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# Lowest energy band function for magnetic steps

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**Abstract.** We study the Schrödinger operator in the plane with a step magnetic field function. The bottom of its spectrum is described by the infimum of the lowest eigenvalue band function, for which we establish the existence and uniqueness of the non-degenerate minimum. We discuss the curvature effects on the localization properties of magnetic ground states, among other applications.

## 1. Introduction

Families of 1D differential operators, dependent on a real parameter, naturally arise within the large field/semi-classical asymptotics of the magnetic Laplacian [10, 11]. Studying the minimum of the ground state energy with respect to the parameter defining the family is central in such problems. Ideally, one aspires to the existence of a unique non-degenerate minimum, but neither this is always the case nor it is easy to confirm such a behavior. So far, examples where the minimum is unique and non-degenerate include the celebrated de Gennes model of harmonic oscillators on the half-axis (see [8, 11, 18] for a discussion in the context of superconductivity), the Montgomery model [15], higher order anharmonic oscillators [13], and superconducting-normal interface operators [22]. Other interesting families of operators appear in [14]. Non-linear models are discussed in [7, 12, 16, 24]. In the present paper, we consider the linear model of magnetic steps studied in [9, 19] and prove the uniqueness and non-degeneracy of the minimum.

### 1.1. The planar magnetic step operator

Let $a \in [-1, 1) \setminus \{0\}$. We define the self-adjoint magnetic Schrödinger operator on the plane

$$\mathcal{L}_a = \partial_{x_2}^2 + (\partial_{x_1} + i\sigma x_2)^2, \quad (x_1, x_2) \in \mathbb{R}^2, \tag{1}$$

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where $\sigma$ is a step function defined as follows:

$$\sigma(x_1, x_2) = \mathbf{1}_{\mathbb{R}_+}(x_2) + a\mathbf{1}_{\mathbb{R}_-}(x_2). \tag{2}$$

The operator $\mathcal{L}_a$ is invariant with respect to translations in the $x_1$-direction; therefore it can be fibered and reduced to a family of 1D Schrödinger operators on $L^2(\mathbb{R})$, $\mathfrak{h}_a[\xi]$, after a Fourier transform along the $x_1$-axis (see [19, 26]). The fiber operators $\mathfrak{h}_a[\xi]$, parametrized by $\xi \in \mathbb{R}$, are defined in Section 1.2.

We have the following link between the spectra of the operators $\mathcal{L}_a$ and $\mathfrak{h}_a[\xi]$ (see [19] and [11, Section 4.3]):

$$\mathrm{sp}\left(\mathcal{L}_a\right) = \overline{\bigcup_{\xi\in\mathbb{R}} \mathrm{sp}(\mathfrak{h}_a[\xi])}. \tag{3}$$

Consequently, the bottom of the spectrum of $\mathcal{L}_a$, denoted by $\beta_a$, can be computed by minimizing the ground state energies of the fibered operators $\mathfrak{h}_a[\xi]$ (see (10) below).

### 1.2. The lowest energy band function

Let $a \in [-1, 1) \setminus \{0\}$. For all $\xi \in \mathbb{R}$, we introduce the operator

$$\mathfrak{h}_a[\xi] = -\frac{d^2}{d\tau^2} + V_a(\xi, \tau),$$

with the potential $V_a(\xi, \tau) = (\xi + \sigma(\tau)\tau)^2$, where

$$\sigma(\tau) = \mathbf{1}_{\mathbb{R}_+}(\tau) + a\mathbf{1}_{\mathbb{R}_-}(\tau). \tag{4}$$

The domain of $\mathfrak{h}_a[\xi]$ is given by

$$\mathrm{Dom}(\mathfrak{h}_a[\xi]) = \left\{u \in B^1(\mathbb{R}) : \left(-\frac{d^2}{d\tau^2} + V_a(\xi, \tau)\right)u \in L^2(\mathbb{R})\right\},$$

where the space $B^n(I)$ is defined for a positive integer $n$ and an open interval $I \subset \mathbb{R}$ as follows:

$$B^n(I) = \left\{u \in L^2(I) : \tau^i \frac{d^j u}{d\tau^j} \in L^2(I), \text{ for all } i, j \in \text{ s.t. } i + j \le n\right\}. \tag{5}$$

The quadratic form associated to $\mathfrak{h}_a[\xi]$ is

$$q_a[\xi](u) = \int_{\mathbb{R}} \left(|u'(\tau)|^2 + V_a(\xi, \tau)|u(t)|^2\right) d\tau \tag{6}$$

defined on $B^1(\mathbb{R})$. The operator $\mathfrak{h}_a[\xi]$ is with compact resolvent. We introduce the lowest eigenvalue of this operator (lowest band function)

$$\mu_a(\xi) = \inf_{\substack{u \in B^1(\mathbb{R}) \\ u \neq 0}} \frac{q_a[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}. \tag{7}$$

This is a simple eigenvalue, to which corresponds a unique positive $L^2$-normalized eigenfunction, $\varphi_{a,\xi}$, i.e., satisfying (see [4, Proposition A.2])

$$\varphi_{a,\xi} > 0, \quad (\mathfrak{h}_a[\xi] - \mu_a[\xi])\varphi_{a,\xi} = 0, \quad \int_{\mathbb{R}} |\varphi_{a,\xi}(\tau)|^2 \, d\tau = 1. \tag{8}$$

Moreover, the above eigenvalue and eigenfunction depend smoothly on $\xi$ (see [8,19]),

$$\xi \mapsto \mu_a(\xi) \text{ and } \xi \mapsto \varphi_{a,\xi} \text{ are in } C^\infty. \tag{9}$$

We introduce the *step constant* (at $a$) as follows:

$$\beta_a := \inf_{\xi \in \mathbb{R}} \mu_a(\xi), \tag{10}$$

along with the celebrated de Gennes constant

$$\Theta_0 := \beta_{-1}. \tag{11}$$

Our main result is the following.

**Theorem 1.1.** *Given $a \in (-1, 0)$, there exists a unique $\zeta_a \in \mathbb{R}$ such that*

$$\beta_a = \mu_a(\zeta_a).$$

*Furthermore, the following holds:*

1. $\zeta_a < 0$ *and satisfies* $\mu_a''(\zeta_a) > 0$;
2. $|a|\Theta_0 < \beta_a < \Theta_0$;
3. *the ground state* $\phi_a := \varphi_{a,\zeta_a}$ *satisfies* $\phi_a'(0) < 0$ *and*

$$\zeta_a = -\sqrt{\beta_a + \phi_a'^2(0)/\phi_a^2(0)}.$$

**Remark 1.2.** 1. The existence of the minimum $\zeta_a$ was known earlier [4, 19]. Our contribution establishes the uniqueness of $\zeta_a$ and that it is a non-degenerate minimum. These new properties were only conjectured in [19] based on numerical computations.

2. The case $a = -1$ is perfectly understood and can be reduced to the study of the de Gennes model (family of harmonic oscillators on the half-axis with Neumann

condition at the origin). In this case, we know the existence of the unique and non-degenerate minimum $\zeta_{-1} = -\sqrt{\Theta_0}$, and that the ground state $\phi_{-1}$ is an even function with a vanishing derivative at the origin ($\phi'_{-1}(0) = 0$).

3. Our comparison result $\beta_a < \Theta_0$ is also new. It was conjectured in [4] based on numerical computations.[1] This comparison has an interesting application to the existence of superconducting magnetic edge states (see Section 4.4).

4. The sign of $\phi'_a(0)$ has an important application too, namely in determining the localization properties of ground states for the Schrödinger operator with magnetic steps and in the large field asymptotics. That will be discussed in Section 4.3.

5. In the case $a \in (0, 1)$, we have $\beta_a = a$ and $\mu_a(\cdot)$ does not achieve a minimum.

The proof of Theorem 1.1 follows the outline below.

- First, we establish the inequality $\beta_a < \Theta_0$ by constructing a test function involving the ground state of a harmonic oscillator on the half-axis with Neumann condition at the origin (see Proposition 3.1). Using the foregoing inequality and a connection to a Robin problem on the half-axis, we establish in Proposition 3.2 the result on the sign of the derivative of the positive ground state, $\phi'_a(0) < 0$.
- Again, observing an interesting relation with a model Robin problem on the half-axis, we compute $\mu''_a(\zeta_a)$ and prove that it is negative (see Proposition 3.3).

The rest of the paper is organized as follows. In Section 2, we recall a model problem of harmonic oscillators on the half-axis with Robin/Neumann condition at the origin, which provides us with the main ingredients of the proof of Theorem 1.1 in Section 3. Section 4 presents consequences of Theorem 1.1 on the large field/semi-classical asymptotics of magnetic Schrödinger operators (Theorem 4.5) and on the non-linear Ginzburg–Landau model of superconductivity (Section 4.4 and Figure 2).

## 2. The Robin model on the half line

We discuss in this section a model operator introduced in [21, 23]. Let $\xi$ and $\gamma$ be two real parameters. We introduce the family of harmonic oscillators on $\mathbb{R}_+$,

$$H[\gamma, \xi] = -\frac{d^2}{d\tau^2} + (\tau + \xi)^2, \tag{12}$$

with the following operator domain (accommodating functions satisfying the Robin condition at the origin)

$$\mathrm{Dom}(H[\gamma, \xi]) = \{u \in B^2(\mathbb{R}_+) \colon u'(0) = \gamma u(0)\}. \tag{13}$$

[1] Many thanks to V. Bonnaillie-Noël for the numerical computations and [4, Figure 5].

The quadratic form associated to $H[\gamma,\xi]$ is

$$B^1(\mathbb{R}_+)\ni u\mapsto q[\gamma,\xi](u)=\int\limits_{\mathbb{R}_+}\left(|u'(\tau)|^2+|(\tau+\xi)u(\tau)|^2\right)d\tau+\gamma|u(0)|^2.$$

The operator $H[\gamma,\xi]$ is with compact resolvent, hence its spectrum is an increasing sequence of eigenvalues $\lambda^j(\gamma,\xi)$, $j\in\mathbb{N}^*$. Furthermore, these eigenvalues are simple (see [11, Section 3.2.1] for the argument). Consequently, we introduce the corresponding orthonormal family of eigenfunctions $u^j_{\gamma,\xi}$ satisfying

$$u^j_{\gamma,\xi}(0)>0. \tag{14}$$

The condition in (14) determines the *normalized* eigenfunction uniquely, because $u^j_{\gamma,\xi}(0)\neq 0$, otherwise it will vanish everywhere by Cauchy's uniqueness theorem, since $(u^j_{\gamma,\xi})'(0)=\gamma u^j_{\gamma,\xi}(0)$ and

$$-\frac{d^2}{d\tau^2}u^j_{\gamma,\xi}+(\tau+\xi)^2u^j_{\gamma,\xi}=\lambda^j(\gamma,\xi)u^j_{\gamma,\xi}\quad\text{on }\mathbb{R}_+.$$

The perturbation theory ensures that the functions

$$\xi\mapsto\lambda^j(\gamma,\xi),\quad \xi\mapsto u^j_{\gamma,\xi},\quad \gamma\mapsto\lambda^j(\gamma,\xi),\quad \gamma\mapsto u^j_{\gamma,\xi} \tag{15}$$

are $C^\infty$. The reader is referred to [25] (for general perturbation theory) and [11, Theorem C.2.2]) for the application in the present context.

The first partial derivatives of the eigenvalues with respect to $\xi$ and $\gamma$ are as follows (see [21, 23])

$$\partial_\xi\lambda^j(\gamma,\xi)=(\lambda^j(\gamma,\xi)-\xi^2+\gamma^2)|u^j_{\gamma,\xi}(0)|^2, \tag{16}$$

$$\partial_\gamma\lambda^j(\gamma,\xi)=|u^j_{\gamma,\xi}(0)|^2. \tag{17}$$

For $\gamma=0$ and $j=1$, (16) will be crucial in the proof of Proposition 3.1 establishing the spectral inequality $\beta_a<\Theta_0$. For $\gamma\neq 0$ and $j\geq 2$, both (16) and (17) will be used in the proof of Proposition 3.3, devoted to the computation of $\mu''_a(\zeta_a)$.

Using the min-max principle, the lowest eigenvalue is defined as follows:

$$\lambda(\gamma,\xi):=\lambda^1(\gamma,\xi)=\inf\mathrm{sp}(H[\gamma,\xi])=\inf_{\substack{u\in B^1(\mathbb{R})\\ u\neq 0}}\frac{q[\gamma,\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}. \tag{18}$$

Note that the *normalized* ground state, $u_{\gamma,\xi}$, does not change sign on $\mathbb{R}_+$, and hence it is positive by our choice in (14).

For $\gamma\in\mathbb{R}$, we introduce the *de Gennes function*,

$$\Theta(\gamma):=\inf_{\xi\in\mathbb{R}}\lambda(\gamma,\xi). \tag{19}$$

**Theorem 2.1** ([8, 21]). *The following statements hold.*

1. *For all* $\xi \in \mathbb{R}$, $\gamma \mapsto \lambda(\gamma, \xi)$ *is increasing.*
2. *For all* $\gamma \in \mathbb{R}$, $\lim_{\xi \to -\infty} \lambda(\gamma, \xi) = 1$ *and* $\lim_{\xi \to +\infty} \lambda(\gamma, \xi) = +\infty$.
3. *For all* $\gamma \in \mathbb{R}$, *the function* $\xi \mapsto \lambda(\gamma, \xi)$ *admits a unique minimum attained at*

$$\xi(\gamma) := -\sqrt{\Theta(\gamma) + \gamma^2}. \tag{20}$$

   *Furthermore, this minimum is non-degenerate,* $\partial_\xi^2 \lambda(\gamma, \xi(\gamma)) > 0$.

4. *For all* $\gamma \in \mathbb{R}$, $-\gamma^2 \leq \Theta(\gamma) < 1$.

**The Neumann realization.** The particular case where $\gamma = 0$ corresponds to the Neumann realization of the operator $H[0, \xi]$, denoted by $H^N[\xi]$, with the associated quadratic form $q^N[\xi] = q[0, \xi]$. The first eigenvalue of $H^N[\xi]$ is denoted by

$$\lambda^N(\xi) = \inf \operatorname{sp}(H^N[\xi]) = \lambda(0, \xi), \tag{21}$$

with the corresponding positive $L^2$-normalized eigenfunction $u_\xi^N := u_{0,\xi}$.

By a symmetry argument [4, 19], we get that the *step* constant $\beta_{-1}$ (in (10)) satisfies

$$\Theta_0 := \beta_{-1} = \Theta(0). \tag{22}$$

This universal value $\Theta_0$ is often named the *de Gennes* constant in the literature [11, 12] and satisfies $\Theta_0 \in (\frac{1}{2}, 1)$. Numerically (see [6]), one finds $\Theta_0 \sim 0.59$. Note that the non-degenerate minimum $\xi_0 := \xi(0)$ of $\mu^N(\cdot)$ satisfies $\xi_0 = -\sqrt{\Theta_0}$.

## 3. The step model on the line

We analyse the band function $\mu_a(\cdot)$ introduced in (7) along with the *positive* normalized ground state $\varphi_{a,\xi}$.

Note that we are focusing on the interesting situation where $a \in (-1, 0)$. As mentioned earlier, for $a \in (0, 1)$, the minimum of $\mu_a(\cdot)$ is not achieved and the step constant $\beta_a = a$, see [4, 19]; while for $a = -1$, the case reduces to the de Gennes model and $\beta_{-1} = \Theta_0$.

### 3.1. Preliminaries

When $a \in (-1, 0)$, it is known that a minimum $\zeta_a$ exists and must be negative ($\zeta_a < 0$) [4, Proposition A.7]. Our Theorem 1.1 sharpens this by establishing that the minimum is unique and non-degenerate. To prove this, new comparison estimates of the step constant $\beta_a$ are needed. These estimates improve the existing ones in the literature [4, 19].

The existence of a minimum is due to the behavior at infinity of the band function $\mu_a(\cdot)$, namely,

$$\lim_{\xi\to-\infty}\mu_a(\xi)=|a| \quad \text{and} \quad \lim_{\xi\to+\infty}\mu_a(\xi)=+\infty,$$

and the following estimates on the step constant,

$$|a|\Theta_0<\beta_a<|a|. \tag{23}$$

Note that the lower bound (23) results from a simple comparison arguments using the min-max principle (see [4, Proposition A.6]). Establishing the upper bound is more tricky and relies on the construction of a trial state related to the Robin model introduced in Section 2 (see e.g., [4, Theorem 2.6]). Finally, we recall the expression for the derivative of $\mu_a(\cdot)$ established in [20] (see also [4, Proposition A.4]).

$$\mu_a'(\xi)=\Big(1-\frac{1}{a}\Big)\big(\varphi_{a,\xi}'(0)^2+(\mu_a(\xi)-\xi^2)\varphi_{a,\xi}(0)^2\big). \tag{24}$$

### 3.2. Comparison with the de Gennes constant

**Proposition 3.1.** *Let $a\in(-1,0)$. For $\beta_a$ and $\Theta_0$ as in (10) and (22) respectively, we have*

$$\beta_a<\Theta_0.$$

*Proof.* If $a\in[-\Theta_0,0)$, then (23) yields that $\beta_a<\Theta_0$ and the conclusion of Proposition 3.1 follows in this particular case.

In the sequel, we fix $a\in(-1,-\Theta_0)$. For all $\xi\in\mathbb{R}$, we denote by $u(\cdot;\xi)=u_\xi^N(\cdot)$ the positive ground state of the de Gennes model (corresponding to the eigenvalue $\lambda^N(\xi)$ in (21)). We introduce the function $g_\xi$ on $\mathbb{R}$ as follows:

$$g_\xi(\tau)=\begin{cases} u(\tau;\xi), & \text{if } t\geq 0,\\ cu(\tau;\xi/\sqrt{|a|}), & \text{if } t<0,\end{cases} \tag{25}$$

with $c=c_\xi:=u(0;\xi)/u(0;\xi/\sqrt{|a|})>0$ so that $g_\xi(0^-)=g_\xi(0^+)$. We observe that $g_\xi$ is in the form domain of the operator $\mathfrak{h}_a[\xi]$. Performing an elementary scaling argument, we get

$$\begin{aligned} q_a[\xi](g_\xi)&=\lambda^N(\xi)\int_{\mathbb{R}_+}|g_\xi(t)|^2\,dt+|a|\lambda^N\Big(\frac{\xi}{\sqrt{|a|}}\Big)\int_{\mathbb{R}_-}|g_\xi(t)|^2\,dt\\ &=\lambda^N(\xi)\int_{\mathbb{R}}|g_\xi(t)|^2\,dt+\Big(|a|\lambda^N\Big(\frac{\xi}{\sqrt{|a|}}\Big)-\lambda^N(\xi)\Big)\int_{\mathbb{R}_-}|g_\xi(t)|^2\,dt.\end{aligned}$$

We choose now $\xi = \xi_0 := -\sqrt{\Theta_0}$ corresponding to $\Theta_0$ in (22). We get $\lambda^N(\xi_0) = \Theta_0$ and

$$q_a[\xi_0](g_{\xi_0}) = \Theta_0 \int_{\mathbb{R}} |g_{\xi_0}(\tau)|^2\, dt + f(|a|) \int_{\mathbb{R}_-} |g_{\xi_0}(\tau)|^2\, d\tau,$$

where $f(x) := x\lambda^N\big(\frac{\xi_0}{\sqrt{x}}\big) - \Theta_0$, for $x \in (\Theta_0, 1)$. By the min-max principle

$$\beta_a \leq \frac{q_a[\xi_0](g_{\xi_0})}{\|g_{\xi_0}\|^2_{L^2(\mathbb{R})}} \leq \Theta_0 + f(|a|)\frac{\int_{\mathbb{R}_-} |g_{\xi_0}(\tau)|^2\, d\tau}{\int_{\mathbb{R}} |g_{\xi_0}(\tau)|^2\, d\tau}.$$

To get that $\beta_a < \Theta_0$, it suffices to prove that $f(x) < 0$, for $x \in (\Theta_0, 1)$.

Let $x \in (\Theta_0, 1)$ and $\alpha = \frac{\xi_0}{\sqrt{x}} \in (-1, \xi_0)$. By (16) (applied for $j = 1$ and $\gamma = 0$), we can write

$$f(x) = x(\lambda^N(\alpha) - \alpha^2) = x\frac{(\lambda^N)'(\alpha)}{|u^N_\alpha(0)|^2}.$$

Since $\alpha \in (-1, \xi_0)$ and $\lambda^N(\cdot)$ is monotone decreasing on the interval $(-1, \xi_0)$, we deduce that $(\lambda^N)'(\alpha) < 0$ and eventually $f(x) < 0$ as required. ■

### 3.3. Variation of the ground state near zero

We pick any $\zeta_a \in \mu_a^{-1}(\beta_a)$ so that $\beta_a = \mu_a(\zeta_a)$, and denote by $\phi_a = \varphi_{a,\zeta_a}$ the positive normalized ground state for $\beta_a$ (suppressing the dependence of the ground state on $\zeta_a$). We determine the sign of the derivative of $\phi_a$ at the origin, thereby yielding that the ground state is a decreasing function in a neighbourhood of 0. This result will be crucial in deriving the sign of some *moments* in Section 4.1 later.

**Proposition 3.2.** *For all $a \in (-1, 0)$ and $\zeta_a \in \mu_a^{-1}(\beta_a)$, the positive normalized ground state $\phi_a = \varphi_{a,\zeta_a}$ satisfies $\phi_a'(0) < 0$.*

*Proof.* The proof relies on a comparison argument involving the Robin model. Let $\gamma_a = \phi_a'(0)/\phi_a(0)$. Since the ground state $\phi_a$ is positive, it suffices to prove that $\gamma_a < 0$. The eigenvalue equation $\mathfrak{h}_a[\zeta_a]\phi_a = \beta_a\phi_a$ written on $\mathbb{R}_+$ is

$$\begin{cases} -\phi_a''(\tau) + (\tau + \zeta_a)^2\phi_a(\tau) = \beta_a\phi_a(\tau), & t > 0, \\ \phi_a'(0) = \gamma_a\phi_a(0), \end{cases} \tag{26}$$

Consequently, $\phi_a$ is an eigenfunction of the Robin operator $H[\gamma_a, \zeta_a]$, defined in (13), with a corresponding eigenvalue $\beta_a$. Using the min-max principle, we have

$$\beta_a \geq \lambda(\gamma_a, \zeta_a) \tag{27}$$

where $\lambda(\gamma_a, \zeta_a)$ is defined in (18).

If $\gamma_a \geq 0$, then by Theorem 2.1, Proposition 3.1 and (22), we get

$$\lambda(\gamma_a, \zeta_a) \geq \lambda(0, \zeta_a) = \lambda^N(\zeta_a) \geq \Theta_0 > \beta_a,$$

thereby contradicting (27). This proves that $\gamma_a < 0$. ∎

### 3.4. Uniqueness and non-degeneracy of the minimum

Now, we establish that the minimum of $\mu_a(\cdot)$ is unique and non-degenerate. The key in our proof is a tricky connection with the Robin model.

**Proposition 3.3.** *For all $a \in (-1, 0)$, there exists $\zeta_a < 0$ such that*

$$\mu_a^{-1}(\beta_a) = \{\zeta_a\} \quad \textit{and} \quad \mu_a''(\zeta_a) > 0,$$

*where $\mu_a(\cdot)$ and $\beta_a$ are the eigenvalues introduced in (7) and (10) respectively.*

*Proof.* First, note that $\mu_a^{-1}(\beta_a) \subset \mathbb{R}_-$ and is non-empty, by [4, Proposition A.7]. Hence, it suffices to prove that any negative critical point is a non-degenerate local minimum.

Let $\eta < 0$ be a critical point of $\mu_a(\cdot)$ (i.e., $\mu_a'(\eta) = 0$). For all $\xi \in \mathbb{R}$, we introduce

$$\gamma(\xi) = \gamma_a(\xi) := \varphi_{\xi,a}'(0)/\varphi_{\xi,a}(0), \tag{28}$$

where $\varphi_{\xi,a}$ is the *positive* normalized ground state of the operator $\mathfrak{h}_a[\xi]$, which is now an eigenfunction for the Robin problem

$$\begin{cases} -\varphi_{\xi,a}''(\tau) + (\tau+\xi)^2\varphi_{\xi,a}(\tau) = \mu_a(\xi)\varphi_{\xi,a}(\tau), & \tau > 0, \\ \varphi_{\xi,a}'(0) = \gamma(\xi)\varphi_{\xi,a}(0). \end{cases} \tag{29}$$

Using this for $\xi = \eta$, we can pick $j = j(\eta) \in \mathbb{N}$ such that $\mu_a(\eta) = \lambda^j(\gamma(\eta), \eta)$, the $j$-th min-max eigenvalue of $H[\gamma(\xi), \xi]$. By the continuity of the involved functions and the simplicity of the eigenvalue $\lambda^j(\gamma(\eta), \eta)$, we can pick $\varepsilon = \varepsilon(\eta) > 0$ such that

$$\mu_a(\xi) = \lambda^j(\gamma(\xi), \xi) \quad \text{for all } \xi \in (\eta - \varepsilon, \eta + \varepsilon). \tag{30}$$

Hence, by (16), (17), and differentiation in (30) with respect to $\xi$ we get

$$\begin{aligned} \mu_a'(\xi) &= \partial_\xi \lambda^j(\gamma(\xi), \xi) \\ &= \big(\lambda^j(\gamma(\xi), \xi) - \xi^2 + \gamma^2(\xi)\big)|u^j_{\gamma(\xi),\xi}(0)|^2 + \gamma'(\xi)|u^j_{\gamma(\xi),\xi}(0)|^2. \end{aligned} \tag{31}$$

Since $\mu_a'(\eta) = 0$, we infer from (24) and (30) that

$$\lambda^j(\gamma(\eta), \eta) - \eta^2 + \gamma(\eta)^2 = \mu_a(\eta) - \eta^2 + \gamma(\eta)^2 = \left(1 - \frac{1}{a}\right)^{-1} \frac{\mu_a'(\eta)}{\varphi_{\eta,a}(0)^2} = 0. \tag{32}$$

Inserting this into (31) after setting $\xi = \eta$, we get (thanks to (14))

$$\gamma'(\eta) = 0. \tag{33}$$

This result will be used in the computation of $\mu_a''(\eta)$ below. In fact, differentiation in (24) with respect to $\xi$ yields

$$\begin{aligned}\mu_a''(\xi) =& \Big(1 - \frac{1}{a}\Big)\Big(\big(\mu_a(\xi) - \xi^2 + \gamma(\xi)^2\big)\partial_\xi \varphi_{\xi,a}^2(0) \\ &+ \big(\mu_a'(\xi) - 2\xi + 2\gamma(\xi)\partial_\xi\gamma(\xi)\big)\varphi_{\xi,a}^2(0)\Big).\end{aligned}$$

Considering again $\xi = \eta$, we get

$$\mu_a''(\eta) = 2\Big(\frac{1}{a} - 1\Big)\eta\varphi_{\eta,a}^2(0).$$

In the above equation, we used (24), (32), and (33). Recall that we take $\eta < 0$ and $a \in (-1, 0)$, hence

$$\mu_a''(\eta) > 0,$$

and this holds for any negative critical point, $\eta$, of $\mu_a(\cdot)$. This finishes the proof. ■

### 3.5. Proof of the main result

Theorem 1.1 now follows by collecting Propositions 3.3, 3.2, and 3.1.

## 4. Applications

### 4.1. Moments

Fix $a \in [-1, 0)$ and consider $\beta_a$ as in (10), the ground state $\phi_a$, and $\zeta_a$ the unique minimum of $\mu_a(\cdot)$ (see Theorem 1.1 and Remark 1.2). We can invert the operator $\mathfrak{h}_a[\zeta_a] - \beta_a$ on the functions orthogonal to the ground state $\phi_a$, thereby leading to the introduction of the regularized resolvent (see e.g., [11, Lemma 3.2.9]):

$$\mathfrak{R}_a(u) = \begin{cases} 0 & \text{if } u \parallel \phi_a, \\ (\mathfrak{h}_a[\zeta_a] - \beta_a)^{-1}u & \text{if } u \perp \phi_a. \end{cases} \tag{34}$$

The construction of certain trial states in Section 4.2 below requires inverting $\mathfrak{h}_a[\zeta_a] - \beta_a$ on functions involving $(\zeta_a + \sigma(\tau)\tau)^n\phi_a(\tau)$, for positive integers $n$, with $\sigma(\cdot)$ introduced in (4). We are then lead to investigate the following *moments*

$$M_n(a) = \int_{-\infty}^{+\infty} \frac{1}{\sigma(\tau)}(\zeta_a + \sigma(\tau)\tau)^n |\phi_a(\tau)|^2\, d\tau,$$

**Proposition 4.1.** *For $a \in [-1, 0)$, we have*

$$M_1(a) = 0, \tag{35}$$

$$M_2(a) = -\frac{1}{2}\beta_a \int_{-\infty}^{+\infty} \frac{1}{\sigma(t)} |\phi_a(\tau)|^2 \, d\tau + \frac{1}{4}\Big(\frac{1}{a} - 1\Big)\zeta_a \phi_a(0)\phi_a'(0), \tag{36}$$

$$M_3(a) = \frac{1}{3}\Big(\frac{1}{a} - 1\Big)\zeta_a \phi_a(0)\phi_a'(0). \tag{37}$$

**Remark 4.2.** 1 (Feynman–Hellmann). We have (see e.g., [4, (A.9)])

$$(\zeta_a + \sigma(\tau)\tau)\phi_a(\tau) \perp \phi_a(\tau) \quad \text{in } L^2(\mathbb{R}). \tag{38}$$

Furthermore, since $M_1(a) = 0$, we get $\frac{1}{\sigma(\tau)}(\zeta_a + \sigma(\tau)\tau)\phi_a \perp \phi_a$. Combined together, we see that

$$(\zeta_a + a\tau)\phi_a \perp \phi_a \quad \text{in } L^2(\mathbb{R}_-), \qquad (\zeta_a + \tau)\phi_a \perp \phi_a \qquad \text{in } L^2(\mathbb{R}_+)$$

which is consistent with (26), since by (24) and (16), $\zeta_a$ is a critical point of the corresponding Robin band function $\lambda^j(\gamma_a, \cdot)$.

2. As a consequence of Theorem 1.1, $M_3(a) = 0$ for $a = -1$, and it is negative for $-1 < a < 0$, which is consistent with [5].

*Proof.* In an analogous manner to [5], we define the operator

$$L = \mathfrak{h}_a[\zeta_a] - \beta_a = -\frac{d^2}{d\tau^2} + (\zeta_a + \sigma(\tau)\tau)^2 - \beta_a.$$

Pick an arbitrary smooth function on $\mathbb{R} \setminus \{0\}$ and set $v = 2p\phi_a' - p'\phi_a$. We check that

$$Lv = \big(p^{(3)} - 4\big((\zeta_a + \sigma\tau)^2 - \beta_a\big)p' - 4\sigma(\zeta_a + \sigma\tau)p\big)\phi_a. \tag{39}$$

Noting that $L\phi_a = 0$, we obtain by an integration by parts,

$$\int_{-\infty}^{+\infty} \phi_a L v \, d\tau = -\phi_a(0)v'(0^-) + \phi_a(0)v'(0^+) + \phi_a'(0)v(0^-) - \phi_a'(0)v(0^+). \tag{40}$$

By taking $p = 1/\sigma^2$, we get $M_1(a) = 0$. Then, inserting $p = \frac{1}{\sigma^2}(\zeta_a + \sigma t)^2$ into (39), we get (36). Finally, the choice $p = \frac{1}{\sigma^2}(\zeta_a + \sigma t)^3$ yields (37). ∎

### 4.2. A model operator in a weighted space

The effective operator $\mathfrak{h}_a[\xi]$, which is the subject of study in Theorem 1.1, is not suitable in the situation of a *curved* magnetic edge, encountered later in the subsequent Section 4.3 (see also [2]). Understanding the influence of the edge's geometry requires a more complicated effective operator which we analyze in the present section. The new effective operator involves the edge's curvature as a parameter, and its ground state energy can be accurately estimated, thanks to the result of Theorem 1.1 on the uniqueness and non-degeneracy of the minimum.

We fix $a \in (-1,0)$, $\delta \in (0, \frac{1}{12})$, $M > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0)$, $Mh^{\frac{1}{2}-\delta} < \frac{1}{3}$. In that way, for $\mathfrak{k} \in [-M, M]$, we can introduce the positive function $a_h = (1 - \mathfrak{k}h^{\frac{1}{2}}\tau)$ and the Hilbert space $L^2((-h^{-\delta}, h^{-\delta}); a_h\, d\tau)$ with the weighted inner product

$$\langle u, v\rangle = \int_{-h^{-\delta}}^{h^{-\delta}} u(\tau)\overline{v(\tau)}(1 - \mathfrak{k}h^{\frac{1}{2}}\tau)\, d\tau.$$

For $\xi \in \mathbb{R}$, we introduce the self-adjoint operator

$$\begin{aligned}\mathcal{H}_{a,\xi,\mathfrak{k},h} = &-\frac{d^2}{d\tau^2} + (\sigma\tau+\xi)^2 + \mathfrak{k}h^{\frac{1}{2}}(1-\mathfrak{k}h^{\frac{1}{2}}\tau)^{-1}\partial_\tau \\ &+ 2\mathfrak{k}h^{\frac{1}{2}}\tau\Big(\sigma\tau+\xi-\mathfrak{k}h^{\frac{1}{2}}\sigma\frac{\tau^2}{2}\Big)^2 - \mathfrak{k}h^{\frac{1}{2}}\sigma\tau^2(\sigma\tau+\xi) + \mathfrak{k}^2h\sigma^2\frac{\tau^4}{4},\end{aligned} \tag{41}$$

where $\sigma(\cdot)$ is the function in (4). The domain of definition of this operator is

$$\mathrm{Dom}(\mathcal{H}_{a,\xi,\mathfrak{k},h}) = \{u \in H^2(-h^{-\delta}, h^{-\delta}) : u(\pm h^{-\delta}) = 0\}. \tag{42}$$

The operator $\mathcal{H}_{a,\xi,\mathfrak{k},h}$ is the Friedrichs extension in $L^2\big((-h^{-\delta}, h^{-\delta}); a_h d\tau\big)$ associated to the quadratic form $q_{a,\xi,\mathfrak{k},h}$ defined by

$$\begin{aligned}&q_{a,\xi,\mathfrak{k},h}(u) \\ &= \int_{-h^{-\delta}}^{h^{-\delta}} \Big(|u'(\tau)|^2 + (1+2\mathfrak{k}h^{\frac{1}{2}}\tau)\Big(\sigma\tau+\xi-\mathfrak{k}h^{\frac{1}{2}}\sigma\frac{\tau^2}{2}\Big)^2 u^2(\tau)\Big)(1-\mathfrak{k}h^{\frac{1}{2}}\tau)\, d\tau.\end{aligned}$$

The operator $\mathcal{H}_{a,\xi,\mathfrak{k},h}$ is with compact resolvent. We denote by $(\lambda_n(\mathcal{H}_{a,\xi,\mathfrak{k},h}))_{n\geq 1}$ its sequence of min-max eigenvalues.

By Theorem 1.1, $\mu_a(\cdot)$ has a unique minimum $\beta_a$ (attained at $\zeta_a$) which is non-degenerate, and the moment $M_3(a)$ in (37) is negative, thereby allowing us to derive the following result on the ground state energy of $\mathcal{H}_{a,\xi,\mathfrak{k},h}$.

**Proposition 4.3.** *Let* $\beta_{a,\mathfrak{k},h} = \inf_{\xi\in\mathbb{R}} \lambda_1(\mathcal{H}_{a,\xi,\mathfrak{k},h})$. *Then, as* $h \to 0_+$,

$$\beta_{a,\mathfrak{k},h} = \beta_a + \mathfrak{k} M_3(a) h^{\frac{1}{2}} + \mathcal{O}(h^{\frac{3}{4}})$$

*uniformly with respect to* $\mathfrak{k} \in [-M, M]$.

*Proof.* We will present the outline of the proof to show the role of Theorem 1.1. A similar approach was detailed in [17, Theorem 11.1]. By the min-max principle, there exists $C > 0$ such that for all $n \geq 1$, $\xi \in \mathbb{R}$ and $h \in (0, h_0)$,

$$|\lambda_n(\mathcal{H}_{a,\xi,\mathfrak{k},h}) - \lambda_n(\mathfrak{h}_a[\xi])| \leq C h^{\frac{1}{2}-2\delta}(1 + \lambda_n(\mathfrak{h}_a[\xi])), \tag{43}$$

where $\mathfrak{h}_a[\xi]$ is the fiber operator in (1). Consequently, we may find a constant $z(a) > 0$ such that

$$\lambda_1(\mathcal{H}_{a,\xi,\mathfrak{k},h}) \geq \beta_a + h^{\frac{1}{2}-2\delta} \quad \text{for } |\xi - \zeta_a| \geq z(a) h^{\frac{1}{4}-\delta}. \tag{44}$$

Note that (44) is a consequence of the fact that $\zeta_a$ is a non-degenerate minimum of $\mu_a(\cdot)$.

Now, we estimate $\lambda_1(\mathcal{H}_{a,\xi,\mathfrak{k},h})$ for $|\xi - \zeta_a| \leq z(a) h^{\frac{1}{4}-\delta} \ll 1$. By (43), the simplicity of the eigenvalues $\lambda_n(\mathfrak{h}_a[\xi])$ and the continuity of the function $\xi \mapsto \lambda_n(\mathfrak{h}_a[\xi])$, we know that as $h \to 0_+$

$$\lambda_1(\mathcal{H}_{a,\xi,\mathfrak{k},h}) = \beta_a + o(1) \quad \text{and} \quad \lambda_2(\mathcal{H}_{a,\xi,\mathfrak{k},h}) = \lambda_2(\mathfrak{h}_a[\zeta_a]) + o(1),$$

with

$$\lambda_2(\mathfrak{h}_a[\zeta_a]) > \lambda_1(\mathfrak{h}_a[\zeta_a]) = \beta_a. \tag{45}$$

One may construct a formal eigen-pair $(\lambda^{\text{app}}_{a,\xi,\mathfrak{k},h}, f^{\text{app}}_{a,\xi,\mathfrak{k},h})$ of the operator $\mathcal{H}_{a,\xi,\mathfrak{k},h}$, with

$$\lambda^{\text{app}}_{a,\xi,\mathfrak{k},h} = c_0 + c_1(\xi - \zeta_a) + c_2(\xi - \zeta_a)^2 + c_3 h^{1/2} \tag{46a}$$

and

$$f^{\text{app}}_{a,\xi,\mathfrak{k},h} = u_0 + (\xi - \zeta_a)u_1 + (\xi - \zeta_a)^2 u_2 + h^{1/2} u_3. \tag{46b}$$

Expanding $R_h := (\mathcal{H}_{a,\xi,\mathfrak{k},h} - \lambda^{\text{app}}_{a,\xi,\mathfrak{k},h}) f^{\text{app}}_{a,\xi,\mathfrak{k},h}$ in powers of $(\xi - \zeta_a)$ and $h^{1/2}$, one can choose $(c_i, u_i)_{0\leq i\leq 3}$ so that the coefficients of the $h^{1/2}$ and the terms $(\xi - \zeta_a)^j$, $j = 0, 1, 2$, vanish. We choose

$$c_0 = \beta_a,$$
$$u_0 = \phi_a,$$

$$c_1 = 0,$$
$$u_1 = -2\mathfrak{R}_a v_1,$$

$$v_1 := (\sigma\tau + \zeta_a)\phi_a \perp \phi_a,$$

$$c_2 = 1 - 4\int_{-\infty}^{+\infty} (\sigma\tau + \zeta_a)\phi_a \mathfrak{R}_a[(\sigma\tau + \zeta_a)\phi_a]\, dt,$$

$$u_2 = \mathfrak{R}_a v_2,$$

$$v_2 := 4(\sigma\tau + \zeta_a)\mathfrak{R}_a[(\sigma\tau + \zeta_a)\phi_a] + (c_2 - 1)\phi_a \perp \phi_a,$$

$$c_3 = \mathfrak{k} M_3(a),$$

$$u_3 = \mathfrak{R}_a v_3,$$

$$v_3 := -\mathfrak{k}\Big(\partial_\tau + \frac{1}{\sigma}(\sigma\tau + \zeta_a)^3 - \frac{\zeta_a^2}{\sigma}(\sigma\tau + \zeta_a)\Big)\phi_a + c_3\phi_a \perp \phi_a,$$

where $\mathfrak{R}_a \in \mathcal{L}(L^2(\mathbb{R}))$ is the regularized resolvent introduced in (34). That the functions $v_1, v_2, v_3$ are orthogonal to $\phi_a$ is ensured by our choice of $c_1, c_2, c_3$, the expressions of the moments in Proposition 4.1, and the first item in Remark 4.2.

Eventually, using $\chi(h^\delta\tau) f^{\mathrm{app}}_{a,\xi,\mathfrak{k},h}$ as a quasi-mode, with $\chi$ a cut-off function introduced to insure the Dirichlet condition at $\tau = \pm h^{-\delta}$, we get by the spectral theorem and (45),

$$\lambda_1(\mathcal{H}_{a,\xi,\mathfrak{k},h}) = c_0 + c_2(\xi - \zeta_a)^2 + c_3 h^{1/2} + \mathcal{O}(\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3, h)). \tag{47}$$

Note that, for $|\xi - \zeta_a| \le z(a) h^{\frac{1}{4}-\delta}$, we have

$$\mathcal{O}(\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3), h) = \mathcal{O}(h^{3(\frac{1}{4}-\delta)}).$$

In order to minimize over $\xi$, we observe that the constant $c_2$ can be expressed in the form[2]

$$c_2 = \frac{1}{2}\mu_a''(\zeta_a),$$

hence $c_2 > 0$ by Theorem 1.1. So, we get from (44) and (45),

$$\inf_{\xi\in\mathbb{R}} \lambda_1(\mathcal{H}_{a,\xi,\mathfrak{k},h}) = c_0 + c_3 h^{1/2} + \mathcal{O}(h^{3(\frac{1}{4}-\delta)}). \tag{48}$$

To improve the error in (48), notice that, by (47), it is enough to minimize over $\{|\xi - \zeta_a| \le h^{\frac{1}{4}}\}$. This finishes the proof of Theorem 4.3. ■

**Remark 4.4.** The approximate eigen-pair $(\lambda^{\mathrm{app}}_{a,\xi,\mathfrak{k},h}, f^{\mathrm{app}}_{a,\xi,\mathfrak{k},h})$ in (46) does not depend on the parameter $\delta$ introduced in (42). Moreover, we have, for $|\xi - \zeta_a| < 1$,

$$\|(\mathcal{H}_{a,\xi,\mathfrak{k},h} - \lambda^{\mathrm{app}}_{a,\xi,\mathfrak{k},h}) f^{\mathrm{app}}_{a,\xi,\mathfrak{k},h}\|_{L^2(\mathbb{R})} = \mathcal{O}(\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3, h)).$$

[2] Using the Feynman–Hellmann formula $\mu_a'(\xi) = \langle(\zeta_a + \sigma(\tau)\tau)\varphi_{a,\xi}, \varphi_{a,\xi}\rangle$, see [4, (A.9)].

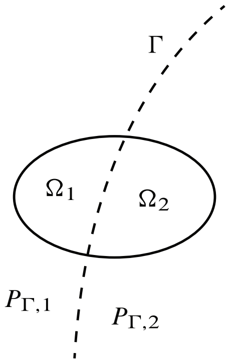


**Figure 1.** The curve $\Gamma$ splits $\mathbb{R}^2$ into two regions, $P_{\Gamma,1}$ and $P_{\Gamma,2}$, and the domain $\Omega$ into two domains $\Omega_1$ and $\Omega_2$.

### 4.3. Magnetic edge and semi-classical ground state energy

With the estimates of the ground state energy of the weighted operator of Section 4.2 in hand, we can study the edge states of a Dirichlet Laplace operator with a magnetic step field, defined in what follows.

**4.3.1. Magnetic edge, the domain and the operator.** Consider a simple smooth planar curve $\Gamma \subset \mathbb{R}^2$ that splits $\mathbb{R}^2$ into two disjoint unbounded open sets, $P_{\Gamma,1}$ and $P_{\Gamma,2}$. We will refer to $\Gamma$ as the *magnetic edge,* since we are going to consider magnetic fields having a jump along $\Gamma$ (see Figure 1).

Now, consider an open bounded simply connected subset $\Omega$ of $\mathbb{R}^2$, with smooth boundary $\partial\Omega$ of class $C^1$, and assume that

1. $\Gamma$ intersects $\partial\Omega$ at two distinct points $p$ and $q$, and the intersection is transversal, i.e., $\mathrm{T}_{\partial\Omega} \times \mathrm{T}_\Gamma \neq 0$ on $\{p, q\}$, where $\mathrm{T}_{\partial\Omega}$ and $\mathrm{T}_\Gamma$ are respectively unit tangent vectors of $\partial\Omega$ and $\Gamma$;
2. $\Omega_1 := \Omega \cap P_{\Gamma,1} \neq \emptyset$ and $\Omega_2 := \Omega \cap P_{\Gamma,2} \neq \emptyset$.

Fix $a \in (-1, 0)$. Let $\mathbf{F}_a \in H^1(\Omega, \mathbb{R}^2)$ be a magnetic potential with the corresponding scalar magnetic field:

$$\operatorname{curl} \mathbf{F}_a = B_a := \mathbf{1}_{\Omega_1} + a\mathbf{1}_{\Omega_2}. \tag{49}$$

We consider the Dirichlet realization of the self-adjoint operator in the domain $\Omega$

$$\mathcal{P}_{h,a} = -(h\nabla - i\mathbf{F}_a)^2 = -h^2\Delta + ih(\operatorname{div} \mathbf{F}_a + \mathbf{F}_a \cdot \nabla) + |\mathbf{F}_a|^2,$$

with domain

$$\operatorname{Dom}(\mathcal{P}_{h,a}) = \{u \in L^2(\Omega) \colon (h\nabla - i\mathbf{F}_a)^j u \in L^2(\Omega),\ j \in \{1, 2\}, u|_{\partial\Omega} = 0\},$$

and quadratic form

$$\mathfrak{q}_{h,a}(u)=\int_{\Omega}|(h\nabla-i\mathbf{F}_a)u|^2\,dx\quad(u\in H_0^1(\Omega)).\tag{50}$$

The bottom of the spectrum of this operator is introduced as follows

$$\lambda_1(\mathscr{P}_{h,a})=\inf_{u\in H_0^1(\Omega)\setminus\{0\}}\frac{\mathfrak{q}_{h,a}(u)}{\|u\|^2_{L^2(\Omega)}}.\tag{51}$$

**4.3.2. Frenet coordinates near the magnetic edge.** We now introduce the Frenet coordinates near $\Gamma$. We refer the reader to [11, Appendix F] and [4] for a similar setup.

Let $s\mapsto M(s)\in\Gamma$ be the arc length parametrization of $\Gamma$ such that

- $\nu(s)$ is the unit normal of $\Gamma$ at the point $M(s)$ pointing to $P_{\Gamma,1}$;
- $T(s)$ is the unit tangent vector to $\Gamma$ at the point $M(s)$, such that $(T(s),\nu(s))$ is a direct frame, i.e., $\det(T(s),\nu(s))=1$.

Now, we define the curvature $k$ of $\Gamma$ as follows $T'(s)=k(s)\nu(s)$. For $\varepsilon>0$, we define the transformation

$$\Phi\colon\mathbb{R}\times(-\varepsilon,\varepsilon)\ni(s,t)\mapsto M(s)+t\nu(s)\in\Gamma_\varepsilon:=\{x\in\mathbb{R}^2\colon\operatorname{dist}(x,\Gamma)<\varepsilon\}\tag{52}$$

and pick $\varepsilon$ sufficiently small so that $\Phi$ is a diffeomorphism.

**4.3.3. Ground state energy and curvature of the magnetic edge.** We introduce the maximal curvature of $\Gamma$ in $\Omega$ as follows

$$k^{\Omega}_{\max}=\max_{x=\Phi(s,0)\in\Gamma\cap\overline{\Omega}}(k(s)).\tag{53}$$

**Theorem 4.5.** *There exist positive constants $c_a,C_a,h_a$ such that the ground state energy in (51) satisfies, for all $h\in(0,h_a)$,*

$$-c_ah^{\frac{5}{3}}\le\lambda_1(\mathscr{P}_{h,a})-(\beta_ah+M_3(a)k^{\Omega}_{\max}h^{\frac{3}{2}})\le C_ah^{\frac{7}{4}}.$$

The proof of Theorem 4.5 can be obtained in a manner similar to that in [17, Proposition 10.8] and [11, Theorem 8.3.2], using in particular the weighted operator in Theorem 4.3, so we omit the details.

We mention two important inequalities that are useful to analyse the bound states of $\mathscr{P}_{h,a}$. The first inequality is

$$\mathfrak{q}_{h,a}(u)\ge\int_{\Omega}\big(U_{h,a}(x)-\mathcal{O}(R_0^{-2}h)\big)|u(x)|^2\,dx\quad(u\in H_0^1(\Omega)),$$

where $R_0 > 1$ is a fixed (arbitrary) constant and

$$U_{h,a}(x) = \begin{cases} |a|h & \text{if } \operatorname{dist}(x,\Gamma) > R_0 h^{1/2}, \\ \beta_a h & \text{if } \operatorname{dist}(x,\Gamma) < R_0 h^{1/2}. \end{cases}$$

The second inequality is

$$\mathfrak{q}_{h,a}(u) \geq \int_\Omega \big(U^\Gamma_{h,a}(x) - \mathcal{O}(h^{\frac{5}{3}})\big)|u(x)|^2\,dx \quad (u \in H^1_0(\Omega)), \tag{54}$$

where

$$U^\Gamma_{h,a}(x) = \begin{cases} |a|h & \text{if } \operatorname{dist}(x,\Gamma) > 2h^{\frac{1}{6}}, \\ \beta_a h + M_3(a)\kappa(s)h^{\frac{3}{2}} & \text{if } \operatorname{dist}(x,\Gamma) < 2h^{\frac{1}{6}} \text{ and } x = \Phi(s,t). \end{cases}$$

### 4.4. Superconductivity along the magnetic edge

The new estimate $\beta_a < \Theta_0$ in Theorem 1.1 gives a more precise description of the nucleation of superconductivity in type-II superconductors subject to magnetic steps fields with certain intensities, considered for instance in [4] (see also [1,3]).

In the context of superconductivity, the set $\Omega$ introduced in Section 4.3 models the horizontal cross section of a cylindrical superconductor, with a large characteristic parameter $\kappa$ and submitted to the magnetic field $HB_a$, where $B_a$ is as in (49), $a \in (-1,0)$, and the parameter $H > 0$ measures the intensity of the magnetic field. The superconducting properties of the sample are described by the minimizing configurations of the following Ginzburg–Landau (GL) energy functional:

$$\begin{aligned} \mathcal{E}_{\kappa,H}(\psi,\mathbf{A}) = &\int_\Omega \Big(|(\nabla - i\kappa H\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\Big)\,dx \\ &+ \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_a|^2\,dx, \end{aligned} \tag{55}$$

where $\psi \in H^1(\Omega;\mathbb{C})$ is the order parameter, and $\mathbf{A} \in H^1(\Omega;\mathbb{R}^2)$ is the induced magnetic field. For a fixed $(\kappa, H)$, the infimum of the energy–the ground state energy–is attained by a minimizer $(\psi^{\mathrm{GL}}, \mathbf{A}^{\mathrm{GL}})_{\kappa,H}$.

In [4], the limit profile of $|\psi^{\mathrm{GL}}|^4$ is determined in the sense of distributions in the regime where $H = b\kappa$ and $\kappa \to +\infty$, with $b > \frac{1}{|a|}$ a fixed constant. More precisely, the following convergence holds

$$\kappa\mathcal{T}^b_\kappa \rightharpoonup \mathcal{T}^b \text{ in } \mathcal{D}'(\mathbb{R}^2), \quad \text{as } \kappa \to +\infty,$$

where

$$C_c^\infty(\mathbb{R}^2) \ni \varphi \mapsto \mathcal{T}_\kappa^b(\varphi) = \int_\Omega |\psi^{\mathrm{GL}}|^4 \varphi \, dx$$

and the limit distribution $\mathcal{T}^b$ is defined via three distributions related to the edges $\Gamma$, $\Gamma_1 = (\partial\Omega_1) \cap (\partial\Omega)$ and $\Gamma_2 = (\partial\Omega_2) \cap (\partial\Omega)$ as follows

$$C_c^\infty(\mathbb{R}^2) \ni \varphi \mapsto \mathcal{T}^b(\varphi) = -2b^{-\frac{1}{2}}(\mathcal{T}_\Gamma^b(\varphi) + \mathcal{T}_{\Gamma_1}^b(\varphi) + \mathcal{T}_{\Gamma_2}^b(\varphi)),$$

with

$$\mathcal{T}_\Gamma^b(\varphi) := \mathrm{e}_a(b) \int_\Gamma \varphi \, ds_\Gamma,$$

$$\mathcal{T}_{\Gamma_1}^b(\varphi) = E_{\mathrm{surf}}(b) \int_{\Gamma_1} \varphi \, ds \quad \mathcal{T}_{\Gamma_2}^b(\varphi) = |a|^{-\frac{1}{2}} E_{\mathrm{surf}}(b|a|) \int_{\Gamma_2} \varphi \, ds.$$

The effective energies $\mathrm{e}_a$ and $E_{\mathrm{surf}}$ correspond respectively to the contribution of the magnetic edge $\Gamma$ and the boundary $\partial\Omega$ (see [4, 7] for the precise definitions). They have the following properties:

- $\mathrm{e}_a(b) = 0$ if and only if $b \geq 1/\beta_a$;
- $E_{\mathrm{surf}}(b) = 0$ if and only if $b \geq 1/\Theta_0$.

Based on the results above, a detailed discussion on the distribution of superconductivity near $\Gamma \cup \partial\Omega$ has been done in [4, Section 1.5]. This discussion mainly relies on the order of the values $|a|\Theta_0$, $\beta_a$ and $\Theta_0$. With the existing estimates in this paper (and [4]), we have

$$|a|\Theta_0 < \beta_a < \min(\Theta_0, |a|) \quad \text{for } a \in (-1, 0).$$

Consequently, we observe that (see Figure 2 for illustration)

- $\mathcal{T}^b = 0$ for $b \geq b_{c,3} := \frac{1}{|a|\Theta_0}$;
- $\mathcal{T}_{\Gamma_1}^b = \mathcal{T}_\Gamma^b = 0$ and $\mathcal{T}_{\Gamma_2}^b \neq 0$ for $b_{c,2} := \frac{1}{\beta_a} \leq b < b_{c,3}$;
- $\mathcal{T}_{\Gamma_1}^b = 0$, $\mathcal{T}_{\Gamma_1}^b \neq 0$ and $\mathcal{T}_{\Gamma_2}^b \neq 0$ for $b_{c,1} := \max(\frac{1}{|a|}, \frac{1}{\Theta_0}) \leq b < b_{c,2}$.

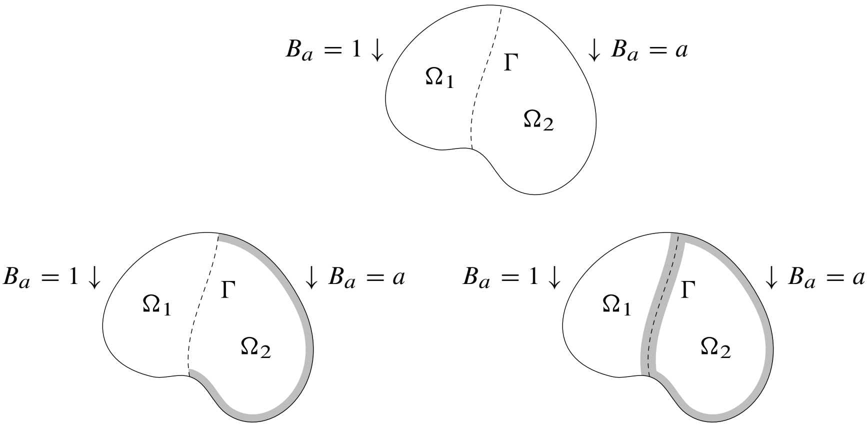


**Figure 2.** Superconductivity localization in the set $\Omega$ submitted to the magnetic field $B_a$, for $a \in (-1, 0)$, with intensity $H = b\kappa$, where respectively $b \geq b_{c,3} := \frac{1}{|a|\Theta_0}$, $b_{c,2} := \frac{1}{\beta_a} \leq b < b_{c,3}$ and $b_{c,1} := \max(\frac{1}{|a|}, \frac{1}{\Theta_0}) \leq b < b_{c,2}$. Only the grey regions carry superconductivity.

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