J. Spectr. Theory 12 (2022), 939–992
DOI 10.4171/JST/432

On the spectrum of the periodic focusing Zakharov–Shabat operator

Gino Biondini, Jeffrey Oregero, and Alexander Tovbis

Abstract. The spectrum of the focusing Zakharov–Shabat operator on the circle is studied, and its explicit dependence on the presence of a semiclassical parameter is also considered. Several new results are obtained. In particular: (i) it is proved that the resolvent set is comprised of two connected components; (ii) new bounds on the location of the Floquet and Dirichlet spectra are obtained, some of which depend explicitly on the value of the semiclassical parameter; (iii) it is proved that the spectrum localizes to a "cross" in the spectral plane in the semiclassical limit. The results are illustrated by discussing several examples in which the spectrum is computed analytically or numerically.

1. Introduction

In this work we investigate the spectrum of the non-self-adjoint Zakharov–Shabat (ZS) scattering problem with a periodic potential. The ZS scattering problem is given by the first-order coupled system of ordinary differential equations (ODEs) [1, 22, 50, 70]:

$$\varepsilon \mathbf{v}' = (-\mathrm{i}\, z\sigma_3 + Q(x))\mathbf{v}, \tag{1.1}$$

where $\mathbf{v}(x; z, \varepsilon) = (v_1, v_2)^T$ (the superscript T denoting matrix transpose), $\sigma_3 = \operatorname{diag}(1, -1)$ is the third Pauli matrix, $Q(x)$ is the matrix-valued function

$$Q(x) = \begin{pmatrix} 0 & q(x) \\ -\overline{q(x)} & 0 \end{pmatrix}, \tag{1.2}$$

$z \in \mathbb{C}$ is the spectral parameter, prime denotes differentiation with respect to the independent variable (here x), overbar denotes complex conjugation, and $0 < \varepsilon \le 1$ is the semiclassical parameter. Unless stated otherwise, throughout this work, the "potential" $q\colon \mathbb{R} \to \mathbb{C}$ is a complex-valued function with minimal period L, i.e.,

$$q(x + L) = q(x), \quad \text{for all } x \in \mathbb{R}. \tag{1.3}$$

2020 Mathematics Subject Classification. Primary 34L05; Secondary 34L20, 37K10, 47B28.
Keywords. Dirac operator, semiclassical limit, nonlinear Schrödinger equation.

Unless stated otherwise, we assume $q \in L^\infty(\mathbb{R})$, the space of essentially bounded Lebesgue measurable functions with essential supremum norm. We say a function $\mathbf{v}(x; z, \varepsilon)$ is solution of equation (1.1) if it is locally absolutely continuous, i.e., if $\mathbf{v} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$, and the equality (1.1) holds almost everywhere (cf. A.1).

The main motivation for studying (1.1) derives from its role in the analysis of the focusing nonlinear Schrödinger (NLS) equation on the circle, which is given by

$$\mathrm{i}\,\varepsilon\partial_t\psi + \varepsilon^2\partial_x^2\psi + 2|\psi|^2\psi = 0, \tag{1.4}$$

where $\psi\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$ is the slowly-varying complex envelope of a quasi-monochromatic, weakly dispersive nonlinear wave packet, and the physical meaning of the variables depends on the context. (E.g., in nonlinear fiber optics, t represents propagation distance while x is a retarded time.) In general, the parameter ε quantifies the relative strength of dispersion compared to nonlinearity. (In the quantum-mechanical setting, ε is also proportional to Planck's constant $\hbar$.)

Specifically, it was shown by Zakharov and Shabat in 1972 that (1.1) makes up the first half of the Lax pair for the focusing NLS equation [70]. This observation is the key to solving the initial value problem for (1.4) by means of the inverse scattering method. The solution $\psi(x, t; \varepsilon)$ of (1.4) with initial data $\psi(x, 0; \varepsilon) := q(x)$ is constructed by computing suitable scattering data generated by the potential $q(x)$ in (1.1). Time evolution according to (1.4) corresponds to an isospectral deformation of the potential in (1.1), and the time evolution of the scattering data can be computed trivially in certain cases. This allows one to obtain the solution $\psi(x, t; \varepsilon)$ of (1.4) by solving an inverse scattering problem, i.e., by reconstructing the potential of the ZS scattering problem from the knowledge of the time-evolved scattering data. As a result, analysis of ZS systems such as (1.1) has become an active area of research (e.g., see [5–7, 16, 25, 29, 30, 35, 36, 40, 44, 47, 56, 62, 63]) and a natural starting point in the study of solutions to the focusing NLS equation.

Importantly, (1.1) and (1.4) depend on the semiclassical parameter ε. Letting $\varepsilon \downarrow 0$ in (1.4) is referred to as the "semiclassical limit" since it allows one to establish a connection between quantum and classical mechanics [45, 46]. Similar phenomena occur in physical applications when dispersive effects are weak compared to nonlinear ones. These situations, referred to as small dispersion limits, can produce a wide variety of physical effects such as supercontinuum generation, dispersive shocks and wave turbulence, to name a few (e.g., see [2, 15, 19, 52, 53, 59, 67–69] and references therein). Semiclassical, or small dispersion, limits are also of mathematical interest. Solutions to equations such as focusing NLS have rich structure which becomes more evident as the semiclassical parameter tends to zero. As a result, much effort has been devoted to the analysis of integrable nonlinear evolution equations in the semiclassical limit (e.g., see [4, 11–13, 20, 32, 34, 37–39, 41, 48, 49, 65, 66] and references therein).

Since (1.1) is a non-self-adjoint eigenvalue problem, its spectrum is in general quite complicated. Much of the research in this area has been devoted to studying (1.1) with zero-background potentials, or more precisely, $q \in L^1(\mathbb{R})$, i.e., Lebesgue integrable. In 1974, Satsuma and Yajima proved that if $q(x) = A \operatorname{sech} x$ with $A \in \mathbb{R}$, then (1.1) has purely imaginary point spectrum [56]. Moreover, when $A \in \mathbb{N}$ there are precisely $2A$ eigenvalues located at $z = \pm\mathrm{i}(n - 1/2)$, $n = 1, \ldots, A$. More recently, one of the authors and S. Venakides extended these results to a one-parameter family of potentials of the form $q(x; \varepsilon) = \operatorname{sech} x \exp(\mathrm{i}\, S(x)/\varepsilon)$, where $S'(x) = \kappa \tanh x$ with $\kappa \in \mathbb{R}$ [63]. More generally, Klaus and Shaw proved that all real, piecewise-smooth, single-lobe potentials with zero background have purely imaginary point spectrum [35, 36]. Here, the term single-lobe means $q(x)$ is nondecreasing when $x < 0$, and is nonincreasing when $x > 0$. Recently, one of the authors and X.-D. Luo generalized the Klaus–Shaw result to real, single-lobe potentials on non-zero background, that is, $q(x) \to q_0$ as $|x| \to \infty$ and $q(x) > q_0$ for all $x \in \mathbb{R}$ [5]. Considerable work has also been devoted to potentials with rapid phase variations, namely, $q(x; \varepsilon) = A(x) \exp(\mathrm{i}\, S(x)/\varepsilon)$, where $A(x)$ and $S(x)$ are both real. Careful numerical experiments by Bronski showed that, in the semiclassical limit, the point spectrum can accumulate on a "Y-shaped" set of curves in the spectral plane, and the number of eigenvalues scales like $O(1/\varepsilon)$ as $\varepsilon \downarrow 0$ [7]. Also, bounds on the point spectrum were derived by Deift, Venakides, and Zhou under the assumptions $A(x) \to 0$, and $S'(x) \to 0$ as $|x| \to \infty$ [7]. (For details of the proof, see the work by DiFranco and Miller [14].) Later, Miller put forth a formal WKB based asymptotic analysis [47]. Moreover, one of the authors and S. Venakides used the idea of the semiclassical limit of the scattering transform to derive spectral information of various potentials [64]. In particular, the Y-shaped spectral curve from [7] was confirmed there. The semiclassical limit of solutions to the focusing NLS equation generated by zero-background potentials was then studied in [4, 20, 32, 34, 41, 65]. Even so, there are still many important open questions.

Many studies have been devoted to analyzing the spectrum of (1.1) with potentials in $L^1(\mathbb{R})$. Many works have also been devoted to studying the spectrum of Hill operators with complex potentials (e.g. see [16, 26, 54, 55, 57, 58, 61] and references therein). However, less is known for the non-self-adjoint ZS system (1.1) with a periodic potential. Some localization results corresponding to (anti)periodic eigenvalues of (1.1) were obtained in [16, 40], and "gap estimates" in weighted Sobolev spaces were obtained in [16, 29, 30]. The spectrum was also rigorously studied in [27, 28, 44, 62], where, among other results, asymptotic statements as $z \to \infty$ were obtained. Moreover, for real-analytic periodic potentials, it was shown in [25] that the set of periodic eigenvalues was discrete and clustered on the real and imaginary axes of the spectral variable in the limit $\varepsilon \downarrow 0$.

The semiclassical limit of (1.1) with real periodic potentials was recently studied by two of the authors using a formal WKB approach, and numerical simulations [6]. This work is partly motivated by that study. The main results of the present work are several statements about the spectrum of (1.1), both in general and in the semiclassical limit. Specifically, the work is organized as follows. We begin in Section 2 by briefly recalling various preliminary notions concerning the spectral problem (1.1). In Section 3 we present various general properties of the spectrum, several of which are new to the best of our knowledge. For example, we show that the resolvent set of (1.1) with a periodic potential is comprised of two connected components, in contrast with the case of Hill's equation, whose resolvent set is path connected. In Section 4 we show that, under fairly general assumptions on the potential, the spectrum localizes to a subset of the real and imaginary axes of the spectral variable in the limit $\varepsilon \downarrow 0$, thus putting the asymptotic and numerical results of [6] into a rigorous mathematical setting. In Section 5 we discuss some further properties acquired by the spectrum when the potential is real and/or symmetric. The proofs of all results discussed in Sections 3, 4, and 5 are given in Section 6. In Section 7, specific examples are analyzed exactly and via careful numerical simulations, and we comment on the similarities between the periodic problem and the infinite line problem when $0 < \varepsilon \ll 1$. Various details about the numerical simulations and some of the calculations are relegated to the appendices.

2. Preliminaries

Equation (1.1) is equivalent to the eigenvalue problem

$$\mathcal{L}^{\varepsilon}\mathbf{v} = z\mathbf{v}, \tag{2.1}$$

where

$$\mathcal{L}^{\varepsilon} := \mathrm{i}\,\sigma_3(\varepsilon\partial_x - Q), \tag{2.2}$$

is a one-dimensional Dirac operator acting in $L^2(\mathbb{R},\mathbb{C}^2)$ with dense domain $H^1(\mathbb{R},\mathbb{C}^2)$, and Q is given by (1.2). Here H^1 is the Sobolev space of square integrable functions with square integrable first derivative. (For a brief discussion on notation and function spaces see A.1.)

Regarding the inverse spectral method for the focusing NLS equation on the circle, one is concerned with the Lax spectrum of the operator $\mathcal{L}^{\varepsilon}$, namely, the set of $z \in \mathbb{C}$ for which a non-trivial solution of (1.1) exists which is bounded for all $x \in \mathbb{R}$. Specifically, the *Lax spectrum* – or, simply, the *spectrum* – is defined as

$$\Sigma_{\mathrm{Lax}} := \{z \in \mathbb{C}\colon \text{there exists } \mathbf{v} \not\equiv 0 \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}) \text{ s.t. } \mathcal{L}^{\varepsilon}\mathbf{v} = z\mathbf{v} \text{ and } \sup_{x\in\mathbb{R}} \|\mathbf{v}(x;z,\varepsilon)\| < \infty\}. \tag{2.3}$$

It is well known that Σ_{Lax} is purely continuous, that is, essential without any eigenvalues and empty residual spectrum [9, 27, 31, 54, 55]. Likewise, consider the operator (2.2), now acting in $L^2([0,L],\mathbb{C}^2)$ with dense domain $H^1([0,L],\mathbb{C}^2)$. Then, for each $\nu \in \mathbb{R}$, the associated *Floquet spectrum* is defined as

$$\begin{aligned}\Sigma_\nu := \{z \in \mathbb{C}\colon &\text{there exists } \mathbf{v} \not\equiv 0 \in H^1([0,L],\mathbb{C}^2)\\ &\text{s.t. } \mathcal{L}^\varepsilon \mathbf{v} = z\mathbf{v} \text{ and } \mathbf{v}(L;z,\varepsilon) = \mathrm{e}^{\mathrm{i}\nu L}\,\mathbf{v}(0;z,\varepsilon)\}.\end{aligned} \tag{2.4}$$

Importantly, $\nu = 2n\pi/L$ corresponds to the periodic spectrum, and $\nu = (2n-1)\pi/L$ corresponds to the antiperiodic spectrum, where $n \in \mathbb{Z}$. Any $z \in \Sigma_\nu$ will be referred to as an eigenvalue of (2.1), and the corresponding $\mathbf{v}(x;z,\varepsilon)$ as a Floquet eigenfunction. Clearly, the eigenvalues depend on ε, i.e., $z = z(\varepsilon)$.

Basic properties of the Lax spectrum that follow from the theory of linear homogeneous ODEs with periodic coefficients are reviewed next to introduce some relevant concepts and to set the notation.

Theorem 2.1 (Floquet, [8, 23]). *Consider the system of linear homogeneous ODEs given by*

$$\mathbf{y}' = A(x)\mathbf{y}, \tag{2.5}$$

where $A \in L^1_{\mathrm{loc}}(\mathbb{R})$ is a $n \times n$ matrix-valued function such that $A(x+L) = A(x)$. Then, any fundamental matrix solution $Y(x)$ of (2.5) can be written in the Floquet normal form

$$Y(x) = \Psi(x)\mathrm{e}^{Rx}, \tag{2.6}$$

where $\Psi(x+L) = \Psi(x)$, Ψ is nonsingular, and R is a constant matrix.

Without loss of generality, one can take R to be in Jordan normal form. Since $\Psi(x)$ is locally absolutely continuous and periodic, the behavior of solutions as $x \to \pm\infty$ is determined by the eigenvalues, called Floquet exponents, of the matrix R. In particular, (i) if the Floquet exponent has non-zero real part then the solution grows exponentially as $x \to \infty$, or as $x \to -\infty$; (ii) if the Floquet exponent has zero real part, but R has non-trivial Jordan blocks then the solution is algebraically growing; (iii) otherwise the Floquet exponent is purely imaginary and the solution remains bounded for all $x \in \mathbb{R}$;

By Theorem 2.1, all Floquet eigenfunctions of the ZS system (1.1) have the form

$$\mathbf{v}(x;z,\varepsilon) = \mathrm{e}^{\mathrm{i}\nu x}\,\mathbf{w}(x;z,\varepsilon), \tag{2.7}$$

where $\mathbf{w}(x+L;z,\varepsilon) = \mathbf{w}(x;z,\varepsilon)$, and $\nu = \nu(\varepsilon) \in \mathbb{R}$ is the quasi-momentum. More generally, one has the so-called normal solutions, that is, solutions of (1.1) such that

$$\mathbf{v}(x+L;z,\varepsilon) = \sigma\mathbf{v}(x;z,\varepsilon), \tag{2.8}$$

and $\sigma = \sigma(\varepsilon)$ is the Floquet multiplier. Thus, a solution of (1.1) is bounded for all $x \in \mathbb{R}$ if and only if $|\sigma| = 1$ in which case one has $\sigma = e^{i\nu L}$. Moreover, the Floquet multipliers are the eigenvalues of a monodromy matrix, which is defined as

$$Y(x+L; z, \varepsilon) = Y(x; z, \varepsilon) M(z; \varepsilon), \tag{2.9}$$

where $Y(x; z, \varepsilon)$ is any fundamental matrix solution of (1.1). Let $\Phi := \Phi(x; z, \varepsilon)$ be the principal matrix solution of (1.1), that is, the solution of (1.1) normalized so that $\Phi(0; z, \varepsilon) \equiv \mathbb{I}$, where $\mathbb{I}$ is the 2×2 identity matrix. Since all monodromy matrices are similar, for the remainder of this work we fix

$$M(z; \varepsilon) = \Phi(L; z, \varepsilon). \tag{2.10}$$

Note also that the well-known symmetries of solutions of (1.1) imply

$$\overline{M(\bar{z}; \varepsilon)} = \sigma_2 M(z; \varepsilon) \sigma_2, \tag{2.11}$$

where σ_2 is the second Pauli matrix (cf. A.1). In turn, (2.11) implies $M(z; \varepsilon)$ can be written as

$$M(z; \varepsilon) = \begin{pmatrix} c(z; \varepsilon) & -\overline{s(\bar{z}; \varepsilon)} \\ s(z; \varepsilon) & \overline{c(\bar{z}; \varepsilon)} \end{pmatrix}, \tag{2.12}$$

where, by (2.10), $c(z; \varepsilon)$ and $s(z; \varepsilon)$ are the components at $x = L$ of the solution $\mathbf{v}(x; z, \varepsilon)$ of (1.1) equaling $(1, 0)^T$ at $x = 0$. Also, since (1.1) is traceless, it follows from Abel's formula that $\det M(z; \varepsilon) \equiv 1$. Hence, the eigenvalues of $M(z; \varepsilon)$ are given by roots of the quadratic equation $\sigma^2 - (2\Delta_\varepsilon)\sigma + 1 = 0$, where we introduced the Floquet discriminant

$$\Delta_\varepsilon := \Delta_\varepsilon(z) = \frac{1}{2} \operatorname{tr} M(z; \varepsilon), \tag{2.13}$$

and "tr" is the matrix trace. Thus, (1.1) has bounded solutions if and only if the following two conditions are simultaneously satisfied:

$$\operatorname{Im} \Delta_\varepsilon(z) = 0, \tag{2.14a}$$

$$-1 \le \operatorname{Re} \Delta_\varepsilon(z) \le 1. \tag{2.14b}$$

(Here "Re" and "Im" denote the real and imaginary components of a complex function, respectively.) Thus, one gets the following equivalent representation of the Lax spectrum, namely,

$$\Sigma_{\text{Lax}} = \{z \in \mathbb{C} \colon \Delta_\varepsilon(z) \in [-1, 1]\}. \tag{2.15}$$

Note (2.12) and (2.13) immediately imply Δ_ε is real-valued along the real z-axis. Further, recall the following:

Lemma 2.2 (Entire, [42,44]). *The Floquet discriminant $\Delta_\varepsilon(z)$ is entire.*

Hence, Δ_ε satisfies the Schwarz reflection principle, i.e., $\Delta_\varepsilon(\bar{z}) = \overline{\Delta_\varepsilon(z)}$.

Additionally, the Floquet spectrum (2.4) has the equivalent representation (see [43] for further details)

$$\Sigma_\nu = \{z \in \mathbb{C} \colon \Delta_\varepsilon(z) = \cos(\nu L),\ \nu \in \mathbb{R}\}. \tag{2.16}$$

Clearly, for $\nu \in \mathbb{R}$ one has $\Sigma_{\nu+2\pi/L} = \Sigma_\nu$. Importantly, the above arguments show that

$$\Sigma_{\mathrm{Lax}} = \bigcup_{\nu \in [0,2\pi/L)} \Sigma_\nu, \tag{2.17}$$

where the eigenfunctions of (2.4) extend naturally to $\mathbb{R}$. That is, the Lax spectrum is the union of all Floquet spectra. (Note however that the terminology used in the literature varies somewhat.)

Recall that $\mathcal{L}^\varepsilon$ is isospectral with respect to time deformations of the potential that obey the NLS equation (1.4). Thus, Σ_ν and Σ_{Lax}, and therefore also Δ_ε, are all conserved with respect to the flow of the NLS equation (1.4).

Also recall that for the defocusing NLS equation [i.e., (1.4) with a negative sign in front of the nonlinear term], the corresponding Zakharov–Shabat spectral problem [i.e., (1.1) with the entry $-\overline{q(x)}$ in $Q(x)$ replaced with $\overline{q(x)}$] is self-adjoint, and therefore the spectrum is confined to the real z-axis. In that case, for periodic potentials the real z-axis decomposes into a (possibly infinite) number of spectral bands [8, 17, 43]. Thus, in the self-adjoint case, knowing the periodic/antiperiodic spectrum and their geometric multiplicities is enough to completely characterize the Lax spectrum.

In contrast, the focusing NLS equation corresponds to the non-self-adjoint case, for which there is no restriction on the location of the spectral bands in the complex plane (apart from constraints such as the Schwarz symmetry). This greatly complicates the analysis; and implies the entire Floquet spectrum is relevant. Nonetheless, one can still introduce the concept of spectral bands and gaps like with self-adjoint problems, as discussed in Section 3.

Importantly, the inverse spectral theory for (1.1) also involves the Dirichlet spectrum. The *Dirichlet spectrum* is defined as

$$\begin{aligned}\Sigma_{\mathrm{Dir}}(x_o) := \{\zeta \in \mathbb{C} \colon &\text{there exists } \mathbf{v} \not\equiv 0 \in H^1([x_o, x_o + L], \mathbb{C}^2) \\ &\text{s.t. } \mathcal{L}^\varepsilon \mathbf{v} = \zeta \mathbf{v} \text{ and } \mathbf{v} \in \mathrm{BC}_{\mathrm{Dir}}(x_o)\},\end{aligned} \tag{2.18}$$

where "$\mathrm{BC}_{\mathrm{Dir}}(x_o)$" are Dirichlet boundary conditions (BCs) with base point x_o, i.e.,

$$v_1(x_o; \zeta, \varepsilon) + v_2(x_o; \zeta, \varepsilon) = 0, \tag{2.19a}$$

$$v_1(x_o + L; \zeta, \varepsilon) + v_2(x_o + L; \zeta, \varepsilon) = 0. \tag{2.19b}$$

Any value $\zeta \in \Sigma_{\mathrm{Dir}}(x_o)$ will be referred to as a Dirichlet eigenvalue of (2.1). Similarly to the Floquet spectrum, one can define $\Sigma_{\mathrm{Dir}}(x_o)$ as the zero set of an analytic function. Consider the following modified fundamental matrix solution of (1.1):

$$\widetilde{\Phi}(x;z,\varepsilon) = \Phi(x;z,\varepsilon)C, \quad C = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & \mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}. \tag{2.20}$$

The monodromy matrix $\widetilde{M}^{\varepsilon}(z)$ associated with $\widetilde{\Phi}$ is $\widetilde{M}^{\varepsilon}(z) = C^{-1}M(z;\varepsilon)C$. Then, it follows easily that (see A.3 for details)

$$\Sigma_{\mathrm{Dir}}(0) = \{\zeta \in \mathbb{C} \colon \widetilde{M}^{\varepsilon}_{21}(\zeta) = 0\}, \tag{2.21}$$

where the subscript "21" denotes the second row first column entry of the corresponding matrix. Unlike the Floquet spectrum, however, the Dirichlet spectrum is dependent on the base point x_o. Similarly, $\Sigma_{\mathrm{Dir}}(x_o)$ is not conserved by the flow of the NLS equation (1.4). Indeed, the Dirichlet spectrum provides angle information in the "action–angle" formalism of an integrable system. Moreover, Floquet and Dirichlet spectra together comprise the set of scattering data from which one can reconstruct the potential (1.3). Namely, one has the following:

Theorem 2.3 (Trace formulae, [44]). *Let $\{z_n\}_{n\in\mathbb{Z}}$ be the sequence of periodic and antiperiodic eigenvalues, $\{\zeta_n(x_o)\}_{n\in\mathbb{Z}}$ the sequences of Dirichlet eigenvalues, and $\{\widehat{\zeta}_n(x_o)\}_{n\in\mathbb{Z}}$ the sequences of auxiliary Dirichlet eigenvalues, defined respectively as follows:*

$$\Delta^2_{\varepsilon}(z_n) - 1 = 0, \quad \widehat{M}^{\varepsilon}_{21}(\zeta_n;x_o) = 0, \quad \check{M}^{\varepsilon}_{21}(\widehat{\zeta}_n;x_o) = 0, \tag{2.22}$$

where $\widehat{M}^{\varepsilon}(z;x_o)$ and $\check{M}^{\varepsilon}(z;x_o)$ are respectively the modified monodromy matrices $\widetilde{M}^{\varepsilon}(z)$ associated with the translated potentials $q(x_o+x)$ and $\mathrm{i}\,q(x_o+x)$. Then,

$$q(x) - \overline{q(x)} = 2\sum_{j\in\mathbb{Z}}(z_{2j} + z_{2j+1} - 2\zeta_j(x)), \tag{2.23a}$$

$$q(x) + \overline{q(x)} = -2\,\mathrm{i}\sum_{j\in\mathbb{Z}}(z_{2j} + z_{2j+1} - 2\widehat{\zeta}_j(x)). \tag{2.23b}$$

The numbering of the eigenvalues in (2.23) is such that, for $|j|$ sufficiently large, take $z_{2j+1} = \bar{z}_{2j}$ and the ζ_j and $\widehat{\zeta}_j$ are the Dirichlet eigenvalues associated with z_{2j} and z_{2j+1} (cf. Theorem 3.14 and [16, 29]).

3. General properties of the spectrum

In this section we discuss some general properties of the Lax spectrum. Some of the results below were known, but here are proven for a broader class of potentials. Other results are new to the best of our knowledge. In this section we assume $\varepsilon > 0$ is fixed. Proofs of all the results in this section are given in Section 6.1.

Owing to (2.14a), the Lax spectrum (2.3) is located along the contour lines

$$\Gamma := \{z \in \mathbb{C} : \operatorname{Im} \Delta_\varepsilon(z) = 0\}. \tag{3.1}$$

Moreover, Γ is the union of an at most countable set of regular analytic curves Γ_n in the complex z-plane [27], each starting from infinity and ending at infinity:

$$\Gamma = \bigcup_{n \in \mathbb{Z}} \Gamma_n. \tag{3.2}$$

(The precise details of the map $n \mapsto \Gamma_n$ are not important for the present purposes.)

Definition 3.1. A *spectral band* is a maximally connected regular analytic arc along Γ_n determined by (2.14b). Each finite portion of Γ_n where $|\operatorname{Re} \Delta_\varepsilon| > 1$ and which is delimited by a band on either side is called a *spectral gap*.

With the above definition, one can talk about bands and gaps along each Γ_n as in a self-adjoint problem. The difference is of course that the bands and gaps are not restricted to lie along the real z-axis as they would be in a self-adjoint problem, but lie instead along arcs of Γ_n. Moreover, different curves $\Gamma_i \neq \Gamma_j$ (and therefore different spectral bands) can intersect at saddle points of Δ_ε. Figure 1 provides a schematic illustration of the Lax spectrum. Note that two curves in Γ can intersect at most once, as a result of the following:

Lemma 3.2. *The set Γ, and thus the Lax spectrum Σ_{Lax}, cannot contain any closed curves in the finite z-plane.*

The equivalent statement to Lemma 3.2 for non-self-adjoint Sturm–Liouville operators is well known [54,55]. As in the self-adjoint case, the band edges still correspond to the periodic and antiperiodic eigenvalues. More precisely, for $\Delta_\varepsilon(z) = 1$, the corresponding eigenfunctions are periodic with period L, while for $\Delta_\varepsilon(z) = -1$, the corresponding eigenfunctions are antiperiodic (i.e., $2L$-periodic). As with Hill's equation, some of the spectral gaps might be closed, in which case, for the purposes of this work, we will count two adjacent bands along a single Γ_n as a single one. On the other hand, two intersecting spectral bands lying on different curves $\Gamma_i \neq \Gamma_j$ will be counted as separate. Given this convention, we say q is a *finite-band potential* if its Lax spectrum is composed of finitely many spectral bands.

We note that the terminology "finite-gap potential" is much more common in the literature, especially in the context of special solutions of infinite-dimensional integrable systems (e.g., see [3]). In the self-adjoint case, every finite-gap potential is also a finite-band potential and vice versa, so the two concepts are equivalent. However, this is not true in the non-self-adjoint case.

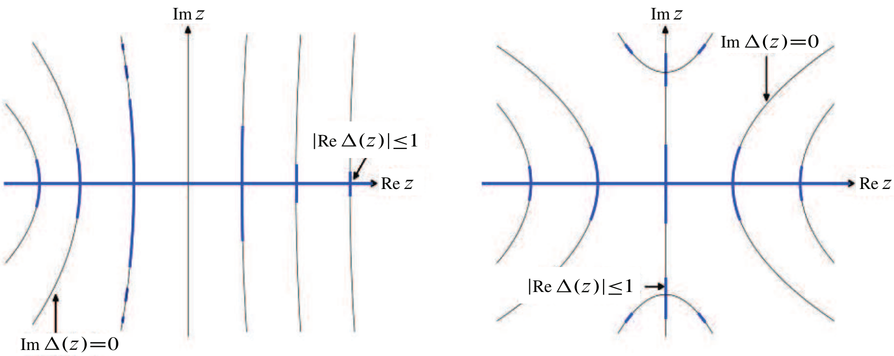

Figure 1. Left. Schematic illustration of the Lax spectrum of the focusing Zakharov–Shabat scattering problem with a generic periodic potential. Right. Schematic illustation of the Lax spectrum of the focusing Zakharov–Shabat scattering problem with a real, even, or odd periodic potential. In these cases elements of the Floquet spectrum come in quartets, i.e., $\{z, \bar{z}, -z, -\bar{z}\}$ (see Lemma 5.1).

The following property is well known, but a proof of it is provided in Section 6.1 for convenience:

Lemma 3.3 (Infinite band, [24, 42, 44]). *The real z-axis is an infinitely long spectral band; that is, $\mathbb{R} \subset \Sigma_{\text{Lax}}$.*

A key to characterize many properties of the spectrum is the asymptotic behavior of the Floquet discriminant as $z \to \infty$.

Lemma 3.4. *If $q \in L^\infty(\mathbb{R})$, then for each fixed $\varepsilon > 0$ the Floquet discriminant Δ_ε has the following asymptotic behavior:*

$$\Delta_\varepsilon(z) = \begin{cases} \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}zL/\varepsilon}(1 + \mathrm{e}^{2\mathrm{i}zL/\varepsilon} + o(1)), & z \to \infty,\ \operatorname{Im} z \geq 0, \\ \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}zL/\varepsilon}(1 + \mathrm{e}^{2\mathrm{i}zL/\varepsilon} + O(1/\operatorname{Im} z)), & \operatorname{Im} z \to \infty,\ \operatorname{Im} z > 0. \end{cases} \tag{3.3}$$

If $q' \in L^1([0, L])$, then for each fixed $\varepsilon > 0$

$$\Delta_\varepsilon(z) = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}zL/\varepsilon}\Big(1 + \mathrm{e}^{2\mathrm{i}zL/\varepsilon} + \frac{1}{2\,\mathrm{i}\,z\varepsilon}(1 - \mathrm{e}^{2\mathrm{i}zL/\varepsilon})\|q\|_2^2\, o\Big(\frac{1}{z}\Big)\Big),$$
$$z \to \infty,\ \operatorname{Im} z \geq 0. \tag{3.4}$$

Moreover,

$$\Delta_\varepsilon(z) = \cos(zL/\varepsilon) - \frac{1}{2z\varepsilon}\sin(zL/\varepsilon)\|q\|_2^2 + o\Big(\frac{\mathrm{e}^{L \operatorname{Im} z}}{z}\Big), \quad z \to \infty,\ \operatorname{Im} z \geq 0. \tag{3.5}$$

If, additionally, $q' \in L^\infty(\mathbb{R})$, then,

$$\Delta_\varepsilon(z) = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}zL/\varepsilon}\Big(1 + \mathrm{e}^{2\mathrm{i}zL/\varepsilon} + \frac{1}{2\,\mathrm{i}\,z\varepsilon}(1-\mathrm{e}^{2\mathrm{i}zL/\varepsilon})\|q\|_2^2 + o\Big(\frac{1}{(\operatorname{Im} z)^2}\Big)\Big),$$
$$\operatorname{Im} z \to \infty,\ \operatorname{Im} z > 0. \tag{3.6}$$

Further, by differentiating (1.1) with respect to z one also has the following:

Lemma 3.5. *If $q \in L^\infty(\mathbb{R})$, then for each fixed $\varepsilon > 0$*

$$\Delta_\varepsilon'(z) = -\frac{L}{\varepsilon}\sin(zL/\varepsilon) + \mathrm{e}^{L\operatorname{Im} z/\varepsilon}\,o(1), \quad z \to \infty,\ \operatorname{Im} z \geq 0. \tag{3.7}$$

If, additionally, $q' \in L^1([0, L])$, then for each fixed $\varepsilon > 0$

$$\Delta_\varepsilon'(z) = -\frac{L}{\varepsilon}\sin(zL/\varepsilon) - \frac{L}{2z\varepsilon^2}\cos(zL/\varepsilon)\|q\|_2^2 + o\Big(\frac{\mathrm{e}^{L\operatorname{Im} z}}{z}\Big),$$
$$z \to \infty,\ \operatorname{Im} z \geq 0. \tag{3.8}$$

Note the asymptotic behavior for $\operatorname{Im} z < 0$ follows from the Schwarz reflection principle. Expansions similar to those in Lemma 3.4 were given in [27,44], but were obtained under the assumptions that the potential is twice differentiable. Lemmas 3.4 and 3.5 extend the validity of the corresponding results to potentials in $L^\infty(\mathbb{R})$. Lemmas 2.2 and 3.4 also immediately imply:

Corollary 3.6. *If $q \in L^\infty(\mathbb{R})$, the Floquet discriminant $\Delta_\varepsilon(z)$ has an essential singularity at infinity.*

The fact that Δ_ε is entire has a further consequence. Recall (2.16), which says that the Floquet spectrum for a given value of $\nu \in \mathbb{R}$ is the set of $z \in \mathbb{C}$ for which $\Delta_\varepsilon(z) = \cos(\nu L)$. Then one easily gets:

Lemma 3.7. *For each $\nu \in \mathbb{R}$, the corresponding Floquet spectrum Σ_ν is discrete.*

Similarly, by (2.21) one has that $\Sigma_{\mathrm{Dir}}(x_o)$ is also discrete. Next, we derive an upper bound on the imaginary component of points in the spectrum.

Lemma 3.8. *Let $q \in L^\infty(\mathbb{R})$. Then, for all $z \in \Sigma_{\mathrm{Lax}}$,*

$$|\operatorname{Im} z| \leq \|q\|_\infty. \tag{3.9}$$

Moreover, for all $\zeta \in \Sigma_{\mathrm{Dir}}(x_o)$,

$$|\operatorname{Im} \zeta| \leq \|q\|_\infty. \tag{3.10}$$

The proof of Lemma 3.8 actually yields a stronger estimate: for all $z \in \Sigma_{\mathrm{Lax}}$,

$$|\operatorname{Im} z| \leq |\operatorname{Re}\langle qv_2, v_1\rangle| / (\|v_1\|_2\|v_2\|_2), \tag{3.11}$$

where $\mathbf{v}(x; z, \varepsilon) = (v_1, v_2)^T$ is an associated Floquet eigenfunction of the ZS problem (1.1), and $\langle\cdot,\cdot\rangle$ is the $L^2([0, L])$ inner product (cf. A.1). Inequality (3.9) refines the bound obtained in Theorem 4.2 in [27]. Importantly, there exists potentials such that estimate (3.9) is sharp (see Section 7.1 for an example).

Next, Lemma 3.4 leads to the following result:

Lemma 3.9. *Let $q \in L^\infty(\mathbb{R})$. Then for each fixed $\varepsilon > 0$ the Floquet discriminant $\Delta_\varepsilon(z)$ has infinitely many simple real zeros, and at most finitely many nonreal zeros. The same is true for the zeros of $\Delta'_\varepsilon(z)$.*

In turn, Lemma 3.2, combined with Lemma 3.4 provides the key to proving the next result. Recall that the resolvent set of an operator is the complement of its spectrum. Thus, we give the following definition:

$$\rho_{\text{Lax}} := \mathbb{C} \setminus \Sigma_{\text{Lax}}. \tag{3.12}$$

Theorem 3.10. *Let $q \in L^\infty(\mathbb{R})$. Then the resolvent set ρ_{Lax} is comprised of two connected components.*

Importantly, Theorem 3.10 should be compared to that for Hill's equation, whose resolvent set is connected [54, 55].

By Lemma 2.2 and Corollary 3.6, one gets that Δ_ε is entire as a function of z with an essential singularity at infinity. It then follows from Picard's theorem that Δ_ε takes on every value infinitely many times with at most one exception. The following result, which is a consequence of Lemma 3.9, clarifies that the lone exception (if one exists) cannot be a value corresponding to the Lax spectrum:

Theorem 3.11. *Let $q \in L^\infty(\mathbb{R})$. Then, for every $\nu \in \mathbb{R}$ the Floquet spectrum Σ_ν is countably infinite.*

Next, we turn our attention to the number of spectral bands. As discussed above, adjacent bands belonging to the same contour Γ_n with a degenerate gap are counted as a single one (and therefore the real axis counts as a single infinitely long band), but bands belonging to different contours $\Gamma_i \neq \Gamma_j$ (which can intersect at most once) are counted as separate. Note that a degenerate gap occurs at a multiple point, i.e., a point $z_n^{\text{m}} \in \mathbb{C}$ such that $\Delta_\varepsilon^2(z_n^{\text{m}}) = 1$, and $\Delta'_\varepsilon(z_n^{\text{m}}) = 0$. Thus, a multiple point is a periodic, or antiperiodic, eigenvalue that is also a critical point of the Floquet discriminant. An important distinction is that in the self-adjoint case necessarily $\Delta''_\varepsilon(z_n^{\text{m}}) \neq 0$, while in the non-self-adjoint case one may have higher order zeros.

Recall that, if the number of spectral bands is finite, we call q a *finite-band potential.* Equivalently, q is a finite-band potential if the number of points $z_n^{\text{c}} \in \mathbb{R}$ such that $\Delta'_\varepsilon(z_n^{\text{c}}) = 0$, and $\Delta_\varepsilon(z_n^{\text{c}}) \in (-1, 1)$ is finite.

Next, let R_N denote the rectangle with vertices $\pm N \pm \mathrm{i}\,\|q\|_\infty$ where $N \in \mathbb{N}$. An important consequence of Lemmas 3.8 and 3.9 is the following:

Theorem 3.12. *Let $q \in L^\infty(\mathbb{R})$ and fix $\varepsilon > 0$. Then, q is a finite-band potential if and only if there exists $N = N(q;\varepsilon) \in \mathbb{N}$ such that $(\Sigma_{\mathrm{Lax}} \setminus \mathbb{R}) \subset R_N$.*

As a special case, Theorem 3.12 implies that, if the Lax spectrum is contained in the "cross" $\mathbb{R} \cup \mathrm{i}\,\mathbb{R}$, the potential is finite-band. Importantly, one can show that, if all periodic and antiperiodic eigenvalues are in $\mathbb{R} \cup \mathrm{i}\,\mathbb{R}$, that is enough to conclude that the potential is finite-band. Specifically:

Theorem 3.13. *Let $q \in L^\infty(\mathbb{R})$ and suppose*

$$\Sigma_{\pi/L} \cup \Sigma_{2\pi/L} \subset \mathbb{R} \cup \mathrm{i}\,\mathbb{R}, \tag{3.13}$$

that is, the periodic/antiperiodic spectrum is only real and purely imaginary. Then q is a finite-band potential.

Note that the converse of Theorem 3.13 does not hold. That is, there are finite-band potentials whose periodic and antiperiodic eigenvalues are not only real or imaginary. (For example, any Galilean transformation of a potential obviously preserves its finite-band nature, but shifts the Lax spectrum horizontally, and therefore moves any imaginary eigenvalues off the imaginary z-axis.) Nonetheless, Theorem 3.13 will be relevant in Section 5, where we study potentials whose spectrum possesses additional symmetries.

Next, we discuss the asymptotic distribution of bands as $z \to \infty$. For this purpose, following [44], we introduce the concept of "spine," defined as a spectral band that intersects the real axis transversally and does not intersect any other band. Recall that any intersection point between two or more bands is a saddle point (or critical point) of the discriminant, that is, a point $z_n^{\mathrm{c}} \in \mathbb{C}$ such that $\Delta'_\varepsilon(z_n^{\mathrm{c}}) = 0$.

Theorem 3.14. *For any $q \in L^\infty(\mathbb{R})$ and each fixed $\varepsilon > 0$, there exists some $N = N(q;\varepsilon) \in \mathbb{N}$, such that all but finitely many bands of the Lax spectrum are spines located outside R_N. Moreover,*

i. *for each of these spines, there exists $n \in \mathbb{Z}$ such that the intersection point between the spine and the real axis is $o(1)$-close to the point $n\pi\varepsilon/L$ as $n \to \pm\infty$;*

ii. *only one spine can be $o(1)$-close to $n\pi\varepsilon/L$ as $n \to \pm\infty$.*

Further, if $q' \in L^1([0, L])$, the intersection point is $O(1/z)$-close.

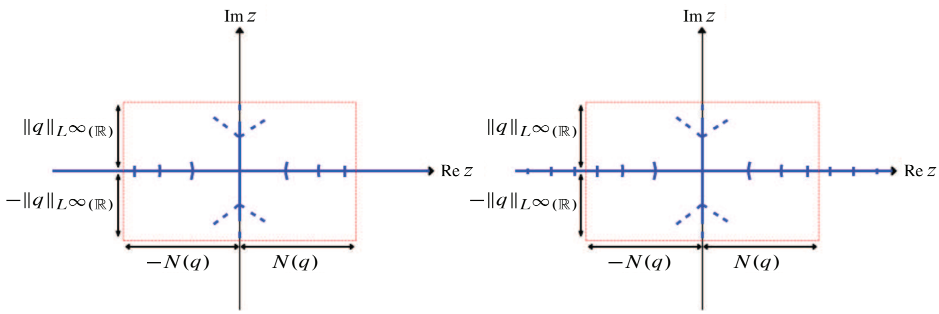

Figure 2. Left. Schematic diagram of the spectrum for a finite-band potential. Right. Schematic diagram of the spectrum for an infinite-band potential.

Related asymptotic results were given in [16, 27, 44]. As a consequence of Theorem 3.14, we have that the potential $q \in L^\infty(\mathbb{R})$ is a finite-band potential if and only if there exists $N \in \mathbb{N}$ such that Σ_{Lax} contains no spines outside R_N. An illustration of Theorem 3.14 is given in Figure 2.

Theorem 3.14 has an important consequence. Since a spectral gap is delimited by the presence of a spectral band on either side of it, the number of gaps associated to any given potential is always finite, since for any potential there is a finite region of the complex plane outside of which all bands are spines (for which there are no associated gaps), and the number of spectral bands (and therefore gaps) inside any finite region of the complex plane is always finite (as a consequence of the fact that $\Delta_\varepsilon(z)$ is entire). In other words, for the focusing Zakharov–Shabat operator on the circle, every potential is a finite-gap potential (cf. [28]). This situation is in marked contrast with the case of the spectral problem for the Hill operator and that for the self-adjoint Zakharov–Shabat operator, for which finite-gap potentials are a special subset. In contrast, for the focusing Zakharov–Shabat operator the only meaningful distinction is that between finite-band and infinite-band potentials (cf. [28]). The key distinction is the existence of finitely many (as opposed to infinitely many) simple periodic/antiperiodic eigenvalues, which defines a Riemann surface of finite-genus [3, 27, 28].

4. Properties of the spectrum in the semiclassical limit

In this section we describe the Lax spectrum in the semiclassical limit for complex-valued potentials that do not depend on ε (see Section 7 for a discussion regarding potentials that depend on ε via a “fast phase.”). Before we do so, however, we dis-

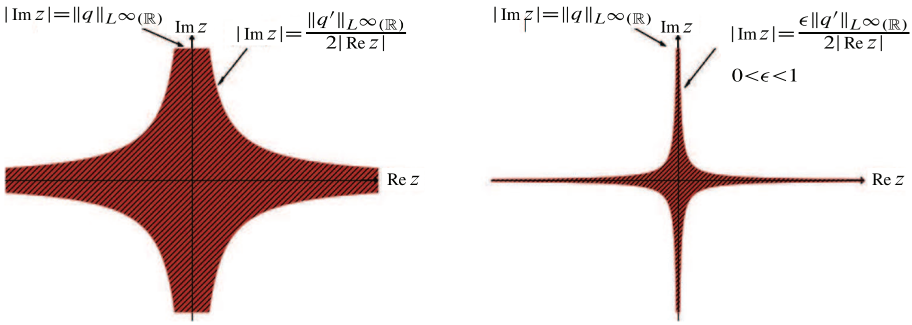

Figure 3. Left. Bounds on the Lax spectrum for $\varepsilon = 1$ (dark red). Right. Bounds on the Lax spectrum when $0 < \varepsilon < 1$ (dark red).

cuss further rigorous bounds on the location of the Lax spectrum under fairly weak assumptions on the potential.

Lemma 4.1. *Suppose $q' \in L^\infty(\mathbb{R})$. If $z \in \Sigma_{\text{Lax}}$, then*

$$|\operatorname{Re} z||\operatorname{Im} z| \le \frac{\varepsilon}{2}\|q'\|_\infty. \tag{4.1}$$

The proof of Lemma 4.1 actually yields the stronger estimate

$$|\operatorname{Re} z||\operatorname{Im} z| \le \frac{\varepsilon}{2}|\operatorname{Im}\langle q'v_2, v_1\rangle|/(\|v_1\|_2\|v_2\|_2), \tag{4.2}$$

where $\mathbf{v}(x; z, \varepsilon) = (v_1, v_2)^T$ is an associated Floquet eigenfunction of the ZS system (1.1), and $\langle\cdot,\cdot\rangle$ denotes the $L^2([0, L])$ inner product (cf. A.1). Thus, if $z \in \Sigma_{\text{Lax}}$, then Lemmas 3.8 and 4.1 together give the bound

$$|\operatorname{Im} z| \le \min\left\{\|q\|_\infty, \frac{\varepsilon\|q'\|_\infty}{2|\operatorname{Re} z|}\right\}. \tag{4.3}$$

Theorem 4.2. *Suppose $q' \in L^\infty(\mathbb{R})$. Then*

$$\Sigma_{\text{Lax}} \subset \Lambda^\varepsilon(q), \tag{4.4}$$

where

$$\Lambda^\varepsilon(q) := \{z \in \mathbb{C} : |\operatorname{Im} z| \le \|q\|_\infty\} \cap \left\{z \in \mathbb{C} : |\operatorname{Re} z||\operatorname{Im} z| \le \frac{\varepsilon}{2}\|q'\|_\infty\right\}. \tag{4.5}$$

That is, the Lax spectrum is contained in the set $\Lambda^\varepsilon(q)$ depending explicitly on ε, the semiclassical parameter. The region $\Lambda^\varepsilon(q)$ is shown in dark red in Figure 3.

Theorem 4.2 complements the localization result of [16] for the periodic and antiperiodic eigenvalues. Moreover, the results in Lemmas 3.8 and 4.1 apply to the entire Lax spectrum, not just the periodic and antiperiodic eigenvalues.

Also, recall that cardinality results for the periodic, antiperiodic, and Dirichlet eigenvalues were obtained in [29, 30, 44]. Namely, there exists $N = N(q) \in \mathbb{N}$ such that in the disc $D(0; (N - 1/2)\pi/L)$ there are exactly $2N - 2$ periodic eigenvalues, $2N$ antiperiodic eigenvalues, and $2N - 1$ Dirichlet eigenvalues. Importantly, for $q \in H^1([0, L])$ one can get an explicit estimate for $N = N(q)$ (see [44] for details). Theorem 4.2 allows one to strengthen those results. For brevity, we let $\varepsilon = 1$ in Theorem 4.3 and Corollary 4.4.

Theorem 4.3. *Suppose $q' \in L^\infty(\mathbb{R})$ and $\varepsilon = 1$. Consider the sets*

$$\Pi_{\mathrm{Lax}}(q) := \Lambda^\varepsilon(q) \cap D(0; (N - 1/2)\pi/L), \tag{4.6a}$$

$$\Pi_{\mathrm{Dir}}(q) := (\mathbb{R} \times \mathrm{i}[-\|q\|_\infty, \|q\|_\infty]) \cap D(0; (N - 1/2)\pi/L). \tag{4.6b}$$

The set $\Pi_{\mathrm{Lax}}(q)$ contains exactly $2N - 2$ periodic eigenvalues and $2N$ antiperiodic eigenvalues. The set $\Pi_{\mathrm{Dir}}(q)$ contains exactly $2N - 1$ Dirichlet eigenvalues.

A consequence of Lemma 3.4 and Lemma 4.1 is the following:

Corollary 4.4. *Fix $\nu \in \mathbb{R}$. Suppose $q' \in L^\infty(\mathbb{R})$ and $\varepsilon = 1$. Then the Floquet eigenvalues $\{z_n(\nu)\}_{n\in\mathbb{Z}}$ can be partially ordered such that $\operatorname{Re} z_n(\nu) \to \pm\infty$ as $n \to \pm\infty$. Moreover,*

$$\operatorname{Im} z_n(\nu) = O(1/\operatorname{Re} z_n(\nu)), \quad \text{as } n \to \pm\infty,\ \operatorname{Re} z \neq 0, \tag{4.7}$$

We now turn to the semiclassical limit. Theorem 4.2 allows us to give a rigorous characterization of the Lax spectrum as $\varepsilon \downarrow 0$. This is important since for certain initial data the focusing NLS equation on the circle appears to admit a coherent structure in the semiclassical limit [6]. Note that spectral confinement to the real and imaginary axes together with the fact that the real z-axis is an infinitely long band implies that all nonlinear excitations emerging from the input have zero velocity.

Corollary 4.5. *Suppose $q' \in L^\infty(\mathbb{R})$. Define*

$$\Sigma_\infty := \mathbb{R} \cup \mathrm{i}[-\|q\|_\infty, \|q\|_\infty]. \tag{4.8}$$

Moreover, let $N_\delta(\Sigma_\infty)$ be a δ-neighborhood of Σ_∞. Then for any $\delta > 0$,

$$\Sigma_{\mathrm{Lax}} \subset N_\delta(\Sigma_\infty), \tag{4.9}$$

for all sufficiently small values of ε. That is, for any $\delta > 0$ there exists an $\varepsilon_* > 0$ such that (4.9) holds for all $0 < \varepsilon \le \varepsilon_*$.

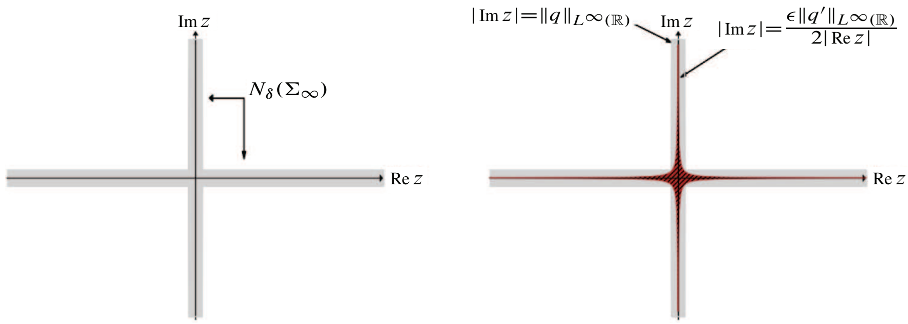

Figure 4. Left. The δ-neighborhood of Σ_∞ (light gray). Right. Bounds on the Lax spectrum for $0 < \varepsilon \le \varepsilon_*$ (dark red) with the δ-neighborhood of Σ_∞ (light gray).

Corollary 4.5 is a direct consequence of Theorem 4.2. Moreover, Corollary 4.5 corroborates and puts on a rigorous footing the numerical results of [6]. Roughly speaking, one can interpret Corollary 4.5 as saying that the Lax spectrum of $\mathcal{L}^\varepsilon$ “converges” to Σ_∞ in the limit $\varepsilon \downarrow 0$. Note, however, that in general the semiclassical limit is a singular limit. This is why it is necessary to state the result in a more indirect way. (It was also shown in [6] through numerical and asymptotic calculations that the number of bands is $O(1/\varepsilon)$ as $\varepsilon \downarrow 0$. However, no rigorous proof of this result exists to the best of our knowledge.) An illustration of the δ-neighborhood $N_\delta(\Sigma_\infty)$ and its relation to the Lax spectrum is given in Figure 4. The proof of Theorem 4.5 and of all results in this section is given in Section 6.2.

5. Properties of the spectrum for real or symmetric potentials

In addition to Schwarz symmetry, the Lax spectrum acquires additional symmetries when the potential in (1.1) is itself symmetric. In particular, if the potential q is real then it follows that the monodromy matrix satisfies

$$\overline{M(-\bar{z};\varepsilon)} = M(z;\varepsilon). \tag{5.1}$$

Next, we list some additional symmetries of the monodromy matrix when the potential is symmetric. These symmetries are easily deduced from the symmetries of the ZS system (1.1) (cf. A.2). Suppose that the potential satisfies a generalized reflection symmetry, that is, there exists $\theta \in \mathbb{R}$ such that $q(-x) = \mathrm{e}^{2\mathrm{i}\theta} q(x)$ for all $x \ge 0$. (Obviously for $\theta = 0 \mod \pi$, and $\theta = \pi/2 \mod \pi$ one has the cases of even and odd potentials, respectively.) Then, one gets

$$\overline{M(-\bar{z};\varepsilon)^{-1}} = (\cos\theta\sigma_1 + \sin\theta\sigma_2) M(z;\varepsilon)(\cos\theta\sigma_1 + \sin\theta\sigma_2), \tag{5.2}$$

where σ_1 and σ_2 are the first and second Pauli matrices, respectively (cf. A.1).

If the potential in (1.1) is PT-symmetric, that is, if $q(-x) = \overline{q(x)}$, then it follows that the monodromy matrix satisfies

$$\overline{M(\bar{z};\varepsilon)^{-1}} = \sigma_3 M(z;\varepsilon)\sigma_3. \tag{5.3}$$

Using the above symmetries, for odd real, or even real potentials we obtain

$$M(z;\varepsilon) = \begin{pmatrix} c(z;\varepsilon) & \pm s(z;\varepsilon) \\ s(z;\varepsilon) & (1 \pm s^2(z;\varepsilon))/c(z;\varepsilon) \end{pmatrix}, \tag{5.4}$$

where $c = M_{11}(z;\varepsilon) \not\equiv 0$, and $s = M_{21}(z;\varepsilon)$, and where the "+" sign holds for odd real potentials and the "–" sign for even real potentials. Using the above symmetries, we get the following:

Lemma 5.1. *Let the potential q satisfy at least one of the following conditions:* (a) *it is real;* (b) *it is even;* (c) *it is odd. Then, Σ_{Lax} is symmetric with respect to the imaginary z-axis, and Γ includes the* $\mathrm{Im}\, z$*-axis, cf.* (3.1).

Next, using Lemma 5.1 we get the following:

Theorem 5.2. *Let the potential q satisfy at least one of the following conditions:* (a) *it is real;* (b) *it is even;* (c) *it is odd. If the periodic and antiperiodic spectra are real and purely imaginary only, then the entire Lax spectrum is contained within the real and imaginary axes, that is, $\Sigma_{\mathrm{Lax}} \subset \Sigma_\infty \subset \mathbb{R} \cup \mathrm{i}\,\mathbb{R}$, cf.* (4.8).

Further, by assuming the potential is real, or symmetric one is able to obtain stronger bounds on the Dirichlet and Floquet spectra:

Lemma 5.3. *Suppose $q' \in L^\infty(\mathbb{R})$. Moreover, let the potential q satisfy at least one of the following conditions:* (a) *it is real;* (b) *it is odd;* (c) *it is PT-symmetric. If $\zeta \in \Sigma_{\mathrm{Dir}}(0)$, then*

$$|\operatorname{Re}\zeta||\operatorname{Im}\zeta| \le \frac{\varepsilon}{2}\|q'\|_\infty. \tag{5.5}$$

Importantly, for q real, the result holds more generally for $\zeta \in \Sigma_{\mathrm{Dir}}(x_o)$. That is, the bound (5.5) is independent of the base point x_o when q is real.

Theorem 5.4. *Assume $q' \in L^\infty(\mathbb{R})$. Let the potential q satisfy at least one of the following conditions:* (a) *it is real;* (b) *it is odd;* (c) *it is PT-symmetric. Then,*

$$\Sigma_{\mathrm{Dir}}(0) \subset \Lambda^\varepsilon(q), \tag{5.6}$$

where

$$\Lambda^\varepsilon(q) := \{\zeta \in \mathbb{C} \colon |\operatorname{Im}\zeta| \le \|q\|_\infty\} \cap \left\{\zeta \in \mathbb{C} \colon |\operatorname{Re}\zeta||\operatorname{Im}\zeta| \le \frac{\varepsilon}{2}\|q'\|_\infty\right\}. \tag{5.7}$$

Thus, for these classes of potentials the Dirichlet spectrum localizes to the real and imaginary axes in the semiclassical limit. An interesting open question is whether the bound obtained in Lemma 5.3 holds more generally for complex-valued potentials. These results should be compared to those pertaining to the Lax spectrum. Finally, we have the following interesting bound:

Lemma 5.5. *Let q be real, strictly positive, and suppose that $q' \in L^\infty(\mathbb{R})$. Then,*

$$|\operatorname{Im} z| \le \frac{\varepsilon}{2}\|(\ln q)'\|_\infty, \tag{5.8}$$

for all $z \in \Sigma_{\mathrm{Lax}} \setminus \mathrm{i}\,\mathbb{R}$. Moreover,

$$|\operatorname{Im} \zeta| \le \frac{\varepsilon}{2}\|(\ln q)'\|_\infty, \tag{5.9}$$

for all $\zeta \in \Sigma_{\mathrm{Dir}}(x_o) \setminus \mathrm{i}\,\mathbb{R}$.

Thus, for strictly positive real potentials $\operatorname{Im} z(\nu;\varepsilon)$ (resp. $\operatorname{Im} \zeta(\varepsilon)$) is uniformly $O(\varepsilon)$ as $\varepsilon \downarrow 0$ for non-purely imaginary $z \in \Sigma_{\mathrm{Lax}}$ (resp. $\zeta \in \Sigma_{\mathrm{Dir}}(x_o)$). (Note by phase invariance this result can easily be extended to strictly negative potentials as well.) This result provides strong justification of the WKB analysis employed by two of the authors in [6]. Moreover, Lemma 5.5 is the analogue for periodic potentials of the classic results about purely imaginary discrete eigenvalues for potentials on the line [5, 35]. Finally, localization along the imaginary axis of the spectral variable as in Lemma 5.5 together with the WKB results obtained in [6] find application in the study of soliton and breather gases in focusing nonlinear media (see [18]).

When the potential in the ZS system (1.1) is real, one can map (1.1) to a non-self-adjoint Hill's equation. Specifically, the invertible change of variables

$$y_\pm(x;\lambda,\varepsilon) = v_1(x;z,\varepsilon) \pm \mathrm{i}\, v_2(x;z,\varepsilon) \tag{5.10}$$

maps (1.1) into linear second-order ODEs

$$(-\varepsilon^2\partial_x^2 + W_\pm(x;\varepsilon))y_\pm = \lambda y_\pm, \tag{5.11}$$

which are of course the celebrated Hill's equation, with spectral parameter $\lambda := z^2$ and complex potentials

$$W_\pm(x;\varepsilon) := -q^2(x) \mp \mathrm{i}\,\varepsilon q'(x). \tag{5.12}$$

Because $W_\pm(x;\varepsilon)$ are not real, in general the eigenvalue problems (5.11) are non-self-adjoint. Note however that, when q is real and even, the potentials in (5.12) are PT-symmetric, i.e., invariant under the combined action of space reflections and complex conjugation [21]. So, the results above for real and even potentials also serve as a further study of spectral problems for the Hill operator with a PT-symmetric potential. Note, estimate (3.9) gives $\operatorname{Re}\lambda \geq -\|q\|_\infty^2$, and the estimate (4.1) gives $\operatorname{Im}\lambda = O(\varepsilon)$ as $\varepsilon \downarrow 0$ in (5.11). Theorem 5.2 implies that, even though real potentials are associated with a Hill operator with a complex potential, if the periodic and antiperiodic Floquet eigenvalues are all real or purely imaginary, then the corresponding non-self-adjoint Hill equation (5.11) has a purely real spectrum. Moreover, this implies that $W_\pm$ is a finite-band potential (recall that for the ZS system the real z-axis is an infinitely long band). Hence, if $q \in L^\infty(\mathbb{R})$, then along the real λ-axis the spectral bands and gaps are confined to the interval $[-\|q\|_\infty^2, 0)$, followed by one infinitely long spectral band along the interval $[0, \infty)$. Note however the thesis of Theorem 5.2 is *not* true if the potential is not real or even or odd, because in that case the spectrum possesses no left-right symmetry. The proofs of all the results in this section are given in Section 6.3.

6. Proofs

6.1. Proofs: General properties of the spectrum

We refer the reader to A.1 for definitions.

Proof of Lemma 3.2. By Lemma 2.2, $\Delta_\varepsilon(z)$ is an entire function differing from a constant. Thus, $\operatorname{Im}\Delta_\varepsilon(z)$ is harmonic. By the maximum principle the set Γ cannot contain any closed curve. Further, along any spectral band we have $\operatorname{Im}\Delta_\varepsilon(z) = 0$. Hence, Σ_{Lax} contains no closed bands in the finite complex z-plane. ∎

Proof of Lemma 3.3. By (2.12), $\Delta_\varepsilon(z) \in \mathbb{R}$ along the real z-axis, and

$$|c(z;\varepsilon)|^2 + |s(z;\varepsilon)|^2 \equiv 1.$$

Further, along the real z-axis $\Delta_\varepsilon(z) = \operatorname{Re} c(z;\varepsilon)$. Thus,

$$|\Delta_\varepsilon(z)| \leq (1 - |s(z;\varepsilon)|^2)^{1/2} \leq 1.$$

Hence, $\mathbb{R} \subset \Sigma_{\text{Lax}}$. ∎

In order to prove Lemma 3.4, we need two intermediate results. Recall that one can write

$$M(z;\varepsilon) = \Phi(L;z,\varepsilon), \tag{6.1}$$

where $\Phi(x;z,\varepsilon)$ is the principal matrix solution of (1.1). Thus, (2.13) implies $\Delta_\varepsilon(z) = \frac{1}{2}\operatorname{tr}\Phi(L;z,\varepsilon)$. Accordingly, Lemma 3.4 is a consequence of the asymptotic behavior of $\Phi(x;z,\varepsilon)$ as $z \to \infty$. Introducing the change of dependent variable

$$\mu(x;z,\varepsilon) := \Phi(x;z,\varepsilon)\,\mathrm{e}^{\mathrm{i}\,zx\sigma_3/\varepsilon}, \tag{6.2}$$

as well as the notation μ_{ij} for $i, j = 1, 2$ to denote matrix entries, we next prove:

Lemma 6.1. *Let $q \in L^\infty([0, L])$, and let* $\operatorname{Im} z \geq 0$. *Then, for each fixed $\varepsilon > 0$*

$$|\mu_{11}(x; z, \varepsilon) - 1| \leq \frac{\mathrm{e}^{M}\, L\|q\|_\infty^2}{2\varepsilon \operatorname{Im} z}, \quad \operatorname{Im} z > 0, \tag{6.3a}$$

where $M = (L/\varepsilon)^2 \|q\|_\infty$. Moreover,

$$\mu_{11}(x; z, \varepsilon) = 1 + o(1), \quad z \to \infty, \operatorname{Im} z \geq 0, \tag{6.3b}$$

uniformly for $x \in [0, L]$. If $q' \in L^1([0, L])$, then for each fixed $\varepsilon > 0$

$$\mu_{11}(x; z, \varepsilon) = 1 + \frac{1}{2\,\mathrm{i}\, z\varepsilon} \int_0^x |q(y)|^2 \,\mathrm{d}\, y + o(1/z), \quad z \to \infty, \operatorname{Im} z \geq 0. \tag{6.4}$$

If, additionally, $q' \in L^\infty([0, L])$, then,

$$\mu_{11}(x; z, \varepsilon) = 1 + \frac{1}{2\,\mathrm{i}\, z\varepsilon} \int_0^x |q(y)|^2 \,\mathrm{d}\, y + O(1/(\operatorname{Im} z)^2), \quad \operatorname{Im} z \to \infty, \operatorname{Im} z > 0. \tag{6.5}$$

Proof of Lemma 6.1. Using (6.2), it follows that $\mu = (\mu_{ij})$ satisfies the following system of ODEs:

$$\varepsilon\mu' = -\mathrm{i}\, z[\sigma_3, \mu] + Q(x)\mu, \tag{6.6}$$

where $[\sigma_3, \mu] := \sigma_3\mu - \mu\sigma_3$. Thus, μ satisfies the matrix Volterra linear integral equation

$$\mu(x; z, \varepsilon) = \mathbb{I} + \int_0^x \mathrm{e}^{-\mathrm{i}\, z(x-y)\sigma_3/\varepsilon}\, Q(y)\mu(y; z, \varepsilon)\, \mathrm{e}^{\mathrm{i}\, z(x-y)\sigma_3/\varepsilon} \,\mathrm{d}\, y/\varepsilon. \tag{6.7}$$

Explicitly, (6.7) yields coupled integral equations for the individual entries of $\mu(x; z, \varepsilon)$. In particular,

$$\mu_{11}(x; z, \varepsilon) = 1 + \int_0^x q(y)\mu_{21}(y; z, \varepsilon) \,\mathrm{d}\, y/\varepsilon, \tag{6.8a}$$

$$\mu_{21}(x; z, \varepsilon) = -\int_0^x \mathrm{e}^{2\,\mathrm{i}\, z(x-y)/\varepsilon}\, \overline{q(y)}\mu_{11}(y; z, \varepsilon) \,\mathrm{d}\, y/\varepsilon, \tag{6.8b}$$

with similar equations for $\mu_{12}(x; z, \varepsilon)$ and $\mu_{22}(x; z, \varepsilon)$. Eliminating μ_{21}, one then obtains

$$\mu_{11}(x; z, \varepsilon) = 1 + K(\mu_{11})(x; z, \varepsilon), \tag{6.9a}$$

where

$$K(f)(x;z,\varepsilon) := \int_0^x k(x,t;z,\varepsilon)\, f(t)\,\mathrm{d}t/\varepsilon, \tag{6.9b}$$

$$k(x,t;z,\varepsilon) := -\overline{q(t)} \int_t^x \mathrm{e}^{2\mathrm{i}z(y-t)/\varepsilon}\, q(y)\,\mathrm{d}y/\varepsilon, \quad x > t. \tag{6.9c}$$

Note that, for all $x \in [0, L]$ and $t \in [0, x]$, $|k(x,t;z,\varepsilon)| \leq L\|q\|_\infty^2/\varepsilon$. Then, introducing the Neumann series

$$\mu_{11}(x;z,\varepsilon) = \sum_{n=0}^{\infty} \mu_{11}^{(n)}(x;z,\varepsilon), \tag{6.10a}$$

$$\mu_{11}^{(0)} = 1, \quad \mu_{11}^{(n+1)}(x;z,\varepsilon) = K(\mu_{11}^{(n)}) = K^{n+1}(\mu_{11}^{(0)}), \tag{6.10b}$$

we have the bound

$$\begin{aligned} |K^n(f)(x;z,\varepsilon)| &\leq \int_0^x |k(x,t_1;z,\varepsilon)| \cdots \int_0^{t_{n-1}} |k(t_{n-1},t_n;z,\varepsilon)||f(t_n)|\,\mathrm{d}t_n \cdots \mathrm{d}t_1/\varepsilon^n \\ &\leq \frac{M^n}{n!}\|f\|_\infty, \end{aligned}$$

where $M := (\frac{L}{\varepsilon})^2\|q\|_\infty^2$. Hence, the series is absolutely convergent, and

$$\sup_{x\in[0,L],\operatorname{Im} z\geq 0} |\mu_{11}(x;z,\varepsilon)| \leq \sum_{n=0}^{\infty} |K^n(1)(x;z,\varepsilon)| \leq \mathrm{e}^M, \tag{6.11}$$

since $\mu_{11}^{(0)} \equiv 1$. Next, from (6.8b) and (6.11) we immediately have $|\mu_{21}(x;z,\varepsilon)| \leq \mathrm{e}^M\, \|q\|_\infty/(2\operatorname{Im} z)$ for all z such that $\operatorname{Im} z > 0$. Thus, using (6.8a) one gets (6.3a).

Next, we consider the case $z \to \infty$ while $\operatorname{Im} z$ remains bounded. The Riemann–Lebesgue lemma applied to (6.9c) implies that, for all $x \in [0, L]$ and $t \in [0, x]$, $k(x, t, z, \varepsilon) = o(1)$ as $z \to \infty$ from $\operatorname{Im} z \geq 0$. Moreover, it is straightforward to show that, as in [17], $q \in L^\infty([0, L])$ is sufficient to ensure that the o symbol is uniform with respect to $x \in [0, L]$. (E.g., this can be done by approximating q in (6.9c) with smooth functions, which are dense in $L^1([0, L])$.) Note that [17] considers a Sturm–Liouville problem, which involves not only q but also q', and absolute continuity of q implies the existence of $q' \in L^1([0, L])$. Here, $q \in L^\infty([0, L])$ implies $q \in L^1([0, L])$. Together with the uniform boundedness of μ_{11} for $\operatorname{Im} z \geq 0$, this gives

$$|K(\mu_{11})(x;z,\varepsilon)| \leq \mathrm{e}^M \int_0^x |k(x,t;z,\varepsilon)|\,\mathrm{d}t/\varepsilon = o(1), \quad z \to \infty,\ \operatorname{Im} z \geq 0. \tag{6.12}$$

Thus, $\mu_{11}(x;z,\varepsilon) = 1 + o(1)$ as $z \to \infty$ from $\operatorname{Im} z \geq 0$, i.e., (6.3b). In turn, using (6.8b) and recalling (6.11) one gets immediately

$$\mu_{21}(x;z,\varepsilon) = o(1), \quad z \to \infty, \operatorname{Im} z \geq 0. \tag{6.13}$$

Now, assume additionally that $q' \in L^1([0,L])$. From (6.6), it follows $\mu_{11}' = q(x)\mu_{21}/\varepsilon$. Then, integrate (6.8b) by parts:

$$\begin{aligned}\mu_{21}(x;z,\varepsilon) = {} & \frac{1}{2\mathrm{i}z}\left(\overline{q(x)}\mu_{11}(x;z,\varepsilon) - \mathrm{e}^{2\mathrm{i}zx/\varepsilon}\overline{q(0)}\right) \\ & - \frac{1}{2\mathrm{i}z}\int_0^x \mathrm{e}^{2\mathrm{i}z(x-y)/\varepsilon}(\overline{q(y)}\mu_{11}(y;z,\varepsilon))'\,\mathrm{d}y,\end{aligned} \tag{6.14}$$

where $\mu_{11}(0;z,\varepsilon) \equiv 1$. This implies, for $q' \in L^\infty([0,L])$,

$$\begin{aligned}\mu_{21}(x;z,\varepsilon) = {} & \frac{1}{2\mathrm{i}z}\left(\overline{q(x)}\mu_{11}(x;z,\varepsilon) - \mathrm{e}^{2\mathrm{i}zx/\varepsilon}\overline{q(0)}\right) \\ & + O\left(\frac{1}{(\operatorname{Im} z)^2}\right), \quad \operatorname{Im} z \to \infty, \operatorname{Im} z > 0.\end{aligned} \tag{6.15}$$

Plugging (6.15) into (6.8a) gives (6.5). Finally, note, by (6.13), and since μ_{11}' is proportional to μ_{21}, it follows that $\mu_{11}'(x;z,\varepsilon) = o(1)$ as $z \to \infty$ from $\operatorname{Im} z \geq 0$. Hence, by (6.14) one gets

$$\begin{aligned}\mu_{21}(x;z,\varepsilon) = {} & \frac{1}{2\mathrm{i}z}\left(\overline{q(x)}\mu_{11}(x;z,\varepsilon) - \mathrm{e}^{2\mathrm{i}zx/\varepsilon}\overline{q(0)}\right) \\ & + o\left(\frac{1}{z}\right), \quad z \to \infty, \operatorname{Im} z \geq 0.\end{aligned} \tag{6.16}$$

Plugging (6.16) into (6.8a) gives (6.4). ∎

Next, since $\mu_{22}(x;z,\varepsilon)$ is unbounded as $\operatorname{Im} z \to \infty$ from $\operatorname{Im} z > 0$ we make a further change of variables, namely,

$$(\tilde{\mu}_{12}, \tilde{\mu}_{22})^T := \mathrm{e}^{2\mathrm{i}zx/\varepsilon}(\mu_{12}, \mu_{22})^T, \tag{6.17}$$

and examine the asymptotic behavior of $\tilde{\mu}_{22}(x;z,\varepsilon)$ as $z \to \infty$ from $\operatorname{Im} z \geq 0$.

Lemma 6.2. *Let $q \in L^\infty([0,L])$, and let $\operatorname{Im} z \geq 0$. Then, for each fixed $\varepsilon > 0$,*

$$|\tilde{\mu}_{22}(x;z,\varepsilon) - \mathrm{e}^{2\mathrm{i}zx/\varepsilon}| \leq \frac{\mathrm{e}^M L\|q\|_\infty^2}{2\varepsilon \operatorname{Im} z}, \quad \operatorname{Im} z > 0, \tag{6.18a}$$

where $M = (L/\varepsilon)^2\|q\|_\infty^2$. Moreover,

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} + o(1), \quad z \to \infty, \operatorname{Im} z \geq 0, \tag{6.18b}$$

uniformly for $x \in [0, L]$. If $q' \in L^1([0, L])$, then, for each fixed $\varepsilon > 0$,

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} - \frac{\mathrm{e}^{2\mathrm{i}zx/\varepsilon}}{2\mathrm{i}z\varepsilon}\int_0^x |q(y)|^2\,\mathrm{d}y + o\Big(\frac{1}{z}\Big), \quad z \to \infty,\ \mathrm{Im}\,z \geq 0. \tag{6.19}$$

If, additionally, $q' \in L^\infty([0, L])$, then,

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} - \frac{\mathrm{e}^{2\mathrm{i}zx/\varepsilon}}{2\mathrm{i}z\varepsilon}\int_0^x |q(y)|^2\,\mathrm{d}y + O\Big(\frac{1}{(\mathrm{Im}\,z)^2}\Big), \quad \mathrm{Im}\,z \to \infty,\ \mathrm{Im}\,z > 0. \tag{6.20}$$

Proof of Lemma 6.2. Using (6.7) and (6.17), we have

$$\tilde{\mu}_{12}(x;z,\varepsilon) = \int_0^x q(y)\tilde{\mu}_{22}(y;z,\varepsilon)\,\mathrm{d}y/\varepsilon, \tag{6.21a}$$

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} - \int_0^x \mathrm{e}^{2\mathrm{i}z(x-y)/\varepsilon}\,\overline{q(y)}\tilde{\mu}_{12}(y;z,\varepsilon)\,\mathrm{d}y/\varepsilon. \tag{6.21b}$$

Eliminating $\tilde{\mu}_{12}$, one then obtains

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} + G(\tilde{\mu}_{22})(x;z,\varepsilon), \tag{6.22a}$$

where

$$G(f)(x;z,\varepsilon) := \int_0^x g(x,t;z,\varepsilon) f(t)\,\mathrm{d}t/\varepsilon, \tag{6.22b}$$

$$g(x,t;z,\varepsilon) := -q(t)\int_t^x \mathrm{e}^{2\mathrm{i}z(x-y)/\varepsilon}\,\overline{q(y)}\,\mathrm{d}y/\varepsilon, \quad x > t. \tag{6.22c}$$

Note that, for all $x \in [0, L]$ and $t \in [0, x]$, $|g(x,t;z,\varepsilon)| \leq L\|q\|_\infty^2/\varepsilon$. Then, introducing the Neumann series

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \sum_{n=0}^\infty \tilde{\mu}_{22}^{(n)}(x;z,\varepsilon), \tag{6.23a}$$

$$\tilde{\mu}_{22}^{(0)} = \mathrm{e}^{2\mathrm{i}zx/\varepsilon}, \quad \tilde{\mu}_{22}^{(n+1)}(x;z,\varepsilon) = G(\tilde{\mu}_{22}^{(n)}) = G^{n+1}(\tilde{\mu}_{22}^{(0)}), \tag{6.23b}$$

we have the bound

$$
\begin{aligned}
|G^n(f)(x;z,\varepsilon)| &\leq \int_0^x |g(x,t_1;z,\varepsilon)| \cdots \int_0^{t_{n-1}} |g(t_{n-1},t_n;z,\varepsilon)||f(t_n)|\,\mathrm{d}t_n \cdots \mathrm{d}t_1/\varepsilon^n \\
&\leq \frac{M^n}{n!}\|f\|_\infty,
\end{aligned}
$$

where $M := (L/\varepsilon)^2\|q\|_\infty^2$. For $\operatorname{Im} z \geq 0$, it follows $|\,\mathrm{e}^{2\mathrm{i}zx/\varepsilon}\,| \leq 1$ so $\tilde{\mu}_{22}(x;z,\varepsilon) = \sum_{n=0}^\infty G^n(\mathrm{e}^{2\mathrm{i}zx/\varepsilon})$ is absolutely convergent, and

$$
\sup_{x\in[0,L],\operatorname{Im} z\geq 0} |\tilde{\mu}_{22}(x;z,\varepsilon)| \leq \sum_{n=0}^{\infty} |G^n(\mathrm{e}^{2\mathrm{i}zx/\varepsilon})(x;z,\varepsilon)| \leq \mathrm{e}^M\,. \tag{6.24}
$$

From (6.21a) and (6.24), we have $|\tilde{\mu}_{12}(x;z,\varepsilon)| \leq \mathrm{e}^M\, L\|q\|_\infty/\varepsilon$. Using (6.21b), one gets (6.18a).

Next, we consider the case $z \to \infty$ while $\operatorname{Im} z$ remains bounded. The Riemann–Lebesgue lemma applied to (6.22c) implies that, for all $x \in [0,L]$ and $t \in [0,x]$,

$$
g(x,t;z,\varepsilon) = o(1), \quad z \to \infty,\ \operatorname{Im} z \geq 0.
$$

Together with the uniform boundedness of $\tilde{\mu}_{22}$ for $\operatorname{Im} z \geq 0$, this gives

$$
|G(\tilde{\mu}_{22})(x;z,\varepsilon)| \leq \mathrm{e}^M \int_0^x |g(x,t;z,\varepsilon)|\,\mathrm{d}t = o(1), \quad z \to \infty,\ \operatorname{Im} z \geq 0. \tag{6.25}
$$

Thus, $\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} + o(1)$ as $z \to \infty$ from $\operatorname{Im} z \geq 0$, i.e., (6.18b). Inserting this expression into (6.21a) then yields, by the Riemann–Lebesgue lemma,

$$
\tilde{\mu}_{12}(x;z,\varepsilon) = \int_0^x q(y)(\mathrm{e}^{2\mathrm{i}zy/\varepsilon} + o(1))\,\mathrm{d}y/\varepsilon = o(1),\ z \to \infty, \quad \operatorname{Im} z \geq 0. \tag{6.26}
$$

Now, assume additionally that $q' \in L^1([0,L])$. From (6.6) it follows $\tilde{\mu}_{12}' = q(x)\tilde{\mu}_{22}/\varepsilon$. Then, integrate (6.21b) by parts:

$$
\begin{aligned}
\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} &+ \frac{1}{2\mathrm{i}z}\overline{q(x)}\tilde{\mu}_{12}(x;z,\varepsilon) \\
&- \frac{1}{2\mathrm{i}z}\int_0^x \mathrm{e}^{2\mathrm{i}z(x-y)/\varepsilon}(\overline{q(y)}\tilde{\mu}_{12}(x;z,\varepsilon))'\,\mathrm{d}y,
\end{aligned} \tag{6.27}
$$

where $\tilde{\mu}_{12}(0;z,\varepsilon) \equiv 0$. Expanding the derivative, using (6.21a), and $q' \in L^\infty([0,L])$ gives

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} - \frac{1}{2\mathrm{i}z}\int_0^x \mathrm{e}^{2\mathrm{i}z(x-y)/\varepsilon}\overline{q(y)}\tilde{\mu}'_{12}(y;z,\varepsilon)\,\mathrm{d}y + O\Big(\frac{1}{(\operatorname{Im} z)^2}\Big), \tag{6.28}$$

as $\operatorname{Im} z \to \infty$ from $\operatorname{Im} z > 0$. Then, using $\tilde{\mu}'_{12} = q(x)(\mathrm{e}^{2\mathrm{i}zx/\varepsilon} + O(1/\operatorname{Im} z))$ as $\operatorname{Im} z \to \infty$ gives (6.20). Moreover, $\tilde{\mu}'_{12} = q(x)(\mathrm{e}^{2\mathrm{i}zx/\varepsilon} + o(1))$ as $z \to \infty$ from $\operatorname{Im} z \geq 0$. Hence, using (6.27) one gets

$$\tilde{\mu}_{22}(x;z,\varepsilon) = \mathrm{e}^{2\mathrm{i}zx/\varepsilon} + \frac{1}{2\mathrm{i}z}\overline{q(x)}\tilde{\mu}_{12}(x;z,\varepsilon) - \frac{\mathrm{e}^{2\mathrm{i}zx/\varepsilon}}{2\mathrm{i}z}\int_0^x |q(y)|^2\,\mathrm{d}y/\varepsilon + o\Big(\frac{1}{z}\Big), \tag{6.29}$$

as $z \to \infty$ from $\operatorname{Im} z \geq 0$. Using (6.26) gives (6.19). ■

We are now ready to prove Lemma 3.4.

Proof of Lemma 3.4. Now that we have the asymptotic behavior of μ_{11}, and $\tilde{\mu}_{22}$, Lemma 3.4 follows by expressing Δ_ε as

$$\Delta_\varepsilon(z) = \frac{1}{2}\operatorname{tr}\Phi(L;z,\varepsilon) = \frac{1}{2}\mathrm{e}^{-\mathrm{i}zL/\varepsilon}(\mu_{11}(L;z,\varepsilon) + \tilde{\mu}_{22}(L;z,\varepsilon)),$$

and combining the above results using the relevant function spaces. ■

Proof of Lemma 3.5. One of the consequences of Lemmas 6.1 and 6.2 is that the normalized matrix fundamental solution $\Phi(x;z,\varepsilon)$, introduced right above equation (2.9), has the asymptotics

$$\Phi(x;z,\varepsilon) = (\mathbb{I} + o(1))\,\mathrm{e}^{-\mathrm{i}zx\sigma_3/\varepsilon}, \quad \text{as } z \to \infty, \tag{6.30}$$

from $\operatorname{Im} z \geq 0$ and bounded, and for each fixed $\varepsilon > 0$.

To prove the remaining formula (3.7), we start with the equation

$$\partial_z\Phi(x;z,\varepsilon) = -\mathrm{i}\,\Phi(x;z,\varepsilon)\int_0^x \Phi^{-1}(y;z,\varepsilon)\sigma_3\Phi(y;z,\varepsilon)\,\mathrm{d}y, \tag{6.31}$$

obtained by differentiating (1.1) (with $\mathbf{v}$ replaced by Φ) with respect to z and then solving the resulting nonhomogeneous ODE for $\partial_z\Phi$. Equation (3.7) follows after substituting (6.30) into (6.31). A further consequence of Lemmas 6.1 and 6.2 is that if $q' \in L^1([0, L])$, then the normalized matrix fundamental solution $\Phi(x; z, \varepsilon)$ has the asymptotics

$$\Phi(x; z, \varepsilon) = \Big[\mathbb{I} + \frac{1}{2\mathrm{i}z}\sigma_3 \int_0^x |q(y)|^2 \,\mathrm{d}y/\varepsilon + \frac{1}{2\mathrm{i}z}\sigma_3(Q(x) - \mathrm{e}^{-2\mathrm{i}zx\sigma_3/\varepsilon} Q(0)) + o\Big(\frac{1}{z}\Big)\Big] \mathrm{e}^{-\mathrm{i}zx\sigma_3/\varepsilon}, \tag{6.32}$$

as $z \to \infty$ from $\operatorname{Im} z \geq 0$ and bounded, and for each fixed $\varepsilon > 0$. Then, substituting (6.32) into (6.31) gives

$$\partial_z \Phi(x; z, \varepsilon) = \Big(-\mathrm{i}\frac{x}{\varepsilon}\sigma_3 - \frac{x}{2\varepsilon^2 z}\int_0^x |q(y)|^2 \,\mathrm{d}y\Big) \mathrm{e}^{-\mathrm{i}zx\sigma_3/\varepsilon} + \frac{x}{2\varepsilon z}\Big(\mathrm{e}^{-\mathrm{i}zx\sigma_3/\varepsilon} Q(0) + \mathrm{e}^{\mathrm{i}zx\sigma_3/\varepsilon} Q(x)\Big) + o\Big(\frac{\mathrm{e}^{L \operatorname{Im} z}}{z}\Big), \tag{6.33}$$

as $z \to \infty$ from $\operatorname{Im} z \geq 0$. Finally, (3.8) is given by $\Delta'_\varepsilon(z) = \frac{1}{2} \operatorname{tr} \partial_z \Phi(L; z, \varepsilon)$. ■

Proof of Corollary 3.6. From Lemmas 2.2 and 3.4, we know Δ_ε is an entire function differing from a constant. Thus, Δ_ε must have either a pole, or an essential singularity at infinity. Suppose Δ_ε has a pole at infinity. Then, it must be a polynomial. This implies there exists an integer $n \geq 1$ such that $\Delta_\varepsilon(z) = O(z^n)$ as $z \to \infty$. By Lemma 3.4, one has $\Delta_\varepsilon(z) = O(\mathrm{e}^{L \operatorname{Im} z})$ as $\operatorname{Im} z \to \infty$ which is a contradiction. Thus, Δ_ε must have an essential singularity at infinity. ■

Proof of Lemma 3.7. Fix $\nu_o \in \mathbb{R}$. Define $f_\varepsilon(z) := \Delta_\varepsilon(z) - \cos(\nu_o L)$. By Lemma 2.2, f_ε is an entire function of z. Further, using (2.16) one gets $\Sigma_{\nu_o} = \{z \in \mathbb{C}\colon f_\varepsilon(z) = 0\}$. Hence, Σ_{ν_o} is the set of zeros of an entire function. Thus, the Floquet spectrum is discrete. ■

Proof of Lemma 3.8. Let $z \in \Sigma_{\mathrm{Lax}}$. This implies $\mathbf{v} \in L^\infty(\mathbb{R})$ is a non-trivial solution of (1.1) bounded for all $x \in \mathbb{R}$. By Floquet's theorem, $\mathbf{v} = \mathrm{e}^{\mathrm{i}\nu x}\, \mathbf{w}$, where $\mathbf{w}(x + L; z, \varepsilon) = \mathbf{w}(x; z, \varepsilon)$, and $\nu \in \mathbb{R}$. Plugging this expression for $\mathbf{v}$ into (1.1) gives a modified scattering problem, namely,

$$\mathrm{i}\,\varepsilon \mathbf{w}' = \begin{pmatrix} z + \varepsilon\nu & \mathrm{i}\,q(x) \\ -\mathrm{i}\,\overline{q(x)} & -z + \varepsilon\nu \end{pmatrix} \mathbf{w}. \tag{6.34}$$

Write (6.34) in component form:

$$\mathrm{i}\,\varepsilon w_1' - \mathrm{i}\,q(x) w_2 = (z + \varepsilon\nu) w_1, \tag{6.35a}$$

$$\mathrm{i}\,\varepsilon w_2' + \mathrm{i}\,\overline{q(x)} w_1 = (-z + \varepsilon\nu) w_2. \tag{6.35b}$$

Multiply (6.35a) by $\bar{w}_1$ and take the complex conjugate. This gives two equations which we integrate over a full period. Thus, we arrive at the integrated expressions

$$\mathrm{i}\int_0^L q(x)\bar{w}_1 w_2 \,\mathrm{d}x = -\mathrm{i}\,\varepsilon\langle w_1, w_1'\rangle - (z+\varepsilon\nu)\langle w_1, w_1\rangle, \tag{6.36a}$$

$$\mathrm{i}\int_0^L \overline{q(x)} w_1 \bar{w}_2 \,\mathrm{d}x = \mathrm{i}\,\varepsilon\langle w_1, w_1'\rangle + (\bar{z}+\varepsilon\nu)\langle w_1, w_1\rangle, \tag{6.36b}$$

where $\langle\cdot,\cdot\rangle$ is the $L^2([0,L])$ inner product of a scalar function (cf. A.1), and boundary terms vanish since $w_1(x+L;z,\varepsilon) = w_1(x;z,\varepsilon)$. Adding (6.36a) to (6.36b), one gets

$$-\operatorname{Im} z\|w_1\|_2^2 = \operatorname{Re}\langle q w_2, w_1\rangle. \tag{6.37}$$

Similarly, multiply (6.35b) by $\bar{w}_2$, take the complex conjugate, and integrate. Thus, we have the integrated expressions

$$\mathrm{i}\int_0^L \overline{q(x)} w_1 \bar{w}_2 \,\mathrm{d}x = \mathrm{i}\,\varepsilon\langle w_2, w_2'\rangle + (-z+\varepsilon\nu)\langle w_2, w_2\rangle, \tag{6.38a}$$

$$\mathrm{i}\int_0^L q(x)\bar{w}_1 w_2 \,\mathrm{d}x = -\mathrm{i}\,\varepsilon\langle w_2, w_2'\rangle + (\bar{z}-\varepsilon\nu)\langle w_2, w_2\rangle, \tag{6.38b}$$

where again boundary terms vanish since $w_2(x+L;z,\varepsilon) = w_2(x;z,\varepsilon)$. Adding (6.38a) and (6.38b), one gets

$$-\operatorname{Im} z\|w_2\|_2^2 = \operatorname{Re}\langle q w_2, w_1\rangle. \tag{6.39}$$

Equating (6.37) and (6.39), we conclude

$$z \in \Sigma_{\mathrm{Lax}} \setminus \mathbb{R} \implies \|w_1\|_2 = \|w_2\|_2. \tag{6.40}$$

Thus, the Cauchy–Schwarz inequality implies

$$0 < |\operatorname{Im} z|\|w_2\|_2^2 \le |\langle q w_2, w_1\rangle| \le \|q\|_\infty \|w_2\|_2^2.$$

Hence, one gets (3.9).

Next, let $\zeta \in \Sigma_{\mathrm{Dir}}(x_o)$. Without loss of generality, take $x_o = 0$. Note that the Dirichlet boundary conditions (2.19) imply

$$|v_1(L;\zeta,\varepsilon)|^2 - |v_1(0;\zeta,\varepsilon)|^2 = |v_2(L;\zeta,\varepsilon)|^2 - |v_2(0;\zeta,\varepsilon)|^2. \tag{6.41}$$

Write (1.1) in component form:

$$\mathrm{i}\,\varepsilon v_1' - \mathrm{i}\,q(x) v_2 = \zeta v_1, \tag{6.42a}$$

$$\mathrm{i}\,\varepsilon v_2' + \mathrm{i}\,\overline{q(x)} v_1 = -\zeta v_2. \tag{6.42b}$$

Multiply (6.42a) by $\bar{v}_1$ and take the complex conjugate. This gives two equations which we integrate over a full period. Thus, we arrive at the integrated expressions

$$\begin{aligned} \mathrm{i}\int_0^L q(x)\bar{v}_1 v_2 \,\mathrm{d}x \\ = \mathrm{i}\,\varepsilon(|v_1(L;\zeta,\varepsilon)|^2 - |v_1(0;\zeta,\varepsilon)|^2) - \mathrm{i}\,\varepsilon\langle v_1, v_1'\rangle - \zeta\|v_1\|_2^2, \end{aligned} \tag{6.43a}$$

and

$$\mathrm{i}\int_0^L \overline{q(x)} v_1 \bar{v}_2 \,\mathrm{d}x = \mathrm{i}\,\varepsilon\langle v_1, v_1'\rangle + \bar{\zeta}\|v_1\|_2^2. \tag{6.43b}$$

Adding (6.43a) to (6.43b), one gets

$$\begin{aligned} \int_0^L q(x)\bar{v}_1 v_2 + \overline{q(x)} v_1 \bar{v}_2 \,\mathrm{d}x \\ = \varepsilon(|v_1(L;\zeta,\varepsilon)|^2 - |v_1(0;\zeta,\varepsilon)|^2) - 2\,\mathrm{Im}\,\zeta\|v_1\|_2^2. \end{aligned} \tag{6.44}$$

Similarly, multiply (6.42b) by $\bar{v}_2$ and take the complex conjugate. This gives two equations which we integrate over a full period. Then, add to get

$$\begin{aligned} \int_0^L q(x)\bar{v}_1 v_2 + \overline{q(x)} v_1 \bar{v}_2 \,\mathrm{d}x \\ = -\varepsilon(|v_2(L;\zeta,\varepsilon)|^2 - |v_2(0;\zeta,\varepsilon)|^2) - 2\,\mathrm{Im}\,\zeta\|v_2\|_2^2. \end{aligned} \tag{6.45}$$

Adding (6.44) to (6.45) gives

$$-\mathrm{Im}\,\zeta\langle \mathbf{v}, \mathbf{v}\rangle = \int_0^L q(x)\bar{v}_1 v_2 + \overline{q(x)} v_1 \bar{v}_2 \,\mathrm{d}x, \tag{6.46}$$

where $\langle \mathbf{v}, \mathbf{v}\rangle$ is the $L^2([0, L], \mathbb{C}^2)$ inner product of a two-component vector function (cf. A.1). Thus, since $2|v_1||v_2| \le |v_1|^2 + |v_2|^2$, it follows

$$|\,\mathrm{Im}\,\zeta|\|\mathbf{v}\|_2^2 \le \|q\|_\infty \int_0^L |\bar{v}_1 v_2| + |v_1 \bar{v}_2| \,\mathrm{d}x \le \|q\|_\infty \|\mathbf{v}\|_2^2.$$

Hence, one gets (3.10). ■

In the following proofs we use the fact that one can rewrite (3.3) as

$$\Delta_\varepsilon(z) = \cos(zL/\varepsilon) + o(1), \quad z \to \infty,\ 0 \le \mathrm{Im}\,z \le \|q\|_\infty, \tag{6.47}$$

for each fixed $\varepsilon > 0$, and that $\Delta_\varepsilon(\bar{z}) = \overline{\Delta_\varepsilon(z)}$.

Lemma 6.3. *Let R_n denote the rectangular region with vertices $\pm\frac{n\pi\varepsilon}{L} \pm \mathrm{i}\,\|q\|_\infty$, where $n \in \mathbb{N}$, and $\varepsilon > 0$ is fixed. Define $R_{k,n} := R_k \setminus R_n$ where $k \in \mathbb{N}$ with $k > n$. Then, for k, n sufficiently large, the functions $\Delta_\varepsilon(z)$ and $\cos(zL/\varepsilon)$ have the same number of zeros inside $R_{k,n}$.*

Proof of Lemma 6.3. Let us fix some arbitrary $\delta \in (0, 1)$ such that $\delta < \sinh(\|q\|_\infty)$ and assume that $k, n \in \mathbb{N}$ are so large that, according to (6.47)

$$|\Delta_\varepsilon(z) - \cos(zL/\varepsilon)| \le \delta, \quad z \in \partial R_{k,n}, \tag{6.48}$$

where $\partial R_{k,n}$ is the boundary of $R_{k,n}$. Then, using $|\cos(x + \mathrm{i}\,y)|^2 = \cos^2 x + \sinh^2 y$, we see that $|\cos(zL/\varepsilon)| \ge 1$, and $|\cos(zL/\varepsilon)| \ge \sinh(\|q\|_\infty)$ on the vertical and the horizontal sides of $\partial R_{k,n}$, respectively. Thus, by inequality (6.48),

$$0 < |\Delta_\varepsilon(z) - \cos(zL/\varepsilon)| < |\cos(zL/\varepsilon)|, \quad z \in \partial R_{k,n}. \tag{6.49}$$

We now apply Rouché's theorem (cf. [60]) for $g(z) := \Delta_\varepsilon(z) - \cos(zL/\varepsilon)$ and $f(z) := \cos(zL/\varepsilon)$ to complete the argument. ∎

Proof of Lemma 3.9. By Lemma 3.8, all zeros of $\Delta_\varepsilon(z)$ are confined to $\mathbf{S} := \{z \in \mathbb{C}\colon |\operatorname{Im} z| \le \|q\|_\infty\}$ (cf. (2.15)). Take $\delta \in (0, 1)$ as in Lemma 6.3 and choose $M > 0$ so large that the inequality (6.48) holds for all $z \in \mathbf{S}$ such that $|\operatorname{Re} z| > M$. Denote by $\mathbf{S}_M$ the set of such z. To prove the lemma, we first prove that $\Delta_\varepsilon(z)$ has infinitely many real zeros and then show that it has no complex (non-real) zeros in $\mathbf{S}_M$.

Consider the points $z_n = n\pi\varepsilon/L$ and $z_{n+1} = (n + 1)\pi\varepsilon/L$, $n \in \mathbb{N}$, where, say, $z_n > M$. The remaining case $z_{-n} < -M$ can be worked out similarly. Since $\Delta_\varepsilon(z)$ is real on $\mathbb{R}$, (6.48) shows that $\Delta_\varepsilon(z_n)\Delta_\varepsilon(z_{n+1}) < 0$ and, thus, there exists a zero of $\Delta_\varepsilon(z)$ on (z_n, z_{n+1}). Thus, there are infinitely many zeros of $\Delta_\varepsilon(z)$ on $\mathbb{R}$.

We can now use Lemma 6.3 to show that there is exactly one zero of $\Delta_\varepsilon(z)$ on the rectangle $R_{n,n+1}$ and, as it was just shown above this zero is real. Thus, there are no non-real zeros in $\mathbf{S}_M$ and we have completed the proof about the zeros of $\Delta_\varepsilon(z)$. In view of (3.7), the proof of zeros of $\Delta'_\varepsilon(z)$ is similar. ∎

Proof of Theorem 3.10. Consider the set $\rho := \mathbf{H} \setminus \Sigma_{\mathrm{Lax}}$, where $\mathbf{H} := \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ is the upper half-plane. Since Σ_{Lax} is Schwarz symmetric, and $\mathbb{R} \subset \Sigma_{\mathrm{Lax}}$ (cf. Lemma 3.3), one gets $\rho_{\mathrm{Lax}} = \rho \cup \bar{\rho}$. To prove the result it is only necessary to show that ρ is connected. First, by Lemma 3.2 no spectral band can be closed in the finite z-plane. Next, we show that the only spectral band extending to infinity is the real z-axis. Suppose to the contrary that there exists a band extending to infinity and intersecting the real z-axis at most once. By Lemma 3.8 and (6.47), it is sufficient to assume the band is confined to $\{z \in \mathbb{C}\colon \operatorname{Re} z \ge 0, 0 \le \operatorname{Im} z \le \|q\|_\infty\}$. Recall that along a spectral band one necessarily has $\operatorname{Im} \Delta_\varepsilon \equiv 0$. Fix $\delta \in (0, 1)$ such that $\delta < \sinh(\|q\|_\infty)$. Then, there exists $N = N(q; \varepsilon) > 0$ such that

$$|\operatorname{Re} \Delta_\varepsilon(z) - \cos(\operatorname{Re} zL/\varepsilon) \cosh(\operatorname{Im} zL/\varepsilon)| < \delta, \tag{6.50}$$

for all $\operatorname{Re} z > N$. Let $n \in \mathbb{N}$ be such that $N < n\pi\varepsilon/L$. As in Lemma 3.9, it then follows that there exists a zero of $\Delta_\varepsilon(z)$ for $\operatorname{Re} z \in (n\pi\varepsilon/L, (n+1)\pi\varepsilon/L)$. Further, by assumption, $\operatorname{Im} z = 0$ for at most one point on the band. This implies that Δ_ε has infinitely many complex zeros which contradicts Lemma 3.9. Thus, we have shown that the real z-axis is the only spectral band extending to infinity. Hence, the set ρ is connected which completes the proof. ∎

Proof of Theorem 3.11. Recall that $\Delta_\varepsilon(z)$ is real-valued along the real z-axis. By Lemma 3.9, there exists a sequence $\{z_n\}$ of simple real zeros of Δ_ε for $|n|$ sufficiently large. Suppose that z_{k-1} and z_k are two consecutive zeros such that $\Delta'_\varepsilon(z_{k-1}) > 0$, and $\Delta'_\varepsilon(z_k) < 0$. Then, by Rolle's theorem, there exists $z_k^{\mathrm{c}} \in (z_{k-1}, z_k)$ such that $\Delta'_\varepsilon(z_k^{\mathrm{c}}) = 0$. Moreover, $\{z_n^{\mathrm{c}}\}$ are simple zeros of Δ'_ε by Lemma 3.9. Also, by Lemma 3.3, necessarily $0 < \Delta_\varepsilon(z_k^{\mathrm{c}}) \le 1$. Suppose $\Delta_\varepsilon(z_k^{\mathrm{c}}) < 1$. Since $\operatorname{Re} \Delta_\varepsilon$ is harmonic, it follows that z_k^{c} is a saddle point; and a band emanates from z_k^{c} into the complex plane along a steepest ascent curve. Further, $\operatorname{Im} \Delta_\varepsilon \equiv 0$ along this steepest ascent curve. Thus, by Theorem 3.10 and continuity, there must exist a band edge along this steepest ascent curve at which one gets $\Delta_\varepsilon(z_k^+) = 1$. Otherwise, $\Delta_\varepsilon(z_k^{\mathrm{c}}) = 1$ (i.e., z_k^{c} is a double point) and no further analysis is required. Next, $\Delta'_\varepsilon(z_k) < 0$ implies $\Delta'_\varepsilon(z_{k+1}) > 0$. In this case, $-1 \le \Delta_\varepsilon(z_{k+1}^{\mathrm{c}}) < 0$, and the argument is completely analogous to that above. Finally, by Theorem 3.14 the bands emanating from the real z-axis do not intersect; and by Lemma 3.7 the Floquet spectrum is discrete. This completes the proof. ∎

Proof of Theorem 3.12. Suppose q is a finite-band potential. Recall $z \in \Sigma_{\mathrm{Lax}}$ implies $|\operatorname{Im} z| \le \|q\|_\infty$ (cf. Lemma 3.8). By Theorem 3.10, the real z-axis is the only band extending to infinity. So, there are finitely many bands and each band (except $\mathbb{R}$) is bounded. Thus, $\Sigma_{\mathrm{Lax}} \setminus \mathbb{R}$ is bounded and the result follows trivially. Next, let $N = N(q; \varepsilon) > 0$ be such that $(\Sigma_{\mathrm{Lax}} \setminus \mathbb{R}) \subset R_N$. Suppose Σ_{Lax} is comprised of infinitely many bands. Then, infinitely many periodic eigenvalues (which correspond to band edges) exist in the closure of R_N. This implies the set of periodic eigenvalues must have a finite limit point which is a contradiction (cf. Lemma 3.7). ∎

Proof of Theorem 3.13. Recall that the periodic and antiperiodic Floquet eigenvalues correspond to the band edges of Σ_{Lax}. Suppose that there exist infinitely many spectral bands. Then, by Theorem 3.12, $(\Sigma_{\mathrm{Lax}} \setminus \mathbb{R}) \cap R_N \neq \emptyset$ for any $N > 0$. This implies that there exist infinitely many periodic, or antiperiodic, eigenvalues along the imaginary z-axis which is a contradiction. The contradiction follows from the fact that the Floquet eigenvalues are discrete with no finite accumulation points, and $|\operatorname{Im} z| \leq \|q\|_\infty$. ■

Proof of Theorem 3.14. Let γ_j denote a generic spectral band. Two possibilities arise: either γ_j has a periodic and an antiperiodic endpoint; or both of its endpoints are periodic (or antiperiodic). If one endpoint is periodic and the other is antiperiodic, $\Delta_\varepsilon(z) = 0$ at some point on γ_j. Conversely, if both endpoints are periodic or antiperiodic, there is a point on γ_j, where $\Delta'_\varepsilon(z) = 0$. In either case, according to Lemma 3.9, there can be no more than finitely many such bands that do not intersect $\mathbb{R}$. In order to prove that these bands are actually spines, it remains to show that, as $z \to \infty$, these bands cannot intersect any other bands. So, let γ_j be such a band, which intersects $\mathbb{R}$ at some point z_o. Then $\Delta'_\varepsilon(z_o) = 0$. Again, by Lemma 3.9 there can only be finitely many such bands that have more than one point where $\Delta'_\varepsilon(z) = 0$ (since the number of complex zeros of Δ'_ε is finite). Hence, there can be only finitely many bands that, in addition to $\mathbb{R}$, also intersects some other band. Thus, we proved that all but finitely many bands are the spines. Moreover, by Lemma 3.5 $\Delta'_\varepsilon(z) = -L\sin(zL/\varepsilon)/\varepsilon + o(1)$ as $z \to \infty$ from $|\operatorname{Im} z| \leq \|q\|_\infty$. Thus, spines are $o(1)$-close to $n\pi\varepsilon/L$ for $|n| \in \mathbb{N}$ sufficiently large. That this is the only $o(1)$-close spine follows from Rouché's theorem. This completes the proof. ■

6.2. Proofs: Semiclassical limit

Proof of Lemma 4.1. Let $z \in \Sigma_{\text{Lax}}$ be such that $|\operatorname{Im} z| > 0$, and $|\operatorname{Re} z| > 0$. Again, we can write $\mathbf{v} = \mathrm{e}^{\mathrm{i}\nu x}\,\mathbf{w}$, where $\mathbf{w}(x+L; z, \varepsilon) = \mathbf{w}(x; z, \varepsilon)$, and $\nu \in \mathbb{R}$. Plugging this expression for $\mathbf{v}$ into (1.1) gives the modified ZS system (6.34). Multiply (6.35a) by $\overline{w}'_1$ and integrate by parts:

$$\mathrm{i}\,\varepsilon\|w'_1\|_2^2 + \mathrm{i}\int_0^L (q(x)w_2)'\,\overline{w}_1\,\mathrm{d}x = (z + \varepsilon\nu)\langle w_1, w'_1\rangle. \tag{6.51}$$

The complex conjugate to (6.51) is

$$-\mathrm{i}\,\varepsilon\|w'_1\|_2^2 - \mathrm{i}\int_0^L (\overline{q(x)}\,\overline{w}_2)'\,w_1\,\mathrm{d}x = (\bar{z} + \varepsilon\nu)\langle w'_1, w_1\rangle. \tag{6.52}$$

Integrate the right-hand side of (6.52) by parts, then add to (6.51) and multiply by ε getting

$$\begin{aligned} &2\,\mathrm{i}\,\varepsilon\operatorname{Im}\langle q' w_2, w_1\rangle \\ &\quad = 2\operatorname{Im} z\langle w_1, \varepsilon w'_1\rangle + \int_0^L \big(\overline{q(x)}\,w_1(\varepsilon\overline{w}'_2) - q(x)\overline{w}_1(\varepsilon w'_2)\big)\,\mathrm{d}x. \end{aligned} \tag{6.53}$$

Using (6.35a), one gets

$$\begin{aligned}&2\,\mathrm{Im}\,z\langle w_1, \varepsilon w_1'\rangle\\ &\quad = 2\,\mathrm{Im}\,z\langle w_1, q w_2\rangle + 2\,\mathrm{i}\,\mathrm{Im}\,z(\bar{z} + \varepsilon\nu)\|w_2\|_2^2.\end{aligned} \tag{6.54}$$

Further, from (6.35b) and recalling equations (6.37), and (6.39), one gets

$$\begin{aligned}&\int_0^L \left(\overline{q(x)}w_1(\varepsilon\overline{w}_2') - q(x)\overline{w}_1(\varepsilon w_2')\right)\mathrm{d}x\\ &= 2\,\mathrm{i}\,\mathrm{Im}\,z\,\mathrm{Im}\langle q w_2, w_1\rangle - 2\,\mathrm{i}\,\mathrm{Im}\,z(\varepsilon\nu - \mathrm{Re}\,z)\|w_2\|_2^2.\end{aligned} \tag{6.55}$$

Substitute (6.54) and (6.55) into (6.53) and simplify to get

$$4\,\mathrm{i}\,\mathrm{Im}\,z\,\mathrm{Re}\,z\|w_2\|_2^2 = 2\,\mathrm{i}\,\varepsilon\,\mathrm{Im}\langle q' w_2, w_1\rangle \tag{6.56}$$

Applying the Cauchy–Schwarz inequality gives

$$0 < 2|\,\mathrm{Re}\,z||\,\mathrm{Im}\,z|\|w_2\|_2^2 \le \varepsilon|\langle q' w_2, w_1\rangle| \le \varepsilon\|q'\|_\infty\|w_2\|_2^2.$$

Hence, one gets (4.1). ∎

Proof of Theorem 4.2. This result follows immediately from Lemmas 3.8 and 4.1. ∎

Proof of Theorem 4.3. This result follows from the cardinality results found in [29, 30] together with Lemmas 3.8 and 4.1. ∎

Proof of Corollary 4.4. Fix $\nu \in \mathbb{R}$. Denote the corresponding countably infinite set of eigenvalues by $\{z_n(\nu)\}_{n\in\mathbb{N}}$. Without loss of generality, assume that $\mathrm{Re}\,z > 0$ and $\mathrm{Im} \ge 0$. By Lemma 2.2 and (2.16), it follows that in any neighborhood of infinity Σ_ν is infinite. Further, $\Sigma_\nu \cap D(0; r)$ is finite for any $r > 0$, where

$$D(0; r) := \{z \in \mathbb{C} \colon |z| \le r\}.$$

This implies that infinity is the only accumulation point of Σ_ν, and that, due to Lemma 3.8, there exists a partially ordered increasing sequence $\{\mathrm{Re}\,z_n(\nu)\}_{n\in\mathbb{N}}$ such that $\mathrm{Re}\,z_n(\nu) \to \infty$ as $n \to \infty$. Finally, that $\mathrm{Im}\,z_n(\nu) = O(1/\,\mathrm{Re}\,z_n(\nu))$ now follows easily from Lemma 4.1. ∎

It now remains to show that Lemmas 3.8 and 4.1 together imply Corollary 4.5.

Proof of Corollary 4.5. Fix $\delta > 0$. Consider the δ-neighborhood defined by $N_\delta(\Sigma_\infty)$. Without loss of generality, assume that $\mathrm{Im}\,z > 0$, and that $|\,\mathrm{Re}\,z| > 0$. Next, consider the curve in the spectral plane defined by $|\,\mathrm{Im}\,z| = \min\left\{\|q\|_\infty, \varepsilon\|q'\|_\infty/2|\,\mathrm{Re}\,z|\right\}$ which bounds the set $\Lambda^\varepsilon(q)$. Then, it is easily seen that there exists $\varepsilon_* > 0$ such that $\Lambda^{\varepsilon_*}(q) \subset N_\delta(\Sigma_\infty)$. Thus, if $z \in \Sigma_{\mathrm{Lax}}$, then $z \in N_\delta(\Sigma_\infty)$ whenever $0 < \varepsilon \le \varepsilon_*$. Hence, $\Sigma_{\mathrm{Lax}} \cap (\mathbb{C} \setminus N_\delta(\Sigma_\infty)) = \emptyset$ which completes the proof. ∎

6.3. Proofs: Real or symmetric potentials

Proof of Lemma 5.1. Let $\mathbf{v}(x; z, \varepsilon)$ be a bounded solution of the ZS system (1.1). Then, (A.7) establishes the $z \mapsto \bar{z}$ symmetry of the spectrum. Similarly, (A.8) and (A.9) establish the $z \mapsto -\bar{z}$ symmetry in the spectrum when the potential is real, even, or odd. Suppose that the potential is real. By (A.8), it follows that the monodromy matrix satisfies the symmetry (5.1). Then, noting $z = -\bar{z}$ if and only if $z = \mathrm{i}\,\eta$, we have $\operatorname{tr} M(z; \varepsilon)$ is real-valued for $z \in \mathrm{i}\,\mathbb{R}$. Similarly, using (A.9), one gets the symmetry (5.2). Thus, if the potential is even ($\theta = 0 \mod \pi$), or odd ($\theta = \pi/2 \mod \pi$), then we have $\operatorname{tr} M(z; \varepsilon)$ is real-valued for $z \in \mathrm{i}\,\mathbb{R}$. Finally, recalling that $\Delta_\varepsilon(z) = \operatorname{tr} M(z; \varepsilon)/2$ it follows that when q is real, even, or odd one gets $\mathrm{i}\,\mathbb{R} \subset \Gamma$. ∎

Proof of Theorem 5.2. Let $z_n^\pm$ be a periodic, or antiperiodic eigenvalue, respectively. This implies $\operatorname{Im} \Delta_\varepsilon(z_n^\pm) = 0$, and $|\operatorname{Re} \Delta_\varepsilon(z_n^\pm)| = 1$. By assumption, $z_n^\pm \in \mathbb{R} \cup \mathrm{i}\,\mathbb{R}$. Further, $z_n^\pm$ belongs to Σ_{Lax}. Suppose $z_n^\pm \in \gamma_j$, where γ_j is a band that leaves the real, or imaginary axis. By analyticity of $\Delta_\varepsilon(z)$, we need consider only two cases. First, suppose there exists another point $z_k^\pm \neq z_n^\pm$ along the spectral band such that $|\operatorname{Re} \Delta_\varepsilon(z_k^\pm)| = 1$. Then, $z_k^\pm \in \mathbb{R} \cup \mathrm{i}\,\mathbb{R}$., Since the real and imaginary z-axes are contours such that $\operatorname{Im} \Delta_\varepsilon(z) = 0$, we get a closed curve in the set Γ, which is a contradiction (see Lemmas 3.2 and 5.1). Second, suppose $|\operatorname{Re} \Delta_\varepsilon(z)| < 1$ for all $z \in \gamma_j$. It then follows that γ_j extends to infinity. By Theorem 3.10, the only element of Σ_{Lax} which has this property is the real z-axis. Again, we have a contradiction. Hence, $\Sigma_{\mathrm{Lax}} \subset \Sigma_\infty \subset \mathbb{R} \cup \mathrm{i}\,\mathbb{R}$. ∎

Proof of Lemma 5.3. Let $\zeta \in \Sigma_{\mathrm{Dir}}(0)$. Without loss of generality, assume $\operatorname{Im} \zeta > 0$, and $\operatorname{Re} \zeta > 0$. Recall that the Dirichlet boundary conditions (2.19) imply

$$|v_1(L; \zeta, \varepsilon)|^2 - |v_1(0; \zeta, \varepsilon)|^2 = |v_2(L; \zeta, \varepsilon)|^2 - |v_2(0; \zeta, \varepsilon)|^2. \tag{6.57}$$

Further, the hypotheses imply $\operatorname{Im} q(0) = 0$. Write (1.1) in component form,

$$\mathrm{i}\,\varepsilon v_1' - \mathrm{i}\,q(x) v_2 = \zeta v_1, \tag{6.58a}$$

$$\mathrm{i}\,\varepsilon v_2' + \mathrm{i}\,\overline{q(x)} v_1 = -\zeta v_2. \tag{6.58b}$$

Multiply (6.58a) by $\bar{v}_1'$ and integrate by parts:

$$\mathrm{i}\,\varepsilon \|v_1'\|_2^2 - \mathrm{i}\,q(0)\big(\overline{v_1(L)} v_2(L) - \overline{v_1(0)} v_2(0)\big) + \mathrm{i} \int_0^L (q(x) v_2)' \bar{v}_1 \,\mathrm{d}x = \zeta \langle v_1, v_1' \rangle. \tag{6.59}$$

Then, using the boundary conditions, one can write

$$\mathrm{i}\,\varepsilon \|v_1'\|_2^2 + \mathrm{i}\,q(0)\big(|v_1(L)|^2 - |v_1(0)|^2\big) + \mathrm{i} \int_0^L (q(x) v_2)' \bar{v}_1 \,\mathrm{d}x = \zeta \langle v_1, v_1' \rangle. \tag{6.60}$$

Take the complex conjugate and add to (6.59):

$$\mathrm{i}\int_0^L (q(x)v_2)'\bar{v}_1 - (\overline{q(x)}\bar{v}_2)'v_1\,\mathrm{d}x = \zeta\langle v_1, v_1'\rangle + \bar{\zeta}\langle v_1', v_1\rangle. \tag{6.61}$$

Note that

$$\bar{\zeta}\langle v_1', v_1\rangle = \bar{\zeta}\big(|v_1(L)|^2 - |v_1(0)|^2\big) - \bar{\zeta}\langle v_1, v_1'\rangle. \tag{6.62}$$

Let $\alpha(\zeta) := \mathrm{i}\,\varepsilon\bar{\zeta}(|v_1(L)|^2 - |v_1(0)|^2)$. Then, using (6.61) and (6.62), one gets

$$\begin{aligned}\alpha(\zeta) &+ \varepsilon\int_0^L q'(x)\bar{v}_1 v_2 - \overline{q'(x)}v_1\bar{v}_2\,\mathrm{d}x\\ &= (2\varepsilon\operatorname{Im}\zeta)\langle v_1, v_1'\rangle + \varepsilon\int_0^L \overline{q(x)}v_1\bar{v}_2' - q(x)\bar{v}_1 v_2'\,\mathrm{d}x,\end{aligned} \tag{6.63}$$

where we multiplied through by ε. Using (6.58b), one then gets

$$\begin{aligned}\varepsilon&\int_0^L \overline{q(x)}v_1\bar{v}_2' - q(x)\bar{v}_1 v_2'\,\mathrm{d}x\\ &= (-2\,\mathrm{i}\operatorname{Re}\zeta)\operatorname{Re}\langle qv_2, v_1\rangle + (2\,\mathrm{i}\operatorname{Im}\zeta)\operatorname{Im}\langle qv_2, v_1\rangle.\end{aligned} \tag{6.64}$$

Then, using (6.44), we get

$$(-2\,\mathrm{i}\operatorname{Re}\zeta)\operatorname{Re}\langle qv_2, v_1\rangle = -\mathrm{i}\operatorname{Re}\zeta\big(\varepsilon(|v_1(L)|^2 - |v_1(0)|^2) - (2\operatorname{Im}\zeta)\|v_1\|_2^2\big). \tag{6.65}$$

Further, by (6.58a)

$$(2\varepsilon\operatorname{Im}\zeta)\langle v_1, v_1'\rangle = (2\operatorname{Im}\zeta)\int_0^L \overline{q(x)}v_1\bar{v}_2\,\mathrm{d}x + 2\big(\mathrm{i}\operatorname{Re}\zeta\operatorname{Im}\zeta + (\operatorname{Im}\zeta)^2\big)\|v_1\|_2^2. \tag{6.66}$$

Finally, using (6.64) and (6.66) and simplifying gives

$$\begin{aligned}\alpha(\zeta) &+ (2\,\mathrm{i}\,\varepsilon)\operatorname{Im}\langle q'v_2, v_1\rangle\\ &= -\mathrm{i}\,\varepsilon\zeta\big(|v_1(L)|^2 - |v_1(0)|^2\big) + (4\,\mathrm{i}\operatorname{Re}\zeta\operatorname{Im}\zeta)\|v_1\|_2^2.\end{aligned} \tag{6.67}$$

Next, we complete the same series of calculations using (6.58b). First, multiply (6.58b) by $\bar{v}_2'$ and integrate by parts.

$$\begin{aligned}\mathrm{i}\,\varepsilon\|v_2'\|_2^2 &+ \mathrm{i}\,\overline{q(0)}\big(v_1(L)\overline{v_2(L)} - v_1(0)\overline{v_2(0)}\big) - \mathrm{i}\int_0^L (\overline{q(x)}v_1)'\bar{v}_2\,\mathrm{d}x\\ &= -\zeta\langle v_2, v_2'\rangle.\end{aligned} \tag{6.68}$$

Then, using the boundary conditions, one can write

$$\mathrm{i}\,\varepsilon\|v_2\|_2^2 - \mathrm{i}\,\overline{q(0)}\big(|v_2(L)|^2 - |v_2(0)|^2\big) - \mathrm{i}\int_0^L (\overline{q(x)}v_1)'\bar{v}_2\,\mathrm{d}x = -\zeta\langle v_2, v_2'\rangle. \quad (6.69)$$

Take the complex conjugate and add

$$\mathrm{i}\int_0^L (q(x)\bar{v}_1)'v_2 - (\overline{q(x)}v_1)'\bar{v}_2\,\mathrm{d}x = -\zeta\langle v_2, v_2'\rangle - \overline{\zeta}\langle v_2', v_2\rangle. \quad (6.70)$$

Note that

$$-\zeta\langle v_2, v_2'\rangle = -\zeta\big(|v_2(L)|^2 - |v_2(0)|^2\big) + \zeta\langle v_2', v_2\rangle. \quad (6.71)$$

Let $\beta(\zeta) := -\mathrm{i}\,\varepsilon\zeta(|v_2(L)|^2 - |v_2(0)|^2)$. Then, using (6.70) and (6.71), one gets

$$\begin{aligned}\beta(\zeta) + \varepsilon\int_0^L q'(x)\bar{v}_1 v_2 - \overline{q'(x)}v_1\bar{v}_2\,\mathrm{d}x \\ = (2\varepsilon\,\mathrm{Im}\,\zeta)\langle v_2', v_2\rangle + \varepsilon\int_0^L \overline{q(x)}v_1'\bar{v}_2 - q(x)\bar{v}_1'v_2\,\mathrm{d}x,\end{aligned} \quad (6.72)$$

where we multiplied through by ε. Using (6.58a) and (6.45) one then gets

$$\begin{aligned}(2\,\mathrm{i})\,\mathrm{Im}\langle v_1', qv_2\rangle = \mathrm{i}\,\mathrm{Re}\,\zeta\big(\varepsilon(|v_1(L)|^2 - |v_1(0)|^2) + (2\,\mathrm{Im}\,\zeta)\|v_2\|_2^2\big) \\ + (2\,\mathrm{i}\,\mathrm{Im}\,\zeta)\,\mathrm{Im}\langle v_1, qv_2\rangle.\end{aligned} \quad (6.73)$$

Further, by (6.58b)

$$(2\varepsilon\,\mathrm{Im}\,\zeta)\langle v_2', v_2\rangle = (-2\,\mathrm{Im}\,z)\langle v_1, qv_2\rangle + 2\big(\mathrm{i}\,\mathrm{Re}\,\zeta\,\mathrm{Im}\,\zeta - (\mathrm{Im}\,\zeta)^2\big)\|v_2\|_2^2. \quad (6.74)$$

Finally, using (6.73) and (6.74) and simplifying gives

$$\begin{aligned}\beta(\zeta) + \varepsilon\int_0^L q'(x)\bar{v}_1 v_2 - \overline{q'(x)}v_1\bar{v}_2\,\mathrm{d}x \\ = \mathrm{i}\,\varepsilon\overline{\zeta}\big(|v_1(L)|^2 - |v_1(0)|^2\big) + (4\,\mathrm{i}\,\mathrm{Re}\,\zeta\,\mathrm{Im}\,\zeta)\|v_2\|_2^2.\end{aligned} \quad (6.75)$$

Add (6.75) to (6.67) to get

$$2\varepsilon\int_0^L q'(x)\bar{v}_1 v_2 - \overline{q'(x)}v_1\bar{v}_2\,\mathrm{d}x = 4\,\mathrm{i}\,\mathrm{Re}\,\zeta\,\mathrm{Im}\,\zeta\int_0^L |v_1|^2 + |v_2|^2\,\mathrm{d}x. \quad (6.76)$$

Noting $2|v_1||v_2| \leq |v_1|^2 + |v_2|^2$, one gets

$$\begin{aligned}4|\operatorname{Re}\zeta||\operatorname{Im}\zeta|\|\mathbf{v}\|_2^2 &= 2\varepsilon\left|\int_0^L q'(x)\bar{v}_1 v_2 - \overline{q'(x)} v_1 \bar{v}_2 \,\mathrm{d}x\right| \\ &\leq 2\varepsilon\|q'\|_\infty \int_0^L |\bar{v}_1 v_2| + |v_1 \bar{v}_2| \,\mathrm{d}x \\ &\leq 2\varepsilon\|q'\|_\infty \|\mathbf{v}\|_2^2,\end{aligned}$$

which in turn yields the desired result. ■

Proof of Theorem 5.4. This result follows from Lemmas 3.8 and 5.3. ■

Proof of Lemma 5.5. We take an approach analogous to the one first employed by Klaus and Shaw for single-lobe potentials decaying as $x \to \pm\infty$. Let $z \in \Sigma_{\mathrm{Lax}} \setminus \mathrm{i}\mathbb{R}$. This implies the solution $\mathbf{v} \not\equiv 0 \in L^\infty(\mathbb{R})$. By Floquet's theorem, $\mathbf{v} = \mathrm{e}^{\mathrm{i}\nu x}\mathbf{w}$, where $\mathbf{w}(x + L; z, \varepsilon) = \mathbf{w}(x; z, \varepsilon)$ and $\nu \in \mathbb{R}$. Plugging this expression for $\mathbf{v}(x; z, \varepsilon)$ into the ZS system gives (6.35) with $\overline{q(x)} = q(x)$. Multiply (6.35a) by $\overline{w}_2$, and (6.35b) by $\overline{w}_1$. Then, subtract and integrate over the period to get the expression

$$\begin{aligned}&\mathrm{i}\,\varepsilon\int_0^L w_1'\overline{w}_2 - \overline{w}_1 w_2' \,\mathrm{d}x - \mathrm{i}\int_0^L q(x)(|w_1|^2 + |w_2|^2)\,\mathrm{d}x \\ &= 2z\operatorname{Re}\langle w_1, w_2\rangle + 2\,\mathrm{i}\,\varepsilon\nu\operatorname{Im}\langle w_1, w_2\rangle.\end{aligned} \tag{6.77}$$

Note that, due to periodicity, integration by parts gives

$$\int_0^L w_1'\overline{w}_2 - \overline{w}_1 w_2' \,\mathrm{d}x = \int_0^L \overline{w}_1' w_2 - w_1 \overline{w}_2' \,\mathrm{d}x. \tag{6.78}$$

Hence,

$$\overline{\int_0^L w_1'\overline{w}_2 - \overline{w}_1 w_2' \,\mathrm{d}x} = \int_0^L w_1'\overline{w}_2 - \overline{w}_1 w_2' \,\mathrm{d}x, \tag{6.79}$$

and it follows that the left-hand side of (6.77) is purely imaginary. Thus,

$$(\operatorname{Re} z)\operatorname{Re}\langle w_1, w_2\rangle = 0.$$

By assumption, $\operatorname{Re} z > 0$ which implies $\operatorname{Re}\langle w_1, w_2\rangle = 0$. Next, multiply (6.35a) by $\overline{w}_1$ and write as

$$\overline{w}_1 w_2 = \frac{\varepsilon w_1'\overline{w}_1}{q(x)} + \mathrm{i}(z + \varepsilon\nu)\frac{|w_1|^2}{q(x)}. \tag{6.80}$$

Take the complex conjugate of (6.80), add, and integrate over the period:

$$2\,\mathrm{Re}\langle w_1, w_2\rangle = \varepsilon \int_0^L \frac{(w_1\bar{w}_1)'}{q(x)}\,\mathrm{d}x - 2\,\mathrm{Im}\,z \int_0^L \frac{|w_1|^2}{q(x)}\,\mathrm{d}x$$

$$= \varepsilon \int_0^L \frac{|w_1|^2 q'(x)}{q^2(x)}\,\mathrm{d}x - 2\,\mathrm{Im}\,z \int_0^L \frac{|w_1|^2}{q(x)}\,\mathrm{d}x$$

Note that, since $q(x) > 0$, it follows easily that

$$\int_0^L \frac{|w_1|^2}{q(x)}\,\mathrm{d}x \geq \min_{x\in[0,L]}\left\{\frac{1}{q(x)}\right\}\|w_1\|_2^2 > 0. \tag{6.81}$$

Finally, since $\mathrm{Re}\langle w_1, w_2\rangle = 0$ one gets

$$|\,\mathrm{Im}\,z| \int_0^L \frac{|w_1|^2}{q(x)}\,\mathrm{d}x = \left|\frac{\varepsilon}{2}\int_0^L \frac{|w_1|^2 q'(x)}{q^2(x)}\,\mathrm{d}x\right| \leq \frac{\varepsilon}{2}\|(\ln q)'\|_\infty \int_0^L \frac{|w_1|^2}{q(x)}\,\mathrm{d}x, \tag{6.82}$$

which gives (5.8).

Next, without loss of generality assume $x_o = 0$. Let $\zeta \in \Sigma_{\mathrm{Dir}}(0) \setminus \mathrm{i}\,\mathbb{R}$. Write the ZS system in component form as in (6.58). Multiply (6.58a) by $\bar{v}_2$, and (6.58b) by $\bar{v}_1$. Subtract and integrate over the period to get the expression

$$\mathrm{i}\,\varepsilon \int_0^L v_1'\bar{v}_2 - \bar{v}_1 v_2'\,\mathrm{d}x - \mathrm{i}\int_0^L q(x)(|v_1|^2 + |v_2|^2)\,\mathrm{d}x = 2\zeta\,\mathrm{Re}\langle v_1, v_2\rangle. \tag{6.83}$$

Then, note integration by parts gives

$$\int_0^L v_1'\bar{v}_2 - \bar{v}_1 v_2'\,\mathrm{d}x = \int_0^L \bar{v}_1' v_2 - v_1\bar{v}_2'\,\mathrm{d}x, \tag{6.84}$$

where since $\mathbf{v}(x; z, \varepsilon)$ is an eigenfunction corresponding to Dirichlet BCs (2.19) it follows

$$(v_1\bar{v}_2 - \bar{v}_1 v_2)|_0^L = -|v_2(L)|^2 + |v_1(L)|^2 + |v_2(0)|^2 - |v_1(0)|^2 = 0. \tag{6.85}$$

Hence,

$$\overline{\int_0^L v_1'\bar{v}_2 - \bar{v}_1 v_2'\,\mathrm{d}x} = \int_0^L v_1'\bar{v}_2 - \bar{v}_1 v_2'\,\mathrm{d}x, \tag{6.86}$$

and it follows that the left-hand side of (6.83) is purely imaginary. The rest of the argument is identical to that above for the Lax spectrum. This completes the proof. ■

7. Examples

In this section we illustrate the results of the previous sections by considering certain interesting classes of periodic potentials and by computing their Lax spectrum analytically, or numerically using Floquet–Hill's method [10]. (We refer the reader to A.6 for some details on Floquet–Hill's method.) All results were checked for numerical convergence.

7.1. Plane wave potentials

As our first example, we consider a potential whose spectrum can be computed exactly (see A.4 for details). Namely, we consider the plane wave potential

$$q(x) = A\,\mathrm{e}^{\mathrm{i}Vx}, \tag{7.1}$$

where $A \in \mathbb{R}$ is constant amplitude, and $V \in \mathbb{R}$ is the wave number.

When $V = 0$, we get a constant background. In this case,

$$\Sigma_{\mathrm{Lax}} = \mathbb{R} \cup \mathrm{i}[-A, A], \tag{7.2}$$

When $V \neq 0$ and $L = 2\pi/|V|$, the Lax spectrum is given by

$$\Sigma_{\mathrm{Lax}} = \mathbb{R} \cup [-\varepsilon V/2 - \mathrm{i}A,\ -\varepsilon V/2 + \mathrm{i}A]. \tag{7.3}$$

Thus, the Lax spectrum is composed of two bands in the complex plane, as shown in Figure 5. Note that in this case the number of bands is not proportional to $1/\varepsilon$ as $\varepsilon \downarrow 0$. Indeed, there are only two spectral bands for any $\varepsilon > 0$. Also, for any $\varepsilon > 0$ and $V \neq 0$, there are no spectral bands on the imaginary axis. It is only in the limit $\varepsilon \downarrow 0$ that the complex band becomes purely imaginary.

7.2. Piecewise continuous potentials

As a second example, we look at whether the assumptions in Theorem 4.5 are necessary or merely sufficient. To this end, suppose that the potential is only piecewise smooth. That is, q and q' have finite left and right limits for all x, and finitely many jump discontinuities on any bounded interval. Specifically, we consider the 2-periodic extension of the signum function

$$q(x) = \operatorname{sgn}(x) := \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{7.4}$$

where $x \in [-1, 1)$. In this case, the potential has a jump discontinuity, and therefore the hypotheses of Lemma 4.1 are not satisfied.

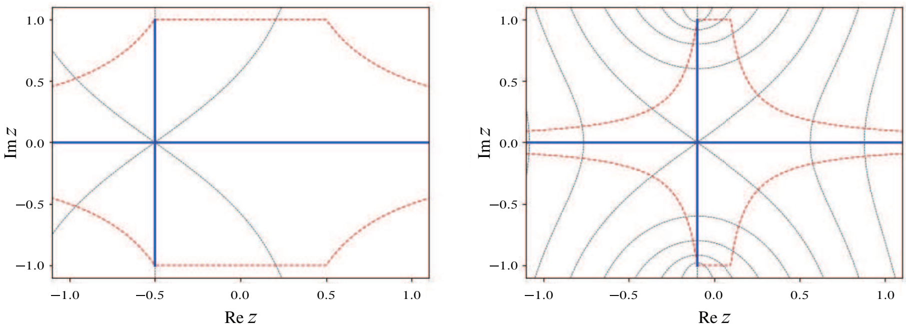

Figure 5. Lax spectrum for the potential $q(x) = \mathrm{e}^{\mathrm{i}x}$ with minimal period $L = 2\pi$ (blue). Contours $\Gamma = \{z \in \mathbb{C} : \operatorname{Im} \Delta_\varepsilon(z) = 0\}$ (black dashed). The curve which bounds the imaginary component of elements in the spectrum $|\operatorname{Im} z| = \min\{\|q\|_\infty, \varepsilon\|q'\|_\infty/2|\operatorname{Re} z|\}$ (red dashed). Left. $\varepsilon = 1$. Right. $\varepsilon = 0.2$.

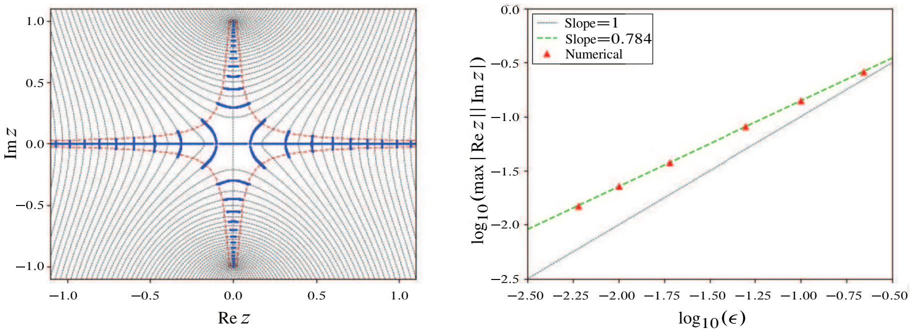

Figure 6. Left. Same as Figure 5 but for the periodic potential $q(x) = \operatorname{sgn}(x)$ with $L = 2$, and $\varepsilon = 0.019$. Right. Convergence of eigenvalues to Σ_∞ as $\varepsilon \downarrow 0$ for "sgn" potential. Numerical computation of $\max_{\nu \in [0, 2\pi/L)} |\operatorname{Re} z(\nu)||\operatorname{Im} z(\nu)|$ as $\varepsilon \downarrow 0$ (red triangles). Least-squares fit through the data points (light green dashed).

On the other hand, the analytically calculated Floquet discriminant is given by (A.32), and the numerically computed Lax spectrum of the ZS system (1.1) for the potential (7.4) is shown in Figure 6. In this example, the Floquet eigenvalues arise in symmetric quartets due to the symmetry of the potential (7.4) (specifically, odd and real).

In this case, one could consider a bound on the imaginary component of the eigenvalue similar to (4.3), but obtained using the numerically calculated eigenvalues at $\varepsilon = 1$, and then examine (numerically) how the spectrum changes as a function of ε. The numerical results indicate that, while the spectrum is still confined to a region of the complex plane similar to (4.5), the size of the region is no longer simply pro-

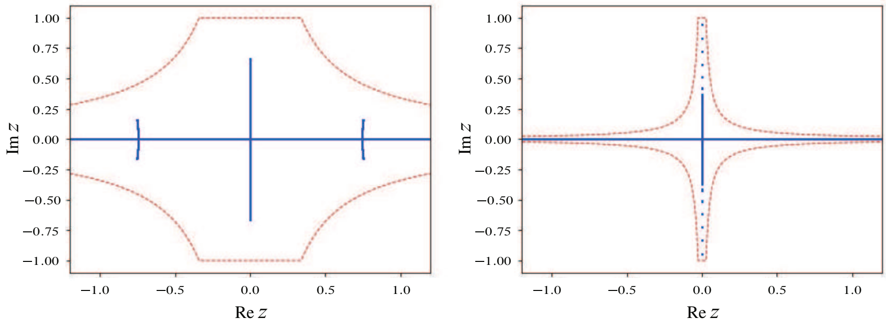

Figure 7. Numerically computed Lax spectrum for the periodic potential $q(x) = \mathrm{e}^{-\sin^2 x}$ (blue). The curve which bounds the imaginary component of the eigenvalue $|\operatorname{Im} z| = \min\{\|q\|_\infty, \varepsilon\|q'\|_\infty/2|\operatorname{Re} z|\}$ (dark red dashed). Left. Semiclassical parameter $\varepsilon = 1$. Right. Semiclassical parameter $\varepsilon = 0.079$.

portional to ε. Indeed, a numerical study shown in Figure 6 (right) suggests that one has $\max_{\nu\in[0,2\pi/L)} |\operatorname{Re} z(\nu)||\operatorname{Im} z(\nu)| = O(\varepsilon^\alpha)$ as $\varepsilon \downarrow 0$, with $\alpha = 0.784 < 1$ (see Figure 6). (Whereas, if the potential satisfied the hypotheses of Corollary 4.5, one would have $\alpha = 1$.) Nonetheless, the numerical results clearly indicate that the spectrum tends to the real and imaginary axes in the limit $\varepsilon \downarrow 0$. An interesting future direction would be to find α rigorously for piecewise continuous potentials. Similar results are also obtained for potentials with steps of arbitrary magnitude (owing to the invariant properties of the scattering problem under scaling transformations) as well as for potentials with asymmetric steps (e.g., $q(x) = 0$ for $x < 0$ and $q(x) = 1$ for $x > 0$). A detailed discussion is omitted for brevity.

7.3. Real-valued periodic single-lobe potentials

Next, we consider a periodic analogue of the Klaus and Shaw single-lobe potentials [35]. Specifically, we take the potential q to be the L-periodic extension of a real-valued continuously differentiable function on the interval $(-L/2, L/2)$. We further assume: (i) $q(x) \geq 0$ for all $x \in \mathbb{R}$, (ii) $q(-x) = q(x)$, and (iii) $q(x)$ is increasing on $[-L/2, 0)$ and is decreasing on $(0, L/2]$. In particular, below we consider two examples:

$$q(x) = \mathrm{e}^{-\sin^2 x}, \qquad L = \pi, \tag{7.5a}$$

$$q(x) = \operatorname{dn}(x; m), \quad L = 2K(m), \tag{7.5b}$$

where $\operatorname{dn}(x; m)$ is a Jacobi elliptic function, $m \in (0, 1)$ the corresponding elliptic parameter, and $K(m)$ the complete elliptic integral of the first kind (see [51] for details). Moreover, when $m = 1$ the problem reduces to that studied in [56].

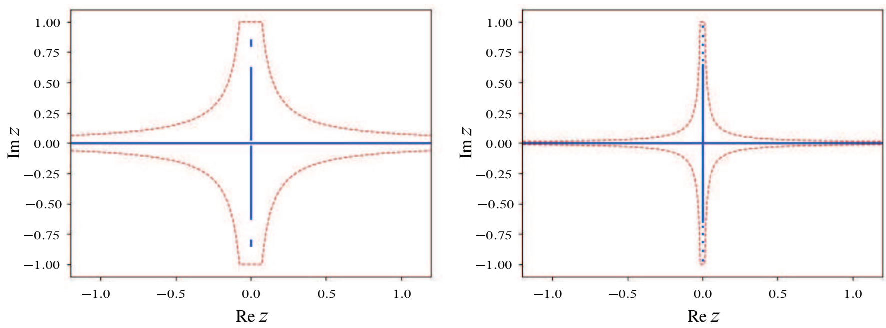

Figure 8. Same as Figure 7, but for the periodic potential $q(x) = \mathrm{dn}(x; m)$, where $m = 0.6$. Left. $\varepsilon = 0.5$. Right. $\varepsilon = 0.1$.

Figures 7 and 8 show the results of numerical computations of the Lax spectrum of the ZS system (1.1) for the two potentials in (7.5). The results clearly demonstrate that, in the case of periodic single-lobe potentials, there exists complex Floquet eigenvalues of (1.1) off of the imaginary axis. This is in contrast to single-lobe potentials decaying as $x \to \pm\infty$, whose eigenvalues were proven to be only real and purely imaginary, both for potentials with zero BCs [35] and for potentials with symmetric non-zero BCs [5]. In contrast, Figures 7 and 8 show that, in the periodic case, the confinement to the real and imaginary axes does not hold in general. Indeed, it would be surprising if such a feature were present in the periodic case as this would form a large class of finite-band potentials by Theorem 3.13. Another interesting feature of the spectrum for single-lobe periodic potentials is the formation of spectral bands and gaps along the intervals $\pm(\mathrm{i}\,q_{\min}, \mathrm{i}\,q_{\max})$ of the spectral plane. Using WKB approximation, it was shown formally in [6] that the number of spectral bands in this interval is $O(1/\varepsilon)$ as $\varepsilon \downarrow 0$. Another interesting question is whether spines can occur along the imaginary axis for single-lobe periodic potentials. The numerical results suggest that the Lax spectrum is purely imaginary outside of a neighborhood of the real z-axis. This property was shown to hold in the limit $\varepsilon \downarrow 0$ (see Theorem 5.5) where the neighborhood was proportional to ε.

7.4. Periodic potentials with rapid phase variations

Here we let the potential depend on the semiclassical parameter, that is, $q = q(x; \varepsilon)$. In particular, we consider a periodic potential with rapidly varying phase, i.e.,

$$q(x; \varepsilon) = A(x)\,\mathrm{e}^{\mathrm{i}\,S(x)/\varepsilon}, \tag{7.6}$$

where $A(x)$ and $S(x)$ are continuously differentiable real functions independent of ε.

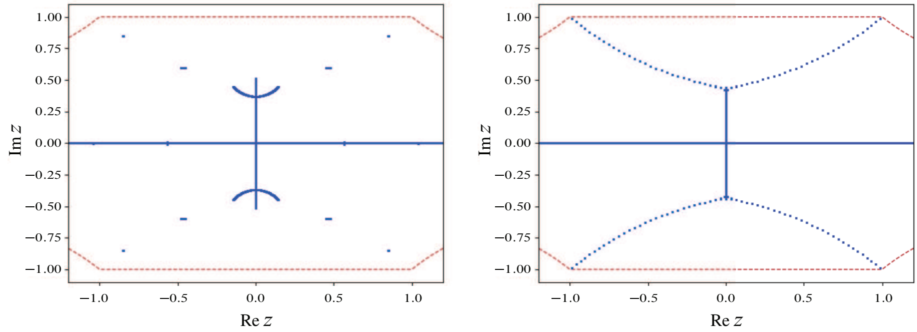

Figure 9. Same as Figure 7 but for the periodic potential $q(x;\varepsilon) = \mathrm{e}^{\mathrm{i}\cos(2x)/\varepsilon}$. Left. $\varepsilon = 0.22$. Right. $\varepsilon = 0.019$.

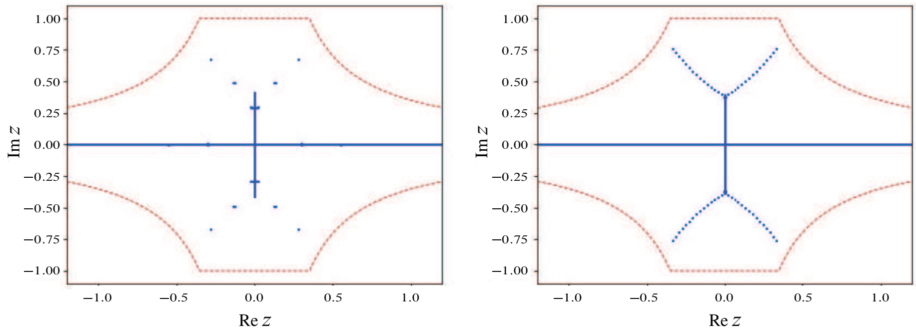

Figure 10. Same as Figure 7 but for the periodic potential $q(x;\varepsilon) = \mathrm{dn}(x;m)\,\mathrm{e}^{\mathrm{i}\,2\,\mathrm{dn}(x;m)/\varepsilon}$, where $m = 0.88$. Left. $\varepsilon = 0.2$. Right. $\varepsilon = 0.03$.

Specifically, we consider the following two potentials:

$$q(x;\varepsilon) = \mathrm{e}^{\mathrm{i}\cos(2x)/\varepsilon}, \qquad L = \pi, \tag{7.7a}$$

$$q(x;\varepsilon) = \mathrm{dn}(x;m)\,\mathrm{e}^{2\,\mathrm{i}\,\mathrm{dn}(x;m)/\varepsilon}, \qquad L = 2K(m). \tag{7.7b}$$

Importantly, the bound (4.3) still holds. The key difference from the previous cases, however, is that now $\|q'\|_\infty = O(1/\varepsilon)$ as $\varepsilon \downarrow 0$. Hence, one does not expect Theorem 4.5 to hold for potentials of the form (7.7). Moreover, using turning point curves and ideas of Deift, Venakides, and Zhou (see [14] for details) one expects that sharper bounds beyond what is available in (4.3) can be obtained. This is an interesting direction for future work. Indeed, the results of numerical computations of the spectrum produced by the potentials (7.7), and depicted in Figures 9 and 10, show that there are complex spectral bands off of the real and imaginary axes that persist in the limit $\varepsilon \downarrow 0$. Importantly, the numerical results are still consistent with the inequality (4.3). More interestingly, the spectrum appears to accumulate on a set of well-defined curves in the complex plane as $\varepsilon \downarrow 0$. This is qualitatively the same Y-shaped curve observed in [7] for the zero-background potential $q(x;\varepsilon) = \operatorname{sech}(2x)\, e^{i\operatorname{sech}(2x)/\varepsilon}$ on the infinite line. This is another example in which the differences in the spectrum due to different choices of boundary conditions seem to become negligible in the semiclassical limit of the focusing problem.

Appendix

A.1. Definitions

At various points in this work, we make use of the Pauli matrices which are defined as

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

Next, we discuss the normed linear spaces used throughout the work. Let g be a Lebesgue measurable function. Then, the essential supremum is defined as

$$\|g\|_\infty := \inf\{C : m(\{x \in \mathbb{R} : |g(x)| > C\}) = 0\}, \tag{A.2}$$

where $[m]$ is the Lebesgue measure. Note we always identify functions equal almost everywhere "a.e." with respect to Lebesgue measure. Specifically, $f \equiv g$ if $m(\{x \in \mathbb{R} : f(x) \neq g(x)\}) = 0$. Thus, a Lebesque-measurable function is in $L^\infty(\mathbb{R})$ if it is bounded a.e. $[m]$. If $g \in L^\infty(\mathbb{R})$ is periodic, one easily gets $g \in L^1_{\text{loc}}(\mathbb{R})$, that is, g is Lebesgue integrable on compact subsets of the real numbers.

Also, we define the inner product

$$\langle f, g \rangle := \int_0^L f(x)\overline{g(x)}\, dx, \tag{A.3}$$

where f, g are scalar Lebesgue measurable functions. This gives

$$\|f\|_2 := \sqrt{\langle f, f \rangle} = \left(\int_0^L |f(x)|^2\, dx\right)^{1/2}, \tag{A.4}$$

and $L^2([0, L])$ is the space of scalar Lebesgue measurable functions which are square integrable, that is, $\|f\|_2 < \infty$. Similarly, we define the inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle := \int_0^L f_1(x)\overline{g_1(x)} + f_2(x)\overline{g_2(x)}\, dx, \tag{A.5}$$

where $\mathbf{f}$, $\mathbf{g}$ are two-component Lebesgue measurable vector functions. This gives

$$\|\mathbf{f}\|_2 := \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} = \left(\int_0^L |f_1(x)|^2 + |f_2(x)|^2 \, \mathrm{d}x \right)^{1/2}, \tag{A.6}$$

and $L^2([0, L], \mathbb{C}^2)$ is the space of two-component vector functions which are square integrable, that is, $\|\mathbf{f}\|_2 < \infty$.

Finally, a function $f\colon [a, b] \to \mathbb{C}$ is absolutely continuous on $[a, b]$ if, for all $\eta > 0$, there exists $\delta > 0$ such that, whenever a finite sequence of pairwise disjoint subintervals $[a_k, b_k]$ of $[a, b]$ satisfies $\sum_k |b_k - a_k| < \delta$, then $\sum_k |f(b_k) - f(a_k)| < \eta$. The collection of such functions is denoted $\mathrm{AC}([a, b])$. Importantly, $f \in \mathrm{AC}([a, b])$ implies f' exists a.e $[m]$, and f' is Lebesgue integrable. A function $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$ if it is absolutely continuous on compact subsets of the real axis.

A.2. Symmetries of the ZS scattering problem

Recall next some symmetries of the solutions of (1.1), all of which are easily verified by direct computation. Let $Y(x; z, \varepsilon)$ be a fundamental matrix solution of the ZS scattering problem (1.1). Then, another solution of (1.1) is

$$\widetilde{Y}(x; z, \varepsilon) = \mathrm{i}\,\sigma_2 \overline{Y(x; \bar{z}, \varepsilon)}. \tag{A.7}$$

Moreover, if the potential is real, a further solution of (1.1) is given by

$$\widetilde{Y}_{(r)}(x; z, \varepsilon) = \overline{Y(x; -\bar{z}, \varepsilon)}. \tag{A.8}$$

Alternatively, suppose the potential satisfies a generalized reflection symmetry, that is, $q(-x) = \mathrm{e}^{2\mathrm{i}\theta} q(x)$ for all $x \geq 0$, for some $\theta \in \mathbb{R}$. (Obviously, for $\theta = 0$, and $\theta = \pi/2$ one has the cases of even and odd potentials, respectively.) Then, another solution of (1.1) is given by

$$\widetilde{Y}_{(g)}(x; z, \varepsilon) = (\cos\theta\sigma_1 + \sin\theta\sigma_2)\overline{Y(-x, -\bar{z}, \varepsilon)}. \tag{A.9}$$

Finally, if the potential is PT-symmetric, i.e., if $q(-x) = \overline{q(x)}$, then another solution of (1.1) is given by

$$\widetilde{Y}_{(pt)}(x; z, \varepsilon) = \sigma_3 \overline{Y(-x; \bar{z}, \varepsilon)}. \tag{A.10}$$

A.3. Comparison between various definitions of Dirichlet spectrum

In this section we show how to connect the Dirichlet spectrum to the zeros of an analytic function. Recall the Dirichlet BCs:

$$v_1(0; \zeta, \varepsilon) + v_2(0; \zeta, \varepsilon) = 0, \quad v_1(L; \zeta, \varepsilon) + v_2(L; \zeta, \varepsilon) = 0, \tag{A.11}$$

where $\{\zeta_n\}$ denote Dirichlet eigenvalues and $x_o = 0$ without loss of generality. Let $\mathbf{v}(x;z,\varepsilon) = \Phi(x;z,\varepsilon)\mathbf{c}(z;\varepsilon)$, where, as before, Φ is a fundamental matrix solution of (1.1) normalized so that $\Phi(0;z,\varepsilon) \equiv \mathbb{I}$, and $\mathbf{c}(z;\varepsilon) = (c_1, c_2)^T$. Clearly, $\mathbf{v}(0;z,\varepsilon) = \mathbf{c}(z;\varepsilon)$. If $\zeta \in \Sigma_{\mathrm{Dir}}(0)$ and $\mathbf{v}(x;\zeta,\varepsilon)$ is the eigenfunction associated to ζ, by the first of (A.11), one can write

$$\mathbf{v}(L;\zeta,\varepsilon) = M(\zeta;\varepsilon)\mathbf{c}(\zeta;\varepsilon) = v_1(0;\zeta,\varepsilon)\begin{pmatrix} M_{11} - M_{12} \\ M_{21} - M_{22} \end{pmatrix}, \tag{A.12}$$

where $M(z;\varepsilon) = \Phi(L;z,\varepsilon)$ is the monodromy matrix associated with Φ, as per (2.9) and (2.10). Using the second of (A.11), we then have

$$(M_{11} - M_{12} + M_{21} - M_{22})|_{z=\zeta} = 0. \tag{A.13}$$

Now, using the modified fundamental matrix solution $\widetilde{\Phi}$ defined in (2.20), recall that the associated monodromy matrix is $\widetilde{M}^{\varepsilon}(z) = C^{-1}M(z;\varepsilon)C$. One therefore gets

$$\widetilde{M}^{\varepsilon}_{21}(\zeta) = \frac{\mathrm{i}}{2}(M_{11} - M_{12} + M_{21} - M_{22})|_{z=\zeta} = 0. \tag{A.14}$$

Hence, $\zeta \in \Sigma_{\mathrm{Dir}}(0)$ if and only if $\widetilde{M}^{\varepsilon}_{21}(\zeta) = 0$, in agreement with (2.21).

Note that some authors define the Dirichlet spectrum via slightly different boundary conditions, namely,

$$v_1(0;\zeta,\varepsilon) - v_2(0;\zeta,\varepsilon) = 0, \quad v_1(L;\zeta,\varepsilon) - v_2(L;\zeta,\varepsilon) = 0, \tag{A.15}$$

(see [16, 29, 30]). Using similar arguments as above, with these BCs one gets

$$(M_{11} - M_{21} + M_{12} - M_{22})|_{z=\zeta} = 0. \tag{A.16}$$

We then introduce the modified fundamental matrix solution $\breve{\Phi}(x;z,\varepsilon) = \Phi(x;z,\varepsilon)\breve{C}$, where

$$\breve{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & -1 \\ \mathrm{i} & 1 \end{pmatrix}. \tag{A.17}$$

For the associated monodromy matrix $\breve{M}^{\varepsilon}(z) = \breve{C}^{-1}M(z;\varepsilon)\breve{C}$, one gets

$$\breve{M}^{\varepsilon}_{21}(\zeta) = -\frac{\mathrm{i}}{2}(M_{11} - M_{21} + M_{12} - M_{22})|_{z=\zeta} = 0. \tag{A.18}$$

Hence, with either choice of BCs, one can relate the Dirichlet spectrum to the zeros of an analytic function which is the row 2 column 1 element of a modified monodromy matrix. The trace formulae (2.23a) and (2.23b) use definition (A.11). The Dirichlet eigenvalues obey certain ODEs (called *Dubrovin equations*) with respect to x_o and t [24, 42].

Importantly, it is straightforward to see that the bounds obtained for the Dirichlet spectrum are independent of which choice of boundary conditions is used.

A.4. Floquet discriminant for constant and plane wave potentials

Here we calculate the Lax spectrum for plane wave potentials, namely, $q(x)$ given by (7.1). In this case, the ZS system (1.1) is given by

$$\varepsilon \mathbf{v}' = \begin{pmatrix} -\mathrm{i}\, z & A\,\mathrm{e}^{\mathrm{i}\, Vx} \\ -A\,\mathrm{e}^{-\mathrm{i}\, Vx} & \mathrm{i}\, z \end{pmatrix} \mathbf{v}. \tag{A.19}$$

As a special case, when $V = 0$ we obtain a constant background potential. In this case, (A.19) is a constant-coefficient differential equation, and it is easy to obtain a fundamental matrix solution as

$$Y(x; z, \varepsilon) = \Big(\mathbb{I} - \frac{\mathrm{i}\, A}{z - \xi}\sigma_1\Big)\,\mathrm{e}^{\mathrm{i}\,\xi x \sigma_3/\varepsilon}, \tag{A.20}$$

where $\mathbb{I}$ is the 2×2 identity matrix, $\xi^2 = A^2 + z^2$, and the Pauli matrices σ_1 and σ_3 are as in (A.1). A simple calculation then yields a monodromy matrix (2.9) as

$$M(z; \varepsilon) = \mathrm{e}^{\mathrm{i}\,\xi L \sigma_3/\varepsilon}, \tag{A.21}$$

and hence,

$$\Delta_\varepsilon(z) = \cos(\xi L/\varepsilon). \tag{A.22}$$

It follows that $z \in \Sigma_{\mathrm{Lax}}$ if and only if $\xi \in \mathbb{R}$. Thus, for a constant background potential one gets

$$\Sigma_{\mathrm{Lax}} = \mathbb{R} \cup [-\mathrm{i}\, A, \mathrm{i}\, A]. \tag{A.23}$$

Additionally, using the similarity transformation (2.20) one gets

$$\widetilde{M}^{\varepsilon}_{21}(z) = \Big(\frac{z - \mathrm{i}\, A}{2\xi}\Big)\sin(\xi L/\varepsilon). \tag{A.24}$$

It then follows that the Dirichlet spectrum is the set of zeros of the analytic function (A.24).

When $V \neq 0$, (A.19) is not constant-coefficient. Nonetheless, the substitution $\mathbf{v} = \mathrm{e}^{\mathrm{i}\, Vx\sigma_3/2}\,\mathbf{u}$ transforms (A.19) to the constant-coefficient system

$$\varepsilon \mathbf{u}' = \begin{pmatrix} -\mathrm{i}(z + \varepsilon V/2) & A \\ -A & \mathrm{i}(z + \varepsilon V/2) \end{pmatrix} \mathbf{u}. \tag{A.25}$$

As before, we obtain a fundamental matrix solution, namely,

$$Y(x; z, \varepsilon) = \mathrm{e}^{\mathrm{i}\, Vx\sigma_3/2}\Big(\mathbb{I} - \frac{\mathrm{i}\, A}{z - \xi + \varepsilon V/2}\sigma_1\Big)\,\mathrm{e}^{\mathrm{i}\,\xi x\sigma_3/\varepsilon}, \tag{A.26}$$

where $\xi^2 = A^2 + (z + \varepsilon V/2)^2$. Simple calculations then yield a monodromy matrix

$$M(z; \varepsilon) = \big(\cos(VL/2) + \sin(VL/2)\big(-\mathrm{i}(z + \varepsilon V/2)\sigma_3 - \mathrm{i}\, A\sigma_2\big)\big)\,\mathrm{e}^{\mathrm{i}\,\xi L\sigma_3/\varepsilon}, \tag{A.27}$$

and hence,

$$\Delta_\varepsilon(z) = \cos(VL/2)\cos(\xi L/\varepsilon) + \frac{z+\varepsilon V/2}{\xi}\sin(VL/2)\sin(\xi L/\varepsilon). \tag{A.28}$$

Take $L = 2\pi/|V|$. Then (A.28) simplifies to $\Delta_\varepsilon(z) = -\cos(2\pi\xi/\varepsilon V)$. It follows that $z \in \Sigma_{\mathrm{Lax}}$ if and only if $\xi \in \mathbb{R}$. Then, the Lax spectrum for a plane wave potential is given by (7.3) (see Figure 5).

A.5. Floquet discriminant for piecewise constant potentials

As one last example, we calculate the trace of the monodromy matrix for $q(x) = A\,\mathrm{sgn}(x)$, where $A > 0$ is a constant, and "sgn" is given by (7.4). In this case, the ZS system (1.1) is given by

$$\varepsilon\mathbf{v}' = \begin{pmatrix} -\mathrm{i}\,z & A\,\mathrm{sgn}(x) \\ -A\,\mathrm{sgn}(x) & \mathrm{i}\,z \end{pmatrix}\mathbf{v}. \tag{A.29}$$

Solving the above system on the intervals $[-L/2, 0)$, and $[0, L/2)$, and imposing the normalization $\Phi(0; z, \varepsilon) \equiv \mathbb{I}$, one finds the matrix solution

$$\Phi(x; z, \varepsilon) = \frac{\xi - z}{2\xi}\Big(\mathbb{I} - \frac{\mathrm{i}\,A}{z-\xi}\sigma_1\Big)\,\mathrm{e}^{\mathrm{i}\,\xi x\sigma_3/\varepsilon}\Big(\mathbb{I} + \frac{\mathrm{i}\,A}{z-\xi}\sigma_1\Big), \tag{A.30}$$

where $\xi^2 = A^2 + z^2$. It follows that

$$M(z;\varepsilon) = \Phi^{-1}(-L/2; z, \varepsilon)\Phi(L/2; z, \varepsilon), \tag{A.31}$$

and, hence,

$$\Delta_\varepsilon(z) = \frac{1}{\xi^2}(A^2 + z^2\cos(\xi L/\varepsilon)). \tag{A.32}$$

A.6. Floquet–Hill's method

Here we provide some details regarding the numerical calculations of the Lax spectrum in the main text. Recall that the Zakharov–Shabat scattering problem is given by the first-order system of ODEs (1.1). Further, recall that we can rewrite (1.1) in the form of an eigenvalue problem (2.1). Since (2.1) is non-self-adjoint, when computing its spectrum numerically one needs an approach capable of accurately calculating eigenvalues in a large region of the complex plane. One such approach is Floquet–Hill's method (see [10] for more details). The method is particularly well-suited for calculating the eigenvalues of linear operators with periodic coefficients, and provides a global approximation of the spectrum.

Recall that Floquet's theorem 2.1 implies that any bounded solution of (1.1) is necessarily of the form

$$\mathbf{v}(x;z,\varepsilon) = \mathrm{e}^{\mathrm{i}\,\nu x}\,\mathbf{w}(x;z,\varepsilon), \tag{A.33}$$

where $\mathbf{w}(x+L;z,\varepsilon) = \mathbf{w}(x;z,\varepsilon)$ and $\nu \in [0, 2\pi/L)$. Plugging (A.33) into the ZS system (1.1) yields the modified eigenvalue problem

$$\mathrm{i}\,\sigma_3(\varepsilon(\partial_x + \mathrm{i}\,\nu) - Q)\mathbf{w} = z\mathbf{w}. \tag{A.34}$$

The key point is that the modified eigenvalue problem acts on periodic functions. Therefore, one can expand (A.34) using a discrete Fourier transform (DFT):

$$\widehat{\mathcal{L}^{\varepsilon}_{\nu,N}}\hat{\mathbf{w}}_N = z_N\hat{\mathbf{w}}_N, \tag{A.35}$$

where

$$\widehat{\mathcal{L}^{\varepsilon}_{\nu,N}} := \begin{pmatrix} -\varepsilon(k+\nu) & -\mathrm{i}\,\mathfrak{T} \\ -\mathrm{i}\,\overline{\mathfrak{T}} & \varepsilon(k+\nu) \end{pmatrix}_{2N\times 2N}. \tag{A.36}$$

Moreover, $k = \operatorname{diag}(k_n)$ with $n \in \{-\frac{N}{2}, \dots, \frac{N}{2} - 1\}$ is a diagonal matrix of Fourier modes such that $k_n = 2n\pi/L$, and $\mathfrak{T}$ is a $N \times N$ Toeplitz matrix representing the convolution operator generated by the DFT of $\widehat{q\mathbf{w}}$.

Floquet–Hill's method then approximates the Floquet spectrum by numerically computing the eigenvalues of the matrix (A.36). Indeed, fix $\nu \in [0, 2\pi/L)$. Then choosing a truncation $N = 2^j$ of the number of Fourier modes of the eigenfunction $\mathbf{w}(x;z,\varepsilon)$ results in a matrix system of dimension $2N$. Moreover, z_N are approximations in the sense that all $z_N \in \Sigma(\widehat{\mathcal{L}^{\varepsilon}_{\nu,N}}) \to z \in \Sigma_\nu$ as $N \to \infty$. By taking an evenly spaced sequence $\nu_i \in [0, 2\pi/L)$ one approximates the entire Lax spectrum, i.e.,

$$\lim_{N\to\infty} \bigcup_{\nu\in[0,2\pi/L)} \Sigma(\widehat{\mathcal{L}^{\varepsilon}_{\nu,N}}) = \Sigma_{\mathrm{Lax}}. \tag{A.37}$$

(For details on convergence, see [33].) Numerical accuracy of the approximation is determined by the number of Fourier modes used in the truncation, and on the method used to compute the eigenvalues of the matrix (A.36). The resolution of the spectral bands is determined by how fine we partition the interval $\nu \in [0, 2\pi/L)$. All computations in the main text were checked for numerical convergence.

Acknowledgment. We thank the anonymous referee for the careful review and the many helpful comments.

Funding. This work was partially supported by the National Science Foundation under grant number DMS-2009487.

References

[1] M. J. Ablowitz, B. Prinari, and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems*. London Math. Soc. Lecture Note Ser. 302, Cambridge University Press, Cambridge, 2004 Zbl 1057.35058 MR 2040621

[2] D. S. Agafontsev and V. E. Zakharov, Integrable turbulence and formation of rogue waves. *Nonlinearity* **28** (2015), no. 8, 2791–2821 Zbl 1330.35391 MR 3382586

[3] E. D. Belekolos, A. I. Bobenko, V. Z. Enol'skii, A. R. Its and V. B. Matveev, "Algebro-geometric approach to nonlinear integrable equations." Springer Series in Nonlinear Dynamics. Springer, Berlin, 1994

[4] M. Bertola and A. Tovbis, Universality for the focusing nonlinear Schrödinger equation at the gradient catastrophe point: rational breathers and poles of the *tritronquée* solution to Painlevé I. *Comm. Pure Appl. Math.* **66** (2013), no. 5, 678–752 Zbl 1355.35169 MR 3028484

[5] G. Biondini and X. Luo, Imaginary eigenvalues of Zakharov–Shabat problems with non-zero background. *Phys. Lett. A* **382** (2018), no. 37, 2632–2637 Zbl 1404.35402 MR 3836750

[6] G. Biondini and J. Oregero, Semiclassical dynamics and coherent soliton condensates in self-focusing nonlinear media with periodic initial conditions. *Stud. Appl. Math.* **145** (2020), no. 3, 325–356 Zbl 1454.35332 MR 4174160

[7] J. C. Bronski, Semiclassical eigenvalue distribution of the Zakharov–Shabat eigenvalue problem. *Phys. D* **97** (1996), no. 4, 376–397 Zbl 1194.81093 MR 1412550

[8] B. M. Brown, M. S. P. Eastham, and K. M. Schmidt, *Periodic differential operators*. Oper. Theory: Adv. Appl. 230, Birkhäuser/Springer, Basel, 2013 Zbl 1267.34001 MR 2978285

[9] L. I. Danilov, Absolute continuity of the spectrum of a periodic Dirac operator. *Differ. Uravn.* **36** (2000), no. 2, 233–240, 287 Zbl 1041.81026 MR 1773794

[10] B. Deconinck and J. N. Kutz, Computing spectra of linear operators using the Floquet-Fourier-Hill method. *J. Comput. Phys.* **219** (2006), no. 1, 296–321 Zbl 1105.65119 MR 2273379

[11] P. Deift and K. T.-R. McLaughlin, A continuum limit of the Toda lattice. *Mem. Amer. Math. Soc.* **131** (1998), no. 624 Zbl 0946.37035 MR 1407901

[12] P. Deift, S. Venakides, and X. Zhou, New results in small dispersion KdV by an extension of the steepest descent method for Riemann-Hilbert problems. *Internat. Math. Res. Notices* (1997), no. 6, 286–299 Zbl 0873.65111 MR 1440305

[13] G. Deng, S. Li, G. Biondini, and S. Trillo, Recurrence due to periodic multisoliton fission in the defocusing nonlinear Schrödinger equation. *Phys. Rev. E* **96** (2017), no. 5, article no. 052213 MR 3825035

[14] J. C. DiFranco and P. D. Miller, The semiclassical modified nonlinear Schrödinger equation. I. Modulation theory and spectral analysis. *Phys. D* **237** (2008), no. 7, 947–997 Zbl 1158.35086 MR 2417083

[15] J. M. Dudley and J. R. Taylor, *Supercontinuum generation in optical fibers.* Cambridge University Press, Cambridge 2010

[16] P. Dzhakov and B. S. Mityagin, Instability zones of one-dimensional periodic Schrödinger and Dirac operators. *Uspekhi Mat. Nauk* **61** (2006), no. 4(370), 77–182; English transl., *Russian Math. Surveys* **61** (2006), no. 4, 663–766 Zbl 1128.47041 MR 2279044

[17] M. S. P. Eastham, *The spectral theory of periodic differential equations*. Texts in Mathematics, Scottish Academic Press, Edinburgh; Hafner Press, New York, 1973 Zbl 0287.34016 MR 3075381

[18] G. El and A. Tovbis, Spectral theory of soliton and breather gases for the focusing nonlinear Schrödinger equation. *Phys. Rev. E* **101** (2020), no. 5, article no. 052207 MR 4115253

[19] G. A. El and M. A. Hoefer, Dispersive shock waves and modulation theory. *Phys. D* **333** (2016), 11–65 Zbl 1415.76001 MR 3523490.

[20] G. A. El, E. G. Khamis, and A. Tovbis, Dam break problem for the focusing nonlinear Schrödinger equation and the generation of rogue waves. *Nonlinearity* **29** (2016), no. 9, 2798–2836 Zbl 1391.35354 MR 3544808

[21] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z .H. Musslimani, S. Rotter and D. N. Christodoulides, Non-Hermitian physics and PT symmetry. *Nature Phys.* **14** (2018), 11–19

[22] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*. Springer Series in Soviet Mathematics, Springer, Berlin, 1987 Zbl 1111.37001 MR 905674

[23] G. Floquet, Sur les équations différentielles linéaires à coefficients périodiques. *Ann. Sci. École Norm. Sup. (2)* **12** (1883), 47–88 Zbl 15.0279.01 MR 1508722

[24] M. G. Forest and J. E. Lee, Geometry and modulation theory for the periodic nonlinear Schrödinger equation. In *Oscillation theory, computation, and methods of compensated compactness (Minneapolis, Minn., 1985)*, pp. 35–69, IMA Vol. Math. Appl. 2, Springer, New York, 1986 Zbl 0615.35003 MR 869821

[25] S. Fujiié and J. Wittsten, Quantization conditions of eigenvalues for semiclassical Zakharov–Shabat systems on the circle. *Discrete Contin. Dyn. Syst.* **38** (2018), no. 8, 3851–3873 Zbl 1400.34138 MR 3814355

[26] F. Gesztesy and R. Weikard, Picard potentials and Hill's equation on a torus. *Acta Math.* **176** (1996), no. 1, 73–107 Zbl 0927.37040 MR 1395670

[27] F. Gesztesy and R. Weikard, A characterization of all elliptic algebro-geometric solutions of the AKNS hierarchy. *Acta Math.* **181** (1998), no. 1, 63–108 Zbl 0955.34073 MR 1654775

[28] F. Gesztesy and R. Weikard, Elliptic algebro-geometric solutions of the KdV and AKNS hierarchies—an analytic approach. *Bull. Amer. Math. Soc. (N.S.)* **35** (1998), no. 4, 271–317 Zbl 0909.34073 MR 1638298

[29] B. Grébert and T. Kappeler, Estimates on periodic and Dirichlet eigenvalues for the Zakharov–Shabat system. *Asymptot. Anal.* **25** (2001), no. 3-4, 201–237 Zbl 0982.34073 MR 1853394

[30] B. Grébert, T. Kappeler, and B. Mityagin, Gap estimates of the spectrum of the Zakharov–Shabat system. *Appl. Math. Lett.* **11** (1998), no. 4, 95–97 Zbl 0940.35186 MR 1630812

[31] P. D. Hislop and I. M. Sigal, *Introduction to spectral theory*. Appl. Math. Sci. 113, Springer, New York, 1996 Zbl 0855.47002 MR 1361167

[32] R. Jenkins and K. D. T.-R. McLaughlin, Semiclassical limit of focusing NLS for a family of square barrier initial data. *Comm. Pure Appl. Math.* **67** (2014), no. 2, 246–320 Zbl 1332.35327 MR 3149844

[33] M. A. Johnson and K. Zumbrun, Convergence of Hill's method for nonselfadjoint operators. *SIAM J. Numer. Anal.* **50** (2012), no. 1, 64–78 Zbl 1246.65134 MR 2888304

[34] S. Kamvissis, K. D. T.-R. McLaughlin, and P. D. Miller, *Semiclassical soliton ensembles for the focusing nonlinear Schrödinger equation*. Ann. Math. Stud. 154, Princeton University Press, Princeton, NJ, 2003 Zbl 1057.35063 MR 1999840

[35] M. Klaus and J. K. Shaw, Purely imaginary eigenvalues of Zakharov–Shabat systems. *Phys. Rev. E (3)* **65** (2002), no. 3, article no. 036607 MR 1905252

[36] M. Klaus and J. K. Shaw, On the eigenvalues of Zakharov–Shabat systems. *SIAM J. Math. Anal.* **34** (2003), no. 4, 759–773 Zbl 1034.34097 MR 1969601

[37] P. D. Lax and C. D. Levermore, The small dispersion limit of the Korteweg–de Vries equation. I. *Comm. Pure Appl. Math.* **36** (1983), no. 3, 253–290 Zbl 0532.35067 MR 697466

[38] P. D. Lax and C. D. Levermore, The small dispersion limit of the Korteweg–de Vries equation. I. *Comm. Pure Appl. Math.* **36** (1983), no. 5, 571–593 Zbl 0527.35073 MR 0716197

[39] P. D. Lax and C. D. Levermore, The small dispersion limit of the Korteweg–de Vries equation. I. *Comm. Pure Appl. Math.* **36** (1983), no. 6, 809–829 Zbl 0527.35074 MR 0720595

[40] Y. Li and D. W. McLaughlin, Morse and Melnikov functions for NLS PDEs. *Comm. Math. Phys.* **162** (1994), no. 1, 175–214 Zbl 0808.35140 MR 1272771

[41] G. D. Lyng and P. D. Miller, The N-soliton of the focusing nonlinear Schrödinger equation for N large. *Comm. Pure Appl. Math.* **60** (2007), no. 7, 951–1026 Zbl 1185.35259 MR 2319053

[42] Y. C. Ma and M. J. Ablowitz, The periodic cubic Schrödinger equation. *Stud. Appl. Math.* **65** (1981), no. 2, 113–158 Zbl 0493.35032 MR 628138

[43] W. Magnus and S. Winkler, *Hill's equation*. Intersci. Tracts Pure Appl. Math. 20, Interscience Publishers, New York etc., 1966 Zbl 0158.09604 MR 0197830

[44] D. W. McLaughlin and E. A. Overman, II, Whiskered tori for integrable PDEs: chaotic behavior in near integrable pde's. In *Surveys in applied mathematics,* Vol. 1, pp. 83–203, Plenum, New York, 1995 Zbl 0843.35116 MR 1366208

[45] A. Messiah, *Quantum mechanics.* Vol. I. North-Holland, Amsterdam; Interscience Publishers, New York, 1961 MR 0129790 Zbl 0102.42602

[46] A. Messiah, *Quantum mechanics.* Vol. II. North-Holland, Amsterdam; Interscience Publishers, New York, 1961 MR 0147125 Zbl 0102.42602

[47] P. D. Miller, Some remarks on a WKB method for the nonselfadjoint Zakharov–Shabat eigenvalue problem with analytic potentials and fast phase. *Phys. D.* **152–153** (2001), 145–162 Zbl 1037.81043 MR 1837905

[48] P. D. Miller and S. Kamvissis, On the semiclassical limit of the focusing nonlinear Schrödinger equation. *Phys. Lett. A* **247** (1998), no. 1-2, 75–86 Zbl 0941.81029 MR 1650432

[49] A. Mussot, C. Naveau, M. Conforti, A. Kudlinski, F. Copie, P. Szriftgiser, and S. Trillo, Fibre multi-wave mixing combs reveal the broken symmetry of Fermi–Pasta–Ulam recurrence. *Nature photonics* **12** (2018), 303–308

[50] S. Novikov, S. V. Manakov, L. P. Pitaevskiĭ, and V. E. Zakharov, *Theory of solitons*. Contemporary Soviet Mathematics, Consultants Bureau, New York, 1984 Zbl 0598.35002 MR 779467

[51] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010 Zbl 1198.00002 MR 2723248

[52] M. Onorato, A. R. Osborne, and M. Serio, Modulational instability in crossing sea states: A possible mechanism for the formation of freak waves. *Phys. Rev. Lett.* **96** (2006), article no. 014503

[53] S. Randoux, P. Walczak, M. Onorato and P. Suret, Intermittency in integrable turbulence. *Phys. Rev. Lett.* **113** (2014), article no. 113902

[54] F. S. Rofe-Beketov, On the spectrum of non-selfadjoint differential operators with periodic coefficients. *Dokl. Akad. Nauk SSSR* **152** (1963), 1312–1315; English transl., *Soviet Math. Dokl.* **4** (1963), 1312–1315 Zbl 0199.14002 MR 0157033

[55] F. S. Rofe-Beketov and A. M. Kholkin, *Spectral analysis of differential operators*. World Scientific Monograph Series in Mathematics 7, World Scientific, Hackensack, NJ, 2005 Zbl 1090.47030 MR 2175241

[56] J. Satsuma and N. Yajima, Initial value problems of one-dimensional self-modulation of nonlinear waves in dispersive media. *Progr. Theoret. Phys. Suppl.* **55** (1974), 284–306 MR 0463733

[57] M. I. Serov, Certain properties of the spectrum of a non-selfadjoint differential operator of the second order. *Dokl. Akad. Nauk SSSR* **131** (1960), 27–29 English transl., *Soviet Math. Dokl.* **1** (1960), 190–192 Zbl 0106.05902 MR 0131016

[58] K. C. Shin, On the shape of spectra for non-self-adjoint periodic Schrödinger operators. *J. Phys. A* **37** (2004), no. 34, 8287–8291 Zbl 1064.81029 MR 2092798

[59] D. R. Solli, C. Ropers, P. Koonath and B. Jalali, Optical rogue waves. *Nature* **450** (2007), 1054–1057

[60] E. C. Titchmarsh, *The theory of functions*. Second edn., Oxford University Press, Oxford, 1939 JFM 65.0302.01 MR 3728294

[61] V. Tkachenko, Spectra of non-selfadjoint Hill's operators and a class of Riemann surfaces. *Ann. of Math. (2)* **143** (1996), no. 2, 181–231 Zbl 0856.34087 MR 1381985

[62] V. Tkachenko, Non-self-adjoint periodic Dirac operators. In *Operator theory, system theory and related topics (Beer-Sheva/Rehovot, 1997)*, pp. 485–512, Oper. Theory Adv. Appl. 123, Birkhäuser, Basel, 2001 Zbl 0996.34070 MR 1821926

[63] A. Tovbis and S. Venakides, The eigenvalue problem for the focusing nonlinear Schrödinger equation: new solvable cases. *Phys. D* **146** (2000), no. 1-4, 150–164 Zbl 0984.34072 MR 1787410

[64] A. Tovbis and S. Venakides, Semiclassical limit of the scattering transform for the focusing nonlinear Schrödinger equation. *Int. Math. Res. Not. IMRN* (2012), no. 10, 2212–2271 Zbl 1239.35151 MR 2923165

[65] A. Tovbis, S. Venakides, and X. Zhou, On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation. *Comm. Pure Appl. Math.* **57** (2004), no. 7, 877–985 Zbl 1060.35137

[66] A. Tovbis, S. Venakides, and X. Zhou, On the long-time limit of semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation: pure radiation case. *Comm. Pure Appl. Math.* **59** (2006), no. 10, 1379–1432 Zbl 1115.35127 MR 2248894

[67] G. B. Whitham, *Linear and nonlinear waves*. Pure Appl. Math., Wiley-Intersci. Ser. Texts Monogr. Tracts, Wiley–Interscience, New York etc., 1974

[68] N. J. Zabusky and M. D. Kruskal, Interaction of solitons in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.* **15** (1965), 240–243 Zbl 1201.35174

[69] V. E. Zakharov, Turbulence in integrable systems. *Stud. Appl. Math.* **122** (2009), no. 3, 219–234 Zbl 1178.37099 MR 2504627

[70] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.* **61** (1971), no. 1, 118–134; English transl., *Soviet Physics JETP* **34** (1972), no. 1, 62–69 MR 0406174

Received 7 October 2020; revised 21 March 2022.

Gino Biondini
Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260, USA; biondini@buffalo.edu

Jeffrey Oregero
Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260, USA; jaoreger@buffalo.edu

Alexander Tovbis
Department of Mathematics, University of Central Florida, 4393 Andromeda Loop N, Orlando, FL 32816, USA; alexander.tovbis@ucf.edu