The wave trace and Birkhoff billiards

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Abstract. The purpose of this article is to develop a Hadamard–Riesz-type parametrix for the wave propagator in bounded planar domains with smooth, strictly convex boundary. This parametrix then allows us to rederive an oscillatory integral representation for the wave trace appearing in the work by Marvizi and Melrose (1982) and compute its principal symbol explicitly in terms of geometric data associated to the billiard map. This results in new formulas for the wave invariants. The order of the principal symbol, which appears to be inconsistent in the works by Marvizi and Melrose (1982) and Popov (1994), is also corrected. In those papers, the principal symbol was never actually computed and to our knowledge, this paper contains the first explicit formulas for the principal symbol of the wave trace. The wave trace formulas we provide are localized near both simple lengths corresponding to nondegenerate periodic orbits and degenerate lengths associated to one parameter families of periodic orbits tangent to a single rational caustic. Existence of a Hadamard–Riesz-type parametrix with explicit symbol and phase calculations in the interior appears to be new in the literature, with the exception of the author's previous work [J. Geom. Anal. 31 (2021), 2238–2295] in the special case of elliptical domains. This allows us to circumvent the symbol calculus used by Duistermaat and Guillemin (1975) and Hezari and Zelditch (2012) when computing trace formulas, which are instead derived from integrating our explicit parametrix over the diagonal.

1. Introduction

The purpose of this paper is to develop a Hadamard–Riesz-type parametrix for the wave propagator in bounded planar domains with smooth, strictly convex boundary. We then use this parametrix to produce asymptotic expansions for the distributional wave trace near isolated lengths in the length spectrum. Let Ω be such a domain and denote by Δ the Dirichlet Laplacian on Ω. The even wave propagator $E(t)$ is defined to be the solution operator for the wave equation

$$\begin{cases} (\partial_t^2 - \Delta)E = 0 & (x \in \Omega), \\ E(0) = \mathrm{Id} & (x \in \Omega), \\ \partial_t E|_{t=0} = 0 & (x \in \Omega), \end{cases} \tag{1}$$

with Dirichlet boundary conditions. In spectral theoretic terms, we can write $E(t) = \cos t\sqrt{-\Delta}$, which is the even part of the half wave propagator $e^{it\sqrt{-\Delta}}$. In [4], it is shown that there exist Lagrangian distributions $E_j(t, x, y)$ such that microlocally away from the tangential rays, the Schwartz kernel of $E(t)$ is given by

$$E(t, x, y) = \sum_{j=-\infty}^{+\infty} E_j(t, x, y) + C^\infty(\mathbb{R} \times \Omega \times \Omega), \tag{2}$$

with E_j corresponding to a wave of j reflections at the boundary. The sum in (2) is locally finite in time. If we restrict our attention to waves which make j reflections and travel approximately once around the boundary, we have the following explicit parametrix for $E_j(t, x, y)$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth, strictly convex boundary. Then there exists $j_0 = j_0(\Omega) \in \mathbb{N}$ sufficiently large such that the following holds: for all $j \geq j_0$, there exists a tubular neighborhood U_j of the diagonal of the boundary $\Delta \partial \Omega \subset \Omega \times \Omega$ such that for all $(x, y) \in U_j$ and t less than but sufficiently close to $|\partial \Omega|$,*

$$E_j(t, x, y) = \sum_{\pm} \sum_{k=1}^{8} (-1)^j e^{\pm i\pi/4} \int_0^\infty e^{\pm i\tau(t - \Psi_j^k(x,y))} b_{j,k,\pm}(\tau, x, y) d\tau$$

$$+ C^\infty(\mathbb{R} \times U_j),$$

microlocally near geodesic loops of rotation number $1/j$. Here, $b_{j,k,\pm} \in S_{cl}^{1/2}(U_j \times \mathbb{R}^1)$ are classical elliptic symbols of order $1/2$ and the functions $\Psi_j^k(x, y)$ ($1 \leq k \leq 8$) are lengths of the 8 billiard orbits connecting y to x in j reflections and approximately one rotation (see Theorem 3.1). The principal term in the asymptotic expansion for $b_{j,k,\pm}$ is given in boundary normal coordinates $x = (\mu, \varphi)$, $y = (\nu, \vartheta)$ (see Section 5.3) by

$$\frac{\tau^{1/2}}{2(1 - \mu\kappa)} \left| \frac{\partial \Psi_j^k}{\partial \mu} \frac{\partial \Psi_j^k}{\partial \vartheta} \frac{\partial^2 \Psi_j^k}{\partial \nu \partial \varphi} + \frac{\partial \Psi_j^k}{\partial \varphi} \frac{\partial \Psi_j^k}{\partial \nu} \frac{\partial^2 \Psi_j^k}{\partial \vartheta \partial \mu} \right.$$

$$\left. - \frac{\partial \Psi_j^k}{\partial \mu} \frac{\partial \Psi_j^k}{\partial \nu} \frac{\partial^2 \Psi_j^k}{\partial \vartheta \partial \varphi} - \frac{\partial \Psi_j^k}{\partial \varphi} \frac{\partial \Psi_j^k}{\partial \vartheta} \frac{\partial^2 \Psi_j^k}{\partial \nu \partial \mu} \right|^{1/2},$$

where κ is the curvature of $\partial \Omega$ at $(0, \varphi)$.

Theorem 1.1 bears a remarkable resemblance to Hadamard's parametrix for the wave propagator on boundaryless manifolds (see [17, 17]), where the phase functions $\pm\tau(t - \Psi_j^k(x, y))$ are replaced by $\pm\tau(t - r(x, y))$, with $r(x, y)$ the geodesic distance from x to y. In that case, for x, y near the diagonal, there is only one geodesic

connecting y to x for small time. In our setting, Theorem 3.1 shows that there are exactly 8 orbits connecting y and x in j reflections and approximately one rotation. The parametrix in Theorem 1.1 is localized strictly away from but near the glancing set. We do not treat contributions of glancing orbits in this paper.

Remark 1.2. For $j \geq j_0(\Omega)$ and $x, y \in \partial\Omega$ sufficiently close to the diagonal, there exist only two orbits connecting y to x in j reflections and approximately one rotation. One is in the clockwise direction and the other is in the counterclockwise direction. In particular, when $x = y \in \partial\Omega$, there exists a unique geodesic loop based at x of rotation number $1/j$. Restricting the functions $\Psi_j^k(x, y)$ appearing in Theorem 1.1 to the diagonal of the boundary yield the *j-loop function*, which we denote by $\Psi_j(q, q)$ (see Definitions 3.5 and 3.6).

As in the case of boundaryless manifolds, we can use the explicit parametrix in Theorem 1.1 to prove trace formulas. It is known that $E(t)$ has a well defined distributional trace

$$\mathrm{Tr} \cos t \sqrt{-\Delta} = \sum_{j=0}^{\infty} \cos t\lambda_j, \tag{3}$$

where λ_j^2 are the Dirichlet eigenvalues of $-\Delta$. The sum in (3) converges in the sense of tempered distributions and has singular support contained in the length spectrum

$$\mathrm{LSP}(\Omega) = \overline{\{\text{lengths of periodic billiard trajectories}\}} \cup \{0\},$$

together with $-\mathrm{LSP}(\Omega)$ (see Section 6). Each periodic billiard orbit in Ω can be classified according to its winding number m and the number of reflections n made at the boundary. Denote the collection of periodic orbits of this type by $\Gamma(m, n)$, normalized so that $m \leq n/2$. $\Gamma(m, n)$ is never empty by a theorem of Birkhoff [3]. The length spectrum can be decomposed accordingly as

$$\mathrm{LSP}(\Omega) = \bigcup_{m,n\in\mathbb{N}} \mathrm{length}(\Gamma(m, n)) \cup \mathbb{N}|\partial\Omega|, \tag{4}$$

where $|\partial\Omega|$ is the length of the boundary. Using the parametrix in Theorem 1.1, we can prove the following theorem.

Theorem 1.3. *Assume the conditions from Theorem* 1.1 *hold and Ω satisfies the non-coincidence condition:*

there exists $\varepsilon_0 > 0$ such that

$$\bigcup_{\substack{m\geq 2 \\ n\geq 1}} \mathrm{length}(\Gamma(m, n)) \cap (|\partial\Omega| - \varepsilon_0, |\partial\Omega|) = \emptyset. \tag{5}$$

For $j \geq j_0$, define $t_j = \inf_{q \in \partial \Omega} \Psi_j(q, q)$ and $T_j = \sup_{q \in \partial \Omega} \Psi_j(q, q)$. Then on any sufficiently small neighborhood of $[t_j, T_j]$, $\operatorname{Tr} E(t)$ has the asymptotic expansion

$$(-1)^j \operatorname{Re}\left\{ e^{i\pi/4} \int\limits_{\partial \Omega} \int\limits_0^\infty e^{i\xi(t - \Psi_j(q,q))} a^j(q, \xi) d\xi dq \right\} + C^\infty(\mathbb{R}),$$

where $a^j(q, \xi)$ is a classical elliptic symbol of order $1/2$ with principal part given by

$$a_0^j(q, \xi) = 4\xi^{1/2} \sin \omega_{j,1}(q, q) \sin^{1/2} \omega_{j,2}(q, q) \left| \frac{\partial \omega_{j,1}}{\partial q'}(q, q) \right|^{1/2} X(q) \cdot N(q).$$

Here, $X(q)$ is the position vector to a boundary point q with respect to a fixed origin and $N(q)$ is the outward unit normal at q. The angles $\omega_{j,1}, \omega_{j,2}$ are the initial and final angles respectively of the unique billiard orbit $\gamma_j(q, q')$ which connects nearby boundary points q and q' in j reflections and approximately one counterclockwise rotation. The function $\Psi_j(q, q')$ is the length of $\gamma_j(q, q')$ and its restriction to the diagonal is the j-loop function.

Remark 1.4. As the position vector X is chosen with respect to an arbitrary interior point p, we can integrate out this symmetry over any measurable subset of the interior in the variable p to obtain a more invariant formula. In particular, we can integrate over open sets, curves and by a limiting argument, the boundary itself in order to obtain a smooth density on $\partial \Omega$. The noncoincidence condition (5) can be weakened and is in particular satisfied for ellipses (see [15]) and nearly circular domains (see [20]).

Remark 1.5. In several cases, one can evaluate the integral appearing in Theorem 1.3 more explicitly. If L_j is an isolated length and the corresponding orbit is nondegenerate or the fixed point set of the time L_j billiard flow is clean in the sense of Bott–Morse (see [8]), then one can apply the method of stationary phase. The case of a one parameter family of orbits tangent to a rational caustic is discussed below. There is an apparent asymmetry between the incident and reflected angles in the symbol $a_0^j(q, \xi)$, but only periodic orbits contribute in the stationary phase computation. For periodic orbits, we have Snell's law $\omega_{j,1} = \omega_{j,2}$. The integral formulas in Theorem 1.3 are valid regardless of how complicated the structure the length spectrum and corresponding orbits may be.

Remark 1.6. The noncoincidence condition (5) was first formulated by Marvizi and Melrose. It is known that the set of domains satisfying this condition is C^∞ dense in the set of all smooth, bounded strictly convex planar domains and moreover contains a C^1 neighborhood of the disk ([36, Proposition 7.2]). It is also satisfied for ellipses ([15, Proposition 4.3]). It is believed by the author to be satisfied by all smooth, bounded, strictly convex planar domains.

Theorem 1.3 provides an explicit formula for the principal term in the parametrix developed in [36]. In contrast to the methods employed in [36], the proof developed in the remainder of this paper uses the explicit parametrix for the wave propagator appearing in Theorem 1.1. Theorem 1.3 also provides clarity on a discrepancy in the literature regarding the order of the wave trace (cf. [36, 48]). Note that in Theorem 1.3 no assumptions are made on the nondegeneracy of orbits. If the length spectrum has high multiplicity, the wave trace is in general quite complicated. However, when periodic orbits come in a one parameter family corresponding to a caustic, we have the following:

Theorem 1.7. *Suppose* Ω, *$j \geq j_0$ are as in Theorem* 1.3 *and* $\mathcal{C}$ *is a caustic for* Ω *of rotation number* $1/j$. *If periodic orbits tangent to* $\mathcal{C}$ *have length* L_j, *then near* $t = L_j$, $\operatorname{Tr} E(t)$ *has the leading asymptotic*

$$c_j \operatorname{Re}\{e^{i\pi/4}(t - L_j - i0)^{-3/2}\},$$

where c_j *is a wave invariant given by the formula*

$$c_j = (-1)^{j+1} 4 \int_{\partial\Omega} \sin^{3/2} \omega_{j,1}(q,q) \left| \frac{\partial \omega_{j,1}}{\partial q'}(q,q) \right|^{1/2} X(q) \cdot N(q) dq.$$

In this case, $\omega_{j,1}(q,q) = \omega_{j,2}(q,q)$ *is the measure of the angle of incidence for the unique periodic orbit of rotation number* $1/j$ *based at* $q \in \partial\Omega$.

While KAM theory provides the existence of irrational caustics, it is shown in [29] that for a fixed j, the set of all smooth convex domains possessing a rational caustic of rotation number $1/j$ is polynomially dense in the variable $1/j$ within the collection of all smooth strictly convex domains, equipped with the C^∞ topology. Exponential density is also proven in the analytic category. As ellipses satisfy the noncoincidence condition (5) (see [15]) and are known to be completely integrable with confocal conic sections as caustics, we have the following corollary to Theorem 1.7.

Corollary 1.8. *For an ellipse* Ω *given by*

$$\Omega = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\},$$

and $L_j \in \mathrm{LSP}(\Omega)$ *sufficiently close to* $|\partial\Omega|$ *corresponding to the length of billiard orbits of rotation number* $1/j$, *the wave invariants in Theorem* 1.7 *are given by*

$$c_j = \int_0^{2\pi} \frac{(-1)^{j+1} 2ab \sin \omega_j \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi} \, d\varphi}{\sqrt{\cos \omega_j (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)(b^2 + (a^2 - b^2) \sin^2 \varphi) G(\zeta_j)} \sqrt{1 - k_{\zeta_j}^2 \sin^2 \varphi}}.$$

Here, $\zeta_j \in [0, b)$ is the parameter of the confocal ellipse

$$\mathcal{C}_{\zeta_j} = \left\{ (x, y) : \frac{x^2}{a^2 - \zeta_j^2} + \frac{y^2}{b^2 - \zeta_j^2} = 1 \right\},$$

to which the orbits of length L_j are tangent and k_{ζ_j} is given by

$$k_{\zeta_j}^2 = \frac{a^2 - b^2}{a^2 - \zeta_j^2}.$$

The parameter ζ_j depends on the rotation number $1/j$ and is defined implicitly by the equation

$$\frac{1}{j} = \frac{F(\arcsin \zeta_j / b; k_{\zeta_j})}{2K(\zeta_j)},$$

where $F(s; k)$ is the elliptic integral

$$\int_0^s \frac{d\tau}{\sqrt{1 - k^2 \sin^2 \tau}}$$

and $K(\zeta_j) = F(\pi/2; k_{\zeta_j})$. The function $G(\zeta_j)$ is defined by

$$\frac{-k_{\zeta_j}^2}{(a^2 - \zeta_j^2)} \int_0^{2\pi} \frac{\sin^2 \tau \, d\tau}{(1 - k_{\zeta_j}^2 \sin^2 \tau)^{3/2}} + (2j + 2) \frac{d}{d\zeta^2} F(\arcsin \zeta/b; k_\zeta) \Big|_{\zeta = \zeta_j},$$

and $\omega_j = \omega_j(\varphi)$ are the angles of reflection for orbits tangent to $\mathcal{C}_{\zeta_j}$, given implicitly by the equation

$$\zeta_j^2 = \sin^2 \omega_j (b^2 + (a^2 - b^2) \sin^2 \varphi).$$

Remark 1.9. Analogous formulas to those appearing in Theorem 1.3, Theorem 1.7 and Corollary 1.8 can also be proved for the Neumann and Robin wave traces as well. The formulas are less succinct but can be easily reproduced by keeping track of boundary terms in Section 6. Alternatively, an earlier version of this paper used Hadamard-type variational formulas which were derived for Robin boundary conditions in the author's previous work [55].

1.1. Schematic outline

The proofs of Theorem 1.3, Theorem 1.7 and Corollary 1.8 use techniques from [55], which are reviewed throughout the paper. In Section 2, we review relevant background on the inverse spectral problem, i.e., determining geometric information from

the Laplace spectrum. The first step in the proof of Theorem 1.3 is to generalize the Hadamard–Riesz-type parametrix for the wave propagator constructed in [55] for ellipses to arbitrary bounded domains with strictly convex, smooth boundary as in Theorem 1.1. This requires a dynamical classification (Theorem 3.1) of the cardinality and structure of all billiard orbits connecting interior points with a fixed number of reflections, analogous to [55, Lemma 5.2]. Section 3 introduces language from dynamical systems necessary to describe the billiard (or broken bicharacteristic) flow, which later appears in the canonical relations of the wave propagator. The proof of Theorem 3.1 is relegated to Section 4, where it is first done in the simple case of the Friedlander model and then broken up into several intermediate lemmas using Lazutkin coordinates for the general case. This material is largely independent from the rest of the paper and is of separate interest from the perspective of dynamical billiards, irrespective of applications to spectral theory. In Section 5, the length functionals corresponding to these orbits allow us to cook up explicit phase functions which parametrize the canonical relations for the wave propagator $e^{it\sqrt{-\Delta}}$, which is a Fourier integral operator microlocally away from the tangential rays. We then carry out the analysis leading to the microlocal parametrix appearing in Theorem 1.1. Construction of a parametrix for the wave propagator in the interior, with principal symbol given explicitly in terms of geometric data, microlocally near transversally reflected, nearly glancing rays appears to be new in the literature, with the exception of the author's previous work [55] in the special case of an ellipse. In Section 6, an integration by parts allows us to compute the localized in time wave trace in terms of a boundary integral. In this case, we can argue that only a select few of the billiard orbits in Theorem 3.1 contribute to the wave trace. Appropriate Maslov factors on each branch of the canonical relations are also computed here. As the order of the principal symbol computed in Section 6 appears to contradict other works in the literature, Section 7 provides an auxiliary confirmation via stationary phase that the order derived here is indeed correct. To our knowledge, this paper contains the first explicit formulas for the principal symbol of the wave trace associated to a convex billiard table near the length of the boundary.

2. Background

The inverse spectral problem has a long history, dating back to Kac in 1966 [28], who asked the famous question "can one hear the shape of a drum?" Mathematically, this corresponds to uniquely determining a domain Ω from the spectrum of its Dirichlet, Neumann, or Robin Laplacian. For bounded domains, the spectrum is purely discrete,

consisting of eigenvalues λ_j^2 satisfying

$$\begin{cases} -\Delta u_j = \lambda_j^2 u_j, & x \in \Omega, \\ Bu_j = 0, & x \in \partial\Omega, \end{cases} \tag{6}$$

where u_j are smooth eigenfunctions on Ω and B is either the restriction operator (Dirichlet boundary conditions), normal differentiation (Neumann boundary conditions), or normal differentiation plus a prescribed function on $\partial\Omega$ (Robin boundary conditions). A variety of approaches in local and global harmonic analysis have been taken to prove partial results in the direction of [28]. One particularly useful strategy is to use the wave group to deduce spectral information about the underlying geometric space, usually a Riemannian manifold. The motivation behind this approach stems from Duistermaat and Hörmander's propagation of singularities theorem, which says that singularities of solutions to the wave equation propagate along (possibly broken) bicharacteristics, which are lifts to $T^*(\mathbb{R} \times \Omega)$ of geodesic or billiard orbits. As linear waves can be superimposed, constructive interference is most pronounced along geodesics which are traversed infinitely often, i.e., periodic orbits. On the trace side, this is reflected in the Poisson relation:

$$\text{SingSupp Tr } e^{it\sqrt{-\Delta}} \subset \pm\overline{\text{LSP}(\Omega)} \cup \{0\}, \tag{7}$$

where the left-hand side is the distributional trace of the half wave propagator (see Section 6) and the right-hand side is the length spectrum of Ω (the closure of all lengths of periodic geodesic or billiard orbits together with $\{0\}$). This is proven in [1] for smooth, strictly convex planar domains with boundary and [44] for more general bounded domains. In particular, the formula (7) generalizes the Poisson summation formula on the torus $\mathbb{R}^n/\mathbb{Z}^n$ from elementary Fourier analysis (see [54], for example).

Asymptotic formulas near the singularities are given by the Selberg trace formula in the case of hyperbolic surfaces [52], the Duistermaat–Guillemin trace theorem for general smooth manifolds under a dynamical nondegeneracy condition [8], and a Poisson summation formula for strictly convex bounded planar domains due to Guillemin and Melrose [16]. However, since these trace formulas involve sums over all periodic orbits of a given length, it is possible that the contributions of distinct orbits having the same length could cancel out and the wave trace is actually smooth near a point in the length spectrum. We say that the length $L \in \mathbb{R}$ of a periodic orbit γ is *simple* if up to time reversal ($t \mapsto -t$), γ is the unique periodic orbit of length L. Without length spectral simplicity, there is no way to deduce Laplace spectral information from the length spectrum alone. It is shown in [44] that generically, smooth convex domains have simple length spectrum associated to only nondegenerate periodic orbits. In that case, the following theorem holds:

Theorem 2.1 ([16, 45]). *Assume γ is a nondegenerate periodic billiard orbit in a bounded, strictly convex domain with smooth boundary and γ has length L which is simple. Then near L, the even wave trace has an asymptotic expansion*

$$\operatorname{Tr}\cos t\sqrt{-\Delta}$$

$$\sim \operatorname{Re}\Big\{a_\gamma(t-L+i0)^{-1} + \sum_{k=0}^{\infty} a_{\gamma k}(t-L+i0)^k \log(t-L+i0)\Big\}, \quad (8)$$

where the coefficients $a_{\gamma k}$ are the wave invariants associated to γ.

Calculations of the wave invariants associated to dynamically convenient orbits have proved extremely useful in the inverse spectral problem associated to (6). For example, the case of rotationally symmetric metrics on S^2 is analyzed in [57]. In [18, 60, 61], the wave invariants associated to bouncing ball orbits are explicitly calculated using Feynman diagrams to analyze the stationary phase computation from which Balian–Bloch (resolvent) formulas are derived (see also [58]). Under mild dynamical conditions and some additional axial symmetry assumptions, these coefficients can be used to determine the Taylor series of a local boundary parametrization. In particular, this allows one to deduce that such domains are spectrally determined amongst a rich class of analytic, symmetric domains. Microlocal parametrices near the glancing set have also been constructed in [1, 10, 45] for demonstrating propagation of singularities, [38, 39] in the context of scattering by a convex obstacle and more recently [25, 26] for proving dispersive estimates. However, there is a lack of precise information on their principal symbols in terms of geometric data and to the author's knowledge, the contributions these tangential rays to the wave trace have not yet been considered in the context of inverse problems.

The wave invariants have also proved useful in variational inverse problems, going back to the seminal papers [13, 14], where the authors proved spectral rigidity for closed manifolds with negative sectional curvature. This was recently generalized to Anosov surfaces in [43]. In the setting of bounded domains, it is proved in [19] that ellipses with Dirichlet/Neumann boundary conditions are infinitesimally spectrally rigid through smooth domains with the same symmetries. These results as well as spectral determination of the Robin function in [15] were generalized to Liouville billiard tables of classical type in [49, 51]. They were also extended to the Robin setting in [55], where both the domain and Robin function were allowed to vary simultaneously. A recent breakthrough was obtained in [20], where the authors showed that ellipses of small eccentricity are spectrally determined. Thorough surveys of the inverse spectral problem are contained in [6, 37, 59, 62]. The present article is inspired by [36], where the following theorem is proved:

Theorem 2.2 ([36]). *If Ω is a bounded and strictly convex planar region, there exists $N = N(\Omega)$ such that if $j > N$, then the contribution $\hat{\sigma}_j$ of E_j to $\hat{\sigma}_D$ is of the form*

$$\hat{\sigma}_j = \frac{1}{2\pi} \int\limits_0^\infty \int\limits_0^L e^{i(t-\mu_j(s))\tau} a(s, \tau) ds d\tau, \tag{9}$$

where in terms of an arc-length coordinate s on $\partial\Omega$,

$$\mu_j(s) = \Psi_j(s, s')|_{s=s'},$$

with $\Psi(s, s') = L(\mathfrak{g})$, with $\mathfrak{g}$ the length of a j-fold geodesic from s to s' and a_j is periodic in s, classical and elliptic of order zero, with principal part of the form $e^{i\pi r_j/4} \alpha_j(s)$, $\alpha_j(s) > 0$.

Here, $E_j(t)$ is the cosine kernel associated to the parametrix for the j reflection wave operator constructed in [4], which is reviewed in Section 5.2. In particular, if L_j is a simple length corresponding to a nondegenerate periodic orbit, then Theorem 2.2 gives an asymptotic expansion for the localized wave trace. In fact, if Ω satisfies the noncoincidence condition (5), i.e., $|\partial\Omega|$ is not a limit point from below of the lengths of orbits of rotation number m/n for $m \geq 2$, then the trace in Theorem 2.2 is a spectral invariant. The purpose of this article is to explicitly calculate the principal symbol $a(s, \tau)$ of (9) in terms of geometric data associated to the billiard map, both in the case of simple lengths corresponding to nondegenerate periodic orbits and also for one parameter families of degenerate periodic orbits, all having the same length associated to a caustic of rational rotation number. It also corrects several errors in literature on the order of $a \in S_{\text{cl}}^{1/2}(\partial\Omega)$.

3. Billiards

Before obtaining a singularity expansion for the wave trace, we first review the relevant background needed on billiards. This will be useful in our discussion of Chazarain's parametrix in Section 5.2. In this section, we denote by Ω a bounded strictly convex region in $\mathbb{R}^2$ with smooth boundary. This means that the curvature of $\partial\Omega$ is a strictly positive function. The billiard map is defined on the coball bundle of the boundary $B^*\partial\Omega = \{(q, \zeta) \in T^*\partial\Omega : |\zeta| < 1\}$, which can be identified with the inward part of the circle (cosphere) bundle $S_{\partial\Omega}^*\mathbb{R}^2$, via the natural orthogonal projection map. We can also identify $B^*\partial\Omega$ with $\mathbb{R}/\ell\mathbb{Z} \times (0, \pi)$, where $\ell = |\partial\Omega|$ is the length of the boundary. Define

$$t_\pm^1(y, \eta) = \inf\{t > 0 : g^{\pm t}(y, \eta) \in \partial\Omega\},$$
$$t_\pm^{-1}(y, \eta) = \sup\{t < 0 : g^{\pm t}(y, \eta) \in \partial\Omega\},$$

where π_1 is projection onto the first factor and $g^{\pm t}$ is the forwards $(+)$ or backwards $(-)$ geodesic flow on $\mathbb{R}^2$, corresponding to the Hamiltonian $H_\pm = \pm|\eta|$ (see Section 5.2). If $(y, \zeta) \in B^*\partial\Omega$ is mapped to the inward pointing covector $(y, \eta) \in T^*_{\partial\Omega}\mathbb{R}^2$ under the inverse projection map, then we define

$$\beta^{\pm 1}(y, \eta) = \widehat{g^{t^1_\pm}(y, \eta)},$$

where a point $\widehat{(x, \xi)}$ is the reflection of ξ through the cotangent line $T^*_x\partial\Omega$, i.e., $\widehat{(x, \xi)}$ has the same footpoint and (co)tangential projection as (x, ξ), but reflected conormal component, so that it is again in the inward facing portion of the circle bundle. We call $\beta := \beta^{+1}$ the *billiard map*. It is well known that β preserves the natural symplectic form induced on $B^*(\partial\Omega)$ and is differentiable there, extending continuously up to the boundary. The maps $\beta^{\pm n}$ are defined via iteration and it is clear that $\beta^{-n} = (\beta^n)^{-1}$ for each $n \in \mathbb{Z}$. Associated to the billiard map is the billiard flow, or broken bicharacteristic flow, which we denote by Φ^t.

Geometrically, a billiard orbit corresponds to a union of line segments which are called *links*. A smooth closed curve $\mathcal{C}$ lying in Ω is called a *caustic* if any link drawn tangent to $\mathcal{C}$ remains tangent to $\mathcal{C}$ after an elastic reflection at the boundary of Ω. By *elastic reflection*, we mean that the angle of incidence equals the angle of reflection at an impact point on the boundary. We can map $\mathcal{C}$ onto the total phase space $B^*\partial\Omega$ to obtain a smooth closed curve which is invariant under β. If the dynamics are integrable, these invariant curves are precisely the Lagrangian tori which foliate phase space. A point P in $B^*\partial\Omega$ is called *q-periodic* $(q \geq 2)$ if $\beta^q(P) = P$. We define the rotation number of a q-periodic orbit P by $\omega(P) = \frac{p}{q}$, where p is the winding number of the orbit generated by P, which we now define. We may consider the modified billiard map $\tilde{\beta} = \Pi^*\beta$, where Π is the natural mapping from $\mathbb{R}/\ell\mathbb{Z} \times [0, \pi]$ to the closure of the coball bundle $\overline{B^*\partial\Omega}$. Pulling back by Π clearly preserves the notion of periodicity. There exists a unique lift $\hat{\beta}$ of the map $\tilde{\beta}$ to the closure of the universal cover $\mathbb{R} \times [0, \pi]$ which is continuous, ℓ periodic in first variable and satisfies $\hat{\beta}(x, 0) = (x, 0)$. Given this normalization, for any point $(x, \vartheta) \in \mathbb{R}/\ell\mathbb{Z} \times [0, \pi]$ in a q periodic orbit of $\tilde{\beta}$, we see that $\hat{\beta}^q(x, \vartheta) = (x + p\ell, \vartheta)$ for some $p \in \mathbb{Z}$. We define this p to be the winding number of the orbit generated by $\Pi(x, \vartheta) \in \overline{B^*\partial\Omega}$. We see that even if a point $\Pi(x, \vartheta)$ generates an orbit which is not periodic in the full phase space but is such that $\pi_1(\tilde{\beta}^q(x, \vartheta)) = x$ for some $q \in \mathbb{Z}$, we can still define a winding number in this case. Such orbits are called *loops* or *geodesic loops*. For a given periodic orbit, the winding number is independent of which point in the orbit is chosen, so we sometimes write $\omega(\gamma) = \omega(P)$ for any $P \in \gamma = \{P, \beta(P), \ldots, \beta^{q-1}(P)\}$. For deeper results and a more thorough introduction to dynamical billiards, we refer the reader to [30, 48, 50, 53].

What will be crucial for us in later sections is a description of all orbits making a fixed number of reflections which connect interior points near the diagonal of the boundary in approximately one rotation. These orbits will allow us to cook up phase functions in Section 5.2 which parametrize the canonical relation of the wave propagator.

Theorem 3.1 (8 orbit theorem). *There exist $C_0 > 0$ and $j_0 = j_0(\Omega)$ sufficiently large such that for $j \geq j_0$ and any two points $x, y \in \text{int}(\Omega)$ which are C_0/j^4 close to the diagonal of the boundary, there exist precisely four distinct, broken geodesics of j reflections making approximately one counterclockwise rotation, emanating from x and terminating at y. Similarly, there exist four such orbits in the clockwise direction. If $x, y \in \partial\Omega$ and are C_0/j^4 close to the diagonal, there exists only one clockwise and one counterclockwise orbit connecting x to y in j reflections and approximately one rotation. In particular, when $x = y \in \partial\Omega$, there is a unique (up to time reversal) geodesic loop based at x of rotation number $1/j$.*

The proof of Theorem 3.1 is based on several lemmas in Section 4 below and is inspired by the author's previous work in [55], where a similar construction was adapted to elliptical billiard tables. As in that paper, the proof actually provides more information on the topological structure of the orbits. The existence of orbits connecting nearby *boundary* points and in particular geodesic loops of small rotation number is well known, although the material in Section 4 below easily reproduces these results. The novelty of Theorem 3.1 is a complete description of orbits connecting *interior points* as opposed to boundary points, which will ultimately allow us to extend microlocal parametrices for the wave propagator from the boundary (as in [36]) to the interior.

We now explain what is meant by approximately one rotation. Let $\xi \in S_x^*\Omega$ be one of the 4 covectors corresponding to the initial condition of a counterclockwise orbit described in Theorem 3.1. Denote by $\hat{x} = \pi_1 g^{t_1^+}(x, \xi)$ the first point of reflection at the boundary (π_1 is projection onto the first factor) and by $\hat{y}$ the $(j + 1)$st point of reflection at the boundary after the orbit reaches y. If x, y are $O(j^{-1})$ close to the diagonal of the boundary, then $|\hat{x} - \hat{y}| = O(j^{-1})$ (see Section 4). Also let ω be the angle of reflection made by the orbit at $\hat{x}$ and note that $\hat{x}, \hat{y}$ and ω all depend implicitly on ξ.

Definition 3.2. We say that an orbit makes *approximately one counterclockwise rotation* if, for each of the initial covectors $\xi \in S_x^*\Omega$ of the 4 counterclockwise orbits provided by Theorem 3.1, we have

$$|\pi_1\hat{\beta}^j(\hat{x}, \omega) - \hat{y} - \ell| \leq \ell/100.$$

Here, $\ell = |\partial\Omega|$ and $\hat{\beta}$ is the lift of the billiard map to the closure of the universal cover $\mathbb{R} \times [0, \pi]$ as described in Section 4.

Remark 3.3. The choice of $\ell/100$ is somewhat arbitrary, but having ℓ in the numerator allows for scale invariance and finding the optimal constant is irrelevant for our purposes. The notion of approximately one clockwise rotation is defined analogously.

Definition 3.4. Of the four counterclockwise orbits emanating from x, two of them become tangent to a level curve of the distance function $d(z) = \mathrm{dist}(z, \partial\Omega)$ before making a reflection at the boundary. We denote these orbits by T orbits (for tangency) and call their first links T *links*. The other two orbits make a reflection at the boundary before becoming tangent to a level curve of d and we call these N *orbits* (for "nontangency"); their first link is called an N *link*. Within either T or N category for the first link, the final link of one of the orbits reaches y before becoming tangent to a level curve (an N link) and the other has a point of tangency before reaching y (a T link). In this way, we obtain four types of counterclockwise orbits from x to y, which we denote by TT, TN, NT, and NN. The same classification also applies to the four clockwise orbits.

See Figure 1 for an example on the ellipse with $j = 4$. These configurations will be important in determining which limiting orbits give rise to geodesic loops of precisely j reflections as $(x, y) \to \Delta\partial\Omega$, where $\Delta : \partial\Omega \to \partial\Omega \times \partial\Omega$ is the diagonal embedding $x \mapsto (x, x)$. In Section 5, we will actually be interested in orbits connecting y to x rather than x to y for reasons related to symplectic geometry and Hörmander's conventions on the theory of Fourier integral operators. As Theorem 3.1 is clearly symmetric in x and y, there is no problem in interchanging the initial and final points of the orbit.

Definition 3.5. For a billiard orbit γ beginning at y and terminating at x, we define the length functional $\Psi(x, y)$ to the Euclidean length of γ. As there are potentially many such γ connecting y and x, $\Psi(x, y)$ is multivalued. For $1 \le k \le 8$, denote by $\Psi_j^k(x, y)$ a branch of the length functional corresponding to one of the orbits of j reflections in Theorem 3.1. It depends only on x, y, j and k. We use the convention that for a fixed number of reflections j, the indices $1 \le k \le 4$ correspond to the counterclockwise orbits TT, TN, NT, NN in that order and the indices $5 \le k \le 8$ correspond to their clockwise counterparts in the same order.

The author learned of a similar function in [36, p. 492], where its restriction to the boundary is defined. In such a case, i.e., if $x, y \in \partial\Omega$, it is stated in [36] and proved in [48] that only a single counterclockwise orbit of j reflections exists between the boundary points if they are sufficiently close and j is sufficiently large. The proof in Section 4 below also shows that as x and y approach the diagonal of the boundary
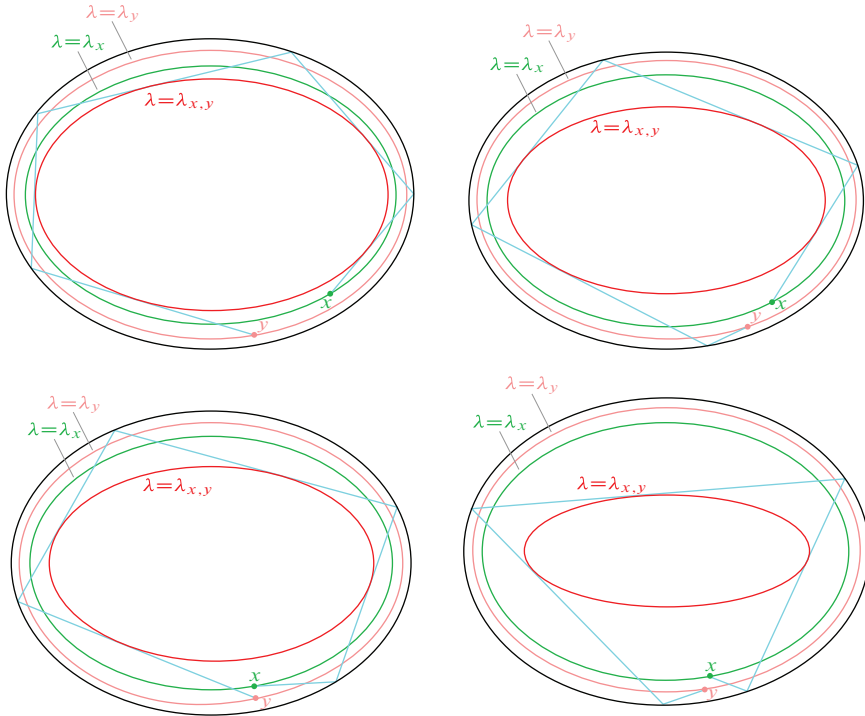
Figure 1. Counterclockwise orbit configurations TT, TN, NT, and NN corresponding to $j = 4$. The green and pink curves are the distance curves on which x and y respectively lie. The red curve is a caustic to which the billiard orbit is tangent in the completely integrable setting.

from the interior of Ω, the corresponding orbits coalesce and converge to the orbits described in [36]. This in fact proves the claims made in [36] and provides an alternative to the methods employed in [48]. The limiting orbits may have a different number of reflections though and this is addressed in the proof of Lemma 6.2.

Definition 3.6. For $x = y = q \in \partial\Omega$, we denote by $\Psi_j(q, q)$ the length of the unique geodesic loop of j reflections based at q. This is the j-loop function.

4. Proof of Theorem 3.1

4.1. Friedlander model

Let us first sketch the proof of 3.1 in the special case of the *Friedlander model*, which can be considered as an approximation to the billiard map near tangential rays. Fol-

lowing [5], the Friedlander operator is defined to be

$$L = -\partial_x^2 - (1 + x)\partial_y^2 \tag{10}$$

on the manifold $M = [0, \infty) \times \mathbb{R}/2\pi\mathbb{Z}$, together with appropriate boundary conditions. The associated classical Hamiltonian is $H(x, y, \xi, \eta) = \xi^2 + (1 + x)\eta^2$, which we restrict to $S^*M = \{(x, y, \xi, \eta) \in T^*M : H(x, y, \xi, \eta) = 1\}$. S^*M is a circle bundle over $[0, \infty) \times \mathbb{R}$ and the fiber over each base point (x, y) is an ellipse, which can be parametrized in elliptical polar coordinates by

$$(\xi, \eta) = \left(\cos\vartheta, \frac{1}{\sqrt{1 + x}}\sin\vartheta\right), \quad 0 \le \vartheta < 2\pi. \tag{11}$$

The integral curves of the Hamiltonian vector field associated to H with initial condition $(x_0, y_0, \xi_0, \eta_0) \in T^*_{(x_0, y_0)}M$ can easily be seen to be

$$\begin{pmatrix} x(t) \\ y(t) \\ \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} x_0 + \xi_0 t - \eta_0^2 t^2/4 \\ y_0 + \eta_0 t + \eta_0(-\eta_0^2 t^3/12 + \xi_0 t^2/2 + x_0 t) \mod 2\pi \\ -\eta_0^2 t^2/4 + \xi_0 t + \xi_0 \\ \eta_0 \end{pmatrix}. \tag{12}$$

If $(x(t), y(t), \xi(t), \eta(t))$ is an orbit of 12 on $[0, T]$ with $x(t) > 0$ for $0 \le t < T$ and $x(T) = 0$, we reflect the covector by the law of equal angles and continue the flow past time T. In other words, the billiard map is discontinuous at $T^*_{\partial M}M$ and we set

$$\xi(T) = -\lim_{t \to T^-}\xi(t), \quad \eta(T) = \lim_{t \to T^-}\eta(t)$$

to extend the flow for all $t > 0$.

Fix the bounce number $j \in \mathbb{N}$ large and let $a = (x_0, y_0)$ with $x_0 > 0$ sufficiently small (in terms of j). If $\vartheta_0 = \pi/2, 3\pi/2$, the corresponding orbit is tangent to the caustic $x = x_0$ and will make less than $1/4$ rotation around ∂M. Denote by $\vartheta(t)$ the angular component of $(\xi(t), \eta(t))$ in elliptical polar coordinates:

$$\vartheta(t) = 2\pi - \arctan\left(-\sqrt{1 + x(t)}\frac{\eta(t)}{\xi(t)}\right). \tag{13}$$

For an initial angle ϑ_0, it can be seen from the formula for $x(t)$ in 12 that the trajectory reaches the y axis at time

$$\begin{aligned} T(\vartheta_0) &= \frac{2\xi_0}{\eta_0^2} + \sqrt{4\frac{\xi_0^2}{\eta_0^4} + 4\frac{x_0}{\eta_0^2}} \\ &= 2(1 + x_0)\frac{\cos\vartheta_0}{\sin^2\vartheta_0} + 2\sqrt{(1 + x_0)^2\frac{\cos^2\vartheta_0}{\sin^4\vartheta_0} + (1 + x_0)\frac{x_0}{\sin^2\vartheta_0}}. \end{aligned} \tag{14}$$
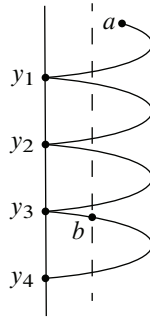
Figure 2. Four bounce forward orbit of the point $a = (x_0, y_0)$, intersecting the caustic $x = \tilde{x}$ at the point b.

One can then directly show that the angle

$$\vartheta_1 := \vartheta(T(\vartheta_0)) = 3\pi - \lim_{t \to T(\vartheta_0)^-} \vartheta(t),$$

from which the reflected trajectory emanates, has a global minimum at $\vartheta_0 = 3\pi/2$ on the interval $[\pi, 2\pi]$: since the orbit is tangent to the caustic $x = x_0$, it is easy to see geometrically that the larger ϑ_1 is, the greater the parameter x of the associated caustic will be. Similarly, the first impact point $y_1 := y(T(\vartheta_0))$ on the universal cover of ∂M is monotone decreasing in ϑ_0 and its speed is bounded for ϑ_0 away from multiples of 2π. Near 2π, the orbits extend far into the right half plane and make too many rotations after j bounces, or at 2π the corresponding orbit is a horizontal half ray. By translation invariance in y, it is clear that links connecting ∂M to itself are vertical translates of one another (Figure 2). Letting y_j denote the y coordinate of the j-th impact point on the boundary, it follows that y_j is also continuous and decreasing as a function of ϑ_1 with approximate speed j:

$$y_j(y_1, \vartheta_1) = y_1 + j(T \sin \vartheta_1 - T^3 \sin^3 \vartheta_1/12 + T^2 \cos \vartheta_1 \sin \vartheta_1/2) \qquad (15)$$

where $T = T(\vartheta_1) = 2 \cos \vartheta_1 / \sin^2 \vartheta_1$ is defined analogously to 14 so that the x coordinate is again 0 after time T since the impact at $(0, y_1)$. Let $b = (\tilde{x}, \tilde{y})$ be sufficiently close to a (in terms of j) and note that the orbit connecting $(0, y_j)$ to $(0, y_{j+1})$ then intersects the caustic $x = \tilde{x}$ exactly twice. If j is large, it is also easy to see that the y coordinates of these intersection points are monotone decreasing in ϑ_1, with approximate speed j. Hence, by the intermediate value theorem and decreasing ϑ_0 on $\pi < \vartheta_0 < 3\pi/2$, we find exactly two angles for which the link $\overline{(0, y_j)(0, y_{j+1})}$ intersects the caustic $x = \tilde{x}$ precisely at the point $b = (\tilde{x}, \tilde{y})$ in approximately one rotation. Similarly, there exist two such angles in $[3\pi/2, 2\pi]$, and 4 more angles in $[0, \pi]$ which generate orbits in the opposite direction (y increasing). This concludes

our sketch of the proof of the 8 orbit theorem in the special case of Friedlander's model. We will now more rigorously prove the claims made above directly for the billiard map on a convex domain by working in Lazutkin coordinates. The required proximity of orbit endpoints a and b to the diagonal and boundary will be made more precise in terms of j below as well. We will use x and y in place of a and b when not working directly with Cartesian coordinates.

4.2. General convex domains

We will construct conic subbundles $C_{\pm}^*(\Omega; j) \subseteq T^*\Omega$ over a tubular neighborhood $\widetilde{U}_j$ of the boundary with the following property: for each $x_0 \in \widetilde{U}_j$, all orbits emanating from x_0 which make j reflections and approximately one rotation in the sense of Section 3 have initial covector in $C_{x_0}^*(\Omega; j)$, the fiber of $\bigcup_{\pm} C_{\pm}^*(\Omega; j)$ over x_0. At each $x_0 \in \widetilde{U}_j$, there also exist distinguished covectors $\xi_0^{\pm} \in T_{x_0}^*(\Omega)\backslash 0$ which are tangent to distance curves foliating a neighborhood of the boundary. If $\widetilde{U}_j$ is chosen to be sufficiently narrow, the j reflection orbits emanating from $\xi_0^{\pm}$ will make less than a quarter rotation. By rotating ξ away from $\xi_0^{\pm}$ in either direction within the fiber of $C_{\pm}^*(\Omega; j)$ at x_0, we will show that the angle of reflection at the first impact point on the boundary increases monotonically. From the boundary, we can then take advantage of the twist property of the billiard map, and show that monotonicity of the incident angles causes orbits of a large number of reflections to wind around Ω. Two of the orbits from x to y will be obtained by perturbing ξ_0^+ in the counterclockwise direction and the other two will be obtained by perturbing ξ_0^+ in the clockwise direction. The four clockwise orbits will then be constructed in a similar manner, rotating ξ_0^- in both the clockwise and counterclockwise fiber directions. The arguments in this section will also provide the additional topological structure of the resulting orbits referenced in Definition 3.4, which can be seen in Figure 1.

4.3. Lazutkin coordinates

Recall that the billiard map is defined on the coball bundle $B^*\partial\Omega$, which can be identified with the collection of inward facing covectors in the circle bundle $S_{\partial\Omega}^*\mathbb{R}^2$. Letting s denote the arc-length parameter on $\partial\Omega$ and ϑ the angle an inward facing covector makes with the positively oriented boundary, we define the modified billiard map $\tilde{\beta}$ in terms of $(s, \vartheta) \in \mathbb{R}/\ell\mathbb{Z} \times (0, \pi)$. In this coordinate system, the modified billiard map is given by $\tilde{\beta}(s_1, \vartheta_1) = (s_2, \vartheta_2)$, where

$$
\begin{cases}
s_2 = s_1 + a_1(s_1)\vartheta_1 + a_2(s_1)\vartheta_1^2 + a_3(s_1)\vartheta_1^3 + F(s_1, \vartheta_1)\vartheta_1^4, \\
\vartheta_2 = \vartheta_1 + b_2(s_1)\vartheta_1^2 + b_3(s_1)\vartheta_1^3 + G(s_1, \vartheta_1)\vartheta_1^4,
\end{cases} \tag{16}
$$

and a_i, b_i, F and G are smooth functions. This is a Taylor expansion in the angular variable near $\vartheta_1 = 0$. There are explicit formulas for the coefficients a_i, b_i ([34]). In particular, $a_1(s_1) = 2\rho(s_1)$ and $b_2 = -2/3\rho'(s_1)$, where $\rho(s_1)$ is the radius of curvature at s_1. In [34], the change of coordinates

$$x = \frac{\int_0^s \rho^{2/3}(t)dt}{\int_0^\ell \rho^{2/3}(t)dt}, \quad \alpha = \frac{4\rho(s)\sin\vartheta/2}{\int_0^\ell \rho^{2/3}(t)dt} \tag{17}$$

was introduced near the circle corresponding $\vartheta = 0$. We call the coordinate system (x, α) in (17) *Lazutkin coordinates.* The advantage of this change of variables is that in these coordinates, the billiard map becomes a small perturbation of the translation map

$$\begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} x_1 + \alpha_1 \\ \alpha_1 \end{pmatrix}, \tag{18}$$

which is completely integrable. The billiard map is given in Lazutkin coordinates by $(x_1, \alpha_1) \mapsto (x_2, \alpha_2)$, where

$$\begin{cases} x_2 = x_1 + \alpha_1 + \alpha_1^3 f(x_1, \alpha_1), \\ \alpha_2 = \alpha_1 + \alpha_1^4 g(x_1, \alpha_1), \end{cases} \tag{19}$$

for some smooth functions f, g. This change of variables was first derived in [34], where it was shown as a consequence of the KAM theorem (see [2, 32, 41, 42]) that there exist an abundance of invariant tori for the billiard map near $\alpha = 0$. We will use these coordinates to prove Theorem 3.1. Without loss of generality, we will often interchange the use of arc-length and Lazutkin coordinates in the domain of the billiard map β.

4.4. Angles of reflection

Before estimating the billiard map and its iterates from the boundary, we need several preliminary estimates on the angles of reflection. In this section, we denote

$$\hat{\beta}^k(x_1, \alpha_1) = (x_k, \alpha_k).$$

Lemma 4.1. *For each $c_1, c_2 > 0$, there exists $j_1 = j_1(\Omega, c_1, c_2) \in \mathbb{N}$ sufficiently large such that for $j \geq j_1$ and $c_1/j \leq \alpha_1 \leq c_2/j$, it follows that*

$$\frac{c_1}{(1 + c_1/j)^k j} \leq \alpha_k = \pi_2\beta^k(x_1, \alpha_1) \leq \frac{c_2(1 + \frac{c_2}{j})^k}{j} \quad (1 \leq k \leq j),$$

where π_2 is the projection onto the angular variable. In particular,

$$c_1 e^{-c_1}/j \leq \pi_2\beta^k(x_1, \alpha_1) \leq c_2 e^{c_2}/j.$$

Proof. Fix c_1, c_2 and choose j_1 large enough that, for all $j \geq j_1$,

$$\frac{c_2^3 e^{3c_2} \sup |g|}{j^3} \leq \frac{c_2}{j}, \quad \frac{c_1^3 \sup |g|}{j^3} \leq \frac{c_1}{j}. \tag{20}$$

We will use Lazutkin coordinates and induction in k to show the estimates in the lemma hold. Let $j \geq j_1$ and suppose the claim is true for $1, \ldots, k-1$. Then

$$\begin{aligned}
\pi_2 \beta^k(x_1, \alpha_1) &= \pi_2 \beta \circ \beta^{k-1}(x_1, \alpha_1) \\
&= \alpha_{k-1} + \alpha_{k-1}^4 g(x_{k-1}, \alpha_{k-1}) \\
&\leq c_2(1 + c_2/j)^{k-1}/j + (c_2(1 + c_2/j)^{k-1}/j)^4 \sup |g| \\
&= \frac{c_2(1 + c_2/j)^{k-1}}{j} \left(1 + \frac{c_2^3(1 + c_2/j)^{3k-3} \sup |g|}{j^3}\right) \\
&\leq \frac{c_2(1 + c_2/j)^{k-1}}{j} \left(1 + \frac{c_2^3 e^{3c_2} \sup |g|}{j^3}\right).
\end{aligned} \tag{21}$$

Given the choice of j_1 in the first line of 20, the upper bound follows also for the k-th iterate. Similarly,

$$\begin{aligned}
\pi_2 \beta^k(x_1, \alpha_1) &= \pi_2 \beta \circ \beta^{k-1}(x_1, \alpha_1) \\
&= \alpha_{k-1} + \alpha_{k-1}^4 g(x_{k-1}, \alpha_{k-1}) \\
&\geq \frac{c_1}{(1 + c_1/j)^{k-1} j} - \left(\frac{c_1}{(1 + c_1/j)^{k-1} j}\right)^4 \sup |g| \\
&= \frac{c_1}{(1 + c_1/j)^{k-1} j} \left(1 - \frac{c_1^3}{(1 + c_1/j)^{3k-3} j^3} \sup |g|\right) \\
&\geq \frac{c_1}{(1 + c_1/j)^{k-1} j} \left(1 - \frac{c_1^3 \sup |g|}{j^3}\right),
\end{aligned} \tag{22}$$

which is greater than

$$\frac{c_1}{(1 + c_1/j)^k j},$$

under the second condition in 20, demonstrating the lower bound for k. ∎

Lemma 4.2. *There exist $C_1, C_2 > 0$ and $j_2 = j_2(\Omega) \in \mathbb{N}$ with the following property: for all $j \geq j_2$ and x, y which are $O(1/j)$ close to the diagonal of the boundary, any orbit γ of j reflections which emanates from x and terminates at y in approximately one rotation (in the sense of Section 3) has angles of reflection α_k in the range*

$$\frac{C_1}{j} \leq \alpha_k \leq \frac{C_2}{j} \quad (1 \leq k \leq j).$$

Proof. Let $\hat{x}, \hat{y}$ denote the 1-st and $(j + 1)$-st points of reflection on the boundary. Recall that by approximately one rotation, we mean that

$$|\hat{\beta}^j(\hat{x}, \alpha_1) - \hat{y} - \ell| \leq \ell/100,$$

where $\hat{\beta}$ is the lift to the universal cover and $\ell = |\partial\Omega|$. Each arc, separated by moments of reflection x_p, x_{p+1} on the boundary, has length $x_{p+1} - x_p$. As γ makes approximately one rotation, we see that there must exist one arc of length

$$x_{p+1} - x_p \geq (99/100)\ell/j.$$

Similarly, there must exist an arc of length

$$x_{m+1} - x_m \leq (101/100)\ell/j.$$

By [35, Proposition 14.1], for each $1 \leq k \leq j$,

$$2\alpha_k \rho_{\min} \leq x_{k+1} - x_k \leq 2\alpha_k \rho_{\max}, \tag{23}$$

where $\rho_{\min}$ and $\rho_{\max}$ are the minimum and maximum radii of curvature respectively for $\partial\Omega$. In particular,

$$\frac{99\ell}{200\rho_{\max}j} \leq \alpha_p, \quad \alpha_m \leq \frac{101\ell}{200\rho_{\min}j}. \tag{24}$$

We now apply Lemma 4.1 to $\beta^{\pm}$ at the points x_p and x_m to conclude that for all $1 \leq k \leq j$,

$$\frac{\exp\left(-\frac{99\ell}{200\rho_{\max}}\right)}{j} \leq \alpha_k \leq \frac{\exp\left(\frac{101\ell}{200\rho_{\min}}\right)}{j}. \tag{25}$$

Here we have used Lemma 4.1 applied to both the billiard map and its inverse, with the constants $c_1 = 99\ell/200\rho_{\max}$ and $c_2 = 101\ell/200\rho_{\min}$ which depend only on Ω. Hence, j_2 may be chosen uniformly for all orbits making approximately one rotation regardless of their initial positions. ∎

We now investigate the angle of reflection at the first point of impact on the boundary. If $d(z) = \text{dist}(z, \partial\Omega)$, then the level curves of d foliate a neighborhood of the boundary. For x_0 in the tubular neighborhood $\tilde{U}_j$ (whose diameter remains to be specified), we define $\xi_0^{\pm} \in \pm T_{x_0}^* d^{-1}(d(x_0)) \cap S_{x_0}^* \Omega$ so that ξ_0^+ points in the counterclockwise direction and ξ_0^- points in the clockwise direction. Let $\xi \in S_{x_0}^* \Omega$ be identified with the corresponding point $\vartheta \in S^1 = \mathbb{R}/\mathbb{Z}$, which denotes the clockwise angular parametrization of the fiber $S_{x_0}^* \Omega$, normalized so that ξ_0^+ and ξ_0^- correspond to 0 and π respectively. Denote by $\alpha_1 \in (0, \pi)$ the angle made between the billiard orbit emanating from (x_0, ξ) and the positively oriented cotangent line at the first point of reflection at the boundary. It is clear that $|\partial\xi/\partial\vartheta| = 1$.
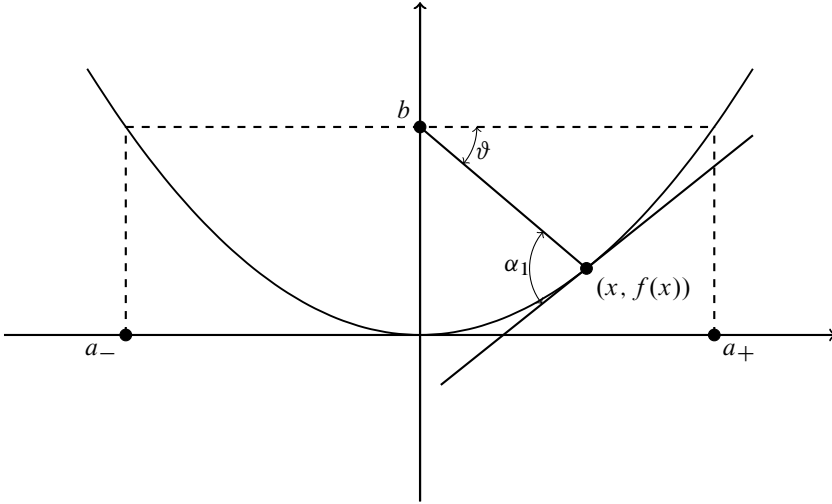
Figure 3. Angular derivative of the billiard flow.

Lemma 4.3. *There exist $c = c(\Omega) > 0$ and $j_3 = j_3(\Omega) \in \mathbb{N}$ such that if $j \geq j_3$ and $d(x_0) = O(1/j^2)$, then*

$$\frac{\partial \alpha_1}{\partial \vartheta} \geq c$$

for all initial conditions $(x_0, \xi(\vartheta))$ which generate a counterclockwise orbit of $j \geq j_0$ reflections, making approximately one rotation.

Proof. Let $x_0 \in \text{int}(\Omega)$ be near the boundary and consider the point $x \in \partial\Omega$ which minimizes the Euclidean distance $|x_0 - x|$ among all boundary points. We then apply an affine change of coordinates $y \mapsto A(y - x)$ with $A \in SO(2)$ so that x is mapped to the origin and the positively oriented unit tangent vector at x is mapped to $(1, 0)$. The vector $x - x_0$ is perpendicular to both $T_x^* \partial\Omega$ and $T_{x_0}^* d^{-1}(|x - x_0|)$ (see Figure 3). Therefore, the point $A(x_0 - x)$ lies on the positive vertical axis and we denote it by $(0, b)$. In these coordinates, the boundary $\partial\Omega$ is locally parametrized by the graph of a smooth convex function f. Since $f'(0) = 0$, we see that $f(t) = \kappa t^2 + R(t)$, with κ denoting the curvature at the point corresponding to $(0, 0)$ and $R(t) = O(|t|^3)$. Dilating by a factor of κ^{-1}, we see that in rescaled coordinates, the function becomes $f(t) = t^2 + R(t)$, with $R(t)$ again of order $O(|t|^3)$.

In graph coordinates, α_1 becomes the angle between $(t, f(t)) - (0, b)$ and the negatively oriented cotangent line $T_{(t, f(t))}^* \partial\Omega$ as illustrated in Figure 3. This shows that

$$\cos \alpha_1 = \frac{(t, f(t) - b) \cdot (1, f'(t))}{\sqrt{t^2 + (f(t) - b)^2} \sqrt{1 + (f'(t))^2}} \geq 0, \tag{26}$$

since $0 \leq \alpha_1 \leq \pi/2$. Recall that $\xi_0^\pm$ are the unit covectors in the positive (+), resp. negative (-), cotangent line $T_{x_0}^* d^{-1}(|x - x_0|)$ corresponding to $\vartheta = 0$ and $\vartheta = \pi$ respectively. For now, we only consider counterclockwise orbits obtained by perturbing the orbit emanating from (x_0, ξ_0^+), as they correspond to $-\pi/2 \leq \vartheta \leq \pi/2$ in the statement of the lemma. Denote the points $a_- = \min\{f^{-1}(b)\}$ and $a_+ = \max\{f^{-1}(b)\}$. We first show that $d\alpha_1/dt < 0$ on $[0, a_+]$ so that the angle of incidence at the first impact point on the boundary is increasing as the initial covector of the trajectory winds clockwise in $S_{x_0}^* \Omega$ away from ξ_0^+, i.e., $d\alpha/dt < 0$ on $[0, a_+]$.

Noting that t and ϑ are negatively correlated, it suffices to show that the logarithmic t derivative of $\cos \alpha_1(t)$ is positive:

$$\frac{d}{dt} \log \cos \alpha_1(t) = \frac{(1 + f''(f - b) + (f')^2)}{t + f'(f - b)} - \frac{t + f'(f - b)}{t^2 + (f - b)^2} - \frac{f' f''}{1 + (f')^2}. \tag{27}$$

Multiplying (27) through by a common denominator, which is positive for t, b small, we obtain

$$\begin{aligned} g(t, b) &:= (t + f'(f - b))(t^2 + (f - b)^2)(1 + (f')^2) \frac{d}{dt} \log \cos \alpha(x) \\ &= (1 + f''(f - b) + (f')^2)(t^2 + (f - b)^2)(1 + (f')^2) \\ &\quad - (t + f'(f - b))^2 (1 + (f')^2) \\ &\quad - f' f''(t + f'(f - b))(t^2 + (f - b)^2). \end{aligned} \tag{28}$$

To show that (28) is positive, we plug in our second order Taylor approximation for f and expand g to fourth order in a parabolic neighborhood of $t = 0, b = 0$:

$$g(t, b) = 3t^4 + 10t^2 b^2 - 2b^3 + b^2 + O((t^2 + b^2)^{5/2}). \tag{29}$$

As $f(t) = t^2 + O(t^3)$, we see that $a_\pm = \pm b^{1/2} + O(b^{3/2})$, so we choose $b \leq 1/j^2$ sufficiently small in order to make (29) positive in a parabolic neighborhood of origin. Examining the initial set up, choosing b small amounts to choosing x_0 close to the boundary, all of which can be done uniformly in x_0 and the curvature κ. Choosing a uniform b gives us a tubular neighborhood of the boundary $\widetilde{U}_j = \{x : 0 < d(x, \partial\Omega) < j^{-2}\}$, as mentioned in the beginning of the section. We will later shrink $\widetilde{U}_j$ by a factor of j^{-2}, following Lemma 4.9.

While the lower bound in (29) is of order $b^2 = O(j^{-4})$, we in fact need uniform bounds for $\partial\alpha_1/\partial\vartheta$ which we now provide. The prefactor in (28) is nonvanishing for $t \neq 0, t \neq a_+$ and in particular, for all t corresponding to an orbit which makes approximately one rotation in j reflections. We first need to compare $\partial/\partial t$ and $\partial/\partial\vartheta$ in terms of b, which depends on j. In graph coordinates, ϑ becomes the angle of the

first link from the horizontal axis (see Figure 3). Hence, $\vartheta = \arctan((b - f(t))/t)$ and

$$\frac{\partial \vartheta}{\partial t} = \frac{1}{1 + (b - f(t))^2/t^2}\Big(-\frac{f'(t)}{t} - \frac{b - f(t)}{t^2}\Big),$$

or, equivalently,

$$\frac{\partial t}{\partial \vartheta} = -\Big(\frac{t^2 + (b - f(t))^2}{tf'(t) + b - f(t)}\Big), \tag{30}$$

where t is defined implicitly as a function of ϑ. Plugging (30) into (27) and using that (29) is bounded below by $b^2/2$ for b sufficiently small, we see that

$$\frac{\partial \alpha}{\partial \vartheta} \geq \frac{b^2(t^2 + (b - f(t))^2)}{2(tf'(t)+b-f(t))(t+f'(t)(f(t)-b))(t^2+(f(t)-b)^2)(1+(f'(t))^2)\tan\alpha_1}$$

$$= \frac{b^2}{2(tf'(t) + b - f(t))(t + f'(t)(f(t) - b))(1 + (f'(t))^2)\tan\alpha_1}. \tag{31}$$

The denominator in (31) is nonvanishing and can be estimated above on $(0, a_+]$ by

$$c'tb \tan\alpha_1 + O(b^{5/2}). \tag{32}$$

for some $c' > 0$, independent of j. Observe that if $t \leq b$ for example, then the denominator is bounded above by $c'b^2$ and $\partial\alpha_1/\partial\vartheta \geq c''$, for some $c'' > 0$ which is also independent of j.

We now find a similar upper bound on $t(\vartheta)$ in terms of b, corresponding to the set of covectors $\xi(\vartheta)$ which produce orbits of j reflections and approximately one rotation. Let $b = rb_0$, where $b_0 = j^{-2}$ for $r \in (0, 1]$. Recall from Lemma 4.2 that there exist constants $C_1, C_2 > 0$ such that for all orbits making approximately one rotation and j reflections, we have $C_1/j \leq \alpha_k \leq C_2/j$ for each $1 \leq k \leq j$. Hence,

$$C_1 b_0^{1/2} \leq \alpha_1 \leq C_2 b_0^{1/2}. \tag{33}$$

Observe that equation (26) gives

$$\log\cos\alpha_1(t) = \log(t + f'(t)(f(t) - b)) - 1/2\log(t^2 + (f(t) - b)^2)$$
$$- 1/2\log(1 + (f'(t))^2). \tag{34}$$

Denote the terms in (34) above by A, B and C. Then,

$$A = \log(t + (2t + O(t^2))(t^2 + O(t^3) - b))$$
$$= \log t + \log(1 + 2t^2 - 2b + O(t^3) + O(b))$$
$$= \log t + 2t^2 + O(t^3) + O(b).$$

Similarly,

$$B = -1/2 \log(t^2 + t^4 - 2t^2 b + b^2 + O(t^5) + O(t^3 b))$$
$$= -\log t - 1/2(t^2 - 2b + b^2/t^2 + O(t^3) + O(tb))$$

and

$$C = -\log(1 + (2t + O(t^2))) = -4t^2 + O(t^3).$$

As $t = O(b^{1/2})$, we have

$$A + B + C = \frac{-b^2}{2t^2} + O(b).$$

On the other hand,

$$A + B + C = \log \cos \alpha_1$$
$$= \log(1 - a_1^2/2 + O(\alpha_1^4)).$$
$$= -\alpha_1^2/2 + O(b_0^{4/N}),$$

which implies that

$$b^2/t^2 = r^2 b_0^2/t^2 \sim \alpha_1^2/2.$$

In particular, (33) implies

$$t \sim \frac{\sqrt{2} r b_0}{\alpha_1} \in \left[\frac{\sqrt{2} r b_0^{1/2}}{C_2}, \frac{\sqrt{2} r b_0^{1/2}}{C_1} \right]. \tag{35}$$

Note also that $\tan \alpha_1 \leq 2\alpha_1 \leq 2C_2 b_0^{1/2}$. Combining this with (35), we see that the denominator in equation (31) is bounded above by

$$c' t b \tan \alpha_1 + O(b^{5/2}) \leq c''' b (r b_0^{1-1/2}) b_0^{1/2} = c''' b^2, \tag{36}$$

for some $c''' > 0$ independent of j. This establishes the claim that $\partial \alpha / \partial \vartheta$ is uniformly bounded below on the set of initial covectors corresponding to orbits of j reflections and approximately one rotation. ∎

4.5. Monotonicity and the twist property

The significance of increasing the angle of reflection at the first impact point on the boundary is that after $j - 1$ reflections at the boundary, the j-th impact point winds around $\partial \Omega$ with approximate speed j. This is essentially the twist property of the billiard map (see [53] for a general definition of twist maps). The following lemma makes this notion quantitative.

Lemma 4.4. *There exists $j_4 = j_4(\Omega) \in \mathbb{N}$ such that if a trajectory makes $j \geq j_4$ reflections at the boundary and approximately one rotation, then the Jacobian of the iterated billiard map satisfies*

$$D\hat{\beta}^j(x_1, \alpha_1) = \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix} + O(1/j)$$

in Lazutkin coordinates lifted to the universal cover. In particular, there exist constants $C_3, C_4, C_5, C_6 > 0$ depending only on Ω such that for any (x_1, α_1), we have

$$C_3 j \leq \frac{\partial}{\partial \alpha_1} \pi_1 \hat{\beta}^j(x_1, \alpha_1) \leq C_4 j,$$

$$C_5 \leq \frac{\partial}{\partial \alpha_1} \pi_2 \hat{\beta}^j(x_1, \alpha_1) \leq C_6,$$

where π_1 and π_2 denote projections onto the first and second components respectively.

Proof. In Lazutkin coordinates,

$$\beta(x_k, \alpha_k) = (x_k + \alpha_k + \alpha_k^3 f(x_k, \alpha_k), \alpha_k + \alpha_k^4 g(x_k, \alpha_k)).$$

Hence,

$$D\hat{\beta}^j(x_1, \alpha_1)$$

$$= \prod_{k=1}^{j} \begin{pmatrix} 1 + \alpha_k^3 \frac{\partial f}{\partial x}(x_k, \alpha_k) & 1 + 3\alpha_k^2 f(x_k, \alpha_k) + \alpha_k^3 \frac{\partial f}{\partial \alpha}(x_k, \alpha_k) \\ \alpha_k^4 \frac{\partial g}{\partial x}(x_k, \alpha_k) & 1 + 4\alpha_k^3 g(x_k, \alpha_k) + \alpha_k^4 \frac{\partial g}{\partial \alpha}(x_k, \alpha_k) \end{pmatrix}, \quad (37)$$

where $(x_k, \alpha_k) = \hat{\beta}^k(x_1, \alpha_1)$ and f, g are extended to be ℓ periodic in x on $\mathbb{R} \times [0, \pi]$. Each of the terms in the product can be decomposed into $(I_2 + N + R_k)$, where I_2 is the 2×2 identity matrix, N is the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (38)$$

and the remainder term is

$$R_k = \begin{pmatrix} \alpha_k^3 \frac{\partial f}{\partial x}(x_k, \alpha_k) & 3\alpha_k^2 f(x_k, \alpha_k) + \alpha_k^3 \frac{\partial f}{\partial \alpha}(x_k, \alpha_k) \\ \alpha_k^4 \frac{\partial g}{\partial x}(x_k, \alpha_k) & 4\alpha_k^3 g(x_k, \alpha_k) + \alpha_k^4 \frac{\partial g}{\partial \alpha}(x_k, \alpha_k) \end{pmatrix}$$

$$= \begin{pmatrix} O(j^{-3}) & O(j^{-2}) \\ O(j^{-4}) & O(j^{-3}) \end{pmatrix},$$

which is small in norm. First choose C and j_4 so that for all $j \geq j_4$ and $1 \leq k \leq j$, $\|R_k\|_{\ell^\infty} \leq Cj^{-2}$ (which is possible by Lemma 4.1). All matrix norms below are assumed to be ℓ^∞. We claim that for $j \geq j_4$ and each $p = 1, \dots, j$,

$$\left\| \prod_{k=1}^{p} \begin{pmatrix} 1 + \alpha_k^3 \dfrac{\partial f}{\partial x}(x_k, \alpha_k) & 1 + 3\alpha_k^2 f(x_k, \alpha_k) + \alpha_k^3 \dfrac{\partial f}{\partial \alpha}(x_k, \alpha_k) \\ \alpha_k^4 \dfrac{\partial g}{\partial x}(x_k, \alpha_k) & 1 + 4\alpha_k^3 g(x_k, \alpha_k) + \alpha_k^4 \dfrac{\partial g}{\partial \alpha}(x_k, \alpha_k) \end{pmatrix} - \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \right\|$$

$$\leq \frac{C(1 + 2p)(1 + C/j^2)^p}{j^2}. \tag{39}$$

The claim is clearly true for $p = 1$. Assuming it holds for $1, \dots, p-1$, we then have

$$\left\| \prod_{k=1}^{p} (I_2 + N + R_k) - (I_2 + pN) \right\|$$

$$= \left\| \left(\prod_{k=1}^{p-1} (I_2 + N + R_k) - (I_2 + (p-1)N) \right)(I_2 + N + R_p) \right.$$

$$\left. - (I_2 + (p-1)N)R_p \right\|$$

$$\leq \left(\frac{C(1 + 2(p-1))(1 + C/j^2)^{p-1}}{j^2} \right)(1 + C/j^2) + \|(I_2 + (p-1)N)R_p\|$$

$$\leq \left(\frac{C(1 + 2p)(1 + C/j^2)^p}{j^2} \right) - \left(\frac{2C(1 + C/j^2)^p}{j^2} \right) + \frac{2C}{j^2}$$

$$\leq \left(\frac{C(1 + 2p)(1 + C/j^2)^p}{j^2} \right).$$

Hence, the claim also follows for p. Having shown this bound for all $1 \leq p \leq j$, $(j \geq j_4)$, it follows from the case $p = j$ that

$$\|D\hat{\beta}^j(x_1, \alpha_1) - (I_2 + jN)\| \leq \left(\frac{C(1 + 2j)(1 + C/j^2)^j}{j^2} \right) \leq \frac{3Ce^C}{j}. \tag{40}$$

Noting that angular derivatives $\partial \pi_1 \hat{\beta}^j / \partial \alpha_1$ and $\partial \pi_2 \hat{\beta}^j / \partial \alpha_1$ are the $(1, 2)$ and $(2, 2)$ components of $D\hat{\beta}^j$ respectively concludes the proof of the lemma. ∎

Remark 4.5. As j increases, Lemma 4.4 shows that the Poincaré map associated to a periodic orbit of rotation number $1/j$ becomes more and more degenerate, corresponding to the accumulation of caustics at the boundary. In fact, there exist higher order Lazutkin coordinates for which the remainder above could be replaced by $O(j^{-N})$ for any $N \in \mathbb{N}$. However, this is not needed in the remainder of the paper.
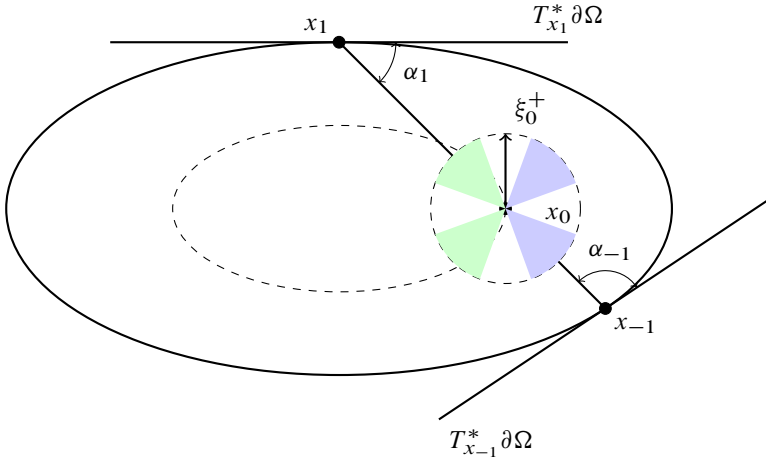
Figure 4. A counterclockwise perturbation of ξ_0^+. The green cones at x_0 are the length one covectors in $C_{x_0,+}^*(\Omega; j)$ and the blue cones at x_0 are the length one covectors in $C_{x_0,-}^*(\Omega; j)$.

The point x_j can be obtained in two ways. The first way is by flowing forwards to the boundary point $x_1 := \pi_1 g^{t_1^+}(x_0, \xi)$, iterating the billiard map j times and projecting onto the first component. The second way is by flowing backwards to the boundary point $x_{-1} := g^{t_1^-}(x_0, \xi)$ and iterating the billiard map $j + 1$ times. If ξ is rotated in the clockwise direction away from ξ_0^+, it is convenient to use the angle α_1 at x_1, since it was shown in Lemma 4.3 that α_1 is increasing in ϑ. If ξ is rotated counterclockwise away from ξ_0^+, it is instead advantageous to use the point (x_{-1}, α_{-1}), since this again corresponds to a *clockwise* perturbation of ξ_0^-. In either regime $\vartheta < \vartheta_0^+$ or $\vartheta > \vartheta_0^+$, we can then take advantage of the monotonicity of x_j in $\alpha_{\pm 1}$ as ξ, or equivalently ϑ, varies. The next lemma shows that the point x_j is monotonically increasing for all orbits making approximately one rotation as $\xi(\vartheta)$ winds in either direction.

Lemma 4.6. *There exist $C_7 > 0$ and $j_5 = j_5(\Omega) \in \mathbb{N}$ such that for all $\vartheta \in (0, \pi/2)$ corresponding to a $j \geq j_5$ reflection orbit emanating from $(x_0, \xi(\vartheta))$ which makes approximately one rotation, we have $\partial x_j / \partial \vartheta \geq C_7 j$. Similarly, for $\vartheta \in (-\pi/2, 0)$ corresponding to an orbit of j reflections and approximately one rotation, we have $\partial x_j / \partial \vartheta \leq -C_7 j$.*

Proof. Denote $x_j = \pi_1 \beta^j(x_1, \alpha_1)$, where π_1 is the projection onto the first component. Then x_j depends only on (x_0, ξ), or equivalently (x_0, ϑ) and can be written as the composition

$$x_j = \pi_1 \beta^j(x_1(x_0, \vartheta), \alpha_1(x_0, \vartheta)).$$

As ϑ increases or decreases from 0, we have

$$\frac{\partial x_j}{\partial \vartheta} = \frac{\partial x_j}{\partial x_1}\frac{\partial x_1}{\partial \vartheta} + \frac{\partial x_j}{\partial \alpha}\frac{\partial \alpha}{\partial \vartheta}. \tag{41}$$

Recall from Lemma 4.4 that

$$C_3 j \leq \frac{\partial x_j}{\partial \alpha} \leq C_4 j, \tag{42}$$

for j large and positive constants C_3 and C_4. This derivative can be made arbitrarily large by choosing j accordingly. Also, Lemma 4.2 showed that all orbits making approximately one rotation with j reflections at the boundary have angles of reflection in the range $C_1/j \leq \alpha \leq C_2/j$ for positive constants C_1 and C_2. We can now use Lemma 4.3, which showed that $|\partial\alpha/\partial\vartheta| \geq c > 0$ independently of j for all ϑ producing an orbit of approximately one rotation in j reflections. Using Lazutkin coordinates and Lemma 4.4, we also have that

$$\frac{\partial x_j}{\partial x_1}\frac{\partial x_1}{\partial \vartheta} = (1 + O(1/j))\frac{\partial x_1}{\partial \vartheta}. \tag{43}$$

In graph coordinates (see proof of Lemma 4.3), one can calculate that

$$\begin{aligned}
\left|\frac{\partial x_1}{\partial \vartheta}\right| &= \left|\frac{\partial x_1}{\partial x}\frac{\partial x}{\partial \vartheta}\right| \\
&= \sqrt{1 + (f'(x))^2}\frac{xf'(x) + b - f}{x^2 + (b - f(x))^2} \\
&\leq 1 + O(b) \leq 2
\end{aligned} \tag{44}$$

which is bounded independently of j. Hence, the term (42) dominates in the expression (41) and x_j winds monotonically around $\partial\Omega$. ∎

At the endpoints $\vartheta = \pm\pi/2$, it is clear that the angle of reflection at $x_{\pm 1}$ is $\pi/2$ since the distance curves are parallel. Angles outside the range $\vartheta \in (-\pi/2, \pi/2)$ correspond to clockwise orbits and Lemma 4.2 showed that for an orbit making approximately one rotation and j reflections at the boundary, all angles of reflection satisfy $C_1/j \leq \alpha_k \leq C_2/j$, $(1 \leq k \leq j)$. If j is large and x_0 is $1/j^4$ close to the boundary as in the statement of Theorem 3.1, it is clear that j reflection orbits emanating from ξ_0^+ (corresponding to $\vartheta = 0$) will make at most a quarter rotation: in graph coordinates $\partial\Omega = \text{graph}\{\kappa t^2 + R(t)\}$ (as in the proof of Lemma 4.3), the orbit with initial covector ξ_0^+ intersects $\partial\Omega$ at $(t_1, j^{-4}) \sim (\kappa^{-1/2}j^{-2}, j^{-4})$, where the tangent line has slope $m \sim 2\kappa^{1/2}j^{-2}$. The angle at this first point of impact point on the boundary is then $\sim \arctan 2\kappa^{1/2}j^{-2} \sim 2\kappa^{1/2}j^{-2}) = O(j^{-2})$. Iterating the billiard

map j times in Lazutkin coordinates then gives

$$x_j = x_1 + \sum_{k=1}^{j-1}(\alpha_k + \alpha_k^3 f(x_k, \alpha_k)).$$

Using Lemma 4.1 with $c_2 = 1$ and j replaced with j^2 then gives

$$|x_j - x_1| \leq (j-1)ej^{-2} + (j-1)e^3 j^{-6} \sup_{B^*\Omega} |f| = O(j^{-1}).$$

In other words, the base point moves $O(1/j)$ along the boundary, which can certainly be made less than $|\partial\Omega|/4$.

It is then clear that the collection of covectors at x_0 whose trajectories make approximately one rotation in j reflections consists of two connected components in $S_{x_0}^*\Omega$:

Definition 4.7. The positive admissible cone $C_{x_0,+}^*(\Omega; j)$ at x_0 is defined to be the set of homogeneous extensions to $T_{x_0}^*\Omega$ of the two components in $S_{x_0}^*(\Omega)$ described above. The negative admissible cone $C_{x_0,-}^*(\Omega; j)$ at x_0 is defined by the same property for orbits making j reflections in the clockwise direction. Their union is denoted by $C_{x_0}^*(\Omega; j) = C_{x_0,+}^*(\Omega; j) \cup C_{x_0,-}^*(\Omega; j)$.

See Figure 4 for an illustration of $C_{x_0}^*(\Omega; j)$.

4.6. Intersection points

We are finally ready to show that the intersection points of the last link with the distance curve of y wind around monotonically as we twist ξ in either direction. We first explain why the last link necessarily intersects $d^{-1}(\text{dist}(y, \partial\Omega))$ twice.

Lemma 4.8. *There exists $j_6 = j_6(\Omega) \in \mathbb{N}$ such that for $j \geq j_6$ and any $N \geq 4$, the distance curve $\{x \in \Omega : \text{dist}(x, \partial\Omega) = j^{-N}\}$ is intersected exactly twice by any link emanating from the boundary at an angle greater than or equal to C_1/j, with C_1 the same constant appearing in Lemma 4.2.*

Proof. For each point p in the distance curve $\{z \in \Omega : d(z, \partial\Omega) = j^{-N}\}$, the tangent line $T_p\{z \in \Omega : d(z, \partial\Omega) = j^{-N}\}$ intersects $\partial\Omega$ exactly twice by convexity. Recall that Lemma 4.2 gave $C_1/j \leq \alpha_k \leq C_2/j$ for each $1 \leq k \leq j$. We now show that the angles of reflection for links which are tangent to the distance curve $d^{-1}(j^{-N})$ are of order $O(j^{-N/2})$. In graph coordinates, $\partial\Omega$ is locally given by $(x, f(x))$ where $f(x) = \kappa x^2 + R(x)$, κ is the curvature of $\partial\Omega$ at the point corresponding to $(0, 0)$ and $R(x) = O(x^3)$. By rescaling, we may assume that $\kappa = 1$. The distance curve

$d^{-1}(j^{-N})$ can be parametrized in graph coordinates by

$$d^{-1}(j^{-N}) = \{(t, f(t)) + rN(t) : t \in \mathbb{R}\},$$

where $r = j^{-N}$ and

$$N(t) = \frac{(-f'(t), 1)}{\sqrt{1 + (f'(t))^2}}$$

is the unit normal to the graph at $(t, f(t))$. There exist precisely two lines through the origin which are tangent to $d^{-1}(j^{-N})$ in graph coordinates. The positive parameter t corresponding to a point of tangency satisfies

$$\frac{f(t) + r/(1 + (f'(t))^2)^{1/2}}{t - rf'(t)/(1 + (f'(t))^2)^{1/2}} = \frac{\partial_t(f(t) + r/(1 + (f'(t))^2)^{1/2})}{\partial_t(t - rf'(t)/(1 + (f'(t))^2)^{1/2})}, \quad (45)$$

where the left-hand side of (45) is the slope of a line connecting the origin to a point on $d^{-1}(r)$ and the right-hand side is the slope of the tangent to $d^{-1}(r)$. Solving for t in terms of r in equation (45) yields $f'(t)t - f(t) = t^2 + O(t^3) = O(r)$. Plugging this into the right-hand side of (45), we see that the angle of tangency satisfies $\tan(\alpha_{\text{tangency}}) = O(r^{1/2})$ and hence $\alpha_{\text{tangency}} = O(j^{-2})$ if $N \geq 4$. The proof is complete by noting that the angles α_k coming from orbits making approximately one rotation and j reflections are bounded below by $C_1/j > O(j^{-4/2})$ for j sufficiently large. ∎

Lemma 4.9. *Under the hypotheses of Lemma 4.8, denote by w_1 and w_2 the arc-length coordinates on $d^{-1}(d(y))$ corresponding to the intersection points of $d^{-1}(d(y))$ and the last link of the j reflection orbit emanating from x. There exist $C_8 > 0$ and $j_7 = j_7(\Omega) \in \mathbb{N}$ such that if $j \geq j_7$ and x, y are $O(1/j^4)$ close the diagonal of the boundary, then $|\partial w_i / \partial \vartheta| \geq C_8 j$ $(i = 1, 2)$ for all ϑ corresponding to $\xi(\vartheta)$ in the cone of admissible covectors at x_0.*

Proof. In graph coordinates, $\partial\Omega$ is again locally parametrized by $(t, f(t))$, where $f(t) = \kappa t^2 + R(t)$, κ is the curvature of $\partial\Omega$ at the point corresponding to $(0, 0)$ and $R(t) = O(t^3)$. By rescaling, we may assume that $\kappa = 1$. The distance curve $d^{-1}(d(y))$ on which y lies can be locally parametrized in graph coordinates by

$$d^{-1}(d(y)) = \{(t, f(t)) + rN(t)\}, \quad (46)$$

where $r = \text{dist}(y, \partial\Omega) = O(j^{-4})$ and

$$N(t) = \frac{(-f'(t), 1)}{\sqrt{1 + (f'(t))^2}}$$

is the unit normal to the graph at $(t, f(t))$. The final link of the orbit connecting $x_j(\vartheta) \in \partial\Omega$ to $x_{j+1}(\vartheta) \in \partial\Omega$ can be parametrized by the line

$$Y = \tan(\alpha_j(\vartheta) + \beta_j(\vartheta))X + f(t_j(\vartheta)) - \tan(\alpha_j(\vartheta) + \beta_j(\vartheta))t_j(\vartheta), \qquad (47)$$

where

$$\beta_j = \arccos\left(\frac{1}{\sqrt{1 + (f'(x_j))^2}}\right)$$

is the angle of the tangent to the graph with the horizontal axis and $t_j = t_j(\vartheta) \in \mathbb{R}$ is defined implicitly by the equation $(t_j, f(t_j)) = x_j \in \partial\Omega$. Noting that the Cartesian coordinates of the parametrization of $d^{-1}(d(y))$ satisfy

$$X = t - \frac{rf'(t)}{\sqrt{1 + (f'(t))^2}}, \qquad Y = f(t) + \frac{r}{\sqrt{1 + (f'(t))^2}}, \qquad (48)$$

we plug these into equation (47) to obtain

$$\left(f(t) + \frac{r}{\sqrt{1 + (f'(t))^2}}\right)$$
$$= \tan(\alpha_j(\vartheta) + \beta_j(\vartheta))\left(t - \frac{rf'(t)}{\sqrt{1 + (f'(t))^2}}\right)$$
$$+ f(t_j(\vartheta)) - \tan(\alpha_j(\vartheta) + \beta_j(\vartheta))t_j(\vartheta). \qquad (49)$$

By Lemma 4.8, there exist precisely two solutions $t = z_1(\vartheta), z_2(\vartheta)$ of equation (49) which correspond to the two intersection points of the last link with the distance curve $d^{-1}(d(y))$. Plugging $z_i(\vartheta)$ into equation (49), differentiating in ϑ and evaluating at ϑ_0 corresponding to $x_j(\xi(\vartheta_0))$ at the origin, we obtain

$$f'(z_i)z_i' - \frac{rf'(z_i)f''(z_i)z_i'}{(1 + f(z_i)^2)^{3/2}}$$
$$= (\alpha_j' + \beta_j')(1 + \tan^2\alpha_j)\left(z_i - \frac{rf'(z_i)}{\sqrt{1 + (f'(z_i))^2}}\right)$$
$$+ \tan\alpha_j\left(z_i' - \frac{rf''(z_i)z_i'}{\sqrt{1 + (f'(z_i))^2}} + \frac{r(f'(z_i))^2 f''(z_i)z_i'}{(1 + (f'(z_i))^2)^{3/2}}\right)$$
$$+ 0 - \tan\alpha_j t_j'. \qquad (50)$$

We now collect all terms with z_i' in (50):

$$A_i z_i' = (\alpha_j' + \beta_j')(1 + \tan^2\alpha_j)\left(z_i - \frac{rf'(z_i)}{\sqrt{1 + (f'(z_i))^2}}\right) - \tan\alpha_j t_j', \qquad (51)$$

where

$$A_i = f'(z_i) - \frac{r f'(z_i) f''(z_i)}{(1 + f(z_i)^2)^{3/2}}$$

$$- \tan \alpha_j \left(1 - \frac{r f''(z_i)}{\sqrt{1 + (f'(z_i))^2}} + \frac{r (f'(z_i))^2 f''(z_i)}{(1 + (f'(z_i))^2)^{3/2}} \right).$$

Before dividing through equation (51) by A_i, we Taylor expand A_i in powers of $1/j$ to show that it is nonvanishing:

$$A_i = 2z_i - \tan \alpha_j + O(j^{-2}). \tag{52}$$

At this point, it is important to distinguish between z_1 and z_2. If $z_1 < z_2$, then insisting that y be $O(j^{-4})$ close to the boundary amounts to setting $z_1 = O(j^{-4})$ and $z_2 = t_{j+1} + O(j^{-4})$. Here, $t_{j+1} = t_{j+1}(\vartheta)$ is defined implicitly by the equation $(t_{j+1}, f(t_{j+1})) = x_{j+1} \in \partial\Omega$. When $i = 1$, $A_1 = -\alpha_j + O(j^{-4}) \leq -C_1/(2j)$ is nonvanishing for j sufficiently large. For $i = 2$, note that $\tan \alpha_j = f(z_2)/z_2 + O(j^{-4})$, which implies that $A_2 = z_2 + O(j^{-2})$ is again nonvanishing. Hence, we may divide through equation (51) by A_i. To estimate the right-hand side of (51), we calculate that

$$\beta_j' = \frac{f'(t_j)}{\sqrt{1 - \frac{1}{1+(f'(t_j))^2}}} \frac{f''(t_j) t_j'}{(1 + (f(t_j)^2))^{3/2}}. \tag{53}$$

The first factor above is well defined by continuity at $t_j = 0$ and equals 1. Hence, $\beta_j' = 2t_j' + O(1/j)$. If $i = 1$, equation (51) becomes

$$(-\tan \alpha_j + O(j^{-4})) z_1' = -\tan \alpha_j t_j' + O(j^{-3}), \tag{54}$$

which implies that $z_1' = t_j' + O(j^{-2})$. When $i = 2$, equation (51) becomes

$$(z_2 + O(j^{-2})) z_2' = t_j' z_2 + O(1/j), \tag{55}$$

which again implies that $z_2' = t_j' + O(1/j)$. Recall that Lemma 4.6 showed x_j' is of order j in Lazutkin coordinates when $\xi(\vartheta) \in C_{x_0}^*(\Omega; j)$. To compare Lazutkin coordinates with t_j, note that in graph coordinates, the arc-length parameter ds on $\partial\Omega$ is given by

$$ds = (1 + (f'(t))^2)^{1/2} dt.$$

As arc-length coordinates and Lazutkin coordinates are comparable independently of j, we conclude that t_j' and hence z_i' are also of order j. We conclude the proof by noting that the arc-length parameter ds' on $d^{-1}(d(y))$ in the parametrization (46) is given by

$$ds' = ((1 + (f'(t))^2)^{1/2} + O(r)) dt,$$

which is also comparable to dt independently of j. ∎

From Lemma 4.9, we conclude that perturbing ξ_0^+ either clockwise or counterclockwise within the cone of admissible covectors (but still such that the corresponding orbit is counterclockwise) results in monotone increasing arc-length coordinates for the intersection points with $d^{-1}(d(y))$. By the intermediate value theorem, such intersection points will then coincide with y exactly 4 times for orbits making approximately one rotation (two intersection points for the clockwise perturbation and two intersections points for the counterclockwise perturbation).

4.7. Clockwise orbits

From Lemma 4.9, we saw that there were precisely 4 counterclockwise orbits connecting $x = x_0$ to y in j reflections and approximately one rotation. The only constraint on x and y was that they were confined to an $O(j^{-4})$ neighborhood of the diagonal of the boundary. By reflecting Ω through the vertical axis, one obtains another smooth strictly convex domain and the reflections of x and y remain $O(j^{-4})$ close to the diagonal of the boundary. Hence, the same procedure produces exactly 4 counterclockwise orbits of j reflections from x to y making approximately one rotation in the reflected domain. Reflecting back through the vertical axis carries these 4 orbits to clockwise orbits in the original domain. This completes the proof of Theorem 3.1.

Remark 4.10. We can take $j_0 = j_0(\Omega) = \max\{j_i : 2 \le i \le 7\}$, with j_i as they appear in Lemmas 4.2, 4.3, 4.4, 4.6, 4.8, and 4.9. Similarly, we can choose a uniform constant C_0 as in the statement of Lemma 3.1. The tubular neighborhood U_j referenced in Theorem 1.1 can be taken to be $\{(x, y) \in \Omega \times \Omega : \mathrm{dist}((x, y), \Delta \partial \Omega) \le j^{-4}\}$ for $j \ge j_0$, where $\Delta : \partial \Omega \times \partial \Omega$ is the diagonal embedding $x \mapsto (x, x)$. For $(x, y) \in U_j$, both x and y satisfy the conditions of Lemma 4.9. The cone bundle $C_x^*(\Omega; j)$ is well defined whenever $\mathrm{dist}(x, \partial \Omega) = O(j^{-4})$.

Remark 4.11. The proof of Theorem 3.1 in this section could actually be extended to a larger region of validity. In particular, the same methods allow us to prove the existence of 8 orbits of rotation number k/j for j sufficiently large in terms of k, connecting points in a comparable open neighborhood of the diagonal of the boundary. Additionally, we could allow x and y to be further away from the diagonal as long as they are both sufficiently close to the boundary. However, we are only concerned with near diagonal terms in this paper for the purposes of deriving trace formulas.

Remark 4.12. If $x \in \partial \Omega$, then $\xi_0^{\pm}$ are tangent to the boundary and perturbing ξ away from ξ_0^+ in the clockwise direction is no longer well defined. Rather, it is equivalent to reflecting and then rotating ξ in the counterclockwise direction. Similarly, ξ_0^- cannot be rotated in the counterclockwise direction. Each of these restrictions reduces the number of j reflection orbits to y in approximately one rotation by 2. If additionally

$y \in \partial\Omega$, the final link only makes one intersection with $d^{-1}(0) = \partial\Omega$ so there are only two orbits of j reflections connecting x to y in approximately one rotation. One is in the counterclockwise direction while the other is in the clockwise direction. In particular, when $x = y$, there is a *unique* geodesic loop (up to parametrization) of j reflections and exactly one rotation (i.e., rotation number $1/j$). The existence of such loops is well known and the proof for boundary points is much simpler, as it only requires Lemma 4.4

5. A parametrix for the wave propagator

In this section, we use microlocal analysis to obtain an oscillatory integral parametrix for the wave propagator in an open subset of $\mathbb{R} \times \Omega \times \Omega$ which contains the diagonal $\mathbb{R} \times \Delta(\partial\Omega)$. The wave kernel $e^{it\sqrt{-\Delta}}$ is actually *not* a Fourier integral operator (FIO) near the tangential rays (see [1, 40]), so we microlocalize the wave kernels near periodic transversal reflecting rays. We begin by reviewing FIOs and Chazarain's parametrix for the wave propagator in Sections 5.1 and 5.2. In Section 5.3, we then cook up oscillatory integrals for each term in Chazarain's parametrix, which microlocally approximate the wave propagator near the orbits described in Theorem 3.1. This approach is inspired by the formulas in [36] and will rely on the symbol calculus in Section 5.2.

5.1. Fourier integral operators

Let X and Y be open sets in $\mathbb{R}^{n_X}$ and $\mathbb{R}^{n_Y}$ respectively. If $a \in S_{1,0}^{\mu}(X \times \mathbb{R}^N)$ is a classical symbol of order μ and $\Theta \in C^\infty(X \times \mathbb{R}^N)$ is a nondegenerate phase function, then the linear operator

$$A(u) = \int\limits_X \int\limits_{\mathbb{R}^N} e^{i\Theta(x,\vartheta)} a(x,\vartheta) u(x) \, d\vartheta \, dx$$

is called a *Lagrangian* or *Fourier integral distribution on X*. Recall that a continuous linear operator $A\colon C_0^\infty(Y) \to \mathcal{D}'(X)$ has an associated Schwartz kernel $K_A \in \mathcal{D}'(X \times Y)$. If K_A is given by a locally finite sum of Lagrangian distributions on $X \times Y$, then we say A is a *Fourier integral operator* (FIO). One can then show that the wavefront set of the kernel is contained in the image of the map $\iota_\Theta\colon (x,y) \mapsto (x, y, d_x\Theta, d_y\Theta)$ when restricted to the critical set $C_\Theta := \{d_\vartheta\Theta = 0\}$. The image of ι_Θ is in fact a conic Lagrangian submanifold $\Lambda_\Theta \subset T^*(X \times Y)$ and the map ι_Θ is a local diffeomorphism from C_Θ onto Λ_Θ. In this case, we say Θ *parametrizes* Λ_Θ. The canonical relation or wavefront relation of A is defined by

$$WF'(A) = \{(x,\xi),(y,\eta)\colon (x,y,\xi,-\eta) \in WF(K_A)\} \subset T^*X \times T^*Y,$$

and describes how the operator A propagates singularities of distributions on which it acts. If ω_X and ω_Y denote the natural symplectic forms on T^*X and T^*Y respectively, one can more invariantly consider FIOs associated to general conic Lagrangian submanifolds $\Lambda \subset T^*X \times T^*Y$ (canonical relations), with respect to the symplectic form $\omega_X - \omega_Y$. The notion of a principal symbol for Fourier integral operators is more subtle than that for pseudodifferential operators. The principal symbol of A is a half density on Λ_Θ given in terms of the parametrization ι_Θ:

$$e = \iota_{\Theta *}(a_0 |dC_\Theta|^{1/2}), \tag{56}$$

where a_0 is the leading order term in the asymptotic expansion for a and $|dC_\Theta|^{1/2}$ is the half density associated with the Gelfand–Leray form on the level set $\{d_\vartheta \Theta = 0\}$. Here, we have ignored Maslov factors coming from the Keller–Maslov line bundle over Λ_Θ. These are nonzero factors $e^{i\sigma\pi/4}$ (σ is known as the Maslov index) which appear in front of the principal symbol as a result of the multiplicity of phase functions parametrizing the canonical relation Λ_Θ, possibly in different coordinate systems. While these factors allow the principal symbol to be defined in a more geometrically invariant way, we defer computation of the Maslov indices until Section 6.3. For a more thorough reference on the global theory of Lagrangian distributions, see [7, 9, 21, 23]. The order of a Fourier integral operator is defined in such a way that when two Fourier integral operators' canonical relations intersect transversally, then the composition is again a Fourier integral operator and order of the composition is the sum of the orders:

$$\text{order}(A) = m = \mu + \frac{1}{2}N - \frac{1}{4}(n_X + n_Y). \tag{57}$$

Recall that here, n_X and n_Y are the dimensions of X and Y respectively. In this case, we write $A \in I^m(X \times Y, \Lambda)$. This convention on orders also generalizes that of pseudodifferential operators, where $X = Y$ and $m = \mu$ coincides with the order of the corresponding symbol class. Sufficient conditions which guarantee that the composition exists are clean or transversal intersection of the two operators' canonical relations. In general, composition of Fourier integral operators and the associated symbol calculus is somewhat complicated, but we will not directly use the composition formula in what follows.

5.2. Chazarain's parametrix

The parametrix developed by Chazarain in [4] provides a microlocal description of the wave kernels near periodic transversal reflecting rays. The parametrices for

$$E(t) = \cos t \sqrt{-\Delta}, \quad S(t) = \frac{\sin t \sqrt{-\Delta}}{\sqrt{-\Delta}}$$

are constructed in the ambient Euclidean space $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$. We only consider $S(t)$, as the formula for $E(t)$ is easily obtained from that of $S(t)$ by differentiating in t. We write $E(t, x, y)$ and $S(t, x, y)$ for the Schwartz kernels of $E(t)$ and $S(t)$ respectively. Following the work in [4], we can find a Lagrangian distribution

$$\widetilde{S}(t, x, y) = \sum_{j=-\infty}^{\infty} S_j(t, x, y), \quad S_j \in I^{-5/4}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \Gamma_\pm^j), \quad (58)$$

which approximates $S(t)$ microlocally away from the tangential rays modulo a smooth kernel. We will describe the canonical relations $\Gamma_\pm^j$ momentarily and in particular, show that the sum in (58) is locally finite. We first explain what is meant by approximating $S(t)$ "microlocally away from the tangential rays." In general, two distributions $f, g \in \mathcal{D}'(\mathbb{R}^n)$ are said to *agree microlocally* near a closed cone $\Lambda_1 \subset T^*\mathbb{R}^n$ if $WF(u - v) \cap \Lambda_1 = \emptyset$. Similarly, using the language from Section 5.1, two operators $A, B : C^\infty(Y) \to C^\infty(X)$ are said to *agree microlocally* near a closed cone $\Lambda_2 \subset T^*X \times T^*Y$ if $WF'(A - B) \cap \Lambda_2 = \emptyset$. This second notion is what we will use to say that our parametrix *approximates $S(t)$ microlocally near the canonical relations $\Gamma_\pm^j$*.

To describe the canonical relations precisely, we first introduce some notation following the presentations in [4, 19]. As the Euclidean wave operator $\square_{\mathbb{R}^2}$ factors into $(\partial_t - \sqrt{\Delta})(\partial_t + \sqrt{-\Delta})$, there are two Hamiltonians corresponding to the symbol $\pm|\eta|$ of $\pm\sqrt{-\Delta}$. Let $H_\pm(y, \eta) = \pm|\eta|$ and $g^{\pm t}$ be the Hamiltonian flow, i.e., the flow map associated to the system

$$\begin{cases} \partial_t y = \frac{\partial H_\pm}{\partial \eta}, \\ \partial_t \eta = -\frac{\partial H_\pm}{\partial y}, \end{cases}$$

which is in fact just the reparametrized geodesic flow on $\mathbb{R}^2$. For $(y, \eta) \in T^*\Omega$ (or $T^*_{\partial\Omega}\mathbb{R}^2$ such that η is transversal to the boundary and inward pointing), recall that in Section 3 we defined

$$t_\pm^1(y, \eta) = \inf_{t>0}\{t : \pi_1 g^{\pm t}(y, \eta) \in \partial\Omega\},$$

$$t_\pm^{-1}(y, \eta) = \sup_{t<0}\{t : \pi_1 g^{\pm t}(y, \eta) \in \partial\Omega\},$$

where π_1 is projection onto the spatial variable. We have $t_\mp^1(y, \eta) = -t_\pm^{-1}(y, \eta)$. We then set

$$\lambda_\pm^1(y, \eta) = g^{\pm t_\pm^1}(y, \eta),$$

$$\lambda_\pm^{-1}(y, \eta) = g^{\pm t_\pm^{-1}}(y, \eta).$$

Also define $\widehat{\lambda_{\pm}^1(y,\eta)}$ to be the reflection of $\lambda_{\pm}^1(y,\eta)$ through the cotangent line at the boundary. In other words, $\widehat{\lambda_{\pm}^1(y,\eta)}$ and $\lambda_{\pm}^1(y,\eta)$ have the same cotangential components but opposite conormal components so that $\widehat{\lambda_{\pm}^1(y,\eta)}$ is inward pointing. The point $\widehat{\lambda_{\pm}^{-1}(y,\eta)}$ is defined analogously. We can then inductively define $t_{\pm}^j(y,\eta)$ and $t_{\pm}^{-j}(y,\eta)$ by the formulas

$$t_{\pm}^j = \inf_{t>0}\{t : \pi_1 g^{\pm t}(\widehat{\lambda_{\pm}^{j-1}(y,\eta)}) \in \partial\Omega\},$$

$$t_{\pm}^{-j} = \sup_{t<0}\{t : \pi_1 g^{\pm t}(\widehat{\lambda_{\pm}^{-(j-1)}(y,\eta)}) \in \partial\Omega\}.$$

The total travel time after j reflections is defined by

$$T_{\pm}^j(y,\eta) = \begin{cases} \sum_{k=1}^{j} t_{\pm}^k(y,\eta), & j > 0, \\ 0, & j = 0, \\ \sum_{k=j}^{-1} t_{\pm}^k(y,\eta), & j < 0. \end{cases}$$

To study how the fundamental solution of $\square_\Omega$ behaves at $\partial\Omega$ when we impose boundary conditions, we propagate the initial data by the free wave propagator on $\mathbb{R}^2$, restrict it to the boundary, reflect, and then propagate again. If such a construction is continued for $j \in \mathbb{Z}$ reflections at the boundary, it is shown in [4] that the FIOs S_j must have canonical relations

$$\Gamma_{\pm}^j = \begin{cases} (t, \tau, g^{\pm t}(y,\eta), y, \eta) : \tau = \pm|\eta|, & j = 0, \\ (t, \tau, g^{\pm(t-T_{\pm}^j(y,\eta))}(\widehat{\lambda_{\pm}^j(y,\eta)}), y, \eta)) : \tau = \pm|\eta|, & j \in \mathbb{Z}\backslash\{0\}. \end{cases}$$

Since $\widetilde{S}(t)$ is a microlocal parametrix, the canonical relation of the true solution operator $S(t)$ is also

$$\Gamma = \bigcup_{j\in\mathbb{Z},\pm}\Gamma_{\pm}^j.$$

Here, $j > 0$ and $j < 0$ correspond to reflections on the inside and outside of the boundary. For $t > 0, \pm\tau > 0$, the canonical relations corresponding to $j > 0$ project onto the interior of Ω when t is small (inside bounces). If $t > 0, \pm\tau > 0$ and $j < 0$, the canonical relations project onto the exterior of Ω, corresponding to outside bounces. In fact, there are four modes of propagation associated to the canonical relations $\Gamma_{\pm}^j$, corresponding to $\pm\tau \geq 0$ (forwards and backwards time) and $\pm j \geq 0$ (inside and outside bounces). See [19] for an explicit local model in the case of one reflection, where Ω is replaced by a half plane. We see that a point $(t, \tau, x, \xi, y, \eta)$ belongs to the canonical relation $\Gamma_{\pm}^j \subset T^*(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)$ if the broken geodesic of j reflections emanating from (y, η) passes through (x, ξ) in time t. For a more thorough discussion of Chazarain's parametrix, see [4, 16].

To be precise, Chazarain actually showed that there exists FIOs S_j such that the sum in (58) is a parametrix for the wave propagator $S(t)$ with canonical relation

$$\Gamma = \bigcup_{j \in \mathbb{Z}, \pm} \Gamma_\pm^j.$$

However, the canonical relations were never parametrized by explicit phase functions and the principal symbols of the operators S_j were not computed in [4]. For the remainder of this section, we concern ourselves with the task of explicitly computing them in terms of geometric and dynamical data associated to the billiard map.

5.3. Oscillatory integral representation

In this section, we cook up an oscillatory integral such that microlocally near the canonical relations $\Gamma_\pm^j$,

$$S_j(t, x, y) = \int_{-\infty}^{\infty} e^{i \Theta_j(t, \tau, x, y)} a_j(\tau, x, y) \, d\tau,$$

where S_j is the jth term in Chazarain's parametrix corresponding to a wave with j reflections and $a_j \in S_{\mathrm{cl}}^\mu$ is a classical symbol of order μ. We will only compute the principal symbol and use L.O.T. to denote lower order terms in the sense of Lagrangian distributions in what follows. Due to the presence of different Maslov factors on each branch of $\Gamma_\pm^j$ corresponding to $\pm\tau > 0$ (cf. Sections 5.1 and 6.3), it is actually more convenient to find operators

$$S_{j,\pm}(t, x, y) = \int_0^{\infty} e^{i \Theta_{j,\pm}(t, \tau, x, y)} a_{j,\pm}(\tau, x, y) \, d\tau, \tag{59}$$

so that $S_j = S_{j,+} + S_{j,-}$ and the phase functions associated to $S_{j,\pm}$ parameterize Γ_+^j and Γ_-^j individually.

We first make precise the notion of microlocalized FIOs. We would like to microlocalize $S(t)$ near periodic orbits of rotation number $1/j$. Oftentimes, it is required that such orbits be nondegenerate, in the sense that 1 is not an eigenvalue of the linearized Poincaré map. However, this assumption is not needed for our trace formulas, which work both for simple nondegenerate orbits as well as degenerate orbits coming in one parameter families as in the case of an ellipse (cf. Theorem 1.7 and Corollary 1.8). We recall that Ω is said to *satisfy the noncoincidence condition* (5) if there exist no periodic orbits of rotation number m/n, $m \geq 2$ having length in a sufficiently small neighborhood of $|\partial\Omega|$. In this case, for j sufficiently large, periodic orbits of rotation number $1/j$ come in isolated families. This follows from the results in [36],

which shows in particular that for j sufficiently large, no two orbits of distinct rotation numbers $1/j$ and $1/k$ can have the same length.

Recall also that for j sufficiently large, there exists a unique geodesic loop of rotation number $1/j$ at each boundary point q, whose length we denote by $\Psi_j(q,q)$ (the j-loop function). It can be shown that periodic billiard orbits arise as critical points of the j-loop function (see [20,55]). As in the statement of Theorem 1.3, denote $t_j = \inf_{q \in \partial\Omega} \Psi_j(q,q)$ and $T_j = \sup_{q \in \partial\Omega} \Psi_j(q,q)$. If Ω satisfies the noncoincidence condition 5, we can find a smooth cutoff function $\chi_1(t)$ which is identically equal to 1 on an open neighborhood of $[t_j, T_j]$ and vanishes in a neighborhood of all other $L \in \mathrm{Lsp}(\Omega)$. As noted in Section 5.2, each propagator S_j has canonical relations $\Gamma_{\pm}^j$. Denote by χ_2 a smooth cutoff function which is identically equal to 1 on $\Gamma^j = \bigcup_{\pm} \Gamma_{\pm}^j$, vanishing near the gliding rays $T^*\partial\Omega$ and conic in the fiber variables τ, ξ and η. Quantizing χ_2 gives a pseudodifferential operator which microlocalizes near the support of χ_2. For a reference, see [22, Chapter 18]. We call such an operator a *microlocal cutoff on $\Gamma_{\pm}^j$*. The composition $\chi_1(t)\chi_2(t, x, y, D_t, D_x, D_y)S(t)$ is then smoothing away from the geodesic loops of rotation number $1/j$ and the trace of this composition is equal to the wave trace modulo C^∞ in an open neighborhood of $[t_j, T_j]$.

We now use Theorem 3.1 to find suitable phase functions $\Theta_{j,\pm}$ which parametrize $\Gamma_{\pm}^j$. Define phase functions Θ_j^k by the formula

$$\Theta_{j,\pm}^k(t, \tau, x, y) = \pm\tau(t - \Psi_j^k(x, y)),$$

where Ψ_j^k are given in Definition 3.5. We then have,

Lemma 5.1. *The phase functions $\Theta_{j,\pm}^k(t, \tau, x, y)$ are smooth in an open neighborhood of the diagonal of the boundary and locally parametrize the canonical relations $\Gamma_{\pm}^j$. In particular, the fibers of both Γ_+^j and Γ_-^j lying over this neighborhood are unions of 8 connected components, which we denote by $\Gamma_{\pm}^{j,k}$ corresponding to $1 \leq k \leq 8$ as in Definition (3.5).*

Proof. For any $x, y \in \Omega$ let

$$\begin{cases} L_{x,y} : \partial\Omega^j \to \mathbb{R}_+, \\ L_{x,y}(q_1, q_2, \ldots q_j) = |x - q_1| + \{\sum_{m=2}^{j} |q_m - q_{m-1}|\} + |q_j - y|, \end{cases} \tag{60}$$

denote the length functional. We first show that billiard trajectories from x to y are in one to one correspondence with critical points of (60) with respect to $q \in \partial\Omega^j$. Let $g \in C^\infty(\mathbb{R}^2)$ be a defining function for $\partial\Omega$ and consider q as a variable in $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2j}$ rather than $\partial\Omega^j$. If q is a critical point of (60), then, as in the method of Lagrange multipliers, by setting $x = q_0$ and $y = q_{j+1}$, we find that for

$1 \leq m \leq j$, there exists $\lambda_m \in \mathbb{R}$ such that

$$\frac{\partial L_{x,y}}{\partial q_m} = \frac{q_m - q_{m-1}}{|q_m - q_{m-1}|} + \frac{q_m - q_{m+1}}{|q_m - q_{m+1}|} = \lambda_m \nabla_{q_m} g.$$

Since $\nabla_{q_m} g \perp \partial\Omega$, this implies that the two unit vectors in the formula for $\partial_{q_m} L_{x,y}$ have opposite tangential components, which is precisely the condition giving elastic collision at the boundary (angle of incidence equals angle of reflection). Similarly, if this condition is satisfied, then q is a critical point for (60).

We now consider the functions Ψ_j^k in Definition 3.5. We have

$$\Psi_j^k(x, y) = |x - q_1^k| + \left\{ \sum_{m=2}^{j} |q_m^k - q_{m-1}^k| \right\} + |q_j^k - y|, \tag{61}$$

where $q_m^k(x, y)$ is the m-th impact point on the boundary for the billiard trajectory corresponding to Ψ_j^k. As opposed to the q_m in the length functional (60), q_m^k will in general have a nontrivial dependence on x and y. Differentiating (61) in x, we obtain

$$\frac{\partial \Psi_j^k}{\partial x_i} = \frac{x - q_1^k}{|x - q_1^k|} \cdot \frac{\partial}{\partial x_i}(x - q_1^k) + \left\{ \sum_{m=2}^{j} \frac{q_m^k - q_{m-1}^k}{|q_m^k - q_{m-1}^k|} \cdot \frac{\partial}{\partial x_i}(q_m^k - q_{m-1}^k) \right\}$$
$$+ \frac{q_j^k - y}{|q_j^k - y|} \cdot \frac{\partial}{\partial x_i}(q_j^k - y). \tag{62}$$

Since for each $x, y \in \Omega$, the path defined by (x, q^k, y) corresponds to a billiard trajectory, we see that all of the terms except the first telescope in (62). Hence,

$$d_x \Psi_j^k = \frac{x - q_1^k}{|x - q_1^k|}. \tag{63}$$

Similarly, differentiating (61) in y, we obtain

$$d_y \Psi_j^k = \frac{y - q_j^k}{|y - q_j^k|}. \tag{64}$$

Geometrically, these gradients are the incident and (reflected) outgoing unit directions of the billiard trajectories described in Theorem 3.1.

We now consider the maps

$$\iota_{\Theta_{j,\pm}^k} : (t, \tau, x, y) \mapsto (t, \pm\tau, x, d_x \Theta_{j,\pm}^k, y, -d_y \Theta_{j,\pm}^k)$$
$$= (t, \pm\tau, x, \mp\tau d_x \Psi_j^k, y, \pm\tau d_y \Psi_j^k) \tag{65}$$

on the critical set

$$C_{\Theta_{j,\pm}^k} = \{t - \Psi_j^k = 0\}.$$

Inserting formulas (63) and (64) into (65) and comparing with the canonical graphs

$$\Gamma_\pm^j = \begin{cases} (t, \tau, g^{\pm t}(y, \eta), y, \eta) \colon \tau = \pm|\eta|, & j = 0, \\ (t, \tau, g^{\pm(t - T_\pm^j(y, \eta))}\widehat{\lambda_j(y, \eta)}, y, \eta) \colon \tau = \pm|\eta|, & j \in \mathbb{Z}\backslash\{0\} \end{cases}$$

from Section 5.2, we see that $\iota_{\Theta_{j,\pm}^k} \colon C_{\Theta_{j,\pm}^k} \to \Gamma_\pm^j$ is a local diffeomorphism (this is why we chose orbits connecting y to x rather than x to y in Definition 3.5). ∎

As our parametrices for the propagators described in [4] are in fact modified by microlocal cutoffs supported away from the tangential rays in $S^*\partial\Omega$, we may assume $\Gamma_\pm^{j,k}$ are smooth nonintersecting Lagrangian submanifolds over the interior. It would be interesting to study mapping properties of the operators S_j^k as the canonical relations degenerate near the glancing set in future work. We now want to derive an explicit oscillatory integral representation (59) for $S(t)$ in a specific coordinate system adapted to Ω. We first need to better understand the forwards and backwards symbols on Γ.

Proposition 5.2. *Let* $e_\pm$ *denote the principal symbol of* $S(t)$ *on* $\Gamma = \bigcup_{j\in\mathbb{Z},\pm} \Gamma_\pm^j$. *Modulo Maslov factors, we then have*

$$e_\pm = \frac{(-1)^j}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2} \quad \text{(Dirichlet)},$$

$$e_\pm = \frac{1}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2} \quad \text{(Neumann/Robin)},$$

where $|dt \wedge dy \wedge d\eta|^{1/2}$ *is the canonical half density.*

Proof. For a proof in the Dirichlet/Neumann case, see [19]. For the Robin boundary conditions, see [55]. ∎

5.4. Proof of Theorem 1.1

As Proposition 5.2 gives $e_\pm = \frac{1}{2\tau i}|dt \wedge dy \wedge d\eta|^{1/2}$ and we now know that the phase functions $\Theta_{j,\pm}^k(t, \tau, x, y) = \pm\tau(t - \Psi_j^k(x, y))$ parametrize connected components of $\Gamma_\pm^j$ lying over an open neighborhood of the diagonal of the boundary, we can compute the principal term in the asymptotic expansion of $a_{j,\pm}$ appearing in the expression (59). Since there are 16 phases for each j, corresponding to $1 \le k \le 8$ and $\pm\tau > 0$, (59) should actually be a sum of 16 oscillatory integrals:

$$S_j(t, x, y) = \sum_\pm \sum_{k=1}^8 \int_0^\infty e^{i\Theta_{j,\pm}^k(t,\tau,x,y)} a_{j,k,\pm}(\tau, x, y)\,d\tau. \tag{66}$$

Recalling formula (56) for the principal symbol of a Fourier integral operator, we must compute the Gelfand–Leray form on the critical set $d_\tau \Theta^k_{j,\pm} = 0$. The Leray measure coming from the Gelfand–Leray form is coordinate invariant and it is ultimately more convenient to first introduce boundary normal coordinates, which we now describe.

Fix a point $p \in \partial\Omega$. For each $q \in \partial\Omega$ near p, we denote by $\gamma_q(\mu)$ the unit speed geodesic with initial condition conormal to $\partial\Omega$ at q and inward pointing. Now, denote by φ the boundary coordinate which parametrizes $\partial\Omega$ with respect to arc-length, such that p is given by $\varphi = 0$. The coordinate φ can be extended smoothly inside Ω so that it is constant along $\gamma_q(\mu)$ for every fixed q near p. For $\varepsilon > 0$ sufficiently small, (μ, φ) is then a smooth coordinate system in an ε neighborhood of $p \in \bar{\Omega}$. In these coordinates, the Euclidean metric is locally given by the warped product

$$g_{\text{Euclid.}} = d\mu^2 + f(\mu, \varphi)d\varphi^2,$$

where f is a locally defined function which is smooth up to the boundary.

Lemma 5.3. *For planar domains, the metric is given by*

$$d\mu^2 + (1 - \kappa\mu)^2 d\varphi^2,$$

where κ is the curvature of the boundary at $(0, \varphi)$.

Proof. Let $|\partial\Omega| = \ell$ and $x(\varphi) = (x_1(\varphi), x_2(\varphi))$ be a parametrization of $\partial\Omega$ with respect to arc-length. The exponential map $\exp: [0, \varepsilon) \times \mathbb{R}/\ell\mathbb{Z} \to \Omega$ is given in boundary normal coordinates by $(\mu, \varphi) \mapsto x + \mu J \dot{x}$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the $\pi/2$ rotation matrix and $\dot{x} = \partial x / \partial\varphi$. We calculate that in these coordinates,

$$\exp^* g_{\text{Euclid.}} = D \exp^T g_{\text{Euclid.}} D \exp = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where

$$A = (\dot{x}_1)^2 + (\dot{x}_1)^2 = 1,$$
$$B = C = -\dot{x}_2\dot{x}_1 + \mu\dot{x}_2\ddot{x}_2 + \dot{x}_1\dot{x}_2 + \mu\dot{x}_1\ddot{x}_1 = \mu(\dot{x}_1\ddot{x}_1 + \dot{x}_2\ddot{x}_2) = 0,$$
$$D = 1 - 2\mu\dot{x}^T J \ddot{x} + \mu^2|\ddot{x}|^2 = (1 - \mu\kappa)^2,$$

and $\kappa = |\ddot{x}|$. ∎

We maintain the notation $f(\mu, \varphi) = (1 - \mu\kappa)^2$ throughout the rest of the paper. This coordinate system is convenient near the boundary because conformal multiples the vector fields $\partial/\partial\mu$ and $\partial/\partial\varphi$ extend the orthogonal and tangential gradients

respectively in a tubular neighborhood of the boundary. As the canonical relations $\Gamma_\pm^j$ involve both x and y variables, we change x variables to (μ, φ) and y variables to (ν, ϑ) according to the procedure described above. In [55], elliptical polar coordinates were instead used of boundary normal coordinates to achieve a similar decomposition into normal and tangential vector fields.

5.5. Proof of Theorem 1.1

Without loss of generality, we use boundary normal coordinates instead of Euclidean coordinates in the domain of Ψ_j^k from here on. We now compute the Gelfand–Leray form in boundary normal coordinates.

Lemma 5.4. *The canonical relation of each operator in* (66) *is parametrized in boundary normal coordinates by*

$$\Gamma_\pm^{j,k} = \{(t, \tau, \mu, \varphi, \xi, \nu, \vartheta, \eta) = (\Psi_j^k, \tau, \mu, \varphi, \mp \tau d_{\mu,\varphi} \Psi_j^k, \nu, \vartheta, \pm \tau d_{\nu,\vartheta} \Psi_j^k)\}.$$

The Gelfand–Leray form on $C_{\Theta_{j,\pm}^k}$ is given by

$$dC_{\Theta_{j,\pm}^k} = \mp f(\mu, \varphi) f(\nu, \vartheta) d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\vartheta.$$

Proof. The first claim follows directly from Lemma 5.1 and the observation that

$$d_\tau \Theta_{j,\pm}^k = \pm(t - \Psi_j^k) = 0$$

on the critical set $C_{\Theta_{j,\pm}^k}$. From this, it is clear that $(\tau, \mu, \varphi, \nu, \vartheta)$ form a smooth coordinate system on $C_{\Theta_{j,\pm}^k}$. The Gelfand–Leray form is uniquely defined on $C_{\Theta_{j,\pm}^k}$ by the condition

$$d(d_\tau \Theta_{j,\pm}^k) \wedge dC_{\Theta_{j,\pm}^k} = f(\mu, \varphi) f(\nu, \vartheta) dt \wedge d\tau \wedge d(\mu, \varphi) \wedge d(\nu, \vartheta), \quad (67)$$

where the right-hand side of (67) coincides with the Euclidean volume form on $\mathbb{R}_t \times \mathbb{R}_\tau \times \Omega \times \Omega$. Hence,

$$dC_{\Theta_{j,\pm}^k} = \mp f(\mu, \varphi) f(\nu, \vartheta) d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\vartheta,$$

as claimed. ∎

We now change variables and use Lemma 5.4 to compute the principal symbol of (66). Dropping the j, k indices on Ψ_j^k in place of differentiation, we have

$$dt = \Psi_\mu d\mu + \Psi_\varphi d\varphi + \Psi_\nu d\nu + \Psi_\vartheta d\vartheta,$$
$$dy = f(\nu, \vartheta) d\nu \wedge d\vartheta,$$
$$d\eta_1 = \Psi_\nu d\tau + \tau(\Psi_{\mu\nu} d\mu + \Psi_{\nu\varphi} d\varphi + \Psi_{\nu\nu} d\nu + \Psi_{\nu\vartheta} d\vartheta),$$
$$d\eta_2 = \Psi_\vartheta d\tau + \tau(\Psi_{\vartheta\mu} d\mu + \Psi_{\vartheta\varphi} d\varphi + \Psi_{\vartheta\nu} d\nu + \Psi_{\vartheta\vartheta} d\vartheta)$$

on the canonical relation $\Gamma_{\pm}^{j,k}$. Wedging all these terms together, we see that

$$
dt \wedge dy \wedge d\eta = \tau (f(\nu, \vartheta)(\Psi_{\mu}\Psi_{\vartheta}\Psi_{\nu\varphi} + \Psi_{\varphi}\Psi_{\nu}\Psi_{\vartheta\mu}
$$
$$
- \Psi_{\mu}\Psi_{\nu}\Psi_{\vartheta\varphi} - \Psi_{\varphi}\Psi_{\vartheta}\Psi_{\nu\mu}))d\tau \wedge d\mu \wedge d\varphi \wedge d\nu \wedge d\vartheta. \quad (68)
$$

Definition 5.5. Denote by $A_j^k(\mu, \varphi, \nu, \vartheta)$ the functions

$$
\frac{1}{f(\mu, \varphi)}(\Psi_{\mu}\Psi_{\vartheta}\Psi_{\nu\varphi} + \Psi_{\varphi}\Psi_{\nu}\Psi_{\vartheta\mu} - \Psi_{\mu}\Psi_{\nu}\Psi_{\vartheta\varphi} - \Psi_{\varphi}\Psi_{\vartheta}\Psi_{\nu\mu}),
$$

where each $\Psi = \Psi_j^k$ depends implicitly on j, k, μ, φ, ν and ϑ.

On each of the canonical relations $\Gamma_{\pm}^{j,k}$ (cf. Lemma 5.1), equation (68) implies that

$$
e_{\pm} = (-1)^j \frac{|dt \wedge dy \wedge d\eta|^{1/2}}{2\tau i}
$$
$$
= \iota_{\Theta_{j,\pm}^k *}\left((-1)^j \frac{1}{2\tau^{1/2}i}|A_j^k(\mu, \varphi, \nu, \vartheta)dC_{\Theta_{j,\pm}^k}|^{1/2}\right) \quad (69)
$$

for Dirichlet boundary conditions and

$$
e_{\pm} = \frac{|dt \wedge dy \wedge d\eta|^{1/2}}{2\tau i} = \iota_{\Theta_{j,\pm}^k *}\left(\frac{1}{2\tau^{1/2}i}|A_j^k(\mu, \varphi, \nu, \vartheta)dC_{\Theta_{j,\pm}^k}|^{1/2}\right) \quad (70)
$$

for Neumann boundary conditions. As a result, we conclude the following description of the operators S_j.

Theorem 5.6. *Microlocally near $\Gamma_{\pm}^j$, the following oscillatory integral is a parametrix for $S_j(t, x, y)$ with Dirichlet boundary conditions in an open neighborhood of $\Delta \partial \Omega \subset \Omega \times \Omega$:*

$$
S_j(t) = (-1)^j \sum_{\pm} \sum_{k=1}^8 \int_0^{\infty} e^{\pm i\tau(t-\Psi_j^k)} \frac{1}{2i\tau^{1/2}}|A_j^k|^{1/2}d\tau + \text{L.O.T.}
$$

Here, L.O.T. *denotes lower order Lagrangian distributions, using the convention* (57).

Definition 5.7. We define the operators $S_{j,\pm}^k(t)$ appearing in Theorem 5.6 by

$$
S_{j,\pm}^k(t) = (-1)^j \sum_{\pm} \sum_{k=1}^8 \int_0^{\infty} e^{\pm i\tau(t-\Psi_j^k)} \frac{1}{2i\tau^{1/2}}|A_j^k|^{1/2}d\tau.
$$

Together with Theorem 3.1 and the choice of j_0, U_j in Remark 4.10, we can immediately derive Theorem 1.1 from Theorem 5.6. To see this, note that

$$
E(t) = \frac{d}{dt}S(t).
$$

On a symbolic level, this implies that the symbol of $E(t)$ is $i\tau e_{\pm}$, with $e_{\pm}$ given by Proposition 5.2. This explains the order of $b_{j,k,\pm} \in S^{1/2}(U_j \times \mathbb{R}^1)$ appearing in Theorem 1.1. Each $(\pm)$ branch of the propagators $S^k_{j,\pm}$ should be multiplied by Maslov factors $e^{\pm i\pi\sigma_j/4}$. We will compute $\sigma_j = 1$ in Section 6.3, which completes the proof of Theorem 1.1. In the next section, we will use this explicit oscillatory integral representation to compute the wave trace.

6. Computing the wave trace

In this section, we use the parametrix in Theorem 1.1, or equivalently Theorem 5.6, to compute an integral formula for the wave trace. Formally, the wave trace is the Fourier transform of the spectral measure $\sum_j \delta(\lambda - \lambda_j)$, where $\{\lambda_j^2\}$ are the Dirichlet (resp. Neumann or Robin) eigenvalues of $-\Delta$ on Ω. This is a distribution of the form

$$\sum_j e^{it\lambda_j}, \tag{71}$$

which can be seen to be weakly convergent by Weyl's law on the asymptotic distribution of Laplace eigenvalues (see [27,56]). The connection between this distribution and the wave equation lies in the fact that (71) is actually the trace of $e^{it\sqrt{-\Delta}}$, the propagator associated to the half wave operator $(\partial_t - i\sqrt{-\Delta})$. Since such a unitary operator is not trace class, we mean that for any Schwartz function φ, the regularized operators

$$\int_{-\infty}^{\infty} \varphi(t) e^{it\sqrt{-\Delta}} dt$$

are of trace class and have trace

$$\sum_j \int_{-\infty}^{\infty} \varphi(t) e^{it\lambda_j} dt.$$

The same holds for the even and odd wave operators

$$E(t) = \cos t\sqrt{-\Delta}, \quad S(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}, \tag{72}$$

and we consider these as they appear more naturally in Chazarain's parametrix (cf. Section 5.2) and $E(t)$ solves the initial boundary value problem (1).

Recall that $\chi_1(t)$ is a cutoff function near the lengths $[t_j, T_j]$ of geodesic loops and Ω is assumed to satisfy the noncoincidence condition (5). Modulo Maslov factors

and a smooth error term, we have

$$\chi_1(t) \operatorname{Tr} E = \int_\Omega E_j(t, x, x) dx, \tag{73}$$

where $E_j = \frac{\partial}{\partial t} S_j$ and S_j is the j bounce wave appearing in Chazarain's parametrix (58).

6.1. Reduction to boundary

As our parametrix in Theorem 5.6 is only valid near the boundary, we want to write the wave trace as an integral over the boundary. There are several ways to do this, one of which involves using Hadamard-type variational formulas from [19, 55] to integrate the radially differentiated wave trace, which via an integration by parts puts the integral (73) on the boundary. Here we use a different technique, suggested by the referee. Let us first establish some notation. Fixing an arbitrary point $O \in \Omega$ to be the origin, we denote by $X(q)$ the position vector of a point $q \in \partial\Omega$ relative to O and $N(q)$ the outward unit normal at q. Let $\nabla^\perp$ and ∇^T denote the unit normal and tangential gradients with respect to $\partial\Omega$ and recall the notation $D_t = -i \partial_t$.

Lemma 6.1. *For each Dirichlet eigenfunction u_j of eigenvalue λ_j^2 in 6, we have*

$$\lambda_j^2 = \frac{1}{2} \int_{\partial\Omega} \langle X, N \rangle |\nabla^\perp u_j(q)|^2 dq, \tag{74}$$

and hence

$$D_t^2 \operatorname{Tr} S(t) = \frac{1}{2} \int_{\partial\Omega} \langle X, N \rangle \nabla_1^\perp \nabla_2^\perp S(t, q, q) dq. \tag{75}$$

Proof. For the first formula, we use the commutator identity $[\Delta, r\partial_r] = 2\Delta$, where $r = |X|$ and ∂_r is the radial vector field, to see that

$$2\lambda_j^2 = -2 \int_\Omega \Delta u_j \overline{u_j} dx = \int_\Omega [-\Delta, r\partial_r] u_j \overline{u_j} dx = \int_{\partial\Omega} r \frac{\partial u_j}{\partial r} \frac{\partial \overline{u_j}}{\partial N} dq.$$

Here we have integrated by parts and used Dirichlet boundary conditions. Decomposing $r\partial_r$ into the normal and tangential vector fields N, ∂_q and again using the boundary condition, we obtain

$$r \frac{\partial u}{\partial r} = \langle X, N \rangle \nabla^\perp u_j.$$

The trace formula follows by writing

$$D_t^2 \operatorname{Tr} S(t) = \sum_j \lambda_j^2 \frac{\sin \lambda_j t}{\lambda_j}$$

$$= \frac{1}{2} \int_{\partial \Omega} \langle X, N \rangle \sum_j \frac{\sin t \lambda_j}{\lambda_j} |\nabla^\perp u_j|^2 dq$$

$$= \frac{1}{2} \int_{\partial \Omega} \langle X, N \rangle \nabla_1^\perp \nabla_2^\perp S(t, q, q) dq. \qquad \blacksquare$$

As we are localizing the wave trace near lengths of geodesic loops with j reflections, it turns out we only need to consider a select few of the operators from Theorem 5.6 in the trace formula. From here on, we only consider Dirichlet boundary conditions. Similar formulas exist in the Neumann and Robin cases. We also use the notation $D_t^{-1} \in \Psi_{\text{ell}}^{-1}(\mathbb{R})$ to denote an elliptic parametrix (Fourier multiplier τ^{-1}) for D_t.

Lemma 6.2. *Modulo Maslov factors and lower order distributions, the even Dirichlet wave trace localized near $[t_j, T_j]$ is given by*

$$\operatorname{Tr} E(t) = \sum_\pm \int_{\partial \Omega} \frac{\langle X, N \rangle}{2} i D_t^{-1} \nabla_1^\perp \nabla_2^\perp (S_{j-1,\pm}^1 + S_{j-1,\pm}^5 + S_{j,\pm}^2$$
$$+ S_{j,\pm}^3 + S_{j,\pm}^6 + S_{j,\pm}^7$$
$$+ S_{j+1,\pm}^4 + S_{j+1,\pm}^8)(t, q, q) \, dq.$$

For completeness, we repeat the proof derived in [55] as it contains substantial geometric insight.

Proof. First note that $E(t) = \partial_t S(t)$, so that

$$\operatorname{Tr} E(t) = i D_t^{-1} D_t^2 \operatorname{Tr} S(t) = \int_{\partial \Omega} \frac{\langle X, N \rangle}{2} i D_t^{-1} \nabla_1^\perp \nabla_2^\perp S(t, q, q) dq.$$

For the localized wave trace, we only need to consider orbits which contribute to the singularities in $[t_j, T_j]$. Recall that for positive time, Theorem 3.1 gives 8 orbits connecting x to y in j reflections and approximately one rotation. These orbits coalesce into geodesic loops as $(x, y) \to \Delta \partial \Omega$. However, as the orbits coalesce within various configurations, not all of the limiting orbits will have j reflections. As Ω satisfies the noncoincidence condition (5), only the limiting geodesic loops of rotation number $1/j$ will contribute to the wave trace near $[t_j, T_j]$. See Figure 1 for visualizing the geometric arguments which follow. We will say that a sequence of billiard orbits γ_n *converges geometrically* to another orbit γ_0 if all impact points on the boundary converge to those of γ_0 in order. Note that the limiting orbit may have a different number

of reflections if certain impact points coalesce in the limit. To demonstrate geometric convergence in this sense, it suffices to show convergence of any two consecutive impact points to two distinct points in the limiting orbit. All other points of reflection on the boundary are then smoothly and implicitly determined by the limiting link formed between these two.

As $(x, y) \to \Delta \partial \Omega$, the two orbits of j reflections in TT configuration ($k = 1, 5$) converge geometrically to a loop of $j + 1$ reflections. The additional vertex appears at the boundary point where x and y coalesce. Similarly, the two NN orbits of j reflections ($k = 4, 8$) can be seen to converge to loops of $j - 1$ reflections. In this case, the first and last points of reflection at the boundary converge to a single impact point. The four orbits of j reflections in TN ($k = 2, 6$) and NT ($k = 3, 7$) configurations preserve exactly j reflections in the limit. Hence, when $x, y \in \text{int } \Omega$ converge to $\Delta \partial \Omega$, only 4 of the 8 orbits contribute to geodesic loops of j reflections. However, in the same limit, two additional TT orbits of $j - 1$ reflections converge to a loop of $(j - 1) + 1 = j$ reflections. Similarly, two NN orbits of $j + 1$ reflections converge to a loop of $(j + 1) - 1 = j$ reflections. Any other orbit from x to y with strictly less than $j - 1$ or strictly more than $j + 1$ reflections at the boundary cannot converge to a loop of j reflections. As we have localized the wave trace near the isolated set of lengths $[t_j, T_j]$ and Ω satisfies the noncoincidence condition (5), only the $4+2+2=8$ orbits which converge geometrically to a loop of exactly j reflections will contribute to singularities here. All additional orbits contribute smooth errors to the wave trace in a small neighborhood of $[t_j, T_j]$.

It should also be clarified that although the parametrices $S_j(t, x, y)$ are constructed in the interior, we can in fact extend them continuously to the diagonal of the boundary and this extension coincides with that of the true propagator $S(t, x, x)$ ($x \in \partial \Omega$) modulo lower order terms. Both propagators agree up to lower order Lagrangian distributions in the interior, microlocally near the canonical relations $\Gamma_{\pm}^j$. The explicit oscillatory integral representation for each $S_j(t, x, y)$ in fact shows that they extend continuously up to the boundary and its diagonal, since the functions $\Psi_j^k(x, y)$ do. The true wave kernel $S(t, x, y)$ also extends continuously up to the boundary as a family of distributions. To see this, note that

$$S(t, x, y) = \sum_j \frac{\sin(t\lambda_j)}{\lambda_j} \psi_j(x)\overline{\psi_j(y)},$$

where $(\psi_j)_{j=1}^{\infty}$ is an L^2 orthonormal basis of Dirichlet or Neumann eigenfunctions corresponding to eigenvalues $(\lambda_j^2)_{j=1}^{\infty}$. Multiplying by a test function $\varphi(t)$ and integ-

rating by parts $4k$ times, we see that

$$\int\limits_{-\infty}^{\infty} S(t,x,y)\varphi(t)dt = \int\limits_{-\infty}^{\infty} \sum_j \frac{\sin(t\lambda_j)}{\lambda_j^{4k+1}} \psi_j(x)\overline{\psi_j(y)}\partial_t^{4k}\varphi(t)dt. \qquad (76)$$

Combining Weyl's law on the asymptotic growth of λ_j (see [27, 56]) and L^∞ bounds for eigenfunctions on manifolds with boundary (see [12]), we see that the integrand in (76) can be made absolutely convergent for k sufficiently large. An application of the dominated convergence theorem then shows that (76) is actually smooth in x, y, so $S(t,x,y)$ has a smooth extension to the diagonal of the boundary as a distribution in t. In particular, both distributions agree up to lower order terms microlocally near the fibers of $\Gamma_\pm^j$ lying over diagonal of the boundary. ∎

Definition 6.3. As shown in the proof of Lemma 6.2 above, for each j, there exist 8 limiting trajectories which converge geometrically to geodesic loops of exactly j reflections. We denote the set of these trajectories by $\mathcal{G}_j(x,y)$ and say that $\gamma_{m,k} \in \mathcal{G}_j$ if $\gamma_{m,k}$ makes $m = j-1, j$ or $j+1$ reflections at the boundary and corresponds to the length functional Ψ_m^k. Since there is a unique geodesic loop of rotation number $1/j$ at each boundary point, the length functionals $\Psi_j^2, \Psi_j^3, \Psi_j^6, \Psi_j^7, \Psi_{j+1}^4, \Psi_{j+1}^8, \Psi_{j-1}^1$ and Ψ_{j-1}^5 corresponding to orbits in $\mathcal{G}_j$ coincide for $x = y \in \partial\Omega$ on the diagonal. We denote their common value by $\Psi_j(x,x)$, which is the j-loop function appearing in Definition 3.6.

6.2. Boundary calculations

As we obtained a rather explicit formula for $S_j(t,x,y)$ in Theorem 5.6, it now remains to differentiate the kernels $S_m^k(t,x,y)$ and substitute them into Lemma 6.2. Using our oscillatory integral representation for $S_m^k(t,x,y)$ in Theorem 5.6, we find that microlocally near $\Gamma_\pm^{m,k}$ and $t \in [t_j, T_j]$, modulo lower order terms in an open neighborhood of the diagonal of the boundary, we have

$$iD_t^{-1}\nabla_1^\perp \nabla_2^\perp S_{m,\pm}^k$$
$$= (-1)^{m+1} \sum_\pm \int\limits_0^\infty e^{\pm i\tau(t-\Psi_m^k)} \frac{\pm(\nabla_1^\perp \Psi_m^k)(\nabla_2^\perp \Psi_m^k)\tau^{3/2}}{2\tau} |A_m^k|^{1/2} d\tau \qquad (77)$$

for Dirichlet boundary conditions. We have only written the terms coming from $\nabla_{1,2}^\perp$ falling on the exponential in equation (77), as all other terms do not contribute positive powers of τ and can be regarded as lower order in the singularity expansion. The operators $\nabla_{1,2}^\perp$ in the integrand of (77) are conformal multiples of the vector fields $\frac{\partial}{\partial\mu}$ and $\frac{\partial}{\partial\nu}$ coming from boundary normal coordinates.

As Lemma 6.2 tells us that the wave trace is given by integrating the normally differentiated sine kernels over the diagonal of the boundary, we want to understand the restriction of (77) to the boundary. We already noted that the j-loop function is well defined for j sufficiently large. The differentiated kernels in equation (77) also have factors of A_j^k and $\nabla_{1,2}^\perp \Psi_j^k$ in the integrand. We now discuss how to extend these derivatives of Ψ_j^k to the diagonal of the boundary in a similar manner. In the proof of Lemma 5.1, the x and y gradients of the functions Ψ_j^k are shown to be

$$d_x \Psi_j^k = \frac{x - q_1^k}{|x - q_1^k|}, \quad d_y \Psi_j^k = \frac{y - q_j^k}{|y - q_j^k|}. \tag{78}$$

Geometrically, these are the incident and reflected outgoing unit directions of the corresponding billiard trajectories at x and y. The expression $\nabla_1^\perp \Psi_j^k$ in (77) can easily be seen to be $\pm \sin \omega_{j,1}^k$, where $\omega_{j,1}^k$ is the angle made between the initial link of the billiard trajectory and the oriented tangent line to the level set of the distance function on which x lies (as in the folliation used in the proof of Theorem 3.1). We use the positively oriented tangent line for $1 \le k \le 4$ and the negatively oriented tangent line for $5 \le k \le 8$. The sign $\pm$ depends on the TT, TN, NT, or NN configuration of the corresponding orbit. Similarly, $\nabla_2^\perp \Psi_j^k = \pm \sin \omega_{j,2}^k$, where $\omega_{j,2}^k$ is the angle made between the final link of the billiard trajectory and the positively ($1 \le k \le 4$) or negatively ($5 \le k \le 8$) oriented tangent line to the distance curve on which y lies. As $x, y \to \Delta \partial \Omega$, the absolute value of the angles associated to trajectories in the $\mathcal{G}_j$ converge to the initial and final angles of reflection of the unique limiting geodesic loop. We are careful to point out that only the absolute values of the angles converge, since for example, the final angles of reflection at y associated to orbits in TN and NN configurations actually converge to the *negative* of the final angle in the limiting trajectory. All limiting loops are automatically in TT configuration.

Lemma 6.4. *On the diagonal of the boundary, the factors A_{j-1}^1, $A_{j-1}^5 A_j^2$, A_j^3, A_j^6, A_j^7, A_{j+1}^4, and A_{j+1}^8 corresponding to orbits in $\mathcal{G}_j$ coincide up to a sign. We denote their common (absolute) value by $|A_j|$, which at a point $(0, \varphi, 0, \varphi)$ in boundary normal coordinates satisfies*

$$|A_j(0, \varphi, 0, \varphi)| = \frac{1}{\sin \omega_{j,2}} \left| \frac{\partial \omega_{j,1}}{\partial \vartheta} \right|.$$

Here, $\omega_{j,1}$ and $\omega_{j,2}$ are the initial and final angles of incidence respectively made by the unique counterclockwise parametrized geodesic loop of rotation number $1/j$ based at $(0, \varphi, 0, \varphi) \in \Delta \partial \Omega$.

Proof. Let (m, k) denote an admissible pair of indices corresponding to an orbit $\gamma_{m,k} \in \mathcal{G}_j$. Recall the notation in the proof of Lemma 5.1 (cf. [55]), where we described

a billiard trajectory by the point $(x, q, y) \in \Omega \times \partial\Omega^m \times \Omega$. Let us first assume $1 \le k \le 4$. If $x \in \Omega$, we denote the positive angle between $q_1 - x$ and the positively oriented tangent line to the leaf of the folliation by distance curves on which x lies by $\omega_{j,1}^k$ (cf. Section 4). Similarly, if $y \in \Omega$, let us also denote the positive angle between $y - q_j$ and the positively oriented tangent line to the distance curve on which y lies by $\omega_{j,2}^k$. Note that $\omega_{j,1}^k$ and $\omega_{j,2}^k$ depend on x, y, j, and k. Using the warped product structure of boundary normal coordinates $(\mu, \varphi, \nu, \vartheta)$, we have

$$\nabla_x^T = (f)^{-1/2} \frac{\partial}{\partial \varphi}, \quad \nabla_y^T = (f)^{-1/2} \frac{\partial}{\partial \vartheta}, \tag{79a}$$

$$\nabla_x^\perp = \frac{\partial}{\partial \mu}, \qquad \nabla_y^\perp = \frac{\partial}{\partial \nu}. \tag{79b}$$

Equation (78), which was derived from the proof of Lemma 5.1 in [55], then shows that

$$\nabla_x^\perp \Psi_j^k(x, y) = \begin{cases} -\sin \omega_{j,1}^k, & k = 1, 2, \\ \sin \omega_{j,1}^k, & k = 3, 4, \end{cases} \tag{80a}$$

$$\nabla_y^\perp \Psi_j^k(x, y) = \begin{cases} -\sin \omega_{j,2}^k, & k = 1, 3, \\ \sin \omega_{j,2}^k, & k = 2, 4, \end{cases} \tag{80b}$$

and

$$\nabla_x^T \Psi_j^k(x, y) = -\cos \omega_{j,1}^k \quad (1 \le k \le 4), \tag{81a}$$

$$\nabla_y^T \Psi_j^k(x, y) = \cos \omega_{j,2}^k \quad (1 \le k \le 4). \tag{81b}$$

Combining (79), (80), and (81), we obtain

$$\nabla_{(\mu,\varphi)} \Psi_j^k(\mu, \varphi, \nu, \vartheta) = \begin{cases} (-\sin \omega_{j,1}^k, -f^{1/2}(\mu, \varphi) \cos \omega_{j,1}^k), & k = 1, \\ (-\sin \omega_{j,1}^k, -f^{1/2}(\mu, \varphi) \cos \omega_{j,1}^k), & k = 2, \\ (\sin \omega_{j,1}^k, -f^{1/2}(\mu, \varphi) \cos \omega_{j,1}^k), & k = 3, \\ (\sin \omega_{j,1}^k, -f^{1/2}(\mu, \varphi) \cos \omega_{j,1}^k), & k = 4, \end{cases} \tag{82a}$$

$$\nabla_{(\nu,\vartheta)} \Psi_j^k(\mu, \varphi, \nu, \vartheta) = \begin{cases} (-\sin \omega_{j,2}^k, f^{1/2}(\nu, \vartheta) \cos \omega_{j,2}^k), & k = 1, \\ (\sin \omega_{j,2}^k, f^{1/2}(\nu, \vartheta) \cos \omega_{j,2}^k), & k = 2, \\ (-\sin \omega_{j,2}^k, f^{1/2}(\nu, \vartheta) \cos \omega_{j,2}^k), & k = 3, \\ (\sin \omega_{j,2}^k, f^{1/2}(\nu, \vartheta) \cos \omega_{j,2}^k), & k = 4. \end{cases} \tag{82b}$$

Using (82) to calculate the $(\mu, \varphi, \nu, \vartheta)$ Hessian of Ψ_j^k, we find

$$\frac{\partial^2 \Psi_j^k}{\partial \vartheta \partial \varphi}(\mu, \varphi, \nu, \vartheta) = f^{1/2}(\mu, \varphi) \sin \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta} \quad (1 \le k \le 4), \tag{83a}$$

$$\frac{\partial^2 \Psi_j^k}{\partial \nu \partial \varphi}(\mu, \varphi, \nu, \vartheta) = f^{1/2}(\mu, \varphi) \sin \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \nu} \quad (1 \le k \le 4), \tag{83b}$$

$$\frac{\partial^2 \Psi_j^k}{\partial \vartheta \partial \mu}(\mu, \varphi, \nu, \vartheta) = \begin{cases} -\cos \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 1, 2, \\ \cos \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 3, 4, \end{cases} \tag{83c}$$

$$\frac{\partial^2 \Psi_j^k}{\partial \nu \partial \mu}(\mu, \varphi, \nu, \vartheta) = \begin{cases} -\cos \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \nu}, & k = 1, 2, \\ \cos \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \nu}, & k = 3, 4. \end{cases} \tag{83d}$$

Inserting (83) into the expression (68) for A_j^k in all possible configurations $(1 \le k \le 4)$, we find that, on the boundary,

$$A_m^k(0, \varphi, 0, \vartheta) = \begin{cases} -\cos \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \nu} - \sin \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 1, \\ -\cos \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \nu} + \sin \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 2, \\ +\cos \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \nu} + \sin \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 3, \\ +\cos \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \nu} - \sin \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 4. \end{cases} \tag{84}$$

Before evaluating this expression on the diagonal of the boundary, we differentiate $\omega_{j,1}^k$ in the direction $L = (y - q)/|y - q|$ of the last link to see that

$$\begin{aligned} 0 &= \nabla_L \omega_{j,1}^k \\ &= \begin{cases} \cos \omega_{j,2}^k \nabla_y^T \omega_{j,1}^k - \sin \omega_{j,2}^k \nabla_y^\perp \omega_{j,1}^k, & k = 1, 3, \\ \cos \omega_{j,2}^k \nabla_y^T \omega_{j,1}^k + \sin \omega_{j,2}^k \nabla_y^\perp \omega_{j,1}^k, & k = 2, 4 \end{cases} \\ &= \begin{cases} \frac{1}{f^{1/2}} \cos \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta} - \sin \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \nu}, & k = 1, 3, \\ \frac{1}{f^{1/2}} \cos \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta} + \sin \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \nu}, & k = 2, 4. \end{cases} \end{aligned}$$

This implies that

$$\sqrt{f(\nu, \vartheta)} \frac{\partial \omega_{j,1}^k}{\partial \nu} = \begin{cases} +\cot \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 1, 3, \\ -\cot \omega_{j,2}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}, & k = 2, 4. \end{cases} \tag{85}$$

Inserting the formula (85) into (84) and evaluating on $\partial\Omega$, we find that

$$|A_m^k(0,\varphi,0,\vartheta)| = \left(\frac{\cos^2 \omega_{j,2}^k}{\sin \omega_{j,2}^k} + \frac{\sin^2 \omega_{j,2}^k}{\sin \omega_{j,2}^k}\right)\left|\frac{\partial \omega_{j,1}^k}{\partial \vartheta}\right| = \frac{1}{\sin \omega_{j,2}^k}\left|\frac{\partial \omega_{j,1}^k}{\partial \vartheta}\right|.$$

Note that for $1 \leq k \leq 4$ and x, y on the boundary, both $\omega_{j,1}^k$ and $\omega_{j,2}^k$ are independent of k. They coincide with the initial and final angles of the unique orbit connecting x to y in j reflections and approximately one counterclockwise rotation. As the orbits corresponding to $5 \leq k \leq 8$ can be viewed as counterclockwise orbits in the reflected domain, we instead defined $\omega_{j,1}^k$ and $\omega_{j,2}^k$ to be the angles made between the initial and final links and the *negatively* oriented tangent lines to the distance curves on which x and y respectively lie. With this convention, we see by symmetry that the roles of $\omega_{j,1}^k$ and $\omega_{j,2}^k$ are interchanged:

$$\nabla_{(\mu,\varphi)}\Psi_j^k(\mu,\varphi,\nu,\vartheta) = \begin{cases} (-\sin \omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos \omega_{j,1}^k), & k = 5, \\ (-\sin \omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos \omega_{j,1}^k), & k = 6, \\ (\sin \omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos \omega_{j,1}^k), & k = 7, \\ (\sin \omega_{j,1}^k, f^{1/2}(\mu,\varphi)\cos \omega_{j,1}^k), & k = 8, \end{cases} \tag{86a}$$

$$\nabla_{(\nu,\vartheta)}\Psi_j^k(\mu,\varphi,\nu,\vartheta) = \begin{cases} (-\sin \omega_{j,2}^k, -f^{1/2}(\nu,\vartheta)\cos \omega_{j,2}^k), & k = 5, \\ (\sin \omega_{j,2}^k, -f^{1/2}(\nu,\vartheta)\cos \omega_{j,2}^k), & k = 6, \\ (-\sin \omega_{j,2}^k, -f^{1/2}(\nu,\vartheta)\cos \omega_{j,2}^k), & k = 7, \\ (\sin \omega_{j,2}^k, -f^{1/2}(\nu,\vartheta)\cos \omega_{j,2}^k), & k = 8. \end{cases} \tag{86b}$$

Replacing (82) by (86), parallel computations to those above then show that for $5 \leq k \leq 8$,

$$|A_m^k(0,\varphi,0,\varphi)| = \frac{1}{\sin \omega_{j,2}^k}\left|\frac{\partial \omega_{j,1}}{\partial \vartheta}\right|.$$

Note that for $5 \leq k \leq 8$, $\omega_{j,1}^{k-4} - \omega_{j,2}^k \to 0$ as $x, y \to \partial\Omega$. Similarly, $\omega_{j,2}^{k-4} - \omega_{j,1}^k \to 0$ as $x, y \to \partial\Omega$. Hence,

$$|A_m^k(0,\varphi,0,\varphi)| = \frac{1}{\sin \omega_{j,1}}\left|\frac{\partial \omega_{j,2}}{\partial \varphi}\right|, \tag{87}$$

for $5 \leq k \leq 8$. Note that the right-hand side of (87) involves angles at boundary points and does not depend on k. Also observe that

$$\frac{\partial^2 \Psi_j^k}{\partial \varphi \partial \vartheta} = \pm \sin \omega_{j,2}^k \frac{\partial \omega_{j,2}^k}{\partial \varphi}, \quad \frac{\partial^2 \Psi_j^k}{\partial \vartheta \partial \varphi} = \pm \sin \omega_{j,1}^k \frac{\partial \omega_{j,1}^k}{\partial \vartheta}. \tag{88}$$

Setting the two equations in (88) equal implies that

$$\frac{1}{\sin \omega_{j,1}} \left| \frac{\partial \omega_{j,2}}{\partial \varphi} \right| = \frac{1}{\sin \omega_{j,2}} \left| \frac{\partial \omega_{j,2}}{\partial \varphi} \right|,$$

which combined with (87), concludes the proof of the lemma. ∎

Remark 6.5. In [55], complete integrability of the ellipse actually implies that $\omega_{j,1} = \omega_{j,2}$ for elliptical billiards; i.e., every geodesic loop is in fact a periodic orbit. In that case, the angular derivative $\partial \omega / \partial \vartheta$ appearing in Lemma 6.4 was explicitly calculated, using action angle coordinates and Jacobi elliptic functions. In general, it may be difficult to compute $\partial \omega_1 / \partial \vartheta$ explicitly despite its relatively simple geometric interpretation.

6.3. Maslov factors

To explicitly compute the Maslov factors $\sigma_j^{\pm}$ on $\Gamma_{\pm}^j$, we use an argument due to Keller ([31]), following the presentation in [11, Appendix B]. The free wave propagator $U(t) = e^{-it\sqrt{-\Delta}}$ on $\mathbb{R}^2$ has an integral kernel given by

$$U(t, x, y) = (2\pi)^{-2} \int_{\mathbb{R}^2_\xi} e^{i((x-y,\xi) - |\xi|t)} d\xi |dx \wedge dy|^{1/2}, \tag{89}$$

considered as a distributional half density (cf. Section 5.1). Let $e_1 = (y - x)/|y - x|$ and $e_2 = Je_1$, where J is a $\pi/2$ counterclockwise rotation. With respect to this basis, we may write $\xi = \tau e_1 + \rho e_2$ and hence

$$U(t, x, y) = (2\pi)^{-2} \int \int e^{i(|x-y|\tau - t\sqrt{\tau^2 + \rho^2})} d\tau d\rho |dx \wedge dy|^{1/2}.$$

We see that stationary points of the phase occur precisely when $|x - y| = t$, $\tau > 0$ and $\rho = 0$. Applying the method of stationary phase in the variable ρ, we find that

$$U(t, x, y) = (2\pi)^{-3/2} \int e^{i(|x-y|-t)\tau} e^{-i\pi/4} \left(\frac{\tau}{t}\right)^{1/2} d\tau |dx \wedge dy|^{1/2} \tag{90}$$

to leading order. The reduction in number of phase variables in formula 90 expresses $U(t, x, y)$ as a classical conormal distribution with principal symbol

$$e^{-i\pi/4} \left(\frac{\tau}{t}\right)^{1/2} |dt \wedge d\tau \wedge ds \wedge dy|^{1/2} \in S^{1/2}(N^*\{|x - y| = t\}) \otimes \Omega_{1/2}, \tag{91}$$

where $\Omega_{1/2}$ is the space of positive half densities on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, s is an arc-length coordinate on the hypersurface $|x - y| = t$ (a distance sphere) and τ is the symplectically dual coordinate to $|x - y| - t$.

Hence, the Maslov indices on $\Gamma_{\pm}^0 = N^*\{|x - y| = t : \pm\tau > 0\}$ are given by $\sigma_0^{\pm} = \pm 1$. It is shown in [16, Section 5] that after a reflection at the boundary, the Maslov factors remain unchanged. Hence, $\sigma_j^{\pm} = \pm 1$ for all $j \in \mathbb{Z}$. Both of the propagators corresponding to Γ_+^j and Γ_-^j contribute to the wave trace singularity near $[t_j, T_j]$, owing to the two modes of propagation:

$$S(t) = \frac{e^{it\sqrt{-\Delta}} - e^{-it\sqrt{-\Delta}}}{2i\sqrt{-\Delta}}. \tag{92}$$

Hence, we multiply each $\pm$ branch of the Lagrangian distributions $S_{j,\pm}$ by $e^{\pm i\pi/4}$, which explains the real parts taken in Theorem 1.3, Theorem 1.7 and Corollary 1.8. Noting that the principal symbol of $\sqrt{-\Delta}$ is $|\xi|$ and $\tau = \pm|\eta| = \pm|\xi|$ on the canonical relations $\Gamma_{\pm}^0$, equations (91), (92) and the convention 57 show that the principal symbol of $S(t)$ on Γ^0 has order

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{4}(3 + 2) - 1 = -\frac{5}{4}.$$

Applying an elliptic parametrix for $(-\Delta_{\mathbb{R}^2})^{-1/2} \in \Psi^{-1}$ to the free sine wave $S(t)$ shows that $E(t) \in I^{-1/4}(\mathbb{R}^2 \times \mathbb{R}^2; N^*(\{|x - y| = t\})')$, corroborating the order appearing in Theorem 1.1.

6.4. Proofs of Theorem 1.3, Theorem 1.7 and Corollary 1.8

Theorem 1.3 readily follows from the formula 77, Lemma 6.4 and the computations in Section 6.3. We now show how Theorem 1.7 and Corollary 1.8 follow directly from Theorem 1.3. Recall from Section 3 that a caustic is a smooth curve $\mathcal{C}$ in Ω such that every link tangent to $\mathcal{C}$ remains tangent to $\mathcal{C}$ after a reflection at the boundary. It is well known (see [30, 53]) that if one periodic orbit is tangent to a caustic $\mathcal{C}$, then every orbit tangent to $\mathcal{C}$ is in fact periodic, with the same period (number of bounces) and winding number. In this case, we say $\mathcal{C}$ is a *rational caustic* and use $\omega(\mathcal{C})$ to denote the (rational) rotation number of any orbit γ tangent to $\mathcal{C}$. Hence, rational caustics correspond to highly degenerate periodic orbits in the sense that they are not isolated and 1 is an eigenvalue of the Poincaré map. Let L_j be the length of a periodic orbit which is tangent to a caustic $\mathcal{C}_j$, making j reflections at the boundary and one rotation. As all orbits tangent to $\mathcal{C}$ are periodic orbits, each $q \in \partial\Omega$ is a critical point of the j-loop function, i.e., $\partial_q \Psi_j(q, q) = 0$. In this case, the length function $\Psi_j(q, q)$ appearing in the phase of $\mathrm{Tr}\cos t\sqrt{-\Delta}$ (as in Theorem 1.3) is the constant function L_j, i.e., every j reflection loop is in fact a periodic orbit of length L_j. Assuming the noncoincidence condition (5) on Ω, all periodic orbits of length L_j arise in this way. Tangency to $\mathcal{C}$ also implies that the angles ω_1 and ω_2 in

the amplitude are equal. Hence, the wave trace in Theorem 1.3 is given by

$$\mathrm{Re}\left\{(-1)^j e^{-i\pi/4} 4 \int_{\partial\Omega} \int_0^\infty e^{i\xi(t-L_j)} |\xi|^{1/2} \sin^{3/2} \omega_1 \left|\frac{\partial\omega_1}{\partial\vartheta}\right|^{1/2} X \cdot N d\xi dq\right\}. \quad (93)$$

As the dq and $d\xi$ integrals can be separated, we obtain the Fourier transform of the homogeneous distribution

$$\chi_+^{3/2}(\xi) = \begin{cases} \xi^{3/2}, & \xi \geq 0, \\ 0, & \xi < 0. \end{cases}$$

One can define χ_+^a similarly to an L_{loc}^1 function for $\mathrm{Re}\,a > -1$ and these distributions fact be analytically continued to a larger region of $a \in \mathbb{C}$. It is shown in [24, Chapter 7] that the Fourier transform of χ_+^a (with dual variable t) is given by $e^{-i\pi(a+1)/2}(t - i0)^{-a-1}$. The proof of Theorem 1.7 is concluded by evaluating the Fourier transform in equation (93) at the point $t - L_j$.

In the special case of an ellipse, the billiard flow is known to be completely integrable and each confocal ellipse is in fact a caustic. Moreover, Poncelet's theorem (see [46, 47]) implies that all periodic orbits tangent to a given confocal ellipse have the same length. Hence, all periodic orbits in the ellipse correspond to rational caustics. It is also known (see [15]) that ellipses satisfy the noncoincidence condition 5 and hence, Theorem 1.7 applies. Calculations from the author's previous work using action-angle coordinates and Jacobi elliptic function theory allow for the explicit computation of $\partial\omega/\partial\vartheta$ appearing in the integrand of (93) (see [55, Section 5.5]). As the boundary of an ellipse

$$\left\{(x, y): \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\right\}$$

is easily parametrized by $(a\cos\varphi, b\sin\varphi)$ for $\varphi \in [0, 2\pi)$, the quantity $X(q) \cdot N(q)dq$ can be explicitly calculated. Combining these observations with Theorem 1.7, we obtain the formula appearing in Corollary 1.8.

7. An auxiliary check on the order of S_j^k

Given a discrepancy in the works [36, 48], we provide an additional check on the order of a_0^j in Theorem 1.3. Assume Ω satisfies the noncoincidence condition (5) and let $\rho \in \mathcal{S}(\mathbb{R})$ be a test function such that $\mathrm{Supp}\,\rho \subset [t_j - \varepsilon, T_j + \varepsilon]$, where ε is sufficiently small to ensure $\mathrm{Supp}\,\rho \cap \mathrm{LSP}(\Omega) = [t_j, T_j]$. Let $L_j \in [t_j, T_j]$ denote the length of a periodic orbit of rotation number $1/j$. We will compute the quantity

$$\mathcal{F}(\rho(t)\,\mathrm{Tr}\cos t\sqrt{-\Delta}) = \int e^{-i\lambda t} \rho(t)\,\mathrm{Tr}\cos t\sqrt{-\Delta}dt \quad (94)$$

in two different ways. Recalling our parametrix for $\cos t \sqrt{-\Delta}$, we see that (94) is given by

$$\sum_{\pm} \int_{\mathbb{R}_t} \int_0^\infty \int_{\partial\Omega} e^{\pm i\tau(t-\Psi_j(q,q))-\lambda t} \rho(t)\tau^m a^j(q) dq d\tau dt, \tag{95}$$

where m is the purported order of $a^j(q)$. Changing variables by $\xi = \tau/\lambda$, we see that (95) becomes

$$\sum_{\pm} \int_{\mathbb{R}_t} \int_0^\infty \int_{\partial\Omega} e^{\pm i\lambda\xi(t-\Psi_j(q,q))-\lambda t} \rho(t)\lambda^{m+1}\xi^m a_0(q) dq d\xi dt. \tag{96}$$

To understand the λ asymptotics of this oscillatory integral, we apply the method of stationary phase. First assume L_j is a simple nondegenerate length. On the critical set, $d_{t,\xi,q}(\xi(t - \Psi_j(q,q)) - t) = 0$, which implies

$$\begin{cases} t = \Psi_j(q,q), \\ \xi = 1, \\ d_q\psi_j(q,q) = 0. \end{cases}$$

Hence, (94) is given by

$$(2\pi)^{3/2}\lambda^{m-1/2} \sum_{q:d_q\Psi_j(q,q)=0} \frac{e^{i\pi/4\,\mathrm{sgn\,Hess}(\Psi_j(q,q))}\rho(\Psi_j(q,q))a_0(q)}{|\partial_q^2(\Psi_j(q,q))|^{1/2}} + o(\lambda^{m-1/2}). \tag{97}$$

Recall that periodic orbits of rotation number $1/j$ arise from critical points of the j-loop function, i.e., $d_q\Psi_j(q,q) = 0$ implies that the geodesic loop of j reflections based at q is actually a periodic orbit. Since L_j was assumed to be simple, corresponding to a unique nondegenerate orbit, there are precisely j such boundary points q and the sum in (97) is actually finite. It is shown in [33, Theorem 3] that nondegeneracy of γ also implies $\partial_q^2(\Psi_j(q,q)) \neq 0$. Now, by the formulas in [16], we know that for a simple length L_j corresponding to a nondegenerate periodic orbit of j reflections, the leading asymptotic of the wave trace modulo Maslov factors is given by

$$\sum \cos t\lambda_j = \frac{L_j}{|\det(I - P_\gamma)|^{1/2}}(t - L_j + i0)^{-1} \mod L^1_{\mathrm{loc}}(\mathbb{R}), \tag{98}$$

where P_γ is the Poincare map associated to the unique periodic orbit γ of length L_j. Formulas in [24] tell us that the Fourier transform of the right-hand side of (98) is a constant multiple of the Heaviside function. Comparing degrees of homogeneity in (97) and (98) immediately implies that $m = 1/2$.

If there are infinitely many critical points of $\Psi_j(q, q)$ in the phase of (97), the analysis is more subtle. For example, Poncelet us theorem for elliptical billiards actually implies that periodic orbits of a fixed length and rotation number come in one parameter families. For j sufficiently large, every boundary point is the base point for a unique periodic orbit tangent to a single confocal ellipse, making j reflections and a single rotation. The lengths of these orbits are independent of the base point, which implies $d_q \Psi_j(q, q)$ vanishes identically. In this case, $\Psi_j(q, q) = L_j$ and applying stationary phase to (96) yields

$$2\pi\lambda^m e^{i\lambda L_j}\rho(L_j)\int\limits_{\partial\Omega} a_0(q)dq + o\left(\lambda^m\right).\tag{99}$$

While the expansion (98) is no longer valid for high length spectral multiplicity, the formulas in [19, 55] show that the *variation* of the wave trace near such a period in the length spectrum is a distribution of the form $d_j \operatorname{Re}(t - L_j + i0)^{-5/2}$ for some constant d_j. Formally, the wave trace has one higher degree of regularity than its variation, which suggests that the wave trace is of the form $c_j(t - L_j + i0)^{-3/2}$, in agreement with Theorem 1.7 and Corollary 1.8. Comparing this with the asymptotics in (99) and formulas in [24] for the Fourier transform of homogeneous distributions, we see again that $m = -1 - (-3/2) = 1/2$. The order of a_0^j was also confirmed in the recent paper [20].

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