(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk, \tag{5.35}$$

and

$$T_\pm^{(6)}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty dz\, K(x,z) \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k)(S(\mp k) - S_\infty)(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk. \tag{5.36}$$

Observe that

$$(\mathcal{F}\mathcal{E}_{\text{even}} Y)(k) = (\mathcal{F}^{-1}\mathcal{E}_{\text{even}} Y)(k), \quad k \in \mathbb{R}. \tag{5.37}$$

It follows from Proposition 5.7, and (5.30)–(5.37) that (3.3) holds for $Y \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^n)$. Finally, approximating $Y \in L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$, by a sequence of functions in $C_0^\infty(\mathbb{R}^+, \mathbb{C}^n)$, it follows that equations (3.3)–(3.6) hold for all $Y \in L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$, and that the wave operators $W_\pm(H, H_0)$ extend to bounded operators on $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$. Here we used the fact that $Q(F_s)$ is bounded in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \le p \le \infty$, since by Theorem 5.4 $F_s \in L^1(\mathbb{R}, \mathbb{C}^n)$, that $\mathbf{K}(K)$ is bounded in $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \le p \le \infty$, because by (5.13), equations (2.1) hold, and that $\mathcal{H}$ is bounded in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 < p < \infty$, see [38, 40]. The operator $\mathcal{R}$ is clearly bounded from $L^p(\mathbb{R}, \mathbb{C}^n)$ into $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \le p \le \infty$. The wave operators $W_\pm(H, H_0)^\dagger$ extend to bounded operators on $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 < p < \infty$, by duality. ■

Proof of Theorem 3.2. Let us take $Y \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^n)$. Recall that $\mathcal{E}_{\text{even}} Y \in C_0^\infty(\mathbb{R}, \mathbb{C}^n)$, and that (5.29) holds. By (5.31), (5.33), $S_\infty = I_n$, and since $\mathcal{F}\mathcal{E}_{\text{even}} Y$ is an even function,

$$
\begin{aligned}
T_\pm^{(1)}(x) + T_\pm^{(3)}(x) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ikx} \chi_{\mathbf{R}^+}(k) (\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbb{R}^+}(k) (\mathcal{F}\mathcal{E}_{\text{even}} Y)(k)\, dk \\
&= \mathcal{E}_{\text{even}} Y(x) = Y(x), \quad x \ge 0,
\end{aligned} \tag{5.38}
$$

where in the second integral in the middle equation in (5.38) we made the change of variable of integration $k \to -k$, and we used both $\mathcal{F}\mathcal{E}_{\text{even}} Y(k) = \mathcal{F}^{-1} \mathcal{E}_{\text{even}} Y(k)$ and $S_\infty = I_n$. We similarly prove, using (5.32) and (5.34),

$$
T_\pm^{(2)}(x) + T_\pm^{(4)}(x) = (\mathbf{K}(K)Y)(x). \tag{5.39}
$$

Hence, by (5.30), (5.38), and (5.39),

$$
(W_\pm Y)(x) = Y(x) + (\mathbf{K}(K)Y)(x) + T_\pm^{(5)}(x) + T_\pm^{(6)}(x). \tag{5.40}
$$

By (5.35), (5.36), and the convolution theorem of the Fourier transform,

$$
T_\pm^{(5)}(x) = (Q(P_\pm)\mathcal{E}_{\text{even}} Y)(x), \quad x \ge 0, \tag{5.41}
$$

and

$$
T_\pm^{(6)}(x) = (\mathbf{K}(K)\mathcal{R}Q(P_\pm)\mathcal{E}_{\text{even}} Y)(x), \tag{5.42}
$$

where $P_\pm$ is defined in (3.7). Equation (3.8) for $Y \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^n)$ follows from (5.40), (5.41), and (5.42). Moreover, as $\mathcal{R}$ is bounded from $L^p(\mathbb{R}, \mathbb{C}^n)$ into $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \le p \le \infty$, $\mathcal{E}_{\text{even}}$ is bounded from $L^p(\mathbb{R}^+, \mathbb{C}^n)$ into $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \le p \le \infty$ and

$\mathbf{K}(K)$ is bounded in $L^p(\mathbb{R}^+, \mathbb{C}^n)$, $1 \le p \le \infty$, because by (5.13) equations (2.1) hold. Moreover, by the Schwarz inequality,

$$\begin{aligned} \|P_\pm\|_{L^1(\mathbb{R}, M^n)} &\le \|(1+|x|^2)^{-1/2}\|_{L^2(\mathbb{R}^+)} \|(1+|x|^2)^{1/2} P_\pm\|_{L^2(\mathbb{R}, M^n)} \\ &\le C \|\chi_{\mathbf{R}^+}(k)(S(\mp k) - S_\infty)\|_{\mathbf{H}^{(1)}(\mathbb{R}, M_n)}. \end{aligned} \tag{5.43}$$

By the definition of $S(k)$ in (5.15) and by [9, Proposition 3.2.4 and Theorems 3.81, and 3.9.15], $S(k)$ is differentiable for $k \in \mathbb{R}$, with continuous derivative for $k \in \mathbb{R} \setminus \{0\}$. Then, since $S(0) = S_\infty$ and by Proposition A.3, $S(k) - S_\infty \in \mathbf{H}^{(1)}(\mathbb{R}^+, M_n)$, we have $\chi_{\mathbf{R}^+}(k)(S(\mp k) - S_\infty) \in \mathbf{H}^{(1)}(\mathbb{R}^+, M_n)$. Hence, by (5.43), $P_\pm \in L^1(\mathbb{R}^+, M^n)$, and then $Q(P_\pm)$ is a bounded operator in $L^p(\mathbb{R}, \mathbb{C}^n)$, $1 \le p \le \infty$. Hence, (3.8) holds for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^n)$ and, moreover, the wave operators $W_\pm(H, H_0)$ extend to bounded operators in $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$.

We now prove that the adjoint wave operators $W_\pm(H, H_0)^\dagger$ extend to bounded operators in $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. By (3.8),

$$\begin{aligned} (W_\pm^\dagger(H, H_0)Y)(x) = Y(x) &+ \big(\mathbf{K}^\dagger(K)Y\big)(x) + \big(\mathcal{E}_{\text{even}}^\dagger Q(P_\pm)^\dagger \mathcal{R}^\dagger Y\big)(x) \\ &+ \big(\mathcal{E}_{\text{even}}^\dagger Q(P_\pm)^\dagger \mathcal{R}^\dagger \mathbf{K}(K)^\dagger Y\big)(x). \end{aligned} \tag{5.44}$$

We have

$$\mathbf{K}(K)^\dagger Y(x) = \int_0^x K^\dagger(y, x) Y(y)\, dy.$$

By (5.13), equation (2.1) holds, and then $\mathbf{K}(K)^\dagger$ is bounded in $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. Further,

$$Q^\dagger(P_\pm)Y(x) = \int_{-\infty}^{\infty} P_\pm^\dagger(y - x) Y(y)\, dy,$$

and as $P_\pm^\dagger \in L^1(\mathbb{R}, M_n)$, it follows that $Q^\dagger(P_\pm)$ in bounded $L^1(\mathbb{R}, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}, \mathbb{C}^n)$. Moreover, $\mathcal{E}_{\text{even}}^\dagger Y(x) = Y(x) + Y(-x)$, and then, $\mathcal{E}_{\text{even}}^\dagger$ is bounded from $L^1(\mathbb{R}, \mathbb{C}^n)$ into $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and from $L^\infty(\mathbb{R}, \mathbb{C}^n)$ into $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. Furthermore,

$$\mathcal{R}^\dagger Y(x) = \begin{cases} Y(x), & x \ge 0, \\ 0, & x < 0, \end{cases}$$

and it follows that $\mathcal{R}^\dagger$ is bounded from $L^1(\mathbb{R}, \mathbb{C}^n)$ into $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and from $L^\infty(\mathbb{R}, \mathbb{C}^n)$ into $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. By (5.44), the adjoint wave operators $W_\pm(H, H_0)^\dagger$ extend to bounded operators in $L^1(\mathbb{R}^+, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^n)$. ■

Remark 5.8. The condition $S_\infty = S(0) = I_n$ is actually necessary in Theorem 3.2, as the following example shows. Consider the scalar case, $n = 1$, with $V = 0$, Dirichlet boundary condition, $Y(0) = 0$, and boundary matrices, $B = -1$, $A = 0$. In this case, by [9, (3.7.5)], $S_\infty = S(0) = -1$. Further, by [9, (4.3.8)], the generalized Fourier maps are given by

$$(\mathbf{F}^{\pm} Y)(k) := \mp 2i \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin kx Y(x)\, dx. \tag{5.45}$$

Hence, by (5.26) and (5.29),

$$W_{\pm}(H, H_0)Y = \pm \mathcal{R}\mathcal{F}^{-1} \operatorname{sign} k \mathcal{F} \mathcal{E}_{\text{even}} Y. \tag{5.46}$$

Moreover, we have

$$\mathcal{F}^{-1} \chi_{\mathbb{R}^-}(k) \mathcal{F} \mathcal{E}_{\text{even}} Y = \mathcal{F} \chi_{\mathbb{R}_+}(k) \mathcal{F}^{-1} \mathcal{E}_{\text{even}} Y. \tag{5.47}$$

Then, by (5.27), (5.28), (5.46), and (5.47),

$$W_{\pm}(H, H_0)Y = \pm i \mathcal{R}\mathcal{H} \mathcal{E}_{\text{even}} Y. \tag{5.48}$$

Finally, since the Hilbert transform is not bounded in $L^1(\mathbb{R}, \mathbb{C})$ and in $L^\infty(\mathbb{R}, \mathbb{C})$ (see [38, 40]), it follows that $W_{\pm}(H, H_0)$ are not bounded in $L^1(\mathbb{R}^+, \mathbb{C})$ and in $L^\infty(\mathbb{R}^+, \mathbb{C})$.

5.3. The L^p-boundedness of the wave operators on the line

Proof of Theorem 4.1. We first prepare some results. Let us denote by H_1 the Hamiltonian $H_{A,B,V}$ with the matrices given in (4.6) with $\Lambda = 0$, and with the potential V identically zero. Note that

$$H_{0,\mathbb{R}} = \mathbf{U} H_1 \mathbf{U}^\dagger. \tag{5.49}$$

By Theorem 5.6, the wave operators $W_{\pm}(H_1, H_0)$ exist and are complete. Then, by [36, Proposition 3], the wave operators $W_{\pm}(H_0, H_1)$ also exists and are complete, and, furthermore,

$$W_{\pm}(H_0, H_1) = W_{\pm}(H_1, H_0)^\dagger. \tag{5.50}$$

Then, by Theorem 3.1, the wave operators $W_{\pm}(H_0, H_1)$, restricted to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^p(\mathbb{R}^+, \mathbb{C}^{2n})$, $1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}^+, \mathbb{C}^{2n})$, $1 < p < \infty$. Further, by the chain rule, see [36, Proposition 2],

$$W_{\pm}(H, H_1) = W_{\pm}(H, H_0) W_{\pm}(H_0, H_1). \tag{5.51}$$

Hence, as $W_\pm(H, H_0)$ and $W_\pm(H_0, H_1)$, restricted to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^p(\mathbb{R}^+, \mathbb{C}^{2n})$, $1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty$, it follows that also $W_\pm(H, H_1)$, restricted to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 < p < \infty$. Finally, by (4.2) and (5.49),

$$W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}}) = \mathbf{U} W_\pm(H, H_1) \mathbf{U}^\dagger, \tag{5.52}$$

and, as $\mathbf{U}$ is bounded from $L^p(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^p(\mathbb{R}, \mathbb{C}^n)$, and $\mathbf{U}^\dagger$ is bounded from $L^p(\mathbb{R}, \mathbb{C}^n)$ into $L^p(\mathbb{R}^+, \mathbb{C}^{2n}), 1 \leq p \leq \infty$, we have that $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})$ restricted to $L^2(\mathbb{R}, \mathbb{C}^n) \cap L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty$, extend uniquely to bounded operators in $L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty$. By duality, the adjoint wave operators $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})^\dagger$ extend uniquely to bounded operators in $L^p(\mathbb{R}, \mathbb{C}^n), 1 < p < \infty$. ■

We now proceed to prove that the wave operators $W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}})$ are bounded in $L^1(\mathbb{R}, \mathbb{C}^n)$ and in $L^\infty(\mathbb{R}, \mathbb{C}^n)$, as stated in Theorem 4.2. We first prepare some results.

Let us denote by A_1, B_1 the matrices (4.6) with $\Lambda = 0$. Then, $H_1 = H_{A_1, B_1, 0}$. Let $\widetilde{A}_1, \widetilde{B}_1$ be the matrices related to A_1, B_1 as in (5.3), (5.4), and (5.5) for some invertible matrices $T_{1,1}, T_{2,1}$ and some unitary matrix $\mathcal{M}_1$. Hence,

$$A_1 = \mathcal{M}_1 \widetilde{A}_1 T_{1,1} \mathcal{M}_1^\dagger T_{2,1}, \quad B_1 = \mathcal{M}_1 \widetilde{B}_1 T_{1,1} \mathcal{M}_1^\dagger T_{2,1}.$$

To simplify the notation, we set $\widetilde{H}_1 := H_{\widetilde{A}_1, \widetilde{B}_1, 0}$. Applying (5.9) to H_1 and $\widetilde{H}_1$ we obtain

$$H_1 = \mathcal{M}_1 \widetilde{H}_1 \mathcal{M}_1^\dagger. \tag{5.53}$$

Let us denote by $\mathbf{F}_1^\pm$, respectively $\widetilde{\mathbf{F}}_1^\pm$, the generalized Fourier maps for H_1 and for $\widetilde{H}_1$ defined in (5.20). Then, by (5.53), we get (see [9, (4.3.35)])

$$\mathbf{F}_1^\pm = \mathcal{M}_1 \widetilde{\mathbf{F}}_1^\pm \mathcal{M}_1^\dagger. \tag{5.54}$$

By (5.26), (5.50), (5.51), and as $F_0 = F_0^\dagger = F_0^{-1}$, $W_\pm(H, H_1) = \left(\mathbf{F}^\pm\right)^\dagger \mathbf{F}_1^\pm$, and using (5.54) we prove

$$W_\pm(H, H_1) = (\mathbf{F}^\pm)^\dagger \mathcal{M}_1 \widetilde{\mathbf{F}}_1^\pm \mathcal{M}_1^\dagger. \tag{5.55}$$

To use (5.55) to study the boundedness of the wave operators, we need to compute explicitly the unitary matrix $\mathcal{M}_1$. For this purpose, we first introduce the following unit vectors in $\mathbb{C}^{2n}$:

$$Y^{(j)} := \left(0, \ldots, \frac{1}{\sqrt{2}}, 0, 0 \ldots, -\frac{1}{\sqrt{2}}, 0, \ldots, 0\right)^T, \quad j = 1, \ldots, n, \tag{5.56}$$

with components that take the value $\frac{1}{\sqrt{2}}$ at the component j; the value $-\frac{1}{\sqrt{2}}$ at the component $j+n$, $1 \leq j \leq n$, and all the other components are zero. Further, we define the following unit vectors in $\mathbb{C}^{2n}$:

$$Y^{(j)} := \left(0, \dots, \frac{1}{\sqrt{2}}, 0, 0 \dots, \frac{1}{\sqrt{2}}, 0, \dots, 0\right)^T, \quad j = n+1, \dots, 2n, \tag{5.57}$$

with components that take the value $\frac{1}{\sqrt{2}}$ at the component $j-n$; the value $\frac{1}{\sqrt{2}}$ at the component j, $n+1 \leq j \leq 2n$, and all the other components are zero.

Proposition 5.9. *Let A_1, B_1 be the matrices (4.6) with $\Lambda = 0$, and let $\widetilde{A}_1, \widetilde{B}_1$ be the matrices related to A_1, B_1 as in (5.3), (5.4), and (5.5) for some invertible matrices $T_{1,1}, T_{2,1}$ and some unitary matrix $\mathcal{M}_1$.*

1. *We have*
$$\widetilde{A}_1 = \begin{Bmatrix} 0_n & 0_n \\ 0_n & -I_n \end{Bmatrix}. \tag{5.58}$$
2. *We have*
$$\widetilde{B}_1 = \begin{Bmatrix} -I_n & 0_n \\ 0_n & 0_n \end{Bmatrix}. \tag{5.59}$$
3. *The boundary conditions (5.4) are given by*
$$Y_j(0) = 0, \quad j = 1, \dots, n, \qquad Y_j'(0) = 0, \quad j = n+1, \dots, 2n. \tag{5.60}$$
That is to say, the first n components of Y satisfy the Dirichlet boundary condition, and the last n components fulfill the Neumann boundary condition.
4. *The unitary matrix $\mathcal{M}_1$ is given by*
$$\mathcal{M}_1 = \{Y^{(1)} Y^{(2)} \dots Y^{(2n)}\}. \tag{5.61}$$
5. *The invertible matrices $T_{1,1}$ and $T_{2,1}$ are given by*
$$T_{1,1} = \begin{Bmatrix} -I_n & 0_n \\ 0_n & iI_n \end{Bmatrix}, \tag{5.62}$$
$$T_{2,1} = \begin{Bmatrix} -I_n & iI_n \\ I_n & iI_n \end{Bmatrix}. \tag{5.63}$$

Proof. We use the notation of the proof of [9, Proposition 3.4.5]. We denote $E := \sqrt{A_1^\dagger A_1 + B_1^\dagger B_1}$ and $U := (B_1 - iA_1)E^{-2}(B_1^\dagger - iA_1^\dagger)$. Then, by (4.6) and a simple computation, we get

$$U = \begin{Bmatrix} 0 & -I_n \\ -I_n & 0 \end{Bmatrix}.$$

It is easily verified that the $Y^{(j)}$, $j = 1, \dots, n$, are eigenvectors of U with eigenvalues 1, and that the vectors $Y^{(j)}$, $j = n+1, \dots, 2n$, are eigenvectors of U with eigenvalue -1. Then, the columns of $\mathcal{M}_1$ are an orthonormal system of eigenvectos of U and, in consequence, $\mathcal{M}_1$ diagonalizes U, as required in [9, (3.4.39)]; $\mathcal{M}_1^\dagger U \mathcal{M}_1 = \mathrm{diag}\{1, \dots, 1, -1, \dots, -1\}$ is the matrix with the first n diagonal entries equal to 1, the second n diagonal entries equal to -1, and all other entries equal to 0. This proves that (4) is satisfied. Using the notation [9, (3.4.41)], with $P = I_{2n}$, we get $\mathcal{M}_1^\dagger U \mathcal{M}_1 = \mathrm{diag}\{e^{2\theta_1}, \dots, e^{2i\theta_{2n}}\} = \mathrm{diag}\{1, \dots, 1, -1, \dots, -1\}$, $0 < \theta_j \leq \pi$, $j = 1, \dots, 2n$. Then, $\theta_1 = \theta_2 = \cdots = \theta_n = \pi$, and $\theta_{n+1} = \theta_{n+2} = \cdots = \theta_{2n} = \pi/2$, and (1)–(3) hold. That (5) holds is immediate, since, by the definition of $T_{1,1}$ and $T_{2,1}$ in [9, p. 103], we have $T_{1,1} := (\widetilde{B}_1 + i\widetilde{A}_1)^{-1}$ and $T_{2,1} := B_1 + iA_1$. ■

Let us denote $\tilde{\psi}_1^+(k,x) := \tilde{\psi}_1(-k,x)$ and $\tilde{\psi}_1^-(k,x) := \tilde{\psi}_1(k,x)$, where $\tilde{\psi}_1(k,x)$ is the physical solution of $\widetilde{H}_1$. By [9, (4.3.6) and (4.3.7)],

$$\tilde{\psi}_1^\pm(k,x) = \{\pm 2i \sin kx, \dots, \pm 2i \sin kx, 2\cos kx, \dots, 2\cos kx\} \tag{5.64}$$

is the diagonal matrix with the first n diagonal components equal to $\pm 2i \sin kx$ and the last n diagonal components equal to $2\cos kx$. Further, by [9, (4.3.8)],

$$(\widetilde{\mathbf{F}}_1^\pm Y)(k) = \sqrt{\frac{1}{2\pi}} \int_0^\infty \tilde{\psi}_1^\pm(k,x)^\dagger Y(x)\, dx. \tag{5.65}$$

By (5.64) and (5.65),

$$(\widetilde{\mathbf{F}}_1^\pm Y)(k) = \mathcal{R}(\pm(\mathcal{F}\mathcal{E}_{\mathrm{odd}} Y_+)(k), (\mathcal{F}\mathcal{E}_{\mathrm{even}} Y_-)(k))^T. \tag{5.66}$$

We denote

$$W_{\pm,\mathcal{M}_1}(H, H_1) := \mathcal{M}_1^\dagger W_\pm(H, H_1)\mathcal{M}_1, \tag{5.67}$$

$$S_{\mathcal{M}_1}(k) := \mathcal{M}_1^\dagger S(k)\mathcal{M}_1, \quad k \in \mathbb{R}. \tag{5.68}$$

and

$$S_{\infty,\mathcal{M}_1} := \mathcal{M}_1^\dagger S_\infty \mathcal{M}_1, \quad k \in \mathbb{R}. \tag{5.69}$$

Using (5.19), (5.22), (5.55), and (5.66)–(5.69) we prove

$$(W_{\pm,\mathcal{M}_1}(H, H_1)Y)(x) := \sum_{j=1}^{6} J_\pm^{(j)}(x), \tag{5.70}$$

where

$$J_\pm^{(1)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\pm ikx} \chi_{\mathbf{R}^+}(k) \big(\pm(\mathcal{F}\mathcal{E}_{\mathrm{odd}} Y_+)(k), (\mathcal{F}\mathcal{E}_{\mathrm{even}} Y_-)(k)\big)^T dk, \tag{5.71}$$

$$J_{\pm}^{(2)}(x) := \frac{1}{\sqrt{2\pi}} \mathcal{M}_1^{\dagger} \int_x^{\infty} dz K(x,z) \mathcal{M}_1 \cdot \int_{\mathbb{R}} e^{\pm ikz} \chi_{\mathbb{R}^+}(k) \left(\pm (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T dk. \tag{5.72}$$

$$J_{\pm}^{(3)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbb{R}^+}(k) S_{\infty,\mathcal{M}_1} \left(\pm (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T dk, \tag{5.73}$$

$$J_{\pm}^{(4)}(x) := \frac{1}{\sqrt{2\pi}} \int_x^{\infty} dz\ \mathcal{M}_1^{\dagger} K(x,z) \mathcal{M}_1 \cdot \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) S_{\infty,\mathcal{M}_1} \left(\pm (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T dk, \tag{5.74}$$

$$J_{\pm}^{(5)}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\mp ikx} \chi_{\mathbf{R}^+}(k) (S_{\mathcal{M}_1}(\mp k) - S_{\infty,\mathcal{M}_1}) \cdot \left(\pm (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T dk, \tag{5.75}$$

$$J_{\pm}^{(6)}(x) := \int_x^{\infty} dz\ \mathcal{M}_1^{\dagger} K(x,z) \mathcal{M}_1 \frac{1}{\sqrt{2\pi}} \cdot \int_{\mathbb{R}} e^{\mp ikz} \chi_{\mathbb{R}^+}(k) (S_{\mathcal{M}_1}(\mp k) - S_{\infty,\mathcal{M}_1}) \cdot \left(\pm (\mathcal{F} \mathcal{E}_{\text{odd}} Y_+)(k), (\mathcal{F} \mathcal{E}_{\text{even}} Y_-)(k) \right)^T dk. \tag{5.76}$$

We denote

$$P_{\pm,\mathcal{M}_1}(x) := \mathcal{M}_1^{\dagger} P_{\pm}(x) \mathcal{M}_1, \tag{5.77}$$

where $P_{\pm}$ is defined in (3.7).

Theorem 5.10. *Suppose that V fulfills (1.3), that $V \in L^1_{\gamma}(\mathbb{R}^+, M_{2n}), \gamma > \frac{5}{2}$, that the constant matrices A, B are given by (4.6) with $\Lambda = 0$, and that*

$$S(0) = S_{\infty} = \begin{Bmatrix} 0_n & I_n \\ I_n & 0_n \end{Bmatrix}, \tag{5.78}$$

where 0_n and I_n are, respectively, the $n \times n$ zero matrix and the $n \times n$ identity matrix. Then, for all $Y \in L^2(\mathbb{R}^+, \mathbb{C}^{2n})$, we have

$$W_{\pm}(H, H_1)Y = Y + \mathbf{K}(K)Y + \mathcal{M}_1 \mathcal{R} Q(P_{\pm,\mathcal{M}_1}) \left(-\mathcal{E}_{\text{odd}}(\mathcal{M}_1^{\dagger} Y)_+, \mathcal{E}_{\text{even}}(\mathcal{M}_1^{\dagger} Y)_- \right)^T$$

$$+\mathbf{K}(K)\mathcal{M}_1\mathcal{R}Q(P_{\pm,\mathcal{M}_1})\big(-\mathcal{E}_{\text{odd}}(\mathcal{M}_1^\dagger Y)_+, \mathcal{E}_{\text{even}}(\mathcal{M}_1^\dagger Y)_-\big)^T. \tag{5.79}$$

The wave operators $W_\pm(H, H_1)$ and $W_\pm(H, H_1)^\dagger$ restricted to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^1(\mathbb{R}^+, \mathbb{C}^{2n})$, respectively to $L^2(\mathbb{R}^+, \mathbb{C}^{2n}) \cap L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$, extend to bounded operators on $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and to bounded operators on $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. The $2n \times 2n$ matrix valued function $K(x, y), x, y \in \mathbb{R}^+$, is defined in (5.12). Moreover, the $2n \times 2n$ matrix valued function $P_{\pm,\mathcal{M}_1}(x)$ is defined in (5.77). The scattering matrix $S(k)$, $k \in \mathbb{R}$, is defined in (5.15) and the quantity S_∞ in (5.16).

Proof of Theorem 5.10. By (5.61), (5.68), (5.69), and (5.78),

$$S_{\mathcal{M}_1}(0) = S_{\infty,\mathcal{M}_1} = \begin{Bmatrix} -I_n & 0_n \\ 0_n & I_n \end{Bmatrix}. \tag{5.80}$$

Then, by (5.70)–(5.77), and (5.80),

$$\begin{aligned} W_{\pm,\mathcal{M}_1}(H, H_1)Y &= Y + \mathcal{M}_1^\dagger \mathbf{K}(K)\mathcal{M}_1 Y + \mathcal{R}Q(P_{\pm,\mathcal{M}_1})(-\mathcal{E}_{\text{odd}}Y_+, \mathcal{E}_{\text{even}}Y_-)^T \\ &\quad + \mathcal{M}_1^\dagger \mathbf{K}(K)\mathcal{M}_1\mathcal{R}Q(P_{\pm,\mathcal{M}_1})(-\mathcal{E}_{\text{odd}}Y_+, \mathcal{E}_{\text{even}}Y_-)^T. \end{aligned} \tag{5.81}$$

Equation (5.79) follows from (5.67) and (5.81). By (5.77),

$$\|P_{\pm,\mathcal{M}_1}\|_{L^1(\mathbb{R},M_{2n})} = \|P_\pm\|_{L^1(\mathbb{R},M_{2n})}. \tag{5.82}$$

As we already proved in the proof of Theorem 3.2 that $P_\pm \in L^1(\mathbb{R}, M_{2n})$, it follows from (5.82) that $P_{\pm,\mathcal{M}_1} \in L^1(\mathbb{R}, M_{2n})$. Hence, $Q(P_{\pm,\mathcal{M}_1})$ is bounded in $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. We already proved in the proof Theorem 3.2 that $\mathbf{K}(K)$ is bounded in $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ and that $\mathcal{R}$ is bounded from $L^1(\mathbb{R}, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. Clearly, $\mathcal{E}_{\text{even}}$ is bounded from $L^1(\mathbb{R},^+ \mathbb{C}^{2n})$ into $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. Moreover, it is clear that $\mathcal{E}_{\text{odd}}$ is also bounded from $L^1(\mathbb{R},^+ \mathbb{C}^{2n})$ into $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. Then, by (5.79) the wave operators $W_\pm(H, H_1)$ extend to bounded operators on $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and to bounded operators on $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. By (5.81) and taking the adjoints, we obtain

$$\begin{aligned} W_{\pm,\mathcal{M}_1}^\dagger(H, H_1)Y &= Y + \mathcal{M}_1^\dagger \mathbf{K}(K)^\dagger \mathcal{M}_1 Y \\ &\quad + \big(-\mathcal{E}_{\text{odd}}^\dagger(Q(P_{\pm,\mathcal{M}_1})^\dagger \mathcal{R}^\dagger Y)_+, \mathcal{E}_{\text{even}}^\dagger(Q(P_{\pm,\mathcal{M}_1})^\dagger \mathcal{R}^\dagger Y)_-\big) \\ &\quad + \big(-\mathcal{E}_{\text{odd}}^\dagger(Q(P_{\pm,\mathcal{M}_1})^\dagger \mathcal{R}^\dagger \mathcal{M}_1^\dagger \mathbf{K}(K)^\dagger \mathcal{M}_1 Y)_+, \\ &\qquad \mathcal{E}_{\text{even}}^\dagger(Q(P_{\pm,\mathcal{M}_1})^\dagger \mathcal{R}^\dagger \mathcal{M}_1^\dagger \mathbf{K}(K)^\dagger \mathcal{M}_1 Y)_-\big). \end{aligned} \tag{5.83}$$

We saw in the proof of Theorem (3.2) that $\mathbf{K}(K)^\dagger$ is bounded in $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$. The fact that $\mathcal{E}_{\text{even}}^\dagger$ and $\mathcal{E}_{\text{odd}}^\dagger$ are bounded from $L^1(\mathbb{R}, \mathbb{C}^{2n})$ into

$L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$, and that $\mathcal{R}^\dagger$ is bounded from $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and from $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$ into $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$ follows immediately. Further, as $\|P^\dagger_{\pm,\mathcal{M}_1}\|_{L^1(\mathbb{R},M_{2n})} = \|P_{\pm,\mathcal{M}_1}\|_{L^1(\mathbb{R},M_{2n})}$ and since we already proved that $P_{\pm,\mathcal{M}_1} \in L^1(\mathbb{R}^+, M_{2n})$, we obtain, $P^\dagger_{\pm,\mathcal{M}_1} \in L^1(\mathbb{R}^+, M_{2n})$, and then, as

$$Q(P_{\pm,\mathcal{M}_1})^\dagger Y(x) = \int_{-\infty}^{\infty} P^\dagger_{\pm,\mathcal{M}_1}(y-x) Y(y)\, dy,$$

it follows that $Q_\pm(P_{\mathcal{M}_1})^\dagger$ is bounded in $L^1(\mathbb{R}, \mathbb{C}^{2n})$ and in $L^\infty(\mathbb{R}, \mathbb{C}^{2n})$. Finally, it follows from (5.83) that the wave operators $W^\dagger_{\pm,\mathcal{M}_1}(H, H_1)$ extend to bounded operators on $L^1(\mathbb{R}^+, \mathbb{C}^{2n})$ and to bounded operators on $L^\infty(\mathbb{R}^+, \mathbb{C}^{2n})$, and by (5.67) this also holds for the wave operators $W^\dagger_\pm(H, H_1)$. ■

Using the unitary transformation **U** given in (4.1), we obtain our result on the boundedness of the wave operators on the line, from Theorem 5.10. However, since Theorem 5.10 involves both $S(0)$ and S_∞, we first introduce some concepts from the stationary scattering theory of matrix Schrödinger operators on the line that we quote from [6]. Under the assumption that $\mathcal{V} \in L^1_1(\mathbb{R}, \mathbb{C}^n)$, the Jost solution from the left $f_l(k, x)$, $x \in \mathbb{R}$, $k \in \overline{\mathbb{C}^+}$, is the $n \times n$ matrix-valued solution to the Schrödinger equation on the line

$$-\frac{d^2}{dx^2} Y(x) + \mathcal{V}(x) Y(x) = k^2 Y(x), \quad x \in \mathbb{R}, \tag{5.84}$$

that satisfies

$$f_l(k, x) = e^{ikx}[I_n + o(1)], \quad f_l'(k, x) = e^{ikx}[ikI_n + o(1)], \quad x \to \infty. \tag{5.85}$$

Further, for $k \in \mathbb{R} \setminus \{0\}$, $f_l(k, x)$ fulfills

$$f_l(k, x) = a_l(k) e^{ikx} + b_l(k) e^{-ikx} + o(1), \quad x \to -\infty. \tag{5.86}$$

Similarly, the Jost solution from the right $f_r(k, x)$, $x \in \mathbb{R}$, $k \in \overline{\mathbb{C}^+}$, is the $n \times n$ matrix-valued solution to the Schrödinger equation (5.84) such that

$$f_r(k, x) = e^{-ikx}[I_n + o(1)], \quad f_r'(k, x) = e^{-ikx}[-ikI_n + o(1)], \quad x \to -\infty. \tag{5.87}$$

Moreover, for $k \in \mathbb{R} \setminus \{0\}$, $f_r(k, x)$ fulfills

$$f_r(k, x) = a_r(k) e^{-ikx} + b_r(k) e^{ikx} + o(1), \quad x \to \infty. \tag{5.88}$$

The transmission coefficient from the left $T_l(k)$ and the transmission coefficient from the right $T_r(k)$ are defined by

$$T_l(k) := \frac{1}{a_l(k)}, \quad T_r(k) := \frac{1}{a_r(k)}. \tag{5.89}$$

The reflection coefficient from the left $L(k)$ and the reflection coefficient from the right $R(k)$ are given by

$$L(k) := \frac{b_l(k)}{a_l(k)}, \quad R(k) := \frac{b_r(k)}{a_r(k)}. \tag{5.90}$$

The physical solution from the left $\Psi_l(k,x)$ is defined as

$$\Psi_l(k,x) := T_l(k) f_l(k,x). \tag{5.91}$$

Then, $\Psi_l(k,x)$ satisfies

$$\Psi_l(k,x) = \begin{cases} T(k)e^{ikx} + o(1), & x \to \infty, \\ e^{ikx} + e^{-ikx} L(k) + o(1), & x \to -\infty. \end{cases} \tag{5.92}$$

The physical solution from the left corresponds to a scattering process where a particle is incident from the left with unit amplitude; it is reflected with amplitude $L(k)$ and it is transmitted with amplitude $T_l(k)$. Similarly, the physical solution from the right $\Psi_r(k,x)$ is defined as

$$\Psi_r(k,x) := T_r(k) f_r(k,x). \tag{5.93}$$

Hence, $\Psi_r(k,x)$ satisfies

$$\Psi_r(k,x) = \begin{cases} e^{-ikx} + e^{ikx} R(k) + o(1) & x \to \infty, \\ T_r(k)e^{-ikx} + o(1) & x \to -\infty. \end{cases} \tag{5.94}$$

The physical solution from the right corresponds to a scattering process where a particle is incident from the right with unit amplitude; it is reflected with amplitude $R(k)$ and it is transmitted with amplitude $T_r(k)$.

The scattering matrix on the line $S_{\mathbb{R}}(k)$ is defined as follows:

$$S_{\mathbb{R}}(k) := \begin{Bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{Bmatrix}. \tag{5.95}$$

Using our results, we can directly define the physical solutions from the left and from the right from the physical solution $\Psi(k,x), k \in \mathbb{R} \setminus \{0\}$, given in (5.19), by means of our unitary transformation $\mathbf{U}$, given in (4.1). We proceed as follows. Let us denote by $\Psi^{(1)}(k,x)$ the $2n \times n$ matrix with the first n columns of $\Psi(k,x)$ and let $\Psi^{(2)}(k,x)$ be the $2n \times n$ matrix with the second n columns of $\Psi(k,x)$. Then, by (5.10) and (5.19),

$$\{\mathbf{U}\Psi^{(1)}\}_{ij} = \begin{cases} e^{-ikx}\delta_{i,j} + e^{ikx}\{S\}_{i,j}(k) + o(1), & x \to \infty, 1 \le i,j \le n, \\ e^{-ikx}\{S\}_{n+i,j}(k) + o(1), & x \to -\infty, 1 \le i,j \le n. \end{cases} \tag{5.96}$$

Further, by (5.94) and (5.96), for $1 \leq i, j \leq n$, we define

$$\{\Psi_r(k,x)\}_{i,j} := \{\mathbf{U}\Psi^{(1)}\}_{ij}, \tag{5.97a}$$
$$\{T_r\}_{i,j}(k) := \{S\}_{n+i,j}(k), \tag{5.97b}$$
$$R_{i,j}(k) := \{S\}_{i,j}(k). \tag{5.97c}$$

Moreover,

$$\{\mathbf{U}\Psi^{(2)}\}_{ij} = \begin{cases} e^{ikx}\{S\}_{i,n+j}(k) + o(1), & x \to \infty, 1 \leq i, j \leq n, \\ e^{ikx}\delta_{i,j} + e^{-ikx}\{S\}_{n+i,n+j}(k) + o(1), & x \to \infty, 1 \leq i, j \leq n. \end{cases} \tag{5.98}$$

Then, by (5.92) and (5.98), for $1 \leq i, j \leq n$, we define

$$\{\Psi_l(k,x)\}_{i,j} := \{\mathbf{U}\Psi^{(2)}\}_{ij}, \tag{5.99a}$$
$$\{T_l\}_{i,j}(k) := \{S\}_{i,n+j}(k). \tag{5.99b}$$
$$L_{i,j}(k) := \{S\}_{n+i,n+j}(k). \tag{5.99c}$$

Moreover, by (5.97), (5.99), for $1 \leq i, j \leq n$, we can directly define the scattering matrix on the line from the scattering matrix on the half line as follows:

$$S_{\mathbb{R}}(k) := \begin{Bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{Bmatrix}, \tag{5.100a}$$

where

$$\{T_r\}_{i,j}(k) := \{S\}_{n+i,j}(k), \tag{5.100b}$$
$$\{R\}_{i,j}(k) := \{S\}_{i,j}(k), \tag{5.100c}$$
$$\{T_l\}_{i,j}(k) := \{S\}_{i,n+j}(k), \tag{5.100d}$$
$$\{L\}_{i,j}(k) := \{S\}_{n+i,n+j}(k). \tag{5.100e}$$

Note that, by (5.100),

$$S(0) = S_\infty = \begin{Bmatrix} 0_n & I_n \\ I_n & 0_n \end{Bmatrix} \iff S_{\mathbb{R}}(0) = S_{\mathbb{R},\infty} = \begin{Bmatrix} I_n & 0_n \\ 0_n & I_n \end{Bmatrix}, \tag{5.101}$$

where we denote

$$S_{\mathbb{R},\infty} = \lim_{|k|\to\infty} S_{\mathbb{R}}(k). \tag{5.102}$$

Proof of Theorem 4.2. By (4.2) and (5.49),

$$W_\pm(H_{\mathbb{R}}, H_{0,\mathbb{R}}) = \mathbf{U}W_\pm(H, H_1)\mathbf{U}^\dagger. \tag{5.103}$$

Then, the theorem follows from Theorem 5.10 and (5.101). ∎

A. The scattering matrix for potentials in $L^1_\gamma(\mathbb{R}^+, M_n), 2 \leq \gamma \leq 3$

In this appendix we always assume that $V \in L^1_\gamma(\mathbb{R}^+, M_n), 2 \leq \gamma \leq 3$. By the definition of $S(k)$ in (5.15) and by [9, Proposition 3.2.4, Theorems 3.81 and 3.9.15], $S(k)$ is differentiable for $k \in \mathbb{R}$, with continuous derivative for $k \in \mathbb{R} \setminus \{0\}$, provided that $V \in L^1_2(\mathbb{R}^+)$. We denote by $\dot{S}(k)$ the derivative of $S(k)$. Moreover, by [9, Theorem 3.10.6],

$$S(k) - S_\infty = O\Big(\frac{1}{|k|}\Big), \quad |k| \to \infty. \tag{A.1}$$

We now consider the high-energy behavior of the derivative of $S(k)$.

Proposition A.1. *Suppose that (1.3) is satisfied, that $V \in L^1_2(\mathbb{R}^+)$, and that the constant matrices A, B satisfy (1.5), and (1.6). Then*

$$\dot{S}(k) = O\Big(\frac{1}{|k|}\Big), \quad |k| \to \infty. \tag{A.2}$$

Proof. By [9, Proposition 3.9.1], $\dot{m}(k,x)$ and $\dot{m}'(k,x)$ exist for $x \in [0,\infty)$ and $k \in \overline{\mathbb{C}^+}$, they are analytic in $k \in \mathbb{C}^+$ and continuous in $k \in \overline{\mathbb{C}^+}$ for each $x \in [0,\infty)$, and they are continuous in $x \in [0,\infty)$ for each $k \in \overline{\mathbb{C}^+}$. Moreover, by [9, (3.2.30)],

$$m(k,0) = I + O\Big(\frac{1}{|k|}\Big), \quad k \to \infty \text{ in } \overline{\mathbb{C}^+}. \tag{A.3}$$

Further, by [9, (3.9.3), (3.94), (3.9.15), and (3.9.17)] and (A.3),

$$\dot{m}(k,0) = O\Big(\frac{1}{|k|}\Big), \quad \dot{m}'(k,0) = O(1), \quad k \to \infty \text{ in } \overline{\mathbb{C}^+}. \tag{A.4}$$

By (5.11),

$$\dot{J}(k) = -\dot{m}(-k,0)^\dagger B - i m(-k,0)^\dagger A + ik\dot{m}(-k,0)^\dagger A + \dot{m}'(-k,0)^\dagger A, \quad k \in \mathbb{R}. \tag{A.5}$$

Then, by (A.3)–(A.5),

$$\dot{J}(k) = O(1)A + O\Big(\frac{1}{|k|}\Big), \quad |k| \to \infty \text{ in } \mathbb{R}. \tag{A.6}$$

Further, by (5.15),

$$\begin{aligned}\dot{S}(k) = {} & \big(\dot{J}(-k)J_0(-k)^{-1}\big)\big(J_0(-k)J_0(k)^{-1}\big)\big(J_0(k)J(k)^{-1}\big) \\ & + \big(J(-k)J_0(-k)^{-1}\big)\big(J_0(-k)J_0(k)^{-1}\big)\big(J_0(k)J(k)^{-1}\big) \\ & \cdot \big(\dot{J}(k)J_0(k)^{-1}\big)\big(J_0(k)J(k)^{-1}\big).\end{aligned} \tag{A.7}$$

By (A.6) and [9, (3.7.11) and (3.7.12)],

$$\dot{J}(-k)J_0(-k)^{-1} = O\Big(\frac{1}{|k|}\Big), \quad |k| \to \infty \text{ in } \mathbb{R}. \tag{A.8}$$

Moreover, by [9, (3.6.3) (3.7.3), and (3.7.4)],

$$J_0(-k)J_0(k)^{-1} = O(1), \quad k \to \infty \text{ in } \mathbb{C}. \tag{A.9}$$

Further, by [9, (3.10.17) and (3.10.18)],

$$J(k)J_0(k)^{-1} = I + O\Big(\frac{1}{|k|}\Big), \quad J_0(k)J(k)^{-1} = I + O\Big(\frac{1}{|k|}\Big), \quad k \to \infty \text{ in } \overline{\mathbb{C}^+}. \tag{A.10}$$

Finally, by (A.7)–(A.10), we obtain

$$\dot{S}(k) = O\Big(\frac{1}{|k|}\Big), \quad |k| \to \infty. \qquad \blacksquare$$

We now study the low-energy behavior of $\dot{S}(k)$.

Proposition A.2. *suppose that* (1.3) *holds and that the constant matrices* A, B *satisfy* (1.5)*, and* (1.6)*.Then*

a. *in the generic case where* $J(0)$ *is invertible, if* $V \in L^1_2(\mathbb{R}^+)$,

$$\dot{S}(k) = O(1), \quad |k| \to 0; \tag{A.11}$$

b. *in the exceptional case where* $J(0)$ *is not invertible, if* $V \in L^1_\gamma(\mathbb{R}^+), 2 \le \gamma \le 3$,

$$\dot{S}(k) = O(|k|^{\gamma-3}), \quad |k| \to 0. \tag{A.12}$$

Proof. As in [9], we denote $m(k,x) := e^{-ikx} f(k,x)$. Note that $m(0,x) = f(0,x)$. By [9, (3.2.13), (3.2.14), (3.2.15), and (3.2.16)] and since

$$|e^z - 1| \le C\frac{|z|}{1+|z|}, \quad z \in \mathbf{C}, \tag{A.13}$$

we have

$$|m(k,x)| \le C, \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+, \tag{A.14}$$

provided that $V \in L^1_1(\mathbb{R}^+)$. By [9, (3.9.15)],

$$|\dot{m}(k,x)| \le C, \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \tag{A.15}$$

By [9, (3.9.17)],

$$|\dot{m}'(k,x)| \le C, \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \tag{A.16}$$

Further, by (A.14) and (A.15),

$$|m(k,x) - m(0,x)| \le C \min[|k|, 1], \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \tag{A.17}$$

Moreover, by [9, (3.9.239)],

$$S(k) = S(0) + k\dot{S}(0) + o(|k|), \quad k \to 0. \tag{A.18}$$

By (5.15),

$$\begin{aligned}\dot{S}(k) &= \dot{J}(-k)J(k)^{-1} + J(-k)J(k)^{-1}\dot{J}(k)J(k)^{-1} \\ &= \dot{J}(-k)J(k)^{-1} - S(k)\dot{J}(k)J(k)^{-1}.\end{aligned} \tag{A.19}$$

If $J(0)$ is invertible, (A.11) follows from Proposition 5.2, (A.5), (A.14), (A.15), (A.16), and the first equality in (A.19). This proves item (a). Let us prove (b). By [9, (3.9.3)],

$$\dot{m}(k,x) = \dot{m}_0(k,x) + \frac{1}{2ik}\int_x^\infty dy\, [e^{2ik(y-x)} - 1]V(y)\dot{m}(k,y), \tag{A.20}$$

where

$$\dot{m}_0(k,x) := \frac{1}{2ik^2}\int_x^\infty dy\, [e^{-2ik(y-x)} - 1 + 2ik(y-x)]e^{2ik(y-x)}V(y)m(k,y). \tag{A.21}$$

Further, taking the limit as $k \to 0$ in (A.20) and (A.21), and using (A.14) and (A.15), we obtain

$$\dot{m}(0,x) = \dot{m}_0(0,x) + \int_x^\infty dy\, (y-x)V(y)\dot{m}(0,y), \tag{A.22}$$

with

$$\dot{m}_0(0,x) := i\int_x^\infty dy\, (y-x)^2 e^{2ik(y-x)}V(y)m(0,y). \tag{A.23}$$

Note that

$$|e^z - 1 - z| \le C\frac{|z|^2}{1+|z|}, \quad z \in \mathbf{C} \tag{A.24}$$

and

$$|e^z - 1 - z - \frac{z^2}{2}| \le C\frac{|z|^3}{1+|z|}, \quad z \in \mathbf{C}. \tag{A.25}$$

It follows from (A.13), (A.17), (A.21), (A.23), (A.24) and (A.25) that

$$|\dot{m}_0(k,x) - \dot{m}_0(0,x)| \leq C \min[|k|^{\gamma-2}, 1], \quad k \in \overline{\mathbb{C}^+},\ x \in \mathbb{R}^+. \tag{A.26}$$

By [9, (3.9.6) and (3.9.7)],

$$\dot{m}(k,x) = \sum_{j=0}^{\infty} \dot{m}_j(k,x), \tag{A.27}$$

where

$$\dot{m}_j(k,x) := \frac{1}{2ik} \int_x^{\infty} dy\, [e^{2ik(y-x)} - 1] V(y) \dot{m}_{j-1}(k,x), \quad j \geq 1, \tag{A.28}$$

and the series in (A.27) is uniformly convergent. Taking the limit as $k \to 0$ in (A.27) and (A.28) we get

$$\dot{m}(0,x) = \sum_{j=0}^{\infty} \dot{m}_j(0,x), \tag{A.29}$$

where

$$\dot{m}_j(0,x) := \int_x^{\infty} dy (y-x) V(y) \dot{m}_{j-1}(0,x), \quad j \geq 1. \tag{A.30}$$

By (A.27) and (A.29),

$$\dot{m}(k,x) - \dot{m}(0,x) = \dot{m}_0(k,x) - \dot{m}_0(0,x) + \sum_{j=1}^{\infty} (\dot{m}_j(k,x) - \dot{m}_j(0,x)), \tag{A.31}$$

and, by (A.28) and (A.30), for $j \geq 1$,

$$\dot{m}_j(k,x) - \dot{m}_j(0,x) = \int_x^{\infty} dy V(y) \Big(\frac{1}{2ik} [e^{2ik(y-x)} - 1] \dot{m}_{j-1}(k,x) \\ - (y-x) \dot{m}_{j-1}(0,x) \Big). \tag{A.32}$$

By [9, (3.9.11)],

$$|\dot{m}_0(k,x)| \leq C, \quad k \in \overline{\mathbb{C}^+},\ x \in \mathbb{R}^+. \tag{A.33}$$

Further, by [9, (3.9.12)],

$$|\dot{m}_j(k,x)| \leq \int_x^{\infty} dy\, y |V(y)| |\dot{m}_{j-1}(k,y)|, \quad j \geq 1,\ k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+. \tag{A.34}$$

Then, by (A.24), and (A.32), for $j \geq 1$,

$$|\dot{m}_j(k,x) - \dot{m}_j(0,x)| \leq C \int_x^\infty dy\, (1+y)^2 |V(y)| [|k||\dot{m}_{j-1}(k,y)| + |\dot{m}_{j-1}(k,y) - \dot{m}_{j-1}(0,y)|]. \quad \text{(A.35)}$$

Without loss of generality, we can take the constant C in (A.26), (A.33), and (A.35) bigger or equal than one. Then, using (A.26), (A.33), (A.34), and (A.35), we prove by mathematical induction that, for $j \geq 0$,

$$|\dot{m}_j(k,x) - \dot{m}_j(0,x)| \leq \min[|k|^{\gamma-2}, 1](j+1)C^{j+1} \frac{1}{j!} \left[\int_x^\infty dy (1+y)^2 |V(y)| \right]^j, \quad \text{(A.36)}$$

for $k \in \overline{\mathbb{C}^+}$, $x \in \mathbb{R}^+$, Then, by (A.31) and (A.36),

$$\begin{aligned} |\dot{m}(k,x) - \dot{m}(0,x)| &\leq C \min[|k|^{\gamma-2}, 1] e^{C \int_x^\infty dy\, (1+y)^2 |V(y)|} \left[1 + C \int_x^\infty dy\, (1+y)^2 |V(y)| \right] \\ &\leq C_1 \min[|k|^{\gamma-2}, 1], \quad k \in \overline{\mathbb{C}^+}, x \in \mathbb{R}^+, \end{aligned} \quad \text{(A.37)}$$

for a constant C_1.

Furthermore, taking the derivative with respect to k in both sides of [9, (3.2.7)] we get

$$\dot{m}'(k,x) = -2i \int_x^\infty dy\, (y-x) e^{2ik(y-x)} V(y) m(k,y) - \int_x^\infty dy\, e^{2ik(y-x)} V(y) \dot{m}(k,y). \quad \text{(A.38)}$$

Taking the limit as $k \to 0$ in (A.38) and using (A.14) and (A.15), we obtain

$$\dot{m}'(0,x) = -2i \int_x^\infty dy\, (y-x) V(y) m(0,y) - \int_x^\infty dy\, V(y) \dot{m}(0,y). \quad \text{(A.39)}$$

By (A.13), (A.14), (A.15), (A.17), (A.37), (A.38), and (A.39), it follows that

$$|\dot{m}'(k,x) - \dot{m}'(0,x)| \leq C \min[|k|^{\gamma-2}, 1], \quad k \in \overline{\mathbb{C}^+},\ x \in \mathbb{R}^+. \quad \text{(A.40)}$$

Furthermore, by (A.5), (A.15), (A.17), (A.37), and (A.40),

$$\dot{J}(k) - \dot{J}(0) = O(|k|^{\gamma-2}), \quad |k| \to 0 \text{ in } \mathbb{R}. \quad \text{(A.41)}$$

By [9, (3.9.237)],

$$J(k)^{-1} = \frac{1}{k}\mathcal{M} + \mathcal{E}_1 + o(1), \quad k \to 0 \text{ in } \overline{\mathbb{C}^+}, \quad \text{(A.42)}$$

where $\mathcal{M}$ and $\mathcal{E}_1$ are constant matrices. Then, by (A.18), the second equality in (A.19), (A.41), and (A.42),

$$\dot{S}(k) = \frac{1}{k}\mathcal{N} + O(|k|^{\gamma-3}), \quad k \to 0, \tag{A.43}$$

where

$$\mathcal{N} := \dot{J}(0)\mathcal{M} - S(0)\dot{J}(0)\mathcal{M}. \tag{A.44}$$

In the case $\gamma = 2$, (A.43) gives us (A.12). When, $2 < \gamma \leq 3$, from (A.43) we obtain, for $\varepsilon > 0$,

$$S(\varepsilon) = S(1) - \int_{\varepsilon}^{1} dk\, \dot{S}(k) = S(1) + \ln \varepsilon \mathcal{N} + O(1), \quad \varepsilon \downarrow 0. \tag{A.45}$$

However, (A.45) is compatible with (A.18) only if $\mathcal{N} = 0$. Hence, by (A.43),

$$\dot{S}(k) = O(|k|^{\gamma-3}), \quad k \to 0. \tag{A.46}$$

This concludes the proof of (A.12). ∎

The results above give us the following proposition:

Proposition A.3. *Suppose that V fulfills (1.3) and that the constant matrices A, B satisfy (1.5) and (1.6). Then, $S(k) - S_\infty \in \mathbf{H}^{(1)}(\mathbb{R}, M_n)$, provided that in the generic case, where $J(0)$ is invertible, $V \in L^1_2(\mathbb{R}^+)$, and in the exceptional case, where $J(0)$ is not invertible, $V \in L^1_\gamma(\mathbb{R}^+)$, $\gamma > \frac{5}{2}$.*

Proof. Since $S(k)$ is differentiable for $k \in \mathbb{R}$, with continuous derivative for $k \in \mathbb{R} \setminus \{0\}$, and it satisfies (A.1) (A.2), (A.11), and (A.12), it follows that it belongs to $\mathbf{H}^{(1)}(\mathbb{R}^+, M_n)$. ∎

Acknowledgement. This paper was partially written while I was visiting the Institut de Mathématique d'Orsay, Université Paris-Sud. I thank Christian Gérard for his kind hospitality.

Funding. The author of this work has been partially supported by projects PAPIIT-DGAPA UNAM IN103918 and IN 100321, and SEP-CONACYT CB 2015, 254062.

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Received 11 May 2021; revised 11 July 2021.

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