

Perturbation determinants and discrete spectra of semi-infinite non-self-adjoint Jacobi operators

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Abstract. We study the trace class perturbations of the half-line, discrete Laplacian and obtain a new bound for the perturbation determinant of the corresponding non-self-adjoint Jacobi operator. Based on this bound, we obtain the Lieb–Thirring inequalities for such operators. The spectral enclosure for the discrete spectrum and embedded eigenvalues are also discussed.

In memory of Sergey Naboko (1950–2020)

Introduction

In the last two decades, there was a splash of activity around the spectral theory of non-self-adjoint perturbations of some classical operators of mathematical physics, such as the Laplace and Dirac operators on the whole space, their fractional powers, and others. Recently, there has been some interest in studying certain discrete models of the above problem. In particular, the structure of the spectrum for compact, non-self-adjoint perturbations of the free Jacobi and the discrete Dirac operators has attracted much attention lately. Actually, the problem concerns the discrete component of the spectrum and the rate of its accumulation to the essential spectrum. Such type of results under the various assumptions on the perturbations are united under a common name *Lieb–Thirring inequalities*. For a nice account of the existing results on the problem for non-self-adjoint, two-sided Jacobi operators, the reader may consult two recent surveys [7] and [10, Section 5.13] and references therein.

The spectral theory of semi-infinite, *self-adjoint* Jacobi matrices is quite popular owing to their tight relation to the theory of orthogonal polynomials on the real line [19]. In contrast, there are only a few papers where semi-infinite, non-self-adjoint Jacobi matrices are examined [1, 2, 4, 8, 13, 14, 18].

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The main object under consideration is a semi-infinite Jacobi matrix

$$J(\{a_j\}, \{b_j\}, \{c_j\})_{j \in \mathbb{N}} = \begin{bmatrix} b_1 & c_1 & & & \\ a_1 & b_2 & c_2 & & \\ & a_2 & b_3 & c_3 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{0.1}$$

with uniformly bounded complex entries, and $a_j c_j \neq 0, j \in \mathbb{N} := \{1, 2, \dots\}$. The spectral theory of the underlying non-self-adjoint Jacobi operator includes, among others, the structure of its spectrum. We denote by J_0 the semi-infinite discrete Laplacian, i.e., $J_0 = J(\{1\}, \{0\}, \{1\})$. If $J - J_0$ is a compact operator, that is,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} c_n = 1, \quad \lim_{n \rightarrow \infty} b_n = 0,$$

the geometric image of the spectrum is plainly evident

$$\sigma(J) = \sigma_{ess}(J_0) \cup \sigma_d(J) = [-2, 2] \cup \sigma_d(J),$$

the discrete component $\sigma_d(J)$, i.e., the set of isolated eigenvalues of finite algebraic multiplicity, is an at most countable set of points in $\mathbb{C} \setminus [-2, 2]$ with the only possible limit points on $[-2, 2]$. To get some quantitative information on the rate of such accumulation, one has to impose some further assumptions on the perturbation. Our case of study in the paper is the *trace class* perturbations of the discrete Laplacian J_0 ,

$$J - J_0 \in \mathcal{S}_1 \Leftrightarrow \sum_{n=1}^{\infty} (|1 - a_n| + |b_n| + |1 - c_n|) < \infty, \tag{0.2}$$

see, e.g., [17, Lemma 2.3].

Our strategy is similar to that in [8, 12]. The key issue is the bound for the *perturbation determinant*

$$L(\lambda, J) := \det(I + (J - J_0)(J_0 - \lambda)^{-1}) \tag{0.3}$$

introduced by M. G. Krein [11] in the late 1950s. The main feature of this analytic function on the resolvent set $\rho(J_0) = \bar{\mathbb{C}} \setminus [-2, 2]$ is that the set of its zeros agrees with the discrete spectrum of the perturbed operator J and, moreover, the multiplicity of each zero equals the algebraic multiplicity of the corresponding eigenvalue. So, the original problem of the spectral theory can be restated as a classical problem of the zero distributions of analytic functions, which goes back to Jensen and Blaschke.

The argument is pursued in two steps. The first one results in the bound for the perturbation determinant, typical for the functions of non-radial growth. The classes of such analytic (and subharmonic) functions in the unit disk were introduced and

studied in [5,9] (for some advances see [6]). The Blaschke-type conditions for the zero sets (Riesz measures) were proved therein, with an important amplification in [15, Theorem 4], better adapted for applications. The second step is just the latter result applied to the bound mentioned above.

Such two-step algorithm is applied, by and large, in [15] to the two-sided Jacobi operators. In our approach to the problem, the argument in the first step is totally different. The point is that for semi-infinite Jacobi operators neither the Fourier transform machinery, nor a simple matrix representation for the resolvent of the free Jacobi operator are at our disposal. Instead, we deal with the associated three-term recurrence relation

$$u_{k-1} + b_k u_k + a_k c_k u_{k+1} = \lambda(z) u_k, \quad k \in \mathbb{N}, \lambda(z) = z + \frac{1}{z}, \quad (0.4)$$

and its solutions. Here $\lambda(\cdot)$ is the Zhukovsky function which maps the unit disk onto the resolvent set $\rho(J_0) = \bar{\mathbb{C}} \setminus [-2, 2]$.

The relation between solutions of (0.4) and eigenvectors of J is straightforward: $(v_k)_{k \geq 1}$ is an eigenvector of J if and only if

$$(u_k)_{k \geq 0}: \quad u_0 = 0, \quad u_1 = v_1, \quad u_k = \frac{v_k}{a_1 \dots a_{k-1}}, \quad k \geq 2$$

is a solution of (0.4).

The solution $u^+ = (u_k^+)_{k \geq 0}$ of (0.4) is called the *Jost solution* if

$$\lim_{k \rightarrow \infty} z^{-k} u_k^+(z) = 1, \quad z \in \mathbb{D}_0 := \mathbb{D} \setminus \{0\}. \quad (0.5)$$

Under certain assumptions on J the Jost solution exists and unique (see Theorem 1.2 below).

We study the Jost solutions by reducing the difference equation (0.4) to a Volterra-type discrete integral equation, the standard idea in analysis, see, e.g., [20, Section 7.5] and [8, 12]. The bounds for the Jost solutions stem from the successive approximations method. The crucial point in such an approach is that the perturbation determinant agrees with the first coordinate u_0^+ of the Jost solution, known as the *Jost function*, see the non-self-adjoint version of Killip and Simon [17, Theorem 2.16] (the calculation there has nothing to do with self-adjointness).

The Lieb–Thirring inequality for the discrete spectrum of semi-infinite Jacobi operators looks as follows.

Theorem 0.1. *Let $J - J_0 \in \mathcal{S}_1$. Then for each $\varepsilon \in (0, 1)$ there is a constant $C(\varepsilon) > 0$ so that*

$$\sum_{\lambda \in \sigma_d(J)} \frac{\text{dist}(\lambda, [-2, 2])}{|\lambda^2 - 4|^{\frac{1-\varepsilon}{2}}} \leq C(\varepsilon) \Delta, \quad \Delta := \sum_{n=1}^{\infty} (|b_n| + |1 - a_n c_n|). \quad (0.6)$$

If J is the discrete Schrödinger operator, that is, $a_n = c_n \equiv 1$, then

$$\sum_{\lambda \in \sigma_d(J)} \frac{\text{dist}(\lambda, [-2, 2])}{|\lambda^2 - 4|^{\frac{1-\varepsilon}{2}}} \leq C(\varepsilon) \|J - J_0\|_1. \tag{0.7}$$

Remark 0.2. The value Δ in (0.6) in place of $\|J - J_0\|_1$ looks quite natural, at least for small perturbations. Indeed, given a Jacobi matrix J , consider a class $S(J)$ of Jacobi matrices

$$S(J) = \{\hat{J} := T^{-1}JT, T = \text{diag}(t_j)_{j \in \mathbb{N}} \text{ is a diagonal isomorphism of } \ell^2(\mathbb{N})\},$$

$$\hat{J} = J(\{a_j r_j\}, \{b_j\}, \{c_j r_j^{-1}\}), \quad r_n = \frac{t_n}{t_{n+1}}, \quad n \in \mathbb{N}.$$

Clearly, $\sigma_d(\hat{J}) = \sigma_d(J)$ since \hat{J} is similar to J . Hence, the left side of (0.6) is constant within the class $S(J)$, and so is Δ , in contrast to $\|J - J_0\|_1$. For the class $S(J_0)$ both sides of (0.6) vanish, whereas $\|J - J_0\|_1, J \in S(J_0)$, can be arbitrarily large.

Next, $|1 - a_n c_n| \leq |1 - a_n| + |1 - c_n| + |1 - a_n||1 - c_n|$, and so

$$\begin{aligned} \Delta &\leq \sum_{n=1}^{\infty} (|b_n| + |1 - a_n| + |1 - c_n|) + \left(\sum_{n=1}^{\infty} |1 - a_n| \right) \left(\sum_{n=1}^{\infty} |1 - c_n| \right) \\ &\leq 3\|J - J_0\|_1 + \|J - J_0\|_1^2. \end{aligned} \tag{0.8}$$

We see that for small perturbations Δ has at least the same order as $\|J - J_0\|_1$.

The bounds for the Jost functions, obtained in the course of the proof of our main statement, provide some new spectral enclosure results.

Theorem 0.3. (i) *Let $J - J_0 \in \mathcal{S}_1$. The discrete spectrum $\sigma_d(J)$ belongs to the following Cassini oval*

$$\sigma_d(J) \subset \left\{ \lambda \in \mathbb{C} \setminus [-2, 2]: |\lambda^2 - 4| \leq \left(\frac{2\Delta}{\log 2} \right)^2 \right\}. \tag{0.9}$$

(ii) *Assume that*

$$\Delta_1 := \sum_{n=1}^{\infty} n(|b_n| + |1 - a_{n-1}c_{n-1}|) < \infty. \tag{0.10}$$

Then the discrete spectrum is missing, $\sigma_d(J) = \emptyset$, as long as $\Delta_1 < \log 2$.

Note that the latter effect is inherent to semi-infinite Jacobi matrices.

The Lieb–Thirring inequality for two-sided Jacobi operators is due to Hansmann and Katriel [15, Theorem 1]. It states that for each $\varepsilon \in (0, 1)$ there is a constant

$C(\varepsilon) > 0$ so that

$$\sum_{\lambda \in \sigma_d(J)} \frac{\text{dist}^{1+\varepsilon}(\lambda, [-2, 2])}{|\lambda^2 - 4|^{\frac{1}{2} + \frac{\varepsilon}{4}}} \leq C(\varepsilon) \|J - J_0\|_1. \tag{0.11}$$

It has been proved recently in [12], that this bound can be refined. Actually, (0.7) holds for two-sided Jacobi operators as well.

The result of Hansmann and Katriel (0.11) is known to be sharp in the sense that (0.11) is false for $\varepsilon = 0$. To prove that, the authors of [3] introduce a special class of two-sided Jacobi operators with rectangular (step) potentials. In Section 3 we deal with an obvious counterpart of this class in semi-infinite setting.

Given $n \in \mathbb{N}$ and $h > 0$, we study a semi-infinite discrete Schrödinger operator with a pure imaginary step potential

$$J_{n,h} := \begin{bmatrix} b_1 & 1 & & & \\ 1 & b_2 & 1 & & \\ & 1 & b_3 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \quad b_j = \begin{cases} ih, & j = 1, \dots, n; \\ 0, & j \geq n + 1. \end{cases} \tag{0.12}$$

It is clear that $J_{n,h} = J_0 + ih\mathcal{P}_n$, \mathcal{P}_n is the orthogonal projection onto the linear span of the first n basis vectors $\{e_1, \dots, e_n\}$.

We write $J_{n,h} = J_{n,\alpha}$ for particular values of $h = h_n = n^{-\alpha}$, $0 < \alpha < 1$. For such operators some quantitative information about the discrete spectrum, which provides sharpness of Theorem 0.1, can be gathered.

Theorem 0.4. *For the discrete Schrödinger operators $J_{n,\alpha}$ the following lower bound holds for large enough n :*

$$\frac{1}{\|J_{n,\alpha} - J_0\|_1} \sum_{\lambda \in \sigma_d(J_{n,\alpha})} \frac{\text{dist}(\lambda, [-2, 2])}{|\lambda^2 - 4|^{1/2}} \geq C \log n. \tag{0.13}$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{\|J_{n,\alpha} - J_0\|_1} \sum_{\lambda \in \sigma_d(J_{n,\alpha})} \frac{\text{dist}(\lambda, [-2, 2])}{|\lambda^2 - 4|^{1/2}} = +\infty. \tag{0.14}$$

The argument in [3] relies on a simple matrix representation for the resolvent of the whole-line free Jacobi matrix and the theory of Kac–Murdock–Szegő matrices, neither of which is available in the semi-infinite case. Instead, we analyze directly the recurrence relations for eigenvectors and apply Rouché’s Theorem to the roots of certain algebraic equations.

1. Jost solutions and discrete Volterra equations

We derive the bounds for the Jost solution u^+ by reducing the difference equation (0.4) to the Volterra-type discrete integral equation. The unity of the first coefficient (0.4) appears to be crucial.

Define a (non-symmetric) Green kernel for $k, m \in \mathbb{Z}$ (as a two-sided Laurent matrix) by

$$G(k, m; z) := \begin{cases} \frac{z^{m-k} - z^{k-m}}{z - z^{-1}}, & m \geq k, \\ 0, & m \leq k, \end{cases} \quad z \in \mathbb{D}_0. \tag{1.1}$$

The basic properties of this kernel can be verified directly

$$\begin{aligned} G(k, m - 1; z) + G(k, m + 1; z) - \left(z + \frac{1}{z}\right)G(k, m; z) &= \delta_{k,m}, \\ G(k - 1, m; z) + G(k + 1, m; z) - \left(z + \frac{1}{z}\right)G(k, m; z) &= \delta_{k,m}. \end{aligned} \tag{1.2}$$

We make use of these properties for $k, m \in \mathbb{N}_0 := \{0, 1, \dots\}$. The kernel

$$T(k, m; z) := -b_m G(k, m; z) + (1 - a_{m-1}c_{m-1})G(k, m - 1; z) \tag{1.3}$$

is a key player of the game. Note that the values of b_0, a_{-1}, c_{-1} are immaterial.

Theorem 1.1. *The Jost solution $u^+ = (u_k^+)_{k \geq 0}$ of the difference equation (0.4) satisfies the discrete Volterra-type equation*

$$u_k^+(z) = z^k + \sum_{m=k+1}^{\infty} T(k, m; z)u_m^+(z), \quad k \in \mathbb{N}_0, \quad z \in \mathbb{D}_0. \tag{1.4}$$

Conversely, each solution $u = (u_k)_{k \geq 0}$ of (1.4) solves (0.4).

Proof. We multiply the first relation (1.2) for G by u_m^+ , relation (0.4) for u^+ by $G(k, m)$, and subtract the later from the former

$$\begin{aligned} [G(k, m + 1)u_m^+ - G(k, m)u_{m-1}^+] + [-b_m G(k, m) + G(k, m - 1)]u_m^+ \\ - a_m c_m G(k, m)u_{m+1}^+ = \delta_{k,m}u_m^+. \end{aligned}$$

Next, we sum up over m from $k + 1$ to N , taking into account that $G(k, k + 1) = 1, G(k, k) = 0$

$$\begin{aligned} G(k, N + 1)u_N^+ + \sum_{m=k+1}^N [-b_m G(k, m) + G(k, m - 1)]u_m^+ \\ - \sum_{m=k}^N a_m c_m G(k, m)u_{m+1}^+ = u_k^+, \end{aligned}$$

or

$$u_k^+ = G(k, N + 1)u_N^+ - a_N c_N G(k, N)u_{N+1}^+ + \sum_{m=k+1}^N T(k, m)u_m^+.$$

The latter equality holds for arbitrary solutions of (0.4). If u^+ is the Jost solution, then, by (1.1) and (0.5),

$$\lim_{N \rightarrow \infty} [G(k, N + 1)u_N^+ - a_N c_N G(k, N)u_{N+1}^+] = z^k,$$

and (1.4) follows.

To prove the converse statement, let $u = (u_k)_{k \geq 0}$ be any solution of (1.4). Then

$$\begin{aligned} u_{k-1} + u_{k+1} &= \left(z + \frac{1}{z}\right)z^k + T(k - 1, k)u_k + T(k - 1, k + 1)u_{k+1} \\ &\quad + \sum_{m=k+2}^{\infty} [T(k - 1, m) + T(k + 1, m)]u_m. \end{aligned}$$

But

$$\begin{aligned} T(k - 1, k)u_k &= -b_k u_k, \\ T(k - 1, k + 1)u_{k+1} &= [-b_{k+1}G(k - 1, k + 1) \\ &\quad + (1 - a_k c_k)G(k - 1, k)]u_{k+1} \\ &= -\left(z + \frac{1}{z}\right)b_{k+1}u_{k+1} + (1 - a_k c_k)u_{k+1} \\ &= \left(z + \frac{1}{z}\right)T(k, k + 1)u_{k+1} + (1 - a_k c_k)u_{k+1}, \\ T(k - 1, m) + T(k + 1, m) &= \left(z + \frac{1}{z}\right)T(k, m). \end{aligned}$$

Finally,

$$u_{k-1} + u_{k+1} = -b_k u_k + (1 - a_k c_k)u_{k+1} + \left(z + \frac{1}{z}\right)\left(z^k + \sum_{m=k+1}^{\infty} T(k, m)u_m\right),$$

which is (0.4). The proof is complete. ■

The further study of the Volterra equation (1.4) relies upon the modified kernel

$$\tilde{T}(k, m; z) := T(k, m; z)z^{m-k}. \tag{1.5}$$

It is easy to verify that $\tilde{T}(k, m; \cdot)$ are polynomials of z . Indeed,

$$z^{m-k}G(k, m; z) = z^{m-k} \frac{z^{m-k} - z^{k-m}}{z - z^{-1}} = z \frac{z^{2(m-k)} - 1}{z^2 - 1}, \quad m \geq k;$$

$$\begin{aligned} z^{m-k} G(k, m-1; z) &= z^{m-k} \frac{z^{m-k-1} - z^{k-m+1}}{z - z^{-1}} \\ &= z^2 \frac{z^{2(m-k-1)} - 1}{z^2 - 1}, \quad m \geq k + 1, \end{aligned}$$

as claimed. The bounds for the kernel \tilde{T} follow from the latter relations

$$\begin{aligned} |z^{m-k} G(k, m; z)| &\leq |z| \cdot \min\left\{(m-k)_+, \frac{2}{|1-z^2|}\right\}, \quad (a)_+ := \max(a, 0); \\ |z^{m-k} G(k, m-1; z)| &\leq |z|^2 \cdot \min\left\{(m-k-1)_+, \frac{2}{|1-z^2|}\right\}, \end{aligned}$$

and so

$$\begin{aligned} |\tilde{T}(k, m; z)| &\leq \delta_m |z| \min\left\{(m-k)_+, \frac{2}{|1-z^2|}\right\}, \quad z \in \bar{\mathbb{D}}, \\ \delta_m &:= |b_m| + |1 - a_{m-1}c_{m-1}|, \quad a_0 = c_0 = 1. \end{aligned} \tag{1.6}$$

In particular,

$$|\tilde{T}(k, m; z)| \leq |\omega(z)| \delta_m, \quad \omega(z) := \frac{2z}{1-z^2}, \quad z \in \mathbb{D}_1 := \bar{\mathbb{D}} \setminus \{\pm 1\}. \tag{1.7}$$

Theorem 1.2. (i) Assume that

$$\Delta = \sum_{n=1}^{\infty} \delta_n < \infty. \tag{1.8}$$

Then the equation (1.4) has a unique solution $u^+ = (u_k^+)_{k \geq 0}$ so that u_k^+ are analytic in \mathbb{D} , continuous in \mathbb{D}_1 , and for $k \in \mathbb{N}_0$ and $z \in \mathbb{D}_1$

$$|u_k^+(z) - z^k| \leq |z|^k (e^{|\omega(z)|s_0(k)} - 1), \quad s_0(k) := \sum_{n=k+1}^{\infty} \delta_n. \tag{1.9}$$

(ii) Assume that (0.10) holds

$$\Delta_1 = \sum_{n=1}^{\infty} n\delta_n < \infty.$$

Then (1.4) has a unique solution $u^+ = (u_k^+)_{k \geq 0}$ so that u_k^+ are analytic in \mathbb{D} , continuous in $\bar{\mathbb{D}}$, and for $k \in \mathbb{N}_0$ and $z \in \bar{\mathbb{D}}$

$$|u_k^+(z) - z^k| \leq |z|^k (e^{|z|s_1(k)} - 1), \quad s_1(k) := \sum_{n=k+1}^{\infty} n\delta_n. \tag{1.10}$$

Proof. It is advisable to introduce new variables in (1.4)

$$f_j(z) := u_j^+(z)z^{-j} - 1, \quad j \in \mathbb{N}_0,$$

so the equation (1.4) turns into

$$\begin{aligned} f_k(z) &= g_k(z) + \sum_{m=k+1}^{\infty} \tilde{T}(k, m; z) f_m(z), \\ g_k(z) &:= \sum_{m=k+1}^{\infty} \tilde{T}(k, m; z), \quad k \in \mathbb{N}_0, z \in \mathbb{D}_1. \end{aligned} \tag{1.11}$$

It is clear from (1.7) and the assumption (1.8) that the latter series converges absolutely and uniformly on compact subsets of \mathbb{D}_1 and so represents an analytic function in \mathbb{D} . Moreover,

$$|g_k(z)| \leq |\omega(z)|s_0(k), \quad k \in \mathbb{N}_0, z \in \mathbb{D}_1.$$

We are going to solve (1.11) by using the successive approximations method. Let

$$f_{k,1}(z) := g_k(z), \quad f_{k,j+1}(z) := \sum_{m=k+1}^{\infty} \tilde{T}(k, m; z) f_{m,j}(z), \quad j \in \mathbb{N}.$$

It is easy to see, by induction, that

$$|f_{k,p}(z)| \leq \frac{(|\omega(z)|s_0(k))^p}{p!}, \quad p \in \mathbb{N}. \tag{1.12}$$

Indeed, once the bound is true for $p = 1$, we assume that it holds for $p = 1, 2, \dots, j$ and $k \in \mathbb{N}_0$. Then

$$|f_{k,j+1}(z)| \leq |\omega(z)| \sum_{m=k+1}^{\infty} \delta_m |f_{m,j}(z)| \leq \frac{|\omega(z)|^{j+1}}{j!} \sum_{m=k+1}^{\infty} \delta_m s_0^j(m).$$

The elementary inequality $(a + b)^{j+1} - a^{j+1} \geq (j + 1)ba^j$, $a, b \geq 0$ gives with $a = s_0(m)$, $b = \delta_m$, $a + b = s_0(m - 1)$

$$\sum_{m=k+1}^{\infty} \delta_m s_0^j(m) \leq \frac{1}{j + 1} \sum_{m=k+1}^{\infty} (s_0^{j+1}(m - 1) - s_0^{j+1}(m)) = \frac{s_0^{j+1}(k)}{j + 1},$$

and so

$$|f_{k,j+1}(z)| \leq \frac{|\omega(z)|^{j+1} s_0^{j+1}(k)}{j!(j + 1)} = \frac{(|\omega(z)|s_0(k))^{j+1}}{(j + 1)!},$$

as claimed.

Hence, the series

$$f_k := \sum_{j=1}^{\infty} f_{k,j}(z)$$

converges absolutely and uniformly on compact subsets of \mathbb{D}_1 and represents an analytic in \mathbb{D} function, continuous in \mathbb{D}_1 . It satisfies (1.11),

$$\begin{aligned} f_k(z) - g_k(z) &= f_k(z) - f_{k,1}(z) = \sum_{j=1}^{\infty} f_{k,j+1}(z) \\ &= \sum_{j=1}^{\infty} \sum_{m=k+1}^{\infty} \tilde{T}(k, m; z) f_{m,j}(z) = \sum_{m=k+1}^{\infty} \tilde{T}(k, m; z) f_m(z). \end{aligned}$$

This solution admits the bound, see (1.12),

$$|f_k(z)| \leq \sum_{j=1}^{\infty} |f_{k,j}(z)| \leq \sum_{j=1}^{\infty} \frac{(|\omega(z)|s_0(k))^j}{j!} = e^{|\omega(z)|s_0(k)} - 1,$$

which is (1.9).

As far as uniqueness goes, suppose that there are two solutions of (1.4)

$$u^+ = (u_k^+)_{k \geq 0}, \quad v^+ = (v_k^+)_{k \geq 0}.$$

Assume further that $z \neq 0$. By (1.7), we have

$$|u_k^+(z) - v_k^+(z)| \leq |\omega(z)| \sum_{m \geq k+1} \delta_m |u_m^+(z) - v_m^+(z)| =: q_k(z). \tag{1.13}$$

Clearly, $q_k \searrow 0$ as $k \rightarrow \infty$.

We fix z and assume first that $q_p > 0$ for all $p \in \mathbb{N}_0$. By (1.13),

$$\frac{q_{p-1} - q_p}{q_p} = \frac{\delta_p |u_p^+(z) - v_p^+(z)| |\omega(z)|}{q_p} \leq |\omega(z)| \delta_p, \quad q_p \leq q_N \prod_{j=p+1}^N (1 + |\omega(z)| \delta_j).$$

The latter product converges as $N \rightarrow \infty$, so $q_p = 0$ in contradiction with our assumption.

Next, let $l \in \mathbb{N}_0$ exist so that $q_l = 0$. Then, in view of monotonicity, $q_{l+1} = q_{l+2} = \dots = 0$, and, in the opposite way, successively,

$$\begin{aligned} |u_l^+ - v_l^+| = 0 &\implies q_{l-1} = 0 \implies |u_{l-1}^+ - v_{l-1}^+| = 0 \implies q_{l-2} = 0 \\ &\implies \dots \implies |u_0^+ - v_0^+| = 0, \end{aligned}$$

so the uniqueness follows.

(ii) The proof goes along the same line of reasoning with the auxiliary bounds for $z \in \bar{\mathbb{D}}$

$$|\tilde{T}(k, m; z)| \leq |z|m\delta_m, \quad |g_k(z)| \leq |z|s_1(k), \quad |f_{k,p}(z)| \leq \frac{(|z|s_1(k))^p}{p!}.$$

The proof is complete. ■

2. Perturbation determinant and the Lieb–Thirring inequality

Under our main assumption (0.2), the perturbation determinant $L(\lambda, J)$ (0.3) is a well-defined analytic function on \mathbb{D} .

Theorem 2.1. *Let $J - J_0 \in \mathcal{S}_1$. Then the bound holds*

$$\log |L(\lambda(z), J)| \leq |\omega(z)|\Delta = \frac{2|z|\Delta}{|1-z||1+z|}, \quad z \in \mathbb{D}. \tag{2.1}$$

Under assumption (0.10),

$$\log |L(\lambda(z), J)| \leq |z|\Delta_1, \quad z \in \bar{\mathbb{D}}. \tag{2.2}$$

Proof. By the non-self-adjoint version of [17, Section 2] (the calculation there is algebraic and so immediately extends to the non-self-djoint case), the Jost solution u^+ of (0.4) equals

$$u_k^+(z) = z^k L(\lambda(z), J^{(k)}), \quad k \in \mathbb{N}_0,$$

where $J^{(k)}$, the k -stripped Jacobi matrix, is obtained from J by dropping the first k rows and columns. So, (2.1) and (2.2) follow directly from (1.9) and (1.10), respectively, with $k = 0$. ■

We are now ready for

Proof of Theorem 0.1. According to [15, Theorem 4], for each $\varepsilon \in (0, 1)$ there is a constant $C(\varepsilon) > 0$ so that the Blaschke-type condition holds for the zero set (divisor) $Z(L)$, L in (2.1), $L(0) = 1$, becomes

$$\sum_{\xi \in Z(L)} (1 - |\xi|) \frac{|\xi^2 - 1|^\varepsilon}{|\xi|^\varepsilon} \leq C(\varepsilon)\Delta,$$

(each zero is taken with its multiplicity). The latter inequality turns into (0.6), when we go over to the Zhukovsky images, taking into account the distortion for the Zhukovsky function [15, Lemma 7]

$$\frac{1}{2} \frac{|1 - z^2|(1 - |z|)}{|z|} \leq \text{dist}(\lambda, [-2, 2]) \leq \frac{1 + \sqrt{2}}{2} \frac{|1 - z^2|(1 - |z|)}{|z|}.$$

For the discrete Schrödinger operators ($a_n = c_n \equiv 1$), one has

$$\Delta = \sum_{n=1}^{\infty} |b_n| = \|J - J_0\|_1,$$

and (0.7) follows. ■

In view of (0.8), it might be worth comparing the key inequality (2.1) with

$$\log |L(z)| \leq \frac{C_{abs}}{|1 - z^2|^2} (\|J - J_0\|_1 + \|J - J_0\|_1^2), \quad z \in \mathbb{D},$$

the result obtained in [17, Theorem 2.8] for the self-adjoint case.

There is yet another consequence of Theorem 1.2 (i), which concerns embedded eigenvalues of the Jacobi operator J . The result is likely to be known (cf. [16] for two-sided discrete Schrödinger operators), so we briefly outline its proof.

Corollary 2.2. *Let $J - J_0 \in \mathcal{S}_1$. Then the operator J has no embedded eigenvalues, i.e., eigenvalues on $(-2, 2)$.*

Proof. Assume on the contrary, that $\lambda = 2 \cos \theta$, $0 < \theta < \pi$, is the eigenvalue of J . In this case, λ is also the eigenvalue for the modified Jacobi operator $\hat{J} := J(\{1\}, \{b_j\}, \{a_j c_j\})$, which is similar to J , see Remark 0.2. Denote by $h = (h_k)_{k \geq 1}$ the corresponding eigenvector, so we have an ℓ^2 -solution $h' = (0, h)$ of (0.4) with $z = e^{i\theta}$.

On the other hand, the Jost solution is known to be continuous in \mathbb{D}_1 , so the second solution $u^+ = (u_k^+)_{k \geq 0}$ of (0.4) comes up. It is clear from (1.9) that

$$|u_k^+(e^{i\theta})| = 1 + o(1), \quad k \rightarrow \infty. \tag{2.3}$$

The Wronskian of these two solutions

$$W(u^+, h') = \prod_{j=k+1}^{\infty} (a_j c_j)^{-1} (u_k^+ h_{k+1} - u_{k+1}^+ h_k)$$

at the point $e^{i\theta}$ is k -independent, and as $h \in \ell^2$, and u^+ is bounded, we see that $W(u^+, h') \equiv 0$. Hence, u^+ and h are to be linearly dependent, that contradicts (2.3) and $h \in \ell^2$. ■

Proof of Theorem 0.3. The bound (1.9) with $k = 0$ means that $L \neq 0$ in \mathbb{D} as long as

$$|\omega(z)| \Delta < \log 2, \quad \frac{|z|}{|1 - z^2|} < \frac{\log 2}{2\Delta}.$$

Since

$$|\lambda^2 - 4| = \left| \frac{1 - z^2}{z} \right|^2,$$

we come to (0.9).

Under assumption $\Delta_1 < \log 2$, the bound (1.10) with $k = 0$ implies $L \neq 0$ on \mathbb{D} , so $\sigma_d(J)$ is missing, as claimed. ■

Remark 2.3. The spectral enclosures are normally derived from the Birman–Schwinger principle. Precisely,

$$\lambda(z) \in \sigma_d(J) \implies \|K(z)\| \leq 1,$$

where K is the Birman–Schwinger operator. In our case, such approach leads to the inclusion

$$\sigma_d(J) \subset \{\lambda \in \mathbb{C} \setminus [-2, 2]: |\lambda^2 - 4| \leq 36^2 \|J - J_0\|_1^2\}.$$

The approach based on the Jost functions for semi-infinite discrete Schrödinger operators gives

$$\sigma_d(J) \subset \{\lambda \in \mathbb{C} \setminus [-2, 2]: |\lambda^2 - 4| \leq \frac{4}{(\log 2)^2} \|J - J_0\|_1^2\}.$$

Note that for two-sided discrete Schrödinger operators the sharp oval which contains the discrete spectrum is known [16]:

$$\sigma_d(J) \subset \{\lambda \in \mathbb{C} \setminus [-2, 2]: |\lambda^2 - 4| \leq \|J - J_0\|_1^2\}.$$

3. Semi-infinite discrete Schrödinger operators with rectangular potentials

We begin with a slight refinement of [3, Lemma 4].

Proposition 3.1. *The discrete spectrum of the operator $J_{n,h}$ (0.12) is contained in the rectangle*

$$\sigma_d(J_{n,h}) \subset \{\lambda \in \mathbb{C}: |\operatorname{Re} \lambda| < 2, 0 < \operatorname{Im} \lambda < h\}. \tag{3.1}$$

Proof. If $J_{n,h}g = \lambda g$, $\|g\| = 1$, then

$$\lambda = \langle J_{n,h}g, g \rangle = \langle J_0g, g \rangle + ih \langle \mathcal{P}_n g, g \rangle.$$

With $g = (g_k)_{k \geq 1}$, $g_0 := 0$, we have

$$\begin{aligned} |\operatorname{Re} \lambda| &= |\langle J_0g, g \rangle| = \left| \sum_{j \geq 1} (g_{j-1} + g_{j+1}) \bar{g}_j \right| = 2 \left| \sum_{j \geq 1} \operatorname{Re}(g_{j-1} \bar{g}_j) \right| \\ &\leq 2 \sum_{j \geq 1} |g_{j-1} g_j| \leq 2 \sum_{j \geq 1} |g_j|^2 = 2. \end{aligned}$$

Actually, the latter inequality is strict or, otherwise, $2|g_{j-1}g_j| = |g_{j-1}|^2 + |g_j|^2$ would imply $|g_{j-1}| = |g_j|$ for all j that is impossible for a nonzero vector $g \in \ell^2$. So, $|\operatorname{Re} \lambda| < 2$, as claimed.

Next, obviously $0 \leq \operatorname{Im} \lambda = h \langle \mathcal{P}_n g, g \rangle \leq h$. Moreover,

$$\operatorname{Im} \lambda = 0 \implies \langle \mathcal{P}_n g, g \rangle = 0 \implies \mathcal{P}_n g = 0,$$

so $J_{n,h} g = J_0 g = \lambda g$, and g is the eigenvector of J_0 that is also impossible. Similarly,

$$\operatorname{Im} \lambda = h \implies \langle \mathcal{P}_n g, g \rangle = \|\mathcal{P}_n g\|^2 = 1 \implies \mathcal{P}_n g = g,$$

and again, the conclusion $J_0 g = (\lambda - ih)g$ is false. ■

Let

$$J_{n,h} g = \lambda g = \left(z + \frac{1}{z}\right)g, \quad z \in \mathbb{D}_0 = \mathbb{D} \setminus \{0\}, \quad g = (g_j)_{j \geq 1} \in \ell^2(\mathbb{N}), \quad g \neq 0.$$

The system of recurrence relations for the coordinates looks as follows

$$\begin{aligned} ihg_1 + g_2 &= \lambda g_1, \\ g_{j-1} + ihg_j + g_{j+1} &= \lambda g_j, \quad j = 2, \dots, n, \\ g_{j-1} + g_{j+1} &= \lambda g_j, \quad j = n + 1, \dots \end{aligned} \tag{3.2}$$

We begin with the characteristic equation for the second series of relations

$$t^2 - (\lambda - ih)t + 1 = (t - t_1)(t - t_2) = 0, \quad t_j = t_j(z, h), \quad j = 1, 2. \tag{3.3}$$

Furthermore,

$$t_1 t_2 = 1, \quad t_1 + t_2 = t_1 + t_1^{-1} = \lambda - ih,$$

and $\operatorname{Im} \lambda < h$ implies $0 < |t_1| < 1 < |t_2|$. Put $\zeta := t_1$, so

$$\zeta + \frac{1}{\zeta} = \lambda - ih = z + \frac{1}{z} - ih, \quad \zeta, z \in \mathbb{D}_0. \tag{3.4}$$

Yet another relation between z and ζ comes from the adjustment in (3.2). Indeed,

$$\begin{aligned} g_j &= a\zeta^j + b\zeta^{-j}, \quad j = 1, \dots, n + 1, \quad |a| + |b| > 0; \\ g_j &= g_n z^{j-n}, \quad j = n, n + 1, \dots, \end{aligned}$$

since $g \in \ell^2$. For $j = 2$ we have $g_2 = a\zeta^2 + b\zeta^{-2}$. On the other hand, the first relation in (3.2) gives

$$g_2 = (\lambda - ih)g_1 = (\zeta + \zeta^{-1})(a\zeta + b\zeta^{-1}) = a\zeta^2 + b\zeta^{-2} + a + b \implies a + b = 0.$$

For $j = n + 1$,

$$g_{n+1} = a\zeta^{n+1} + b\zeta^{-n-1} = (a\zeta^n + b\zeta^{-n})z \implies a\zeta^n(z - \zeta) + b\zeta^{-n}(z - \zeta^{-1}) = 0.$$

The system of two homogeneous, linear equations has a nontrivial solution if and only if its determinant vanishes

$$\begin{vmatrix} 1 & 1 \\ \zeta^n(z - \zeta) & \zeta^{-n}(z - \zeta^{-1}) \end{vmatrix} = \zeta^{-n}(z - \zeta^{-1}) - \zeta^n(z - \zeta) = 0. \tag{3.5}$$

Hence, if $\lambda \in \sigma_d(J_{n,h})$, then (3.4) and (3.5) hold.

Conversely, let a pair $z, \zeta \in \mathbb{D}_0$ satisfy (3.4) – (3.5), so that ζ is the root of the characteristic equation (3.3) in \mathbb{D} . It is a matter of direct calculation to check that $g = (g_j)_{j \geq 1}$ with

$$g_j := \zeta^j - \zeta^{-j}, \quad j = 1, 2, \dots, n + 1; \quad g_j = g_n z^{j-n}, \quad j = n + 1, \dots$$

is the eigenvector of $J_{n,h}$ with the eigenvalue λ .

Let us express z from (3.5)

$$z = \frac{\zeta^{n+1} - \zeta^{-n-1}}{\zeta^n - \zeta^{-n}} = \frac{\zeta^{2n+2} - 1}{\zeta^{2n+1} - \zeta} \in \mathbb{D} \tag{3.6}$$

and plug this expression into (3.4). We come to the main algebraic equation associated with $J_{n,h}$

$$\zeta + \frac{1}{\zeta} + ih = \frac{\zeta^{2n+2} - 1}{\zeta^{2n+1} - \zeta} + \frac{\zeta^{2n+1} - \zeta}{\zeta^{2n+2} - 1},$$

or

$$P_n(\zeta) := \zeta^{2n-1}(\zeta^2 - 1)^2 - ih(1 - \zeta^{2n})(1 - \zeta^{2n+2}) = 0. \tag{3.7}$$

Conversely, let ζ be a root of (3.7) in \mathbb{D} so that (3.6) holds. Then (3.5) is true, and (3.4) is a simple consequence of the equality

$$(\zeta^{2n+2} - 1)^2 + (\zeta^{2n+1} - \zeta)^2 - (\zeta + \zeta^{-1})(\zeta^{2n+2} - 1)(\zeta^{2n+1} - \zeta) = \zeta^{2n}(\zeta^2 - 1)^2.$$

Thereby, we come to the following result.

Proposition 3.2. *Let $z \in \mathbb{D}_0$. The number $\lambda = z + z^{-1} \in \sigma_d(J_{n,h})$ if and only if there is a root $\zeta \in \mathbb{D}_0$ of the polynomial P_n (3.7) so that (3.6) holds.*

Our next goal is to show that the number of such roots is large enough (proportional to n).

From now on, we put $h := n^{-\alpha}$, $0 < \alpha < 1$ is fixed, and write $J_{n,h} = J_{n,\alpha}$. The parameter n is assumed to be large.

Take $0 < c_2 < \alpha < c_1 < 1$ and put

$$r_j = r_j(n) := 1 - c_j \frac{\log n}{2n + 1}, \quad j = 1, 2, r_1 < r_2. \tag{3.8}$$

Clearly,

$$r_j^{2n+1} = n^{-c_j} \left(1 + O\left(\frac{\log^2 n}{n}\right) \right), \quad j = 1, 2. \tag{3.9}$$

Proposition 3.3. *The circular segment*

$$S = S(n, a) := \left\{ w = r e^{i\varphi} : r_1 < r < r_2, a \frac{\pi}{2} < \varphi < \left(\frac{1}{2} - a\right) \frac{\pi}{2} \right\}, \quad 0 < a < \frac{1}{4},$$

contains at least $N = (1/4 - a)n + O(1)$ simple roots of P_n (3.7).

Proof. Write $P_n = P_{n,2} - P_{n,1}$ with

$$\begin{aligned} P_{n,1}(w) &= 2w^{2n+1} + i n^{-\alpha}, \\ P_{n,2}(w) &= w^{2n+1}(w^2 + w^{-2}) + i n^{-\alpha}(w^{2n} + w^{2n+2} - w^{4n+2}). \end{aligned}$$

We wish to compare the roots of P_n with the roots of a simple binomial $P_{n,1}$. The latter are known explicitly

$$\begin{aligned} P_{n,1}(w) = 0 &\iff w = w_{n,k} = \gamma_n \exp\left\{ \frac{i\pi}{2(2n+1)}(4k-1) \right\}, \quad k = 1, \dots, 2n+1, \\ \gamma_n = |w_{n,k}| &= \left(\frac{n^{-\alpha}}{2}\right)^{1/2n+1} = 1 - \alpha \frac{\log n}{2n+1} + O\left(\frac{1}{n}\right). \end{aligned}$$

Define the values $\varphi_k = \varphi_k(n)$ by

$$\varphi_k := \frac{\pi(4k+1)}{2(2n+1)} \implies \varphi_{k-1} < \arg w_{n,k} < \varphi_k, \quad k = 1, \dots, 2n+1, \tag{3.10}$$

and the segments $S_k = S_k(n)$ by

$$S_k := \{ w = r e^{i\varphi} : r_1 < r < r_2, \varphi_{k-1} < \varphi < \varphi_k \}.$$

Since $r_1(n) < \gamma_n < r_2(n)$ for large n , each segment S_k contains exactly one root $w_{n,k}$ of $P_{n,1}$. As it turns out, $P_{n,1}$ dominates $P_{n,2}$ on the boundary ∂S_k for certain values of k , specified below.

We have

$$P_{n,1}(r e^{i\varphi}) = 2r^{2n+1} \cos(2n+1)\varphi + i(2r^{2n+1} \sin(2n+1)\varphi + n^{-\alpha}),$$

and so

$$|P_{n,1}(r e^{i\varphi})| \geq |2r^{2n+1} \sin(2n+1)\varphi + n^{-\alpha}|. \tag{3.11}$$

For the second polynomials,

$$P_{n,2}(re^{i\varphi}) = r^{2n+1}e^{i(2n+1)\varphi} \left(r^2e^{2i\varphi} + r^{-2}e^{-2i\varphi} + in^{-\alpha}(re^{i\varphi} + r^{-1}e^{-i\varphi} - r^{2n+1}e^{i(2n+1)\varphi}) \right),$$

and since

$$\begin{aligned} r^2e^{2i\varphi} + r^{-2}e^{-2i\varphi} &= (r^2 - 1)e^{2i\varphi} + 2\cos 2\varphi + (r^{-2} - 1)e^{-2i\varphi} \\ &= 2\cos 2\varphi + O\left(\frac{\log n}{n}\right), \quad r_1 \leq r \leq r_2, \end{aligned} \tag{3.12}$$

we come to the bound

$$|P_{n,2}(re^{i\varphi})| \leq r^{2n+1}(2|\cos 2\varphi| + O(n^{-\alpha})), \quad r_1 \leq r \leq r_2, \tag{3.13}$$

with O uniform in r and φ .

Let first $\varphi = \varphi_k, r_1 \leq r \leq r_2$. By (3.10), $\sin(2n + 1)\varphi_k = 1$ for all k , so (3.11) and (3.13) imply

$$\begin{aligned} |P_{n,1}(re^{i\varphi_k})| &\geq 2r^{2n+1} + n^{-\alpha}, \\ |P_{n,2}(re^{i\varphi_k})| &\leq 2r^{2n+1}|\cos 2\varphi_k| + o(n^{-\alpha}), \end{aligned}$$

(see (3.9) for the last term). Hence, $|P_{n,1}(re^{i\varphi_k})| > |P_{n,2}(re^{i\varphi_k})|$ for all k , that is, the domination holds on the radial sides of S_k .

Next, let $r = r_2, \varphi_{k-1} \leq \varphi \leq \varphi_k$, be the exterior arc of S_k . By (3.9),

$$\begin{aligned} |P_{n,1}(re^{i\varphi})| &= |2r^{2n+1}e^{i(2n+1)\varphi} + in^{-\alpha}| = r^{2n+1}(2 + O(n^{-(\alpha-c_2)})), \\ |P_{n,2}(re^{i\varphi_k})| &\leq r^{2n+1}(2|\cos 2\varphi_k| + O(n^{-\alpha})). \end{aligned}$$

We choose k in such a way that the value $|\cos 2\varphi_k|$ is separated from 1. An elementary calculation shows that for

$$\begin{aligned} k \in \Lambda &= \Lambda_{n,a} := [a'_n, a''_n], \\ a'_n &= a\frac{2n+1}{4} + 1, \quad a''_n = \left(\frac{1}{2} - a\right)\frac{2n+1}{4} - 1, \end{aligned} \tag{3.14}$$

the inequality holds

$$a\pi < 2\varphi_{k-1} < 2\varphi_k < \left(\frac{1}{2} - a\right)\pi. \tag{3.15}$$

So, again

$$|P_{n,1}(r_2e^{i\varphi})| > |P_{n,2}(r_2e^{i\varphi})|, \quad \varphi_{k-1} \leq \varphi \leq \varphi_k, \quad k \in \Lambda.$$

Finally, along the interior arc of S_k , that is, $r = r_1, \varphi_{k-1} \leq \varphi \leq \varphi_k$, we have $r_1^{2n+1} = O(n^{-c_1}) = o(n^{-\alpha})$, and so

$$|P_{n,1}(r_1e^{i\varphi})| \geq n^{-\alpha}, \quad |P_{n,2}(r_1e^{i\varphi})| = O(n^{-c_1}),$$

as needed.

By Rouché’s Theorem, each region S_k with $k \in \Lambda$ contains one simple root of P_n . The number of such regions is $N = (1/4 - a)n + O(1)$, and, by (3.15), all of them lie in S . The proof is complete. ■

In the rest of the paper we focus on the roots of P_n just found. Denote them by $\zeta_k = \zeta_k(n) = \rho_k e^{i\theta_k}$, $\rho_k = \rho_k(n)$, $\theta_k = \theta_k(n)$. The equality (3.7) with $\zeta = \zeta_k$ reads

$$-i n^\alpha \zeta_k^{2n+1} (\zeta_k - \zeta_k^{-1})^2 = (1 - \zeta_k^{2n})(1 - \zeta_k^{2n+2}).$$

As in (3.12) above,

$$(\zeta_k - \zeta_k^{-1})^2 = -4 \sin^2 \theta_k \left(1 + O\left(\frac{\log n}{n}\right)\right).$$

Since $\rho_k^{2n+1} = O(n^{-c_2})$, we end up with

$$4i \sin^2 \theta_k n^\alpha \rho_k^{2n+1} e^{i(2n+1)\theta_k} = 1 + O(n^{-c_2}). \tag{3.16}$$

Take the absolute value and the real part in (3.16):

$$\begin{aligned} 4 \sin^2 \theta_k n^\alpha \rho_k^{2n+1} &= 1 + O(n^{-c_2}), \\ -4 \sin^2 \theta_k n^\alpha \rho_k^{2n+1} \sin(2n + 1)\theta_k &= 1 + O(n^{-c_2}), \end{aligned} \tag{3.17}$$

with O uniform in $k \in \Lambda$. By plugging the first equality in (3.17) into the second one, we have

$$\sin(2n + 1)\theta_k = -1 + O(n^{-c_2}). \tag{3.18}$$

Regarding θ_k themselves, it is clear from the construction of S_k that

$$\theta_k = \frac{\pi}{2(2n + 1)}(4k + 1) + O\left(\frac{1}{n}\right), \quad k \in \Lambda. \tag{3.19}$$

For the range of arguments in S , we have

$$0 < 4 \sin^2 a \frac{\pi}{2} \leq 4 \sin^2 \theta_k \leq 4 \sin^2 \left(\frac{1}{2} - a\right) \frac{\pi}{2} < 2, \tag{3.20}$$

so it follows from (3.17) that

$$\rho_k = 1 - \alpha \frac{\log n}{2n + 1} + O\left(\frac{1}{n}\right), \tag{3.21}$$

with O uniform in $k \in \Lambda$ (cf. with $\gamma_n = |w_{n,k}|$).

Proof of Theorem 0.4. In view of Proposition 3.2, to make sure that the roots $\{\zeta_k\}_{k \in \Lambda}$ generate eigenvalues of $J_{n,\alpha}$, we have to check the relation (3.6). Indeed, let

$$z_k = z_k(n) := \frac{\zeta_k^{2n+2} - 1}{\zeta_k^{2n+1} - \zeta_k}.$$

Then

$$1 - |z_k|^2 = \frac{(\rho_k^2 - 1)(1 - \rho_k^{4n+2}) - 4\rho_k^{2n+2} \sin(2n + 1)\theta_k \sin \theta_k}{|\zeta_k^{2n+1} - \zeta_k|^2}.$$

By (3.21), the first term in the numerator is

$$(\rho_k^2 - 1)(1 - \rho_k^{4n+2}) = O\left(\frac{\log n}{n}\right).$$

As for the second one, the first equality in (3.17) and (3.20) imply

$$\rho_k^{2n+1} = \frac{n^{-\alpha}}{4 \sin^2 \theta_k} (1 + O(n^{-c_2})) > \frac{n^{-\alpha}}{4},$$

and in view of (3.18) and (3.20), the second term is

$$-4 \sin(2n + 1)\theta_k \sin \theta_k > c > 0.$$

Hence, $1 - |z_k|^2 > 0$ for large enough n , as claimed.

Therefore, the numbers

$$\lambda_k = \lambda_k(n) := \zeta_k + \zeta_k^{-1} + in^{-\alpha} = \rho_k e^{i\theta_k} + \rho_k^{-1} e^{-i\theta_k} + in^{-\alpha}$$

are *distinct* eigenvalues of $J_{n,\alpha}$ for all large enough n and $k \in \Lambda$ (3.14). We have, as in (3.12) above,

$$\lambda_k = 2 \cos \theta_k + in^{-\alpha} + O\left(\frac{\log n}{n}\right), \tag{3.22}$$

and so, by Proposition 3.1,

$$\text{dist}(\lambda_k, [-2, 2]) = \text{Im } \lambda_{n,k} = n^{-\alpha}(1 + o(1)). \tag{3.23}$$

As we restrict ourselves with the eigenvalues $\lambda_k, k \in \Lambda$,

$$\sum_{\lambda \in \sigma_d(J_{n,\alpha})} \frac{\text{dist}(\lambda, [-2, 2])}{|\lambda^2 - 4|^{1/2}} \geq \sum_{k \in \Lambda} \frac{\text{dist}(\lambda_k, [-2, 2])}{|\lambda_k^2 - 4|^{1/2}} =: I_{n,a}.$$

Next, by the location of the discrete spectrum and (3.22), we see that

$$\begin{aligned} |\lambda_k + 2|^{1/2} &\leq (4 + n^{-\alpha})^{1/2} < \sqrt{5}, \\ |\lambda_k - 2|^{1/2} &\leq 2 \sin \frac{\theta_k}{2} + n^{-\alpha/2}(1 + o(1)) \leq \theta_k + n^{-\alpha/2}(1 + o(1)), \end{aligned} \tag{3.24}$$

and so (3.23) and (3.24) imply (see also (3.19))

$$\begin{aligned} I_{n,a} &\geq \frac{n^{-\alpha}(1 + o(1))}{\sqrt{5}} \sum_{k \in \Lambda} \left[\frac{\pi}{2(2n + 1)} (4k + 1) + n^{-\alpha/2}(1 + o(1)) \right]^{-1} \\ &\geq \frac{n^{1-\alpha}(1 + o(1))}{\pi \sqrt{5}} \sum_{k \in \Lambda} \frac{1}{k + u_n}, \quad u_n := \frac{n^{1-\alpha/2}}{\pi} (1 + o(1)) + 1. \end{aligned}$$

An elementary bound

$$\sum_{m_1}^{m_2} \frac{1}{k + u_n} > \int_{m_1}^{m_2} \frac{dx}{x + u_n} = \log \frac{m_2 + u_n}{m_1 + u_n}$$

with $m_1 := [a'_n] + 1, m_2 := [a''_n]$ (see (3.14)) gives

$$I_{n,a} \geq \frac{n^{1-\alpha}(1 + o(1))}{\pi\sqrt{5}} \log \frac{1 - 2a + 4n^{-\alpha/2}(1 + o(1))}{a + 2n^{-\alpha/2}(1 + o(1))}. \tag{3.25}$$

Up to this point, we have tacitly assumed that the value $a \in (0, 1/4)$ from Proposition 3.3 is fixed. As a matter of fact, a can vary with n . For example, if $a = n^{-\alpha/4}$, (3.25) yields

$$I_{n,a} \geq Cn^{1-\alpha} \log n.$$

The bound (0.13) now follows from

$$\|J_{n,\alpha} - J_0\|_1 = \sum_{j=1}^n |b_j(n)| = n^{1-\alpha}. \quad \blacksquare$$

Remark 3.4. The Jost solution $u^+ = (u_j^+)_{j \geq 0}$ for Jacobi operators with the step potential can be found explicitly. Indeed, the recurrence relation

$$\begin{aligned} u_{k-1}^+(z) + ihu_k^+(z) + u_{k+1}^+(z) &= \left(z + \frac{1}{z}\right)u_k^+(z), \quad k = 1, 2, \dots, n; \\ u_k^+(z) &= z^k, \quad k = n, n + 1 \dots \end{aligned}$$

can be resolved, and we come to the following expression for the Jost function u_0^+ (which is the same as the perturbation determinant)

$$L(z, J_{n,h}) = z^n U_n \left(\frac{z + z^{-1} - ih}{2} \right) - z^{n+1} U_{n-1} \left(\frac{z + z^{-1} - ih}{2} \right),$$

U_k is the Chebyshev polynomial of the second kind. It might be a challenging problem analyzing the roots of the polynomial on the right side directly.

References

[1] A. I. Aptekarev, V. Kaliaguine, and W. Van Assche, Criterion for the resolvent set of nonsymmetric tridiagonal operators. *Proc. Amer. Math. Soc.* **123** (1995), no. 8, 2423–2430
 Zbl [0824.47017](#) MR [1254830](#)

[2] B. Beckermann, Complex Jacobi matrices. *J. Comput. Appl. Math.* **127** (2001), no. 1–2, 17–65. Zbl [0977.30007](#) MR [1808568](#)

- [3] S. Bögli and F. Štampach, On Lieb–Thirring inequalities for one-dimensional non-self-adjoint Jacobi and Schrödinger operators. *J. Spectr. Theory* **11** (2021), no. 3, 1391–1413
Zbl [07481672](#) MR [4322041](#)
- [4] A. Borichev, R. L. Frank, and A. Volberg, Counting eigenvalues of Schrödinger operators with fast decaying complex potentials. *Adv. Math.* **397** (2022), article no. 108115
Zbl [07472307](#) MR [4377931](#)
- [5] A. Borichev, L. Golinskii, and S. Kupin, A Blaschke-type condition and its application to complex Jacobi matrices. *Bull. Lond. Math. Soc.* **41** (2009), no. 1, 117–123
Zbl [1175.30007](#) MR [2481997](#)
- [6] A. Borichev, L. Golinskii, and S. Kupin, On zeros of analytic functions satisfying non-radial growth conditions. *Rev. Mat. Iberoam.* **34** (2018), no. 3, 1153–1176
Zbl [1401.30004](#) MR [3850283](#)
- [7] M. Demuth, M. Hansmann, and G. Katriel, Eigenvalues of non-selfadjoint operators: a comparison of two approaches. In *Mathematical physics, spectral theory and stochastic analysis*, pp. 107–163, Oper. Theory Adv. Appl. 232, Birkhäuser, Basel, 2013
Zbl [1280.47005](#) MR [3077277](#)
- [8] I. Egorova and L. Golinskii, On the location of the discrete spectrum for complex Jacobi matrices. *Proc. Amer. Math. Soc.* **133** (2005), no. 12, 3635–3641 Zbl [1090.47018](#)
MR [2163601](#)
- [9] S. Favorov and L. Golinskii, A Blaschke-type condition for analytic and subharmonic functions and application to contraction operators. In *Linear and complex analysis*, pp. 37–47, Amer. Math. Soc. Transl. Ser. (2) 226, Amer. Math. Soc., Providence, RI, 2009 Zbl [1188.30064](#) MR [2500508](#)
- [10] R. L. Frank, The Lieb–Thirring inequalities: recent results and open problems. In *Nine mathematical challenges – an elucidation*, pp. 45–86, Proc. Sympos. Pure Math. 104, Amer. Math. Soc., Providence, RI, 2021 MR [4337417](#)
- [11] I. C. Gohberg and M. G. Kreĭn, Introduction to the theory of linear nonselfadjoint operators. Translations of Mathematical Monographs 18, American Mathematical Society, Providence, R.I., 1969 Zbl [0181.13504](#) MR [0246142](#)
- [12] L. Golinskii, Volterra-type discrete integral equations and spectra of non-self-adjoint Jacobi operators. *Integral Equations Operator Theory* **93** (2021), no. 6, article id. 63
Zbl [07445581](#) MR [4346505](#)
- [13] L. Golinskii and S. Kupin, Lieb–Thirring bounds for complex Jacobi matrices. *Lett. Math. Phys.* **82** (2007), no. 1, 79–90 Zbl [1136.47018](#) MR [2367876](#)
- [14] L. B. Golinskii and I. E. Egorova, On limit sets for the discrete spectrum of complex Jacobi matrices. *Mat. Sb.* **196** (2005), no. 6, 43–70; English translation in *Sb. Math.* **196** (2005), no. 5–6, 817–844. Zbl [1098.47029](#) MR [2164551](#)
- [15] M. Hansmann and G. Katriel, Inequalities for the eigenvalues of non-selfadjoint Jacobi operators. *Complex Anal. Oper. Theory* **5** (2011), no. 1, 197–218 Zbl [1222.47039](#)
MR [2773062](#)
- [16] O. O. Ibrogimov and F. Štampach, Spectral enclosures for non-self-adjoint discrete Schrödinger operators. *Integral Equations Operator Theory* **91** (2019), no. 6, article no. 53 Zbl [1439.39012](#) MR [4032211](#)

- [17] R. Killip and B. Simon, Sum rules for Jacobi matrices and their applications to spectral theory. *Ann. of Math. (2)* **158** (2003), no. 1, 253–321 Zbl [1050.47025](#) MR [1999923](#)
- [18] V. È. Ljance, Spectrum and resolvent of a nonselfadjoint difference operator. *Ukrain. Mat. Ž.* **20** (1968), 489–503 MR [0261379](#)
- [19] B. Simon, *Szegő's theorem and its descendants*. M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011 Zbl [1230.33001](#) MR [2743058](#)
- [20] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*. Math. Surv. Monogr. 72, American Mathematical Society, Providence, RI, 2000 Zbl [1056.39029](#) MR [1711536](#)

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