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On the Landis conjecture for the fractional Schrödinger equation

Pu-Zhao Kow

Abstract. In this paper, we study a Landis-type conjecture for the general fractional Schrödinger equation $((-P)^s + q)u = 0$. As a byproduct, we also prove the additivity and boundedness of the linear operator $(-P)^s$ for non-smooth coefficents. For differentiable potentials q, if a solution decays at a rate $\exp(-|x|^{1+})$, then the solution vanishes identically. For non-differentiable potentials q, if a solution decays at a rate $\exp(-|x|^{\frac{4s}{4s-1}+})$, then the solution must again be trivial. The proof relies on delicate Carleman estimates. This study is an extension of the work by Rüland and Wang (2019).

1. Introduction

In this work, we study a Landis-type conjecture for the fractional Schrödinger equation

$$((-P)^s + q)u = 0 \quad \text{in } \mathbb{R}^n, \quad \text{where } P = \sum_{j,k=1}^{n} \partial_j a_{jk}(x)\partial_k \tag{1.1}$$

with $s \in (0,1)$ and $|q(x)| \le 1$. Here, the operator $(-P)^s$ is defined as

$$(-P)^s u := \int_0^\infty \lambda^s \,\mathrm{d}E_\lambda u = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{tP} - 1)u \,\frac{\mathrm{d}t}{t^{1+s}} \tag{1.2}$$

for all

$$u \in \mathrm{dom}((-P)^s) := \Big\{ u \in L^2(\mathbb{R}^n) \colon \int_0^\infty \lambda^{2s} \,\mathrm{d}\|E_\lambda u\|^2 < \infty \Big\}$$

where $\{E_\lambda\}$ is the spectral resolution of $-P$ (each $\{E_\lambda\}$ is a projection in $L^2(\mathbb{R}^n)$) and $\{e^{tP}\}_{t\ge 0}$ is the heat-diffusion semigroup generated by $-P$, see, e.g., [11, 34].

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The Landis conjecture was proposed by E.M. Landis in the 60's [21]. He conjectured the following statement. Let $|q(x)| \leq 1$ and let u be a solution to (1.1) with $P = \Delta$ and $s = 1$. If $|u(x)| \leq C_0$ and $|u(x)| \leq \exp(-C|x|^{1+})$, then $u \equiv 0$. However, this statement is false. In [26], Meshkov constructed a (complex-valued) potential q and a (complex-valued) nontrivial u with $|u(x)| \leq C\exp(-C|x|^{\frac{4}{3}})$. In the same literature, he also showed that if $|u(x)| \leq C\exp(-C|x|^{\frac{4}{3}+})$, then $u \equiv 0$. In other words, the exponent $\frac{4}{3}+$ is optimal. In [1], Bourgain and Kenig derived a quantitative form of Meshkov's result, which is based on the Carleman method; their result then extended by Davey in [4], including the drift term. Following, in [22], Lin and Wang further extend Davey's result by replacing Δ by P.

The results mentioned above allowing *complex-valued* solutions. It is also interesting to study the real-version of Landis conjecture, which proposed by Kenig in [20, Question 1]. The case when $n = 1$ and $n = 2$ were resolved in [24, 28], respectively. To the best of the author's knowledge, the real-version of Landis conjecture is still open for $n \geq 3$. Here we also refer some related works [5–8, 19].

In [31], Rüland and Wang consider the Landis conjecture of the fractional Schrödinger equation (1.1) with $P = \Delta$ and $0 < s < 1$. For the case when $s = 1/2$, in [3], we remark that Cassano proved the Landis conjecture for the Dirac equation. In some sense, the Dirac operator is the square root of the Laplacian operator, that is, the phenomena are similar when $s = 1/2$.

1.1. Main results

We assume that the second order elliptic operator P satisfies the elliptic condition

$$\lambda|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leq \lambda^{-1}|\xi|^2 \quad \text{for some constant } 0 < \lambda \leq 1. \tag{1.3}$$

Assume that $a_{jk} = a_{kj} \in \mathcal{C}^{0,1}(\mathbb{R}^n)$ for all $1 \leq j,k \leq n$, and satisfy

$$\max_{1\leq j,k\leq n} \sup_{|x|\geq 1} |a_{jk}(x) - \delta_{jk}(x)| + \max_{1\leq j,k\leq n} \sup_{|x|\geq 1} |x||\nabla a_{jk}(x)| \leq \varepsilon \tag{1.4}$$

for some sufficiently small $\varepsilon > 0$ and

$$\max_{1\leq j,k\leq n} \sup_{|x|\geq 1} |\nabla^2 a_{jk}(x)| \leq C \tag{1.5}$$

for some positive constant C.

In this paper, we prove the following Landis-type conjecture for the fractional Schrödinger equations.

Theorem 1.1. *Let $s \in (0,1)$ and assume that $u \in \mathrm{dom}((-P)^s)$ is a solution to equation (1.1) with (1.3), (1.4), and (1.5). We assume that the potential $q \in \mathcal{C}^1(\mathbb{R}^n)$*

satisfies $|q(x)| \leq 1$ *and*

$$|x||\nabla q(x)| \leq 1.$$

If u *further satisfies*

$$\int_{\mathbb{R}^n} e^{|x|^\alpha} |u|^2 \, \mathrm{d}x \leq C < \infty \quad \text{for some } \alpha > 1,$$

then $u \equiv 0$.

We also have the following result for non-differentiable potential q.

Theorem 1.2. *Let* $s \in (1/4, 1)$ *and assume that* $u \in \mathrm{dom}((-P)^s)$ *is a solution to* (1.1) *with* (1.3), (1.4), *and* (1.5). *Now, we assume that the potential* q *satisfies* $|q(x)| \leq 1$. *If* u *satisfies*

$$\int_{\mathbb{R}^n} e^{|x|^\alpha} |u|^2 \, \mathrm{d}x \leq C < \infty \quad \text{for some } \alpha > \frac{4s}{4s-1},$$

then $u \equiv 0$.

Remark 1.3. When $s = \frac{1}{2}$, Theorem 1.1 and Theorem 1.2 still hold without (1.5).

Remark 1.4. We prove Theorem 1.2 using the splitting arguments in [31]. Similarly to [31], we assume $s \in (\frac{1}{4}, 1)$ due to the sub-sllipticity nature. We also see that, as $s \to 1$, the exponent $\frac{4s}{4s-1}$ in Theorem 1.2 tends to $\frac{4}{3}$, which is the optimal exponent for the classical Schrödinger equation.

Remark 1.5. The condition (1.4) allows small perturbations of Laplacian only, which works as a sufficient condition in deriving Carleman estimate. In [10], they also imposed similar assumption to prove the strong unique continuation property for (1.1). In contrast to the works [6, 28], which studied the *real-version* of Landis conjecture, such condition is not needed, since their proofs did not involve any Carleman estimate.

1.2. Main ideas

The main method of proving Theorem 1.1 and 1.2 is Carleman estimates. However, due to the non-locality of $(-P)^s$, the techniques here are much complicated than those for the classical case, i.e., $s = 1$. One of the major tricks is to localize $(-P)^s$, which is motivated by Caffarelli and Silvestre's fundamental work [2]. Here we will use the Caffarelli–Silvestre-type extension of $(-P)^s$ proved in [33, 34]. After localizing $(-P)^s$, we will derive a Carleman estimate on $\mathbb{R}^{n+1}_+$ mimicking the one proved in [30]. This Carleman estimate enables passing of the boundary decay to the bulk decay.

1.3. Main difficulties: regularity of $(-P)^s$

Using the Fourier transform, it is easy to see that

$$(-\Delta)^\alpha(-\Delta)^\beta = (-\Delta)^{\alpha+\beta} \quad \text{and} \quad (-\Delta)^s \in \mathcal{L}(\dot{H}^{\beta+s}(\mathbb{R}^n), \dot{H}^{\beta-s}(\mathbb{R}^n)).$$

However, extension of these properties to $(-P)^s$ is not trivial. We establish the additivity property of $(-P)^s$ by introducing the Balakrishnan definition of $(-P)^s$, which is equivalent to (1.2), see, e.g., [25] or [37, Section IX.11]. The continuity of the map $(-P)^s\colon H^{2s}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ can be also obtained by the Balakrishnan operator, as well as the interpolation of the single operator $-P$. Here, we shall not interpolate on the family of the operator $(-P)^s$, see also [12] for the interpolation theory of the analytic family of multilinear operators.

Remark 1.6. In [32], R. T. Seeley showed that the operator $(-P)^s$ is a pseudo-differential operator of order $2s$ if a_{jk} are smooth. In this case, we can apply the theory of pseudo-differential operator, see, e.g., [36]. As a byproduct, we loosened the smoothness hypothesis that required by theories of the pseudo-differential operator. Moreover, the boundary value theories for the fractional Laplacian have been elaborated in recent years, see, e.g., [13–17]. In [17], Grubb calculated the first few terms in the symbol of $(-P)^s$.[1]

1.4. Main difficulties: Carleman estimates

In [31], Rüland and J.-N. Wang proved their Carleman estimates by estimating a certain commutator term, see [31, (31)–(33)]. In our case, we shall approximate P by Δ. However, we face difficulties while controlling the remainder terms. Here, we solve this problem using the ideas in [27]. It is also interesting to mention that the terms of second derivative in the Carleman estimate should be $\widetilde{\nabla}(\nabla\tilde{u})$ rather than $\widetilde{\nabla}^2\tilde{u}$, where $\widetilde{\nabla} = (\nabla, \partial_{n+1})$ is the gradient operator on $\mathbb{R}^{n+1}$, and $\tilde{u}$ is the Caffarelli–Silvestre-type extension of u.

1.5. Organization of the paper

In Section 2, we localize the operator $(-P)^s$ and solve the problems described in Paragraph 1.3. Following, in Section 3, we show that the decay of u implies the decay of the Caffarelli–Silvestre-type extension $\tilde{u}$ of u. Then, we derive some delicate Carleman estimates on $\mathbb{R}^n_+$ in Section 4. Finally, we prove Theorem 1.1 and Theorem 1.2 in Section 5.

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[1] I would like to thank Prof Gerd Grubb for bringing these issues to my attention and for pointing out several related references.

2. Caffarelli–Silvestre-type extension

Let $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+ = \{(x', x_{n+1}) : x_{n+1} > 0\}$, and we write $x = (x', x_{n+1})$ with $x' \in \mathbb{R}^n$ and $x_{n+1} \in \mathbb{R}_+$. We also denote $\nabla' = (\partial_1, \ldots, \partial_n)$ and $\nabla = (\nabla', \partial_{n+1})$. For $x_0 \in \mathbb{R}^n \times \{0\}$, we denote the half balls in $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^n \times \{0\}$ by

$$B_r^+(x_0) := \{x \in \mathbb{R}^{n+1}_+ : |x - x_0| \le r\},$$
$$B_r'(x_0) := \{(x', 0) \in \mathbb{R}^n \times \{0\} : |(x', 0) - x_0| \le r\},$$

$B_r^+(0) = B_r^+$, and $B_r'(0) = B_r'$. We define the annulus

$$A^+_{r,R} := \{x \in \mathbb{R}^{n+1}_+ : r \le |x| \le R\},$$
$$A'_{r,R} := \{(x', 0) \in \mathbb{R}^n \times \{0\} : r \le |(x', 0)| \le R\}.$$

We consider the following Sobolev spaces:

$$L^2(D, x_{n+1}^{1-2s}) := \left\{ v : D \to \mathbb{R} : \int_D x_{n+1}^{1-2s} |v|^2 \, \mathrm{d}x < \infty \right\},$$
$$\dot{H}^1(D, x_{n+1}^{1-2s}) := \left\{ v : D \to \mathbb{R} : \int_D x_{n+1}^{1-2s} |\nabla v|^2 \, \mathrm{d}x < \infty \right\},$$
$$H^1(D, x_{n+1}^{1-2s}) := \left\{ v : D \to \mathbb{R} : \int_D x_{n+1}^{1-2s} (|v|^2 + |\nabla v|^2) \, \mathrm{d}x < \infty \right\},$$

where D is a relative open set in $\overline{\mathbb{R}^{n+1}_+}$.

For $s \in (0,1)$, let $\tilde{u}$ be a solution to the following degenerate elliptic equation:

$$[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P]\tilde{u} = 0 \quad \text{in } \mathbb{R}^{n+1}_+, \tag{2.1}$$
$$\tilde{u} = u \quad \text{on } \mathbb{R}^n \times \{0\}. \tag{2.2}$$

Refer to [34, equation (1.8) in Theorem 1.1], the fractional elliptic operator $(-P)^s$ satisfies

$$(-P)^s u(x') = c_s \lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}(x), \tag{2.3}$$

with

$$c_s = \frac{4^s \Gamma(s)}{2s\Gamma(-s)} < 0 \quad (\text{in particular, } c_{1/2} = -1),$$

see also [33]. The following lemma is a special case of [10, Proposition 2.1]:

Lemma 2.1. *Let $0 < s < 1$, and assuming that $a_{jk} = a_{kj} \in \mathscr{C}^{0,1}(\mathbb{R}^n)$ satisfies the elliptic condition (1.3). Then, there exists an extension operator*

$$\mathsf{E}_s : \mathrm{dom}((-P)^s) \to H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s}) \cap \mathscr{C}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+)$$

such that $\tilde{u} = \mathsf{E}_s(u)$ is a solution of (2.1) and the boundary conditions (2.2) and (2.3) are attained as $L^2(\mathbb{R}^n)$-limits.

The proof of Lemma 2.1 is same as in [33, 34]. The following estimate also holds true:

$$\|\tilde{u}(\bullet, x_{n+1})\|_{L^2(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)} \quad \text{for all } \ x_{n+1} > 0. \tag{2.4}$$

with $\tilde{u} = \mathsf{E}_s(u)$, see [34, p. 2097] or [33, p. 48–49]. From [38, Proposition 2.6], indeed

$$\mathsf{E}_s\colon H^s(\mathbb{R}^n) \to H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s}) \tag{2.5}$$

is a bounded linear operator. Using [23, Remark 7.4], we know that

$$\mathscr{C}_c^\infty(\overline{\mathbb{R}^{n+1}_+}) \text{ is dense in } H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s}),$$

thus, given any $v \in H^s(\mathbb{R}^n)$, we have $\tilde{v} = \mathsf{E}_s(v) \in H^1_{\mathrm{loc}}(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ and

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n \times \{0\}} ((-P)^s u) v \, \mathrm{d}x' \right| \\
&\quad \equiv \left| \int_{\mathbb{R}^n \times \{0\}} \big(\lim_{x_{n+1} \to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}\big) v \, \mathrm{d}x' \right| \\
&\quad = \left| \int_{\mathbb{R}^{n+1}_+} x_{n+1}^{1-2s} \partial_{n+1}^{1-2s} \partial_{n+1} \tilde{u} \partial_{n+1} \tilde{v} \, \mathrm{d}x + \int_{\mathbb{R}^{n+1}_+} A(x') \nabla' \tilde{u} \cdot \nabla' \tilde{v} \, \mathrm{d}x \right| \\
&\quad \le \lambda^{-1} \|\nabla \tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} \|\nabla \tilde{v}\|_{L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} \\
&\quad \equiv \lambda^{-1} \|\mathsf{E}_s(u)\|_{\dot{H}^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} \|\mathsf{E}_s(v)\|_{\dot{H}^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})} \\
&\quad \le C \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{H^s(\mathbb{R}^n)} \quad \text{using (2.5)}.
\end{aligned}$$

Therefore, by arbitrariness of $v \in H^s(\mathbb{R}^n)$, we conclude the following lemma:

Lemma 2.2. *Let $0 < s < 1$ and a_{jk} given as in Lemma 2.1. Then $(-P)^s\colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ is a bounded linear operator.*

Note that

$$Pu = \sum_{j,k=1}^n a_{jk} \partial_j \partial_k u + \sum_{j,k=1}^n (\partial_j a_{jk}) \partial_k u.$$

Since a_{jk} is uniformly Lipschitz, then

$$\| - Pu\|_{L^2(\mathbb{R}^n)} \le C \|u\|_{H^2(\mathbb{R}^n)}. \tag{2.6}$$

We here also remark that $\mathrm{dom}(-P) = H^2(\mathbb{R}^n)$ is the maximal extension such that $-P$ is self-adjoint and densely defined in $L^2(\mathbb{R}^n)$, see [11, equation (2.8)]. Given any $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, we see that

$$\langle Pu, \phi \rangle = (u, P\phi)_{L^2(\mathbb{R}^n)} \le \|u\|_{L^2(\mathbb{R}^n)} \|P\phi\|_{L^2(\mathbb{R}^n)} \le C \|u\|_{L^2(\mathbb{R}^n)} \|\phi\|_{H^2(\mathbb{R}^n)},$$

where $\langle \bullet, \bullet \rangle$ is the $H^{-2}(\mathbb{R}^n) \oplus H^2(\mathbb{R}^n)$ duality pair. Since

$$\mathcal{C}_c^\infty(\mathbb{R}^n) \text{ is dense in } H^\gamma(\mathbb{R}^n) \text{ for each } \gamma \in \mathbb{R} \quad \text{(see, e.g., [23, Remark 7.4])},$$

then we know that

$$\|Pu\|_{H^{-2}(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}. \tag{2.7}$$

We shall prove the followings:

Lemma 2.3. *Let $0 < s < 1$ and a_{jk} given as in Lemma 2.1. We have the inequality*

$$\|(-P)^s u\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^{2s}(\mathbb{R}^n)}. \tag{2.8}$$

Moreover, we have

$$\|(-P)^s u\|_{H^{-2s}(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}. \tag{2.9}$$

Remark 2.4. Using the duality argument as in (2.7), we know that (2.8) and (2.9) are equivalent.

In order to prove Lemma 2.3, we introduce the Balakrishnan operator as in [25, Definition 3.1.1 and Definition 5.1.1].

Definition 2.5. Let $\alpha \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Re z > 0\}$.

(1) If $0 < \Re\alpha < 1$, then $\mathrm{dom}((-P)_B^\alpha) = \mathrm{dom}(-P)$ and

$$(-P)_B^\alpha \phi = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda - P)^{-1}(-P)\phi \, d\lambda.$$

(2) If $\Re\alpha = 1$, then $\mathrm{dom}((-P)_B^\alpha) = \mathrm{dom}((-P)^2)$ and

$$(-P)_B^\alpha \phi = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}\Big[(\lambda - P)^{-1} - \frac{\lambda}{\lambda^2+1}\Big](-P)\phi \, d\lambda + \sin\frac{\alpha\pi}{2}(-P)\phi.$$

(3) If $n < \Re\alpha < n+1$ for $n \in \mathbb{N}$, then $\mathrm{dom}((-P)_B^\alpha) = \mathrm{dom}((-P)^{n+1})$ and

$$(-P)_B^\alpha \phi = (-P)_B^{\alpha-n}(-P)^n \phi.$$

(4) If $\Re\alpha = n+1$ for $n \in \mathbb{N}$, then $\mathrm{dom}((-P)_B^\alpha) = \mathrm{dom}((-P)^{n+2})$ and

$$(-P)_B^\alpha \phi = (-P)_B^{\alpha-n}(-P)^n \phi.$$

The following proposition, which can be found at [25, Theorem 6.1.6], shows that $(-P)_B^s$ and $(-P)^s$ are equivalent.

Proposition 2.6. *Let* $0 < s < 1$. *If* $u \in \mathrm{dom}((-P)^s_B)$, *then the strong limit*

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{\infty} (1 - e^{tP}) u \, \frac{dt}{t^{1+s}}$$

exists, and

$$(-P)^s_B u = c'_s \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{\infty} (1 - e^{tP}) u \, \frac{dt}{t^{1+s}} \quad \textit{for some positive constant } c'_s,$$

where $\{e^{tP}\}_{t \geq 0}$ *is the heat-diffusion semigroup generated by* $-P$.

Here and after, we shall not distinguish between $(-P)^s$ and $(-P)^s_B$, as well as $\mathrm{dom}((-P)^s)$ and $\mathrm{dom}((-P)^s_B)$. Using [25, Theorem 5.1.2], we have the following fact:

$$\text{if } u \in \mathrm{dom}((-P)^{\alpha+\beta}), \text{ then } (-P)^{\beta} u \in \mathrm{dom}((-P)^{\alpha}),$$

and the following identity holds:

$$(-P)^{\alpha}(-P)^{\beta} u = (-P)^{\alpha+\beta} u \quad \text{for all } u \in \mathrm{dom}((-P)^{\alpha+\beta}) \tag{2.10}$$

for all $\alpha, \beta \in \mathbb{C}$ with $\Re\alpha > 0$ and $\Re\beta > 0$. Since $(-P)^s$ is self-adjoint in $L^2(\mathbb{R}^n)$, then

$$\|(-P)^s u\|^2_{L^2(\mathbb{R}^n)} = ((-P)^{2s} u, u)_{L^2(\mathbb{R}^n)}.$$

Now, we are ready to prove Lemma 2.3.

Proof of Lemma 2.3. We first consider the case when $0 < s \leq 1/2$. Since $(-P)^s$ is self-adjoint, by observing that $(-P)^{2s} = (-P)^s(-P)^s$ (using (2.10)), Lemma 2.2 immediate implies

$$\begin{aligned} \|(-P)^s u\|^2_{L^2(\mathbb{R}^n)} &= ((-P)^{2s} u, u)_{L^2(\mathbb{R}^n)} \\ &\leq \|(-P)^{2s} u\|_{H^{-2s}(\mathbb{R}^n)} \|u\|_{H^{2s}(\mathbb{R}^n)} \\ &\leq C \|u\|^2_{H^{2s}(\mathbb{R}^n)}. \end{aligned} \tag{2.11}$$

When $1/2 < s < 1$, by observing that $(-P)^{2s} = (-P)^{2s-1}(-P) = (-P)(-P)^{2s-1}$ (using (2.10)) and $0 < 2s - 1 < 1$, using Lemma 2.2 we can easily show that

$$\begin{aligned} \|(-P)^{2s} u\|_{H^{1-2s}(\mathbb{R}^n)} &\leq C \|u\|_{H^{1+2s}(\mathbb{R}^n)} \\ \|(-P)^{2s} u\|_{H^{-1-2s}(\mathbb{R}^n)} &\leq C \|u\|_{H^{-1+2s}(\mathbb{R}^n)}. \end{aligned}$$

By interpolating the above two inequalities, we conclude that (2.11) holds for all $0 < s < 1$, and we complete the proof of Lemma 2.3. ∎

3. Boundary decay implies bulk decay

Firstly, we translate the decay behavior on $\mathbb{R}^n$ to decay behavior which is also holds on $\mathbb{R}^{n+1}_+$.

Proposition 3.1. *Let $s \in (0,1)$ and $u \in H^s(\mathbb{R}^n)$ be a solution to (1.1), with (1.3) and (1.4). For $s \neq \frac{1}{2}$, we further assume (1.5). Assume that $|q(x)| \leq 1$ and there exists $\alpha > 1$ such that*

$$\int_{\mathbb{R}^n} e^{|x|^\alpha} |u|^2 \, dx \leq C < \infty.$$

Then there exist constants $C_1, C_2 > 0$ so that the Caffarelli–Silvestre-type extension $\tilde{u}(x)$ satisfies

$$|\tilde{u}(x)| \leq C_1 e^{-C_2 |x|^\alpha} \quad \text{for all } x \in \mathbb{R}^{n+1}_+.$$

The ideas of proving Proposition 3.1 is similar to [31, Proposition 2.2]. The proof of [31, Proposition 2.2] utilized [30, Propositions 5.10–5.12]. The extension of such propositions involving many details, especially the Carleman estimate in [30, Propositions 5.7]. For sake of readability, here we present the details of the proofs.

In order to obtain the interior decay, similar to [31, Proposition 2.3], we need the following three-ball inequalities.

Lemma 3.2. *Let $s \in (0,1)$ and $\tilde{u} \in H^1(B_4^+, x_{n+1}^{1-2s})$ be a solution to*

$$[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P]\tilde{u} = 0 \quad \text{in } \mathbb{R}^{n+1}_+$$

with (1.3). Assume that $r \in (0,1)$ and $\bar{x}_0 = (\bar{x}_0', 5r) \in B_2^+$. Then, there exists $\alpha = \alpha(n,s) \in (0,1)$ such that

$$\|\tilde{u}\|_{L^\infty(B_{2r}^+(\bar{x}_0))} \leq C \|\tilde{u}\|^\alpha_{L^\infty(B_r^+(\bar{x}_0))} \|\tilde{u}\|^{1-\alpha}_{L^\infty(B_{4r}^+(\bar{x}_0))}.$$

Proof. As $(\bar{x}_0)_{n+1} = 5r$, this follows from a standard interior L^2 three ball inequalities together with L^∞-L^2 estimates for uniformly elliptic equations. ■

Also, we need the following boundary-bulk propagation of smallness estimation:

Lemma 3.3. *Let $s \in (0,1)$ and let $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ be a solution to (2.1) with (1.3) and $q \in L^\infty(\mathbb{R}^n)$. We assume that*

$$\max_{1 \leq j,k \leq n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1 \leq j,k \leq n} \|\nabla' a_{jk}\|_\infty \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$. For $s \neq \frac{1}{2}$, we further assume

$$\max_{1 \leq j,k \leq n} \|(\nabla')^2 a_{jk}\|_\infty \leq C$$

for some positive constant C. Assume that $x_0 \in \mathbb{R}^n \times \{0\}$. Then

(a) *there exist* $\alpha = \alpha(n,s) \in (0,1)$ *and* $c = c(n,s) \in (0,1)$ *such that*

$$\begin{aligned}
&\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{cr}^+(x_0))} \\
&\quad \le C\big[\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{16r}^+(x_0))} + r^{1-s}\|u\|_{L^2(B_{16r}'(x_0))}\big]^{\alpha} \\
&\qquad \times \big[r^{s+1}\|\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B_{16r}'(x_0))} \\
&\qquad\quad + r^{1-s}\|u\|_{L^2(B_{16r}'(x_0))}\big]^{1-\alpha} \\
&\qquad + C\big[\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{16r}^+(x_0))} + r^{1-s}\|u\|_{L^2(B_{16r}'(x_0))}\big]^{\frac{2s}{1+s}} \\
&\qquad \times \big[r^{s+1}\|\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B_{16r}'(x_0))} \\
&\qquad\quad + r^{1-s}\|u\|_{L^2(B_{16r}'(x_0))}\big]^{\frac{1-s}{1+s}};
\end{aligned}$$

(b) *there exist* $\alpha = \alpha(n,s) \in (0,1)$ *and* $c = c(n,s) \in (0,1)$ *such that*

$$\begin{aligned}
&\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^\infty(B_{\frac{cr}{2}}^+(x_0))} \\
&\quad \le C r^{-\frac{n}{2}}\big[r^{s-1}\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{16r}^+(x_0))} + \|u\|_{L^2(B_{16r}'(x_0))}\big]^{\alpha} \\
&\qquad \times \big[r^{2s}\|\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B_{16r}'(x_0))} + \|u\|_{L^2(B_{16r}'(x_0))}\big]^{1-\alpha} \\
&\qquad + C r^{-\frac{n}{2}}\big[r^{s-1}\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{16r}^+(x_0))} + \|u\|_{L^2(B_{16r}'(x_0))}\big]^{\frac{2s}{1+s}} \\
&\qquad \times \big[r^{2s}\|\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B_{16r}'(x_0))} + \|u\|_{L^2(B_{16r}'(x_0))}\big]^{\frac{1-s}{1+s}} \\
&\qquad + C r^{-\frac{n}{2}} r^{s}\|qu\|_{L^2(B_{16r}'(x_0))}^{\frac{1}{2}}\|u\|_{L^2(B_{16r}'(x_0))}^{\frac{1}{2}}.
\end{aligned}$$

Using Lemma 3.2 and Lemma 3.3, and imitating the chain-ball argument in [31], we can obtain Proposition 3.1.

3.1. Proof of the part (a) of Lemma 3.3 for the case $s \in [1/2, 1)$

We first prove the following extension of the Carleman estimate in [30, Proposition 5.7].

Lemma 3.4. *Let* $s \in [\frac{1}{2}, 1)$ *and let* $w \in H^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s})$ *with* $\operatorname{supp}(w) \subset B_{1/2}^+$ *be a solution to*

$$\Big[\partial_{n+1} x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k\Big] w = f \quad \text{in } \mathbb{R}_+^{n+1},$$

$$w = 0 \quad \text{on } \mathbb{R}^n \times \{0\}.$$

Suppose that

$$\phi(x) = \phi(x', x_{n+1}) := -\frac{|x'|^2}{4} + 2\Big(-\frac{1}{2-2s}x_{n+1}^{2-2s} + \frac{1}{2}x_{n+1}^2\Big).$$

We assume that

$$\max_{1\le j,k\le n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1\le j,k\le n} \|\nabla' a_{jk}\|_\infty \le \varepsilon$$

for some sufficiently small $\varepsilon > 0$. *For* $s \neq \frac{1}{2}$, *we further assume*

$$\max_{1\le j,k\le n} \|(\nabla')^2 a_{jk}\|_\infty \le C$$

for some positive constant C. *Assume additionally that*

$$\begin{aligned}\|x_{n+1}^{\frac{2s-1}{2}} f\|_{L^2(\mathbb{R}^{n+1}_+)} &+ \lim_{x_{x+1}\to 0} \|\Delta' w\|_{L^2(\mathbb{R}^n\times\{0\})}\\ &+ \lim_{x_{n+1}\to 0} \|\nabla' w\|_{L^2(\mathbb{R}^n\times\{0\})}\\ &+ \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s}\partial_{n+1} w\|_{L^2(\mathbb{R}^n\times\{0\})} < \infty.\end{aligned}$$

Then there exist $\tau_0 > 1$ *and a constant* C *such that*

$$\begin{aligned}&\tau^3\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}} w\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\ &\le C\Big(\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}} f\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\ &\qquad + \tau^{-1}\lim_{x_{n+1}\to 0}\|e^{\tau\phi}\Delta' w\|^2_{L^2(\mathbb{R}^n\times\{0\})}\\ &\qquad + \tau\lim_{x_{n+1}\to 0}\|e^{\tau\phi}x'\cdot\nabla' w\|^2_{L^2(\mathbb{R}^n\times\{0\})}\\ &\qquad + \tau\lim_{x_{n+1}\to 0}\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1} w\|^2_{L^2(\mathbb{R}^n\times\{0\})}\Big)\end{aligned}$$

for all $\tau \ge \tau_0$.

Proof. Now, we prove the Carleman estimate for $s \in (\frac{1}{2}, 1)$, as the case $s = \frac{1}{2}$ is naturally included in our estimates.

Step 1: Conjugation. Let $\tilde{u} = x_{n+1}^{\frac{1-2s}{2}} w$, we have

$$x_{n+1}^{\frac{2s-1}{2}} f = \Delta\tilde{u} + \mathring{c}_s x_{n+1}^{-2}\tilde{u} + \sum_{j,k=1}^{n}(a_{jk} - \delta_{jk})\partial_j\partial_k\tilde{u},$$

where $\mathring{c}_s = \frac{1-4s^2}{4}$. Let $u = e^{\tau\phi}\tilde{u}$, we have

$$\begin{aligned} e^{\tau\phi} x_{n+1}^{\frac{2s-1}{2}} f &= [\Delta + \tau^2|\nabla\phi|^2 + \mathring{c}_s x_{n+1}^{-2} - \tau\Delta\phi - 2\tau\nabla\phi\cdot\nabla]u \\ &\quad + \sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\partial_j\partial_k u \\ &\quad - \tau\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})[(\partial_k\phi)\partial_j + (\partial_j\phi)\partial_k]u \\ &\quad + \sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})[\tau^2(\partial_k\phi)(\partial_j\phi) - \tau(\partial_j\partial_k\phi)]u. \end{aligned}$$

We write $L^+ = S + A + \text{(I)} + \text{(II)} + \text{(III)}$, where

$$\begin{aligned} S &= \Delta + \tau^2|\nabla\phi|^2 + \mathring{c}_s x_{n+1}^{-2}, \quad A = -2\tau\nabla\phi\cdot\nabla - \tau\Delta\phi, \\ \text{(I)} &= \sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\partial_j\partial_k \\ \text{(II)} &= -\tau\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})[(\partial_k\phi)\partial_j + (\partial_j\phi)\partial_k] \\ \text{(III)} &= \sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})[\tau^2(\partial_k\phi)(\partial_j\phi) - \tau(\partial_j\partial_k\phi)]. \end{aligned}$$

We now define $L^- := S - A + \text{(I)} - \text{(II)} + \text{(III)}$,

$$\mathcal{D} := \|L^+u\|^2 - \|L^-u\|^2 \quad \text{and} \quad \mathcal{S} := \|L^+u\|^2 + \|L^-u\|^2,$$

where

$$\begin{aligned} \|\bullet\| &= \|\bullet\|_{L^2(\mathbb{R}^{n+1}_+)}, \quad \|\bullet\|_0 = \|\bullet\|_{L^2(\mathbb{R}^n\times\{0\})} \\ \langle\bullet,\bullet\rangle &= \langle\bullet,\bullet\rangle_{L^2(\mathbb{R}^{n+1}_+)}, \quad \langle\bullet,\bullet\rangle_0 = \langle\bullet,\bullet\rangle_{L^2(\mathbb{R}^n\times\{0\})} \end{aligned}$$

and we omit the notations "$\lim_{x_{n+1}\to 0}$" in $\|\bullet\|_0$ and $\langle\bullet,\bullet\rangle_0$.

Step 2: Estimating the bulk contributions.

Step 2.1: Estimating the difference $\mathcal{D}$. Observe that $\mathcal{D} = 4\langle Su, Au\rangle + R$, where

$$\begin{aligned} R &= 4\langle Su, \text{(II)}\,u\rangle + 4\langle Au, \text{(I)}\,u\rangle + 4\langle Au, \text{(III)}\,u\rangle \\ &\quad + 4\langle \text{(I)}\,u, \text{(II)}\,u\rangle + 4\langle \text{(II)}\,u, \text{(III)}\,u\rangle. \end{aligned}$$

Step 2.1.1: Computation the principal term. Note that

$$
\begin{aligned}
2\langle Su, Au\rangle = {} & \langle [S,A]u,u\rangle + 2\tau\langle Su, (\partial_{n+1}\phi)u\rangle_0 \\
& - \langle Au, \partial_{n+1}u\rangle_0 + \langle \partial_{n+1}(Au), u\rangle_0. \qquad (3.1)
\end{aligned}
$$

Observe that $[S,A] = [S,A]_1 + [S,A]_2$, where

$$
\begin{aligned}
[S,A]_1 &= [\Delta' + \tau^2|\nabla'\phi|^2, -2\tau\nabla'\phi\cdot\nabla' - \tau\Delta'\phi], \\
[S,A]_2 &= \Big[\partial_{n+1}^2 + \tau^2(\partial_{n+1}\phi)^2 + \frac{1-4s^2}{4}x_{n+1}^{-2}, -2\tau\partial_{n+1}\phi\partial_{n+1} - \tau\partial_{n+1}^2\phi\Big].
\end{aligned}
$$

The following identity can be found in [30, equation (5.20) of Proposition 5.7]:

$$
\langle [S,A]_1u,u\rangle = -\frac{1}{2}\tau^3\||x'|u\|^2 - 2\tau\|\nabla'u\|^2. \qquad (3.2)
$$

For our purpose, we need to refine the estimate [30, equation (5.22) of Proposition 5.7]. The following identity can be found in [30, equation (5.19) of Proposition 5.7]:

$$
\begin{aligned}
\langle [S,A]_2u,u\rangle = {} & 4\tau^3\langle u, (\partial_{n+1}\phi)^2(\partial_{n+1}^2\phi)u\rangle \\
& + 4\tau\langle \partial_{n+1}u, (\partial_{n+1}^2\phi)\partial_{n+1}u\rangle \\
& - \tau\langle u, (\partial_{n+1}^4\phi)u\rangle \\
& - 4\mathring{c}_s\tau\langle u, x_{n+1}^{-3}(\partial_{n+1}\phi)u\rangle \\
& + 4\tau\langle (\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0.
\end{aligned}
$$

From [30, equation (5.21) of Proposition 5.7], we have

$$
(\partial_{n+1}\phi)^2(\partial_{n+1}^2\phi) = 8(x_{n+1}^{1-2s} - x_{n+1})^2((2s-1)x_{n+1}^{-2s} + 1)
$$

and

$$
\begin{aligned}
& -\tau\langle u, (\partial_{n+1}^4\phi)u\rangle + (2s+1)(2s-1)\tau\langle u, x_{n+1}^{-3}(\partial_{n+1}\phi)u\rangle \\
& \quad = 2\tau(1-2s)(1+2s)^2\|x_{n+1}^{-1-s}u\|^2 - 8\tau\mathring{c}_s\|x_{n+1}^{-1}u\|^2.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\langle [S,A]_2u,u\rangle = {} & 32\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
& + 32(2s-1)\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\
& + 8\tau(2s-1)\|x_{n+1}^{-s}\partial_{n+1}u\|^2 \\
& + 2\tau(1-2s)(1+2s)^2\|x_{n+1}^{-1-s}u\|^2 \\
& + 8\tau\|\partial_{n+1}u\|^2 - 8\tau\mathring{c}_s\|x_{n+1}^{-1}u\|^2 \\
& + 4\tau\langle (\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0. \qquad (3.3)
\end{aligned}
$$

Combining (3.1), (3.2), and (3.3), we reach

$$\begin{aligned}4\langle Su, Au\rangle &= 64\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\ &\quad + 64(2s-1)\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\ &\quad + 16\tau(2s-1)\|x_{n+1}^{-s}\partial_{n+1}u\|^2 \\ &\quad + 16\tau\|\partial_{n+1}u\|^2 \\ &\quad - 8\tau\mathring{c}_s\|x_{n+1}^{-1}u\|^2 \\ &\quad + 4\tau(1-2s)(1+2s)^2\|x_{n+1}^{-1-s}u\|^2 \\ &\quad - \tau^3\||x'|u\|^2 - 4\tau\|\nabla' u\|^2 \\ &\quad + 8\tau\langle(\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0 \\ &\quad + 4\tau\langle Su, (\partial_{n+1}\phi)u\rangle_0 \\ &\quad - 2\langle Au, \partial_{n+1}u\rangle_0 \\ &\quad + 2\langle\partial_{n+1}(Au), u\rangle_0. \end{aligned} \tag{3.4}$$

Step 2.1.2: Estimating the remainder. Using integration by parts, we can estimate R from below:

$$\begin{aligned}R \geq -C\varepsilon\big[&\tau\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 + \tau\|\partial_{n+1}u\|^2 + \tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\ &+ \tau\|x_{n+1}^{-1}u\|^2 + \tau\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 + \tau\|x_{n+1}^{\frac{2s-1}{2}}|x'|\nabla' u\|_0^2 \\ &+ \tau^3\|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}}u\|_0^2\big].\end{aligned}$$

Here we would like to highlight some features when estimating the second term of R, that is, $\langle Au, \text{(I)}\, u\rangle$. Note that

$$\begin{aligned}&\langle -2\tau\partial_{n+1}\phi\partial_{n+1}u, (a_{jk} - \delta_{jk})\partial_j\partial_k u\rangle \\ &\quad = \tau\langle\partial_{n+1}\phi\partial_{n+1}u, (\partial_j a_{jk})\partial_k u\rangle - \boxed{\tau\langle\partial_{n+1}^2\phi\partial_j u, (a_{jk} - \delta_{jk})\partial_k u\rangle} \\ &\qquad + \tau\langle\partial_{n+1}\phi\partial_j u, (\partial_k a_{jk})\partial_{n+1}u\rangle - \tau\langle\partial_{n+1}\phi\partial_j u, (a_{jk} - \delta_{jk})\partial_k u\rangle_0 \end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\langle -\tau(\partial_{n+1}^2\phi)u, (a_{jk} - \delta_{jk})\partial_j\partial_k u\rangle \\ &\quad = \tau\langle(\partial_{n+1}^2\phi)u, (\partial_j a_{jk})\partial_k u\rangle + \tau\langle\partial_{n+1}^2\phi\partial_j u, (a_{jk} - \delta_{jk})\partial_k u\rangle \qquad (3.6) \\ &\quad = -\frac{\tau}{2}\langle(\partial_{n+1}^2\phi)u, (\partial_j\partial_k a_{jk})u\rangle + \boxed{\tau\langle\partial_{n+1}^2\phi\partial_j u, (a_{jk} - \delta_{jk})\partial_k u\rangle}. \qquad (3.7)\end{aligned}$$

So, summing up (3.5) and (3.7), we note that the problematic term

$$\tau\langle\partial_{n+1}^2\phi\partial_j u, (a_{jk} - \delta_{jk})\partial_k u\rangle$$

is canceled. It problematic because $\partial_{n+1}^2\phi$ has singularity x_{n+1}^{-2s} for $s \in (1/2, 1)$. However, when $s = \frac{1}{2}$, $\partial_{n+1}^2\phi$ has no singularity. In this case, we consider (3.6) rather than (3.7). This is the reason why we can loosen the second derivative assumption for the case $s = \frac{1}{2}$.

Step 2.1.3: Combining the commutator and the remainder. Using the Hardy inequality in Lemma A.1, we reach

$$\|x_{n+1}^{-s-1}u\|^2 \le \frac{4}{(2s+1)^2}\|x_{n+1}^{-s}\partial_{n+1}u\|^2 + \frac{2}{2s+1}\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2,$$

thus

$$\begin{aligned}&16\tau(2s-1)\|x_{n+1}^{-s}\partial_{n+1}u\|^2 - 4\tau(2s-1)(2s+1)^2\|x_{n+1}^{-1-s}u\|^2\\ &\quad\ge -8\tau(2s-1)(2s+1)\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2.\end{aligned}$$

Therefore, choosing sufficiently small $\varepsilon > 0$, we reach

$$\begin{aligned}\mathcal{D} \ge\ & 64\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 + \frac{639}{10}(2s-1)\tau^3\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2\\ &+ \frac{159}{10}\tau\|\partial_{n+1}u\|^2 + \frac{39}{10}\tau(2s-1)(2s+1)\|x_{n+1}^{-1}u\|^2 - 4\tau\|\nabla' u\|^2\\ &- C\varepsilon\tau\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 + 8\tau\langle(\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0 + 4\tau\langle Su, \partial_{n+1}\phi\rangle_0\\ &- 2\langle Au, \partial_{n+1}u\rangle_0 + 2\langle\partial_{n+1}(Au), u\rangle_0 - \tau\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2\\ &- \tau\|x_{n+1}^{\frac{2s-1}{2}}|x'|\nabla' u\|_0^2 - \tau^3\|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}}u\|_0^2\\ &- 8\tau(2s-1)(2s+1)\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2.\end{aligned} \tag{3.8}$$

Step 2.2: Estimating the sum $\mathcal{S}$. Observe that

$$\mathcal{S} \ge 2\|Su\|^2 + 2\|Au\|^2 - C\varepsilon\Big[\sum_{j,k=1}^{n}\|\partial_j\partial_k u\|^2 + \tau^2\|\nabla' u\|^2 + \tau^4\|u\|^2\Big].$$

Since $\mathring{c}_s < 0$, then

$$\begin{aligned}2\|Su\|^2 &= 2\|\Delta' u + (\partial_{n+1}^2 u + \tau^2|\nabla\phi|^2 u + \mathring{c}_s x_{n+1}^{-2}u)\|^2\\ &= 2\|\Delta' u\|^2 + 4\langle\Delta' u, \partial_{n+1}^2 u\rangle + 4\tau^2\langle\Delta' u, |\nabla\phi|^2 u\rangle\\ &\quad + 4\mathring{c}_s\langle\Delta' u, x_{n+1}^{-2}u\rangle + 2\|\partial_{n+1}^2 u + \tau^2|\nabla\phi|^2 u + \mathring{c}_s x_{n+1}^{-2}u\|^2\\ &= 2\sum_{j,k=1}^{n}\|\partial_j\partial_k u\|^2 + 4\langle\Delta' u, \partial_{n+1}^2 u\rangle + 4\tau^2\langle\Delta' u, |\nabla\phi|^2 u\rangle\\ &\quad - 4\mathring{c}_s\langle\nabla' u, x_{n+1}^{-2}\nabla' u\rangle + 2\|\partial_{n+1}^2 u + \tau^2|\nabla\phi|^2 u + \mathring{c}_s x_{n+1}^{-2}u\|^2\\ &\ge 2\sum_{j,k=1}^{n}\|\partial_j\partial_k u\|^2 + 4\langle\Delta' u, \partial_{n+1}^2 u\rangle + 4\tau^2\langle\Delta' u, |\nabla\phi|^2 u\rangle.\end{aligned}$$

Since

$$4\langle \Delta' u, \partial_{n+1}^2 u\rangle = 4\langle \nabla'\partial_{n+1}u, \nabla'\partial_{n+1}u\rangle - 4\langle \Delta' u, \partial_{n+1}u\rangle_0$$

and for $\varepsilon_0 > 0$, we have

$$\begin{aligned}
&4\tau^2\langle \Delta' u, |\nabla\phi|^2 u\rangle \\
&\quad = \tau^2\langle \Delta' u, |x'|^2 u\rangle + 16\tau^2\langle \Delta' u, (x_{n+1}^{1-2s} - x_{n+1})^2 u\rangle \\
&\quad \geq -\tau^2(1+\varepsilon_0)\|\nabla' u\|^2 - \tau^2 C\varepsilon_0^{-1}\|u\|^2 - 16\tau^2\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{S} &\geq 2\|Su\|^2 + 2\|Au\|^2 - C\varepsilon\Big[\sum_{j,k=1}^{n}\|\partial_j\partial_k u\|^2 + \tau^2\|\nabla' u\|^2 + \tau^4\|u\|^2\Big] \\
&\geq 2\|\nabla(\nabla' u)\|^2 - \tau^2(1+\varepsilon_0)\|\nabla' u\|^2 \\
&\quad - \tau^2 C\varepsilon_0^{-1}\|u\|^2 - 16\tau^2\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 \\
&\quad - C\varepsilon\Big[\sum_{j,k=1}^{n}\|\partial_j\partial_k u\|^2 + \tau^2\|\nabla' u\|^2 + \tau^4\|u\|^2\Big] - 4\langle \Delta' u, \partial_{n+1}u\rangle_0. \qquad (3.9)
\end{aligned}$$

Step 2.3: Combining the difference $\mathcal{D}$ and the sum $\mathcal{S}$. After combining (3.8) and (3.9), we choose small $\varepsilon > 0$, and consequently choose small $\varepsilon_0 > 0$ and large τ, hence

$$\begin{aligned}
\Big(\tau + s + \frac{1}{2}\Big)\|L^+ u\|^2 &\geq \frac{9}{10}(2s-1)\|\nabla(\nabla' u)\|^2 + \|Su\|^2 \\
&\quad + 64\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
&\quad + \frac{639}{10}(2s-1)\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\
&\quad + \frac{159}{10}\tau^2\|\partial_{n+1}u\|^2 - 4\tau^2\|\nabla' u\|^2 \\
&\quad - \frac{171}{20}(2s-1)\tau^2\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 \\
&\quad + \frac{39}{10}\tau^2(2s-1)(2s+1)\|x_{n+1}^{-1}u\|^2 \\
&\quad + 8\tau^2\langle(\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0 + 4\tau^2\langle Su, \partial_{n+1}\phi\rangle_0 \\
&\quad - 2\tau\langle Au, \partial_{n+1}u\rangle_0 + 2\tau\langle\partial_{n+1}(Au), u\rangle_0 \\
&\quad - \tau^2\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 - \tau^2\|x_{n+1}^{\frac{2s-1}{2}}|x'|\nabla' u\|_0^2 \\
&\quad - \tau^4\|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}}u\|_0^2 \\
&\quad - 8\tau^2(2s-1)(2s+1)\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2 \\
&\quad - 2(2s-1)\langle \Delta' u, \partial_{n+1}u\rangle_0. \qquad (3.10)
\end{aligned}$$

Step 2.4: Obtaining gradient estimates. Since $\operatorname{supp}(u) \subset B^+_{1/2}$ and $s > \frac{1}{2}$, thus

$$0 \le (x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^s = x_{n+1}^{1-s} - x_{n+1}^{1+s} \le x_{n+1}^{1-s} \le 1,$$

and hence

$$\begin{aligned}
&\frac{172}{20}(2s-1)\tau^2\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 \\
&\quad = -\frac{172}{20}(2s-1)\tau^2\langle (x_{n+1}^{1-2s} - x_{n+1})x^s\Delta' u, (x_{n+1}^{1-2s} - x_{n+1})x^{-s}u\rangle \\
&\quad \le \frac{86}{20}(2s-1)\delta\|(x_{n+1}^{1-2s} - x_{n+1})x^s\Delta' u\|^2 \\
&\qquad + \frac{86}{20}(2s-1)\tau^4\delta^{-1}\|(x_{n+1}^{1-2s} - x_{n+1})x^{-s}u\|^2 \\
&\quad \le \frac{86}{20}(2s-1)\delta\|\Delta' u\|^2 + \frac{86}{20}(2s-1)\tau^4\delta^{-1}\|(x_{n+1}^{1-2s} - x_{n+1})x^{-s}u\|^2.
\end{aligned}$$

Choose $\delta = \frac{8}{43}$, we reach

$$\begin{aligned}
&\frac{172}{20}(2s-1)\tau^2\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 \\
&\quad \le \frac{8}{10}(2s-1)\|\Delta' u\|^2 + 23.1125(2s-1)\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})x^{-s}u\|^2. \qquad (3.11)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\frac{41}{10}\tau^2\langle Su, u\rangle &= \frac{41}{10}\tau^2\|\nabla u\|^2 - \frac{41}{10}\tau^4\||\nabla\phi|u\|^2 + \frac{41}{5}(2s+1)(2s-1)\tau\|x_{n+1}^{-1}u\|^2 \\
&\quad + \frac{41}{10}\tau^2\langle \partial_{n+1}u, u\rangle_0 \\
&\ge \frac{41}{10}\tau^2\|\nabla u\|^2 - \frac{41}{10}\tau^4\Big(\frac{1}{16}\|u\|^2 + 4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2\Big) \\
&\quad + \frac{41}{5}(2s+1)(2s-1)\tau\|x_{n+1}^{-1}u\|^2 + \frac{41}{10}\tau^2\langle \partial_{n+1}u, u\rangle_0 \\
&= \frac{41}{10}\tau^2\|\nabla u\|^2 - \frac{41}{160}\tau^4\|u\|^2 - \frac{164}{10}\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
&\quad + \frac{41}{5}(2s+1)(2s-1)\tau\|x_{n+1}^{-1}u\|^2 \\
&\quad + \frac{41}{5}(2s+1)(2s-1)\tau\|x_{n+1}^{-1}u\|^2 + \frac{41}{10}\tau^2\langle \partial_{n+1}u, u\rangle_0. \qquad (3.12)
\end{aligned}$$

Define $\psi_s(x_{n+1}) := x_{n+1}^{1-2s} - x_{n+1}$. Since $\operatorname{supp}(u) \subset B^+_{1/2}$, so $0 \le x_{n+1} \le 1/2$, for $s \in (1/2, 1)$, the derivative can be easily estimated

$$\psi_s'(x_{n+1}) = (1-2s)x_{n+1}^{-2s} - 1 < 0 \quad \text{for } 0 \le x_{n+1} \le 1/2.$$

Since $\psi_s(x_{n+1})$ is decreasing on $[0, 1/2]$, for $s \in (1/2, 1)$,

$$\inf_{0\le x_{n+1}\le 1/2}(x_{n+1}^{1-2s} - x_{n+1}) = \inf_{0\le x_{n+1}\le 1/2}\psi_s(x_{n+1}) = \psi_s\Big(\frac{1}{2}\Big) = \frac{1}{2}(4^s - 1) \ge \frac{1}{2}.$$

Combining this with (3.12), we reach the estimate

$$\begin{aligned}
&\frac{41}{10}\tau^2\|\nabla' u\|^2 + \frac{41}{5}(2s+1)(2s-1)\tau^2\|x_{n+1}^{-1}u\|^2 + \frac{41}{10}\tau^2\langle\partial_{n+1}u, u\rangle_0 \\
&\quad\le \frac{41}{10}\tau^2\|\nabla u\|^2 + \frac{41}{5}(2s+1)(2s-1)\tau^2\|x_{n+1}^{-1}u\|^2 + \frac{41}{10}\tau^2\langle\partial_{n+1}u, u\rangle_0 \\
&\quad\le \frac{41}{10}\tau^2\langle Su, u\rangle + \frac{41}{160}\tau^4\|u\|^2 + \frac{164}{10}\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
&\quad\le \frac{41}{20}\delta\|Su\|^2 + \frac{41}{20}\delta^{-1}\tau^4\|u\|^2 + \frac{41}{160}\tau^4\|u\|^2 + \frac{164}{10}\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
&\quad\le \frac{41}{20}\delta\|Su\|^2 + \frac{82}{10}\delta^{-1}\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 + \frac{41}{40}\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
&\qquad + \frac{164}{10}\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2.
\end{aligned}$$

Choosing $\delta = \frac{20}{41}$, hence

$$\begin{aligned}
&\frac{41}{10}\tau^2\|\nabla' u\|^2 + \frac{41}{5}(2s+1)(2s-1)\tau^2\|x_{n+1}^{-1}u\|^2 + \frac{41}{10}\tau^2\langle\partial_{n+1}u, u\rangle_0 \\
&\quad\le \|Su\|^2 + 34.235\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2. \qquad (3.13)
\end{aligned}$$

Step 2.5: Plugging gradient estimates into (3.10)**.** Combining (3.10), (3.11), and (3.13), we reach

$$\begin{aligned}
\Big(\tau + s + \frac{1}{2}\Big)\|L^+u\|^2 \ge\ & \frac{1}{10}(2s-1)\|\nabla(\nabla' u)\|^2 + 29.765\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})u\|^2 \\
& + 40.7875(2s-1)\tau^4\|(x_{n+1}^{1-2s} - x_{n+1})x_{n+1}^{-s}u\|^2 \\
& + \frac{159}{10}\tau^2\|\partial_{n+1}u\|^2 + \frac{1}{10}\tau^2\|\nabla' u\|^2 \\
& + \frac{1}{20}(2s-1)\tau^2\|(x_{n+1}^{1-2s} - x_{n+1})\nabla' u\|^2 \\
& + 12.1\tau^2(2s-1)(2s+1)\|x_{n+1}^{-1}u\|^2 \\
& + 8\tau^2\langle(\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0 + 4\tau^2\langle Su, (\partial_{n+1}\phi)u\rangle_0 \\
& - 2\tau\langle Au, \partial_{n+1}u\rangle_0 + 2\tau\langle\partial_{n+1}(Au), u\rangle_0 \\
& - \tau^2\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 - \tau^2\|x_{n+1}^{\frac{2s-1}{2}}|x'|\nabla' u\|_0^2 \\
& - \tau^4\|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}}u\|_0^2 \\
& - 8\tau^2(2s-1)(2s+1)\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2 \\
& - 4(2s-1)\langle\Delta' u, \partial_{n+1}u\rangle_0 + \frac{41}{10}\tau^2\langle\partial_{n+1}u, u\rangle_0. \qquad (3.14)
\end{aligned}$$

Hence, we reach

$$
\begin{aligned}
2\tau\|L^+u\|^2 \geq\, & 25\tau^4\|(x_{n+1}^{1-2s}-x_{n+1})u\|^2 + \frac{1}{10}\tau^2\|\nabla u\|^2 \\
& + 12\tau^2(2s-1)(2s+1)\|x_{n+1}^{-1}u\|^2 + 8\tau^2\langle(\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0 \\
& + 4\tau^2\langle Su, (\partial_{n+1}\phi)u\rangle_0 - 2\tau\langle Au, \partial_{n+1}u\rangle_0 + 2\tau\langle\partial_{n+1}(Au), u\rangle_0 \\
& - \tau^2\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 - \tau^2\|x_{n+1}^{\frac{2s-1}{2}}|x'|\nabla' u\|_0^2 \\
& - \tau^4\|(x_{n+1}^{1-2s}-x_{n+1})^{\frac{1}{2}}u\|_0^2 - 8\tau^2(2s-1)(2s+1)\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2 \\
& - 4(2s-1)\langle\Delta' u, \partial_{n+1}u\rangle_0 + \frac{41}{10}\tau^2\langle\partial_{n+1}u, u\rangle_0. \qquad (3.15)
\end{aligned}
$$

Since $u = e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}w$, we estimate that

$$
\begin{aligned}
\|\nabla u\|^2 \geq\, & \frac{1}{2}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2 - 2\tau^2\|e^{\tau\phi}|\nabla\phi|x_{n+1}^{\frac{1-2s}{2}}w\|^2 \\
& - 2\Big(\frac{2s-1}{2}\Big)^2\|e^{\tau\phi}x_{n+1}^{-\frac{1+2s}{2}}w\|^2 \\
\geq\, & \frac{1}{2}\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2 - 16\tau^2\|(x_{n+1}^{1-2s}-x_{n+1})u\|^2 - (2s-1)^2\|x_{n+1}^{-1}u\|^2.
\end{aligned}
$$

Step 3: Estimating the boundary contributions. We want to show that

$$
\|e^{\tau\phi}x_{n+1}^{-2s}w\|_0 \leq C_s\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0 < \infty. \qquad (3.16)
$$

Indeed, since $w(x',0)\equiv 0$, thus

$$
\begin{aligned}
x_{n+1}^{-2s}w(x',x_{n+1}) &= x_{n+1}^{1-2s}\int_0^1 \partial_{n+1}w(x',tx_{n+1})\,dt \\
&= \int_0^1 (tx_{n+1})^{1-2s}\partial_{n+1}w(x',tx_{n+1})t^{2s-1}\,dt.
\end{aligned}
$$

Multiplying above equation by $e^{\tau\phi}$, taking the L^2-norm with respect to x' and using the fact that $\partial_{n+1}\phi < 0$ on supp(w) gives

$$
\begin{aligned}
& \|e^{\tau\phi}x_{n+1}^{-2s}w(\bullet,x_{n+1})\|_0 \\
& \leq \sup_{t\in(0,1)}\|e^{\tau\phi(\bullet,tx_{n+1})}(tx_{n+1})^{1-2s}\partial_{n+1}w(\bullet,tx_{n+1})\|_0\int_0^1 t^{2s-1}\,dt.
\end{aligned}
$$

Taking $x_{n+1}\to 0$ proves (3.16).

We observe that

$$\begin{aligned}
&4\tau^2\langle Su, (\partial_{n+1}\phi)u\rangle_0 - 2\tau\langle Au, \partial_{n+1}u\rangle_0 + 2\tau\langle\partial_{n+1}(Au), u\rangle_0 \\
&\quad = 8\tau^2\langle\partial_{n+1}u, \nabla'\phi\cdot\nabla'u\rangle_0 + 4\tau^2\langle(\partial_{n+1}u)^2, \partial_{n+1}\phi\rangle_0 \\
&\qquad - 4\tau^2\langle(\partial_{n+1}\phi), |\nabla'u|^2\rangle_0 + 4\tau^2\langle(\Delta'\phi - \partial_{n+1}^2\phi)u, \partial_{n+1}u\rangle_0 \\
&\qquad - 2\tau^2\langle(\partial_{n+1}^3\phi)u, u\rangle_0 + 4\tau^4\langle(\partial_{n+1}\phi)|\nabla\phi|^2u, u\rangle_0 \\
&\qquad - \tau^2(2s+1)(2s-1)\langle x_{n+1}^{-2}u, (\partial_{n+1}\phi)u\rangle_0 \\
&\quad \geq 8\tau^2\langle\partial_{n+1}u, \nabla'\phi\cdot\nabla'u\rangle_0 + 4\tau^2\langle(\partial_{n+1}u)^2, \partial_{n+1}\phi\rangle_0 \\
&\qquad + 4\tau^2\langle(\Delta'\phi - \partial_{n+1}^2\phi)u, \partial_{n+1}u\rangle_0 + 4\tau^4\langle(\partial_{n+1}\phi)|\nabla\phi|^2u, u\rangle_0.
\end{aligned}$$

Note that (3.16) imply

$$\begin{aligned}
\partial_{n+1}u &= e^{\tau\phi}\left(x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}w - \frac{2s-1}{2}x_{n+1}^{-\frac{1+2s}{2}}w\right) + x_{n+1}^{\frac{3-2s}{2}}R, \\
\nabla'u &= e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla'w + x_{n+1}^{s+\frac{1}{2}}R',
\end{aligned}$$

where $\|R\|_0 \leq C\tau$ and $\|R'\|_0 \leq C\tau$.

Hence,

$$\begin{aligned}
&|\langle\partial_{n+1}u, \nabla'\phi\cdot\nabla'u\rangle_0| \\
&\quad = \left|\left\langle e^{\tau\phi}\left(x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}w - \frac{2s-1}{2}x_{n+1}^{-\frac{1+2s}{2}}w\right), e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla'\phi\cdot\nabla'w\right\rangle_0\right| \\
&\quad = \left|\left\langle e^{\tau\phi}\left(x_{n+1}^{1-2s}\partial_{n+1}w - \frac{2s-1}{2}x_{n+1}^{-2}w\right), \frac{1}{2}e^{\tau\phi}x'\cdot\nabla'w\right\rangle_0\right| \\
&\quad \leq \frac{1}{2}|\langle e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w, e^{\tau\phi}x'\cdot\nabla'w\rangle_0| + \frac{2s-1}{4}|\langle e^{\tau\phi}x_{n+1}^{-2}w, e^{\tau\phi}x'\cdot\nabla'w\rangle_0|.
\end{aligned}$$

Using (3.16), we reach

$$|\langle\partial_{n+1}u, \nabla'\phi\cdot\nabla'u\rangle_0| \leq \|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0\|e^{\tau\phi}x'\cdot\nabla'w\|_0.$$

Similarly, using (3.16), we have

$$\begin{aligned}
|\langle(\partial_{n+1}u)^2, \partial_{n+1}\phi\rangle_0| + |\langle(\Delta'\phi - \partial_{n+1}^2\phi)u, \partial_{n+1}u\rangle_0| &\leq C\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0^2, \\
|\langle(\partial_{n+1}\phi)|\nabla\phi|^2u, u\rangle_0| &\leq C\|e^{\tau\phi}x_{n+1}^{2-4s}w\|_0^2 \to 0.
\end{aligned}$$

Also,

$$\begin{aligned}
|\langle(\partial_{n+1}^2\phi)\partial_{n+1}u, u\rangle_0| &\leq C\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0^2, \\
\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 &= \|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w - \frac{2s-1}{2}e^{\tau\phi}x_{n+1}^{-2s}w\|_0^2, \\
&\leq C\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0^2,
\end{aligned}$$

$$\|x_{n+1}^{\frac{2s-1}{2}}|x'|\nabla' u\|_0^2 = \|e^{\tau\phi}|x'|\nabla' w\|_0^2,$$
$$\|(x_{n+1}^{1-2s} - x_{n+1})^{\frac{1}{2}}u\|_0^2 \to 0,$$
$$\|x_{n+1}^{-\frac{1}{2}-s}u\|_0^2 = \|e^{\tau\phi}x_{n+1}^{-2s}w\|_0^2 \le C\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0^2,$$
$$|\langle \partial_{n+1}u, u\rangle_0| \to 0.$$

Finally, we also have

$$\begin{aligned}
|\langle \Delta' u, \partial_{n+1}u\rangle_0| &\le \|x_{n+1}^{\frac{2s-1}{2}}\Delta' u\|_0^2 + \|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 \\
&= \left\| -\frac{n\tau}{2}e^{\tau\phi}w + \frac{\tau^2}{4}|x'|^2 e^{\tau\phi}w - \tau e^{\tau\phi}x'\cdot\nabla' w + e^{\tau\phi}\Delta' w\right\|_0^2 \\
&\quad + \|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u\|_0^2 \\
&\le C\|e^{\tau\phi}\Delta' w\|_0^2 + C\tau^2\|e^{\tau\phi}x'\cdot\nabla' w\|_0^2 + C\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0^2.
\end{aligned}$$

Step 4: Conclusion. Put them together, we reach

$$\begin{aligned}
&\tau^3\|u\|^2 + \tau\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2 \\
&\quad \le C(\|L^+u\|^2 + \tau^{-1}\|e^{\tau\phi}\Delta' w\|_0^2 + \tau\|e^{\tau\phi}x'\cdot\nabla' w\|_0^2 + \tau\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}w\|_0^2),
\end{aligned}$$

which is our desired result. ■

As in [30], we introduce the following sets for $s \in [\frac{1}{2}, 1)$:

$$C_{s,r}^+ := \left\{(x', x_{n+1}) \in \mathbb{R}_+^{n+1} : x_{n+1} \le \left[(1-s)\left(r - \frac{|x'|^2}{4}\right)\right]^{\frac{1}{2-2s}}\right\},$$
$$C_{s,r}' := \left\{(x', 0) \in \mathbb{R}^n \times \{0\} : 0 \le \left[(1-s)\left(r - \frac{|x'|^2}{4}\right)\right]^{\frac{1}{2-2s}}\right\}.$$

With this notation, we infer the following analogous to [30, Proposition 5.10]:

Lemma 3.5. *Let $s \in [\frac{1}{2}, 1)$. Suppose that $\widetilde{w} \in H^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s})$ is a solution to*

$$\left[\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}\sum_{j,k=1}^n \partial_j a_{jk}\partial_k\right]\widetilde{w} = 0 \quad \text{in } \mathbb{R}_+^{n+1},$$
$$\widetilde{w} = w \quad \text{on } \mathbb{R}^n \times \{0\},$$

with $w = 0$ on B_1'. We assume that

$$\max_{1\le j,k\le n}\|a_{jk} - \delta_{jk}\|_\infty + \max_{1\le j,k\le n}\|\nabla' a_{jk}\|_\infty \le \varepsilon$$

for some sufficiently small $\varepsilon > 0$. For $s \neq \frac{1}{2}$, we further assume

$$\max_{1\leq j,k\leq n} \|(\nabla')^2 a_{jk}\|_\infty \leq C$$

for some positive constant C. Then, there exists $\alpha = \alpha(n,s) \in (0,1)$ such that

$$\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/8}^+)} \leq C\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/2}^+)}^{\alpha} \cdot \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{L^2(C_{s,1/2}')}^{1-\alpha}.$$

Proof. We may assume that $\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/2}^+)} > 0$ and

$$\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/2}^+)} \geq c_0 \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{L^2(C_{s,1/2}')}^{1-\alpha}$$

for some sufficiently large constant $c_0 > 0$. Otherwise the result is trivial.

Let η be a smooth cut-off function satisfies

$$\eta(x) = \begin{cases} 1 & \text{in } C_{s,3/16}^+, \\ 0 & \text{in } \mathbb{R}_+^{n+1} \setminus C_{s,1/4}^+, \end{cases}$$

and $|\partial_{n+1}\eta| \leq Cx_{n+1}$ in $\mathbb{R}_+^{n+1}$ with $\partial_{n+1}\eta = 0$ on $\mathbb{R}^n \times \{0\}$. Define $\overline{w} = \eta\widetilde{w}$. Note that $\overline{w}$ satisfies $\mathrm{supp}(\overline{w}) \subset B_{1/2}^+$ and it solves

$$\begin{aligned} \Big[\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}\sum_{j,k=1}^n a_{jk}\partial_j\partial_k\Big]\overline{w} &= f \quad \text{in } \mathbb{R}_+^{n+1}, \\ \overline{w} &= 0 \quad \text{on } \mathbb{R}^n \times \{0\}, \end{aligned}$$

where

$$\begin{aligned} f &= \partial_{n+1}(x_{n+1}^{1-2s}\partial_{n+1}\eta)\widetilde{w} + x_{n+1}^{1-2s}\sum_{j,k=1}^n \partial_j(a_{jk}\partial_k\eta)\widetilde{w} + 2x_{n+1}^{1-2s}\partial_{n+1}\eta\partial_{n+1}\widetilde{w} \\ &\quad + x_{n+1}^{1-2s}\sum_{j,k=1}^n a_{jk}\partial_k\eta\partial_j\widetilde{w} + x_{n+1}^{1-2s}\sum_{j,k=1}^n a_{jk}\partial_j\eta\partial_k\widetilde{w} - x_{n+1}^{1-2s}\sum_{j,k=1}^n (\partial_j a_{jk})\partial_k\overline{w}. \end{aligned}$$

Since η and $\nabla\eta$ are bounded, together with $|\partial_{n+1}\eta| \leq Cx_{n+1}$, we know that

$$\|x_{n+1}^{\frac{2s-1}{2}} f\|_{L^2(\mathbb{R}_+^{n+1})} \leq C(\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/4}^+)} + \|x_{n+1}^{\frac{1-2s}{2}} \nabla\widetilde{w}\|_{L^2(C_{s,1/4}^+)}) < \infty.$$

Moreover, since $w|_{B_1'} = 0$ and $\mathrm{supp}(\eta) \subset B_1'$ on $\mathbb{R}^n \times \{0\}$, then

$$\lim_{x_{n+1}\to 0} \nabla'\overline{w} = 0, \qquad \lim_{x_{n+1}\to 0} \Delta'\overline{w} = 0$$

and also

$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\overline{w} = \eta \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}.$$

So, by the Carleman estimate in Lemma 3.4, there exists $\tau_0 > 1$ such that

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\overline{w}\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\nabla\overline{w}\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
&\quad\le C(\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}} f\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \lim_{x_{n+1}\to 0}\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}\overline{w}\|^2_{L^2(\mathbb{R}^n\times\{0\})})
\end{aligned}$$

for all $\tau \ge \tau_0$. Then, for large τ_0, the last term of f was absorbed by the gradient term in the left-hand-side, so we have

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}\overline{w}\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
&\quad\le C(\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}} g\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \lim_{x_{n+1}\to 0}\|e^{\tau\phi}x_{n+1}^{1-2s}\partial_{n+1}\overline{w}\|^2_{L^2(\mathbb{R}^n\times\{0\})}),
\end{aligned}$$

where $g = f + x_{n+1}^{1-2s}\sum_{j,k=1}^n(\partial_j a_{jk})\partial_k\overline{w}$.

Let

$$\phi_- := \inf_{x\in C^+_{s,1/8}}\phi(x) \quad \text{and} \quad \phi_+ := \sup_{x\in C^+_{s,1/4}\setminus C^+_{s,3/16}}\phi(x).$$

Hence,

$$\begin{aligned}
&\tau^3 e^{2\tau\phi_-}\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|^2_{L^2(C^+_{s,1/8})} \\
&\quad\le C[e^{2\tau\phi_+}\|x_{n+1}^{\frac{2s-1}{2}} g\|^2_{L^2(C^+_{s,1/4}\setminus C^+_{s,3/16})} + \tau \lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|^2_{L^2(C'_{s,1/4})}].
\end{aligned}$$

Dividing above equation by τ, since $\tau \ge 1$ and applying Caccioppoli's inequality (Lemma A.6), we obtain

$$\begin{aligned}
\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(C^+_{s,1/8})} &\le C[e^{\tau(\phi_+-\phi_-)}\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(C^+_{s,1/2})} \\
&\quad + e^{-\tau\phi_-}\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{L^2(C'_{s,1/2})}].
\end{aligned}$$

Observe that

$$-\frac{|x'|^2}{4} \ge \frac{1}{1-s}x_{n+1}^{2-2s} - \frac{1}{8} \quad \text{in } C^+_{s,1/8},$$

and also since $s \ge \frac{1}{2}$,

$$x_{n+1}^2 \le \Big[(1-s)\Big(\frac{1}{4} - \frac{|x'|^2}{4}\Big)\Big]^{\frac{1}{1-s}} \le \frac{1}{8^{\frac{1}{1-s}}} \le \frac{1}{64}$$

and

$$-\frac{|x'|^2}{4} \leq \frac{1}{1-s} x_{n+1}^{2-2s} - \frac{3}{16} \quad \text{in } C_{s,1/4}^+ \setminus C_{s,3/16}^+,$$

so $\phi_- \geq -\frac{1}{8}$ and $\phi_+ \leq -\frac{11}{64}$, that is, $\phi_+ - \phi_- \leq -\frac{19}{64} < 0$. So, we can choose τ (which is large) to satisfy

$$e^{\tau(\phi_+ - \phi_-)} = \frac{\lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \widetilde{w}\|_{L^2(C_{s,1/2}')}^{1-\alpha}}{\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/2}^+)}^{1-\alpha}} \leq \frac{1}{c_0}$$

for large c_0, where $\alpha \in (0, 1)$ will be chosen later. Note that

$$e^{-\tau\phi_-} = \frac{\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C_{s,1/2}^+)}^{\frac{\phi_-}{\phi_+ - \phi_-}(1-\alpha)}}{\lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \widetilde{w}\|_{L^2(C_{s,1/2}')}^{\frac{\phi_-}{\phi_+ - \phi_-}(1-\alpha)}}.$$

Finally, choosing $\alpha \in (0, 1)$ satisfies $\alpha = \frac{\phi_-}{\phi_+ - \phi_-}(1-\alpha)$ will implies our desired result. ■

For our purpose, we only need the following simplified version of the Lemma above:

Corollary 3.6. *Let $s \in [\frac{1}{2}, 1)$. Suppose that $\widetilde{w} \in H^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s})$ is a solution to*

$$\left[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k\right] \widetilde{w} = 0 \quad \text{in } \mathbb{R}_+^{n+1},$$
$$\widetilde{w} = w \quad \text{on } \mathbb{R}^n \times \{0\},$$

with $w = 0$ on B_1'. We assume that

$$\max_{1\leq j,k\leq n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1\leq j,k\leq n} \|\nabla' a_{jk}\|_\infty \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$. For $s \neq \frac{1}{2}$, we further assume

$$\max_{1\leq j,k\leq n} \|(\nabla')^2 a_{jk}\|_\infty \leq C$$

for some positive constant C. Then, there exist $\alpha = \alpha(n, s) \in (0, 1)$, $c = c(n, s) \in (0, 1)$, and a constant C such that

$$\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(B_c^+)} \leq C \|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(B_2^+)}^{\alpha} \cdot \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \widetilde{w}\|_{L^2(B_2')}^{1-\alpha}.$$

Now, we are ready to prove the part (a) of Lemma 3.3 for the case when $s \in [\frac{1}{2}, 1)$.

Proof of the part (a) *of Lemma* 3.3 *for* $s \in [\frac{1}{2}, 1)$. In order to invoke the estimation from Corollary 3.6, we split our solution u into two parts $\tilde{u} = u_1 + u_2$, where $u_1 := \mathsf{E}_s(\zeta u)$ satisfies

$$\begin{aligned} \Big[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k\Big] u_1 = 0 \quad & \text{in } \mathbb{R}_+^{n+1}, \\ u_1 = \zeta u \quad & \text{on } \mathbb{R}^n \times \{0\}, \end{aligned}$$

where $\zeta \in \mathcal{C}_0^\infty(B'_{16})$ is a smooth cut-off function with $\zeta = 1$ on B'_8. Since $u_1 := \mathsf{E}_s(\zeta u)$, from (2.4) we have

$$\int_{\mathbb{R}^n} |u_1(x', x_{n+1})|^2 \, dx' \le \|u_1\|^2_{L^2(\mathbb{R}^n \times \{0\})} \le \|u\|^2_{L^2(B'_{16})}.$$

So,

$$\begin{aligned} \|x_{n+1}^{\frac{1-2s}{2}} u_1\|^2_{L^2(B_{10}^+)} &\le \int_0^{10} \int_{\mathbb{R}^n} x_{n+1}^{1-2s} |u_1(x', x_{n+1})|^2 \, dx' \, dx_{n+1} \\ &\le \Big(\int_0^{10} x_{n+1}^{1-2s} \, dx_{n+1}\Big) \|u\|^2_{L^2(B'_{16})} = C \|u\|^2_{L^2(B'_{16})}. \end{aligned} \tag{3.17}$$

Note that u_2 satisfies

$$\begin{aligned} \Big[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k\Big] u_2 = 0 \quad & \text{in } \mathbb{R}_+^{n+1}, \\ u_2 = u - \zeta u \quad & \text{on } \mathbb{R}^n \times \{0\}. \end{aligned}$$

Since $u_2 = 0$ on B'_8, by Corollary 3.6, there exist $\alpha = \alpha(n, s) \in (0, 1)$, $c = c(n, s) \in (0, 1)$, and a constant C such that

$$\|x_{n+1}^{\frac{1-2s}{2}} u_2\|_{L^2(B_c^+)} \le C \|x_{n+1}^{\frac{1-2s}{2}} u_2\|^\alpha_{L^2(B_2^+)} \cdot \lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \partial_{n+1} u_2\|^{1-\alpha}_{L^2(B'_2)}. \tag{3.18}$$

Let η be a smooth, radial cut-off function with $\eta = 1$ in B_2^+ and $\eta = 0$ outside B_4^+. Plug $w = \eta x_{n+1}^{1-2s} \partial_{n+1} u_2$ into the trace characterization lemma (Lemma A.5), we reach

$$\begin{aligned} &\lim_{x_{n+1} \to 0} \|x_{n+1}^{1-2s} \partial_{n+1} u_2\|_{L^2(B'_2)} \\ &\le C \Big[\mu^{1-s} (\|x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} u_2\|_{L^2(B_4^+)} \\ &\qquad + \|x_{n+1}^{\frac{2s-1}{2}} \nabla (\eta x_{n+1}^{1-2s} \partial_{n+1} u_2)\|_{L^2(\mathbb{R}_+^{n+1})}) \\ &\qquad + \mu^{-2s} \lim_{x_{n+1} \to 0} \|\eta x_{n+1}^{1-2s} \partial_{n+1} u_2\|_{H^{-2s}(\mathbb{R}^n \times \{0\})}\Big]. \end{aligned} \tag{3.19}$$

We first control the boundary term of (3.19). Since η is a bounded multiplier on $H^{2s}(\mathbb{R}^n)$, using duality, we have

$$\begin{aligned}\|\eta v\|_{H^{-2s}(\mathbb{R}^n\times\{0\})} &= \sup_{\|\varphi\|_{H^{2s}(\mathbb{R}^n\times\{0\})}=1} |\langle v,\eta\varphi\rangle_{L^2(\mathbb{R}^n\times\{0\})}| \\ &\le \|v\|_{H^{-2s}(B_8')} \sup_{\|\varphi\|_{H^{2s}(\mathbb{R}^n\times\{0\})}=1} \|\eta\varphi\|_{H^{2s}(\mathbb{R}^n\times\{0\})} \\ &\le C\|v\|_{H^{-2s}(B_8')} \sup_{\|\varphi\|_{H^{2s}(\mathbb{R}^n\times\{0\})}=1} \|\varphi\|_{H^{2s}(\mathbb{R}^n\times\{0\})} \\ &= C\|v\|_{H^{-2s}(B_8')}.\end{aligned}$$

Plug $v = x_{n+1}^{1-2s}\partial_{n+1}u_2$, we have

$$\lim_{x_{n+1}\to 0}\|\eta x_{n+1}^{1-2s}\partial_{n+1}u_2\|_{H^{-2s}(\mathbb{R}^n\times\{0\})} \le C\cdot \lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}u_2\|_{H^{-2s}(B_8')}. \tag{3.20}$$

Applying the Caccioppoli's inequality in Lemma A.6, with zero Dirichlet condition and zero inhomogeneous terms, we have

$$\|x_{n+1}^{\frac{1-2s}{2}}\nabla u_2\|_{L^2(B_4^+)} \le C\|x_{n+1}^{\frac{1-2s}{2}}u_2\|_{L^2(B_8^+)}. \tag{3.21}$$

Also, we have

$$\begin{aligned}&\|x_{n+1}^{\frac{2s-1}{2}}\nabla(\eta x_{n+1}^{1-2s}\partial_{n+1}u_2)\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\le \|x_{n+1}^{\frac{1-2s}{2}}(\nabla\eta)\partial_{n+1}u_2\|_{L^2(\mathbb{R}^{n+1}_+)} + \|x_{n+1}^{\frac{1-2s}{2}}\eta\nabla'\partial_{n+1}u_2\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\quad + \|x_{n+1}^{\frac{2s-1}{2}}\eta\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1}u_2\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\le C\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}u_2\|_{L^2(B_4^+)} + \|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}(\nabla' u_2)\|_{L^2(B_4^+)} \\ &\quad + \left\|x_{n+1}^{\frac{1-2s}{2}}\sum_{j,k=1}^n\partial_j a_{jk}\partial_k u_2\right\|_{L^2(B_4^+)} \\ &\le C\Big[\|x_{n+1}^{\frac{1-2s}{2}}\nabla u_2\|_{L^2(B_4^+)} + \|x_{n+1}^{\frac{1-2s}{2}}\nabla(\nabla' u_2)\|_{L^2(B_4^+)}\Big],\end{aligned}$$

where the last inequality follows by the boundedness assumptions of a_{jk}. Observe that

$$\begin{aligned}0 &= \nabla'\Big[\partial_{n+1}(x_{n+1}^{1-2s}\partial_{n+1}u_2) + x_{n+1}^{1-2s}\sum_{j,k=1}^n\partial_j(a_{jk}\partial_k u_2)\Big] \\ &= \Big[\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}\sum_{j,k=1}^n\partial_j a_{jk}\partial_k\Big](\nabla' u_2) \\ &\quad + x_{n+1}^{1-2s}\sum_{j=1}^n\partial_j\Big(\sum_{k=1}^n\nabla' a_{jk}\partial_k u_2\Big).\end{aligned}$$

Applying Caccioppoli's inequality in Lemma A.6 on $\nabla' u_2$ with zero Dirichlet condition and $f_j = \sum_{k=1}^n \nabla' a_{jk}\partial_k u_2$, since $\|\nabla' a_{jk}\|_\infty \le \varepsilon$, we have

$$\begin{aligned}&\|x_{n+1}^{\frac{1-2s}{2}}\partial_{n+1}\nabla(\nabla' u_2)\|_{L^2(B_4^+)}\\ &\quad\le C'\|x_{n+1}^{\frac{1-2s}{2}}\nabla' u_2\|_{L^2(B_6^+)} \le C\|x_{n+1}^{\frac{1-2s}{2}} u_2\|_{L^2(B_8^+)},\end{aligned}\tag{3.22}$$

where the second inequality follows by (3.21). Hence, we reach

$$\|x_{n+1}^{\frac{2s-1}{2}}\nabla(\eta x_{n+1}^{1-2s}\partial_{n+1}u_2)\|_{L^2(\mathbb{R}^{n+1}_+)} \le C\|x_{n+1}^{\frac{1-2s}{2}} u_2\|_{L^2(B_8^+)}.\tag{3.23}$$

Plugging (3.20), (3.21), and (3.23) into (3.19), and optimizing the result estimate in $\mu > 0$ gives

$$\begin{aligned}&\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}u_2\|_{L^2(B_2')}\\ &\quad\le C\|x_{n+1}^{\frac{1-2s}{2}} u_2\|_{L^2(B_8^+)}^{\frac{2s}{1+s}}\cdot\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}u_2\|_{H^{-2s}(B_8')}^{\frac{1-s}{1+s}}.\end{aligned}$$

Plugging this into (3.18) leads to

$$\begin{aligned}&\|x_{n+1}^{\frac{1-2s}{2}} u_2\|_{L^2(B_c^+)}\\ &\quad\le C\|x_{n+1}^{\frac{1-2s}{2}} u_2\|_{L^2(B_8^+)}^{\tilde\alpha}\cdot\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}u_2\|_{H^{-2s}(B_8')}^{1-\tilde\alpha}\\ &\quad\le C\big(\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|_{L^2(B_8^+)}+\|x_{n+1}^{\frac{1-2s}{2}} u_1\|_{L^2(B_8^+)}\big)^{\tilde\alpha}\\ &\qquad\times\big(\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\tilde u\|_{H^{-2s}(B_8')}+\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}u_1\|_{H^{-2s}(B_8')}\big)^{1-\tilde\alpha},\end{aligned}\tag{3.24}$$

where $\tilde\alpha = \frac{1-s}{1+s}\alpha + \frac{2s}{1+s}$. Then we have

$$\begin{aligned}\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}u_1\|_{H^{-2s}(\mathbb{R}^n\times\{0\})} &= \|(-P)^s u_1\|_{H^{-2s}(\mathbb{R}^n\times\{0\})}\\ &\le C\|u_1\|_{L^2(\mathbb{R}^n\times\{0\})} \le C\|\tilde u\|_{L^2(B_{16}')},\end{aligned}\tag{3.25}$$

where the second inequality follows by Lemma 2.3.

By combining (3.17), (3.24), and (3.25), we reach

$$\begin{aligned}\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|_{L^2(B_c^+)} &\le C\big(\|x_{n+1}^{\frac{1-2s}{2}}\tilde u\|_{L^2(B_{16}^+)}+\|\tilde u\|_{L^2(B_{16}')}\big)^{\tilde\alpha}\\ &\quad\times\big(\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\tilde u\|_{L^2(B_{16}')}+\|u\|_{L^2(B_{16}')}\big)^{1-\tilde\alpha},\end{aligned}\tag{3.26}$$

which is our desired claim of (a). ■

Indeed, by combining (3.26) with the Caccioppoli's inequality (Lemma A.6), we reach

$$\begin{aligned}
&\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B_{\tilde{c}}^+)} + \|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(B_{\tilde{c}}^+)} \\
&\quad \leq C\big(\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B_{16}^+)} + \|\tilde{u}\|_{L^2(B_{16}')}\big)^{\tilde{\alpha}} \\
&\qquad \times \big(\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B_{16}')} + \|u\|_{L^2(B_{16}')}\big)^{1-\tilde{\alpha}} \\
&\qquad + \lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B_{16}')}^{\frac{1}{2}}\|u\|_{L^2(B_{16}')}^{\frac{1}{2}}, \qquad (3.27)
\end{aligned}$$

with $\tilde{c} = c/2$. Slightly modify the proof of (3.24), we can obtain the following analogue of [30, Proposition 5.11]:

Lemma 3.7. *Let $s \in [\frac{1}{2}, 1)$ and $\widetilde{w}$ is the Caffarelli–Silvestre-type extension of some $f \in H^\gamma(\mathbb{R}^n)$ as in (2.1), where $\gamma \in \mathbb{R}$ with $f|_{C_{s,1}'} = 0$. We assume that*

$$\max_{1\leq j,k\leq n}\|a_{jk} - \delta_{jk}\|_\infty + \max_{1\leq j,k\leq n}\|\nabla' a_{jk}\|_\infty \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$. For $s \neq \frac{1}{2}$, we further assume

$$\max_{1\leq j,k\leq n}\|(\nabla')^2 a_{jk}\|_\infty \leq C$$

for some positive constant C. Then, there exist $C = C(n,s)$ and $\alpha = \alpha(n,s) \in (0,1)$ such that

$$\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(C_{s,1/8}^+)} \leq C\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(C_{s,1}^+)}^\alpha \cdot \lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{H^{-s}(C_{s,1/2}')}^{1-\alpha}.$$

Proof. Let η be a smooth cut-off function supported in $C_{s,1/2}^+$ with $\eta = 1$ in $C_{s,1/4}^+$. Using this cut-off function, and following the ideas in the proof of (3.24), by using Lemma A.4 rather than Lemma A.5, we can obtain the above inequality. ∎

3.2. Proof of the part (a) of Lemma 3.3 for the case $s \in (0, 1/2)$

Let $\widetilde{w}$ solves (2.1). If we define $\bar{s} := 1 - s \in (1/2, 1)$,

$$v(x) = x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}(x) \quad \text{and} \quad f = \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w} = c_s^{-1}(-P)^s u, \quad (3.28)$$

then

$$\begin{aligned}
\Big[\partial_{n+1}x_{n+1}^{1-2\bar{s}}\partial_{n+1} + x_{n+1}^{1-2\bar{s}}\sum_{j,k=1}^n \partial_j a_{jk}\partial_k\Big]v = 0 \quad &\text{in } \mathbb{R}_+^{n+1}, \\
v = f \quad &\text{on } \mathbb{R}^n \times \{0\}.
\end{aligned}$$

Using this observation, and follows the ideas in [30, Proposition 5.12], we can obtain an analogue of Lemma 3.7:

Lemma 3.8. *Let $s \in (0, 1/2)$ and let $x_0 \in \mathbb{R}^n \times \{0\}$. Suppose*

$$\Big[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k\Big] \widetilde{w} = 0 \quad \text{in } \mathbb{R}^{n+1}_+,$$
$$\widetilde{w} = w \quad \text{on } \mathbb{R}^n \times \{0\},$$

with $w = 0$ on $C'_{\bar{s},2}$. We assume that

$$\max_{1\le j,k\le n} \|a_{jk} - \delta_{jk}\|_\infty + \max_{1\le j,k\le n} \|\nabla' a_{jk}\|_\infty \le \varepsilon$$

for some sufficiently small $\varepsilon > 0$. We further assume

$$\max_{1\le j,k\le n} \|(\nabla')^2 a_{jk}\|_\infty \le C$$

for some positive constant C. Then there exist $C = C(n, s)$ and $\alpha = \alpha(n, s) \in (0, 1)$ such that

$$\begin{aligned}
&\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C^+_{\bar{s},1/8})} \\
&\quad \le C \max\{\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C^+_{\bar{s},2})}, \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \widetilde{w}\|_{H^{-s}(C'_{\bar{s},2})}\}^\alpha \\
&\qquad \times \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \widetilde{w}\|^{1-\alpha}_{H^{-s}(C'_{\bar{s},2})}.
\end{aligned}$$

Proof. Let v and f as in (3.28). Let $\tilde{v}$ be the Caffarelli–Silvestre-type extension of ηf as in (2.1), where η is a cut-off function satisfies

$$\eta = \begin{cases} 1 & \text{in } C^+_{\bar{s},1}, \\ 0 & \text{outside } C^+_{\bar{s},2}, \end{cases}$$

with $|\partial_{n+1}\eta| \le C x_{n+1}$. As consequences, the function $\bar{v} := v - \tilde{v}$ is the Caffarelli–Silvestre extension of $(1-\eta) f$ and solves

$$\Big[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k\Big] \bar{v} = 0 \quad \text{in } \mathbb{R}^{n+1}_+,$$
$$\bar{v} = 0 \quad \text{on } C'_{\bar{s},1}.$$

Hence, by Lemma 3.7 and since $\bar{s} = 1 - s$, we have

$$\begin{aligned}
\|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|_{L^2(C^+_{\bar{s},1/8})} &\le C \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|^\alpha_{L^2(C^+_{\bar{s},1})} \cdot \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2\bar{s}} \partial_{n+1} \bar{v}\|^{1-\alpha}_{H^{-\bar{s}}(C'_{\bar{s},1/2})} \\
&= C \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|^\alpha_{L^2(C^+_{\bar{s},1})} \cdot \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2\bar{s}} \partial_{n+1} \bar{v}\|^{1-\alpha}_{H^{-1+s}(C'_{\bar{s},1/2})}.
\end{aligned}$$

Since $\widetilde{w} = 0$ on $C'_{\bar{s},2}$, thus

$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2\bar{s}} \partial_{n+1} v|_{C'_{\bar{s},1/2}} = \lim_{x_{n+1}\to 0} x_{n+1}^{1-2\bar{s}} (\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} \widetilde{w})|_{C'_{\bar{s},1/2}}$$
$$= - \lim_{x_{n+1}\to 0} \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \widetilde{w}|_{C'_{\bar{s},1/2}} = 0.$$

Hence,

$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2\bar{s}} \partial_{n+1} \bar{v}|_{C'_{\bar{s},1/2}} = \lim_{x_{n+1}\to 0} x_{n+1}^{1-2\bar{s}} \partial_{n+1} \tilde{v}|_{C'_{\bar{s},1/2}},$$

and thus

$$\|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|_{L^2(C^+_{\bar{s},1/8})} \leq C \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|^{\alpha}_{L^2(C^+_{\bar{s},1})} \cdot \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2\bar{s}} \partial_{n+1} \tilde{v}\|^{1-\alpha}_{H^{-1+s}(C'_{\bar{s},1/2})}.$$

Using

$$\lim_{x_{n+1}\to 0} x_{n+1}^{1-2\bar{s}} \partial_{n+1} \tilde{v} = -c_{\bar{s}} (-P)^{\bar{s}} (\eta f) = -c_{\bar{s}} (-P)^{1-s} (\eta f),$$

we have

$$\lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2\bar{s}} \partial_{n+1} \tilde{v}\|_{H^{-1+s}(C'_{\bar{s},1/2})}$$
$$le C \|(-P)^{1-s} (\eta f)\|_{H^{-1+s}(\mathbb{R}^n)} r \leq C \|\eta f\|_{H^{1-s}(\mathbb{R}^n)}, \tag{3.29}$$

where the last inequality follows by Lemma 2.2. Thus,

$$\|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|_{L^2(C^+_{\bar{s},1/8})} \leq C \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|^{\alpha}_{L^2(C^+_{\bar{s},1})} \|\eta f\|^{1-\alpha}_{H^{1-s}(\mathbb{R}^n)}. \tag{3.30}$$

Firstly, we estimate the right-hand side of (3.30) by

$$\begin{aligned} \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|_{L^2(C^+_{\bar{s},1})} &\leq \|x_{n+1}^{\frac{1-2\bar{s}}{2}} v\|_{L^2(C^+_{\bar{s},1})} + \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \tilde{v}\|_{L^2(C^+_{\bar{s},1})} \\ &\leq \|x_{n+1}^{\frac{1-2\bar{s}}{2}} v\|_{L^2(C^+_{\bar{s},1})} + C \|\eta f\|_{H^{\bar{s}}(\mathbb{R}^n \times \{0\})} \\ &= \|x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \widetilde{w}\|_{L^2(C^+_{\bar{s},1})} + C \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \\ &\leq C [\|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C^+_{\bar{s},2})} + \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}], \end{aligned}$$

where the second inequality follows by (2.4) and the last one is followed by the Caccioppoli's inequality in Lemma A.6. Similarly, we can estimate the left-hand side of (3.30) by

$$\begin{aligned} \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \bar{v}\|_{L^2(C^+_{\bar{s},1/8})} &\geq \|x_{n+1}^{\frac{1-2\bar{s}}{2}} v\|_{L^2(C^+_{\bar{s},1/8})} - \|x_{n+1}^{\frac{1-2\bar{s}}{2}} \tilde{v}\|_{L^2(C^+_{\bar{s},1/8})} \\ &\geq \|x_{n+1}^{\frac{1-2s}{2}} \partial_{n+1} \widetilde{w}\|_{L^2(C^+_{\bar{s},1/8})} - C \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})} \\ &\geq c \|x_{n+1}^{\frac{1-2s}{2}} \widetilde{w}\|_{L^2(C^+_{\bar{s},1/8})} - C \|\eta f\|_{H^{1-s}(\mathbb{R}^n \times \{0\})}, \end{aligned}$$

where the last inequality is followed by Poincaré inequality. Thus, (3.30) becomes

$$\begin{aligned} &\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(C^+_{\bar{s},1/8})} \\ &\quad\le C[\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(C^+_{\bar{s},2})}+\|\eta f\|_{H^{1-s}(\mathbb{R}^n\times\{0\})}]^{\alpha}\|\eta f\|^{1-\alpha}_{H^{1-s}(\mathbb{R}^n)}. \end{aligned} \tag{3.31}$$

Next, we estimate the boundary contribution $\|\eta f\|_{H^{1-s}(\mathbb{R}^n\times\{0\})}$. Using the interpolation inequality in Lemma A.4, we have

$$\begin{aligned} &\|\eta f\|_{H^{\beta}(\mathbb{R}^n\times\{0\})} \\ &\quad=\|\langle D'\rangle^{\beta}\eta f\|_{L^2(\mathbb{R}^n\times\{0\})} \\ &\quad\le C\mu^{1-s}(\|x_{n+1}^{\frac{2s-1}{2}}\langle D'\rangle^{\beta}(\eta v)\|_{L^2(\mathbb{R}^{n+1}_+)}+\|x_{n+1}^{\frac{2s-1}{2}}\nabla(\langle D'\rangle^{\beta}(\eta v))\|_{L^2(\mathbb{R}^{n+1}_+)}) \\ &\qquad+C\mu^{-s}\|\langle D'\rangle^{\beta}(\eta f)\|_{H^{-s}(\mathbb{R}^n\times\{0\})}. \end{aligned}$$

Using $\|\langle D'\rangle^{\beta}u\|_{L^2}\le\|u\|_{L^2}+\|\nabla' u\|_{L^2}$ for $\beta\le 1$, we have

$$\begin{aligned} &\|x_{n+1}^{\frac{2s-1}{2}}\langle D'\rangle^{\beta}(\eta v)\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\quad\le\|x_{n+1}^{\frac{2s-1}{2}}\eta v\|_{L^2(\mathbb{R}^{n+1}_+)}+\|x_{n+1}^{\frac{2s-1}{2}}\nabla'(\eta v)\|_{L^2(\mathbb{R}^{n+1}_+)}, \\ &\|x_{n+1}^{\frac{2s-1}{2}}\nabla(\langle D'\rangle^{\beta}(\eta v))\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\quad\le\|x_{n+1}^{\frac{2s-1}{2}}\nabla(\eta v)\|_{L^2(\mathbb{R}^{n+1}_+)}+\|x_{n+1}^{\frac{2s-1}{2}}\nabla\nabla'(\eta v)\|_{L^2(\mathbb{R}^{n+1}_+)}. \end{aligned}$$

Using (3.21) and (3.22), we know that

$$\begin{aligned} &\|x_{n+1}^{\frac{2s-1}{2}}\langle D'\rangle^{\beta}(\eta v)\|_{L^2(\mathbb{R}^{n+1}_+)}+\|x_{n+1}^{\frac{2s-1}{2}}\nabla(\langle D'\rangle^{\beta}(\eta v))\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &\quad\le C\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C^+_{\bar{s},2})}, \end{aligned}$$

hence

$$\|\eta f\|_{H^{\beta}(\mathbb{R}^n\times\{0\})}\le C[\mu^{1-s}\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C^+_{\bar{s},2})}+\mu^{-s}\|\eta f\|_{H^{\beta-s}(\mathbb{R}^n\times\{0\})}]. \tag{3.32}$$

Choosing $\mu>0$ in (3.32) such that the right contributions become equal, i.e.

$$\mu=\frac{\|\eta f\|_{H^{\beta-s}(\mathbb{R}^n\times\{0\})}}{\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C^+_{\bar{s},2})}}.$$

Here, using unique continuation, we notice $\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C^+_{\bar{s},2})}\neq 0$, unless $\widetilde{w}$ vanishes globally. Using this choice of $\mu>0$, we reach the multiplicative estimate

$$\|\eta f\|_{H^{\beta}(\mathbb{R}^n\times\{0\})}\le C\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|^{s}_{L^2(C^+_{\bar{s},2})}\|\eta f\|^{1-s}_{H^{\beta-s}(\mathbb{R}^n\times\{0\})}. \tag{3.33}$$

Starting from $\beta = 1 - s$, if we iterate (3.33) for k times, we reach

$$\|\eta f\|_{H^{1-s}(\mathbb{R}^n\times\{0\})} \leq C\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C_{\bar{s},2}^+)}^{\gamma}\|\eta f\|_{H^{1-s-ks}(\mathbb{R}^n\times\{0\})}^{1-\gamma}.$$

Choose $k \in \mathbb{N}$ be the smallest integer such that $1 - ks < 0$, we reach

$$\begin{aligned}\|\eta f\|_{H^{1-s}(\mathbb{R}^n\times\{0\})} &\leq C\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C_{\bar{s},2}^+)}^{\gamma}\|\eta f\|_{H^{-s}(\mathbb{R}^n\times\{0\})}^{1-\gamma} \\ &\leq C\|x_{n+1}^{\frac{2s-1}{2}}\widetilde{w}\|_{L^2(C_{\bar{s},2}^+)}^{\gamma}\|f\|_{H^{-s}(C'_{\bar{s},2})}^{1-\gamma}. \end{aligned} \tag{3.34}$$

Inserting (3.34) into (3.31) gives our desired result. ∎

For our purpose, we only need the following version of inequality:

Corollary 3.9. *Let $s \in (0, 1/2)$ and let $x_0 \in \mathbb{R}^n \times \{0\}$. Suppose*

$$\begin{aligned}\Big[\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}\sum_{j,k=1}^{n}\partial_j a_{jk}\partial_k\Big]\widetilde{w} &= 0 \quad \text{in } \mathbb{R}_+^{n+1}, \\ \widetilde{w} &= w \quad \text{on } \mathbb{R}^n \times \{0\},\end{aligned}$$

with $w = 0$ on $C'_{\bar{s},2}$. We assume that

$$\max_{1\leq j,k\leq n}\|a_{jk} - \delta_{jk}\|_\infty + \max_{1\leq j,k\leq n}\|\nabla' a_{jk}\|_\infty \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$. We further assume

$$\max_{1\leq j,k\leq n}\|(\nabla')^2 a_{jk}\|_\infty \leq C$$

for some positive constant C. Then there exist $C = C(n, s)$, $c = c(n, s)$ and $\alpha = \alpha(n, s) \in (0, 1)$ such that

$$\begin{aligned}&\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(B_c^+)} \\ &\quad\leq C\max\{\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(B_2^+)}, \lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{H^{-s}(B_2')}\}^{\alpha} \\ &\qquad\times\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{H^{-s}(B_2')}^{1-\alpha} \\ &\quad\leq C\big[\|x_{n+1}^{\frac{1-2s}{2}}\widetilde{w}\|_{L^2(B_2^+)}^{\alpha}\cdot\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{H^{-s}(B_2')}^{1-\alpha} \\ &\qquad+\lim_{x_{n+1}\to 0}\|x_{n+1}^{1-2s}\partial_{n+1}\widetilde{w}\|_{H^{-s}(B_2')}\big].\end{aligned}$$

Now, we are ready to proof the part (a) of Lemma 3.3 for the case $s \in (0, 1/2)$.

Proof of the part (a) of Lemma 3.3 for $s \in (0, \frac{1}{2})$. The case $s \in (0, 1/2)$ is similar to the case $s \in (1/2, 1)$. As above, the estimation for u_1 is a direct result of (2.4). For u_2, we use Corollary 3.9 and the interpolation inequality in Lemma A.5. With this estimation, the analogues of (3.26) and (3.27) are followed by combining the estimates in splitting argument as above. Note that (3.27) becomes

$$\begin{aligned}
&\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{\bar{c}}^+)} + \|x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u}\|_{L^2(B_{\bar{c}}^+)} \\
&\quad \leq C\big(\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{16}^+)} + \|\tilde{u}\|_{L^2(B'_{16})}\big)^{\alpha} \\
&\qquad \times \big(\lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}\|_{L^2(B'_{16})} + \|u\|_{L^2(B'_{16})}\big)^{1-\alpha} \\
&\quad + C\big(\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{16}^+)} + \|\tilde{u}\|_{L^2(B'_{16})}\big)^{\frac{2s}{1+s}} \\
&\qquad \times \big(\lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}\|_{L^2(B'_{16})} + \|u\|_{L^2(B'_{16})}\big)^{\frac{1-s}{1+s}} \\
&\quad + \lim_{x_{n+1}\to 0} \|x_{n+1}^{1-2s} \partial_{n+1} \tilde{u}\|_{L^2(B'_{16})}^{\frac{1}{2}} \|u\|_{L^2(B'_{16})}^{\frac{1}{2}},
\end{aligned} \tag{3.35}$$

which is our desired result. ∎

Finally, combining (3.35) and Lemma A.7, we can immediately obtain the part (b) of Lemma 3.3.

4. Carleman estimate

4.1. A Carleman estimate with differentiability assumption

Modifying the arguments in [27], we can proof the following Carleman estimate.

Theorem 4.1. *Let $s \in (0, 1)$ and let $\tilde{u} \in H^1(\mathbb{R}_+^{n+1}, x_{n+1}^{1-2s})$ with $\operatorname{supp}(\tilde{u}) \subset \mathbb{R}_+^{n+1} \setminus B_1^+$ be a solution to*

$$\begin{aligned}
\Big[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k\Big] \tilde{u} &= f \qquad \text{in } \mathbb{R}_+^{n+1}, \\
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} \tilde{u} &= V \tilde{u} \quad \text{on } \mathbb{R}^n \times \{0\},
\end{aligned}$$

where $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$, $f \in L^2(\mathbb{R}_+^{n+1}, x_{n+1}^{2s-1})$ with compact support in $\mathbb{R}_+^{n+1}$, and $V \in \mathcal{C}^1(\mathbb{R}^n)$. Assume that

$$\max_{1\leq j,k\leq n} \sup_{|x'|\geq 1} |a_{jk}(x') - \delta_{jk}(x')| + \max_{1\leq j,k\leq n} \sup_{|x'|\geq 1} |x'||\nabla' a_{jk}(x')| \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$. Let further $\phi(x) = |x|^{\alpha}$ for $\alpha \geq 1$. Then there exist constants $C = C(n,s,\alpha)$ and $\tau_0 = \tau_0(n,s,\alpha)$ such that

$$\begin{aligned}
&\tau^3 \|e^{\tau\phi}|x|^{\frac{3\alpha}{2}-1} x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau \|e^{\tau\phi}|x|^{\frac{\alpha}{2}} x_{n+1}^{\frac{1-2s}{2}} \nabla\tilde{u}\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&\quad + \tau^{-1} \|e^{\tau\phi}|x|^{-\frac{\alpha}{2}+1} x_{n+1}^{\frac{1-2s}{2}} \nabla(\nabla'\tilde{u})\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&\qquad \leq C\big[\|e^{\tau\phi} x_{n+1}^{\frac{2s-1}{2}} |x| f\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau \|e^{\tau\phi}|x|^{\frac{\alpha}{2}}(|V|^{\frac{1}{2}} + |x'|^{\frac{1}{2}}|\nabla' V|^{\frac{1}{2}})\tilde{u}\|_{L^2(\mathbb{R}^n\times\{0\})}^2 \\
&\qquad\quad + \tau^{-1} \|e^{\tau\phi}|x|^{-\frac{\alpha}{2}+1}(|V|^{\frac{1}{2}} + |x'|^{\frac{1}{2}}|\nabla' V|^{\frac{1}{2}})\nabla'\tilde{u}\|_{L^2(\mathbb{R}^n\times\{0\})}^2\big].
\end{aligned}$$

for all $\tau \geq \tau_0$. Here, $\nabla' = (\partial_1, \ldots, \partial_n)$ and $\nabla = (\partial_1, \ldots, \partial_n, \partial_{n+1})$.

Proof of Theorem 4.1. We proceed in eight steps.

Step 1: Changing the coordinates. Write $x = e^t\omega$ with $t \in \mathbb{R}$ and $\omega \in \mathcal{S}^n_+$, we have

$$\partial_j = e^{-t}(\omega_j\partial_t + \Omega_j) \quad \text{for all } j = 1, \ldots, n+1.$$

Since

$$\Omega_k\omega_j = \delta_{jk} - \omega_k\omega_j, \tag{4.1}$$

so

$$\partial_j\partial_k = e^{-2t}(\omega_j\omega_k\partial_t^2 + \omega_j\Omega_k\partial_t + \omega_k\Omega_j\partial_t + (\delta_{jk} - 2\omega_j\omega_k)\partial_t + \Omega_j\Omega_k - \omega_j\Omega_k).$$

Since ∂_j and ∂_k commute, then

$$\Omega_j\Omega_k - \omega_j\Omega_k = \Omega_k\Omega_j - \omega_k\Omega_j,$$

that is, Ω_j and Ω_k commute up to some lower order terms. Write

$$\partial_j\partial_k = \frac{1}{2}(\partial_j\partial_k + \partial_k\partial_j);$$

we reach

$$\begin{aligned}
\partial_j\partial_k =& e^{-2t}\bigg(\omega_j\omega_k\partial_t^2 + \omega_j\Omega_k\partial_t + \omega_k\Omega_j\partial_t + (\delta_{jk} - 2\omega_j\omega_k)\partial_t \\
&+ \frac{1}{2}\Omega_j\Omega_k + \frac{1}{2}\Omega_k\Omega_j - \frac{1}{2}\omega_j\Omega_k - \frac{1}{2}\omega_k\Omega_j\bigg).
\end{aligned}$$

Also, the vector fields have the following properties

$$\begin{aligned}
&\sum_{j=1}^{n+1}\omega_j\Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^{n+1}\Omega_j\omega_j = n \quad \text{in } \mathcal{S}^n_+, \\
&\sum_{j=1}^{n}\omega_j\Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^{n}\Omega_j\omega_j = n \quad \text{on } \partial\mathcal{S}^n_+.
\end{aligned}$$

Using this coordinate,

$$
\begin{aligned}
f = {} & e^{-(1+2s)t}\Big[\omega_{n+1}^{1-2s}\partial_t^2 + \omega_{n+1}^{1-2s}(n-2s)\partial_t + \sum_{j=1}^{n+1}\Omega_j\omega_{n+1}^{1-2s}\Omega_j\Big]\tilde{u} \\
& + e^{-(1+2s)t}\omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[\omega_j\omega_k\partial_t^2 + \omega_j\Omega_k\partial_t \\
& \qquad + \omega_k\Omega_j\partial_t + \frac{1}{2}\Omega_j\Omega_k + \frac{1}{2}\Omega_k\Omega_j\Big]\tilde{u} \\
& + e^{-(1+2s)t}\omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[(\delta_{jk}-2\omega_j\omega_k)\partial_t - \frac{1}{2}\omega_j\Omega_k \\
& \qquad - \frac{1}{2}\omega_k\Omega_j\Big]\tilde{u} \quad \text{in } \mathcal{S}_+^n\times\mathbb{R}.
\end{aligned}
$$

Next, let $\bar{u} = e^{\frac{n-2s}{2}t}\tilde{u}$ and $\tilde{f} = e^{\frac{n-2s}{2}t}e^{(1+2s)t}f = e^{\frac{n+2+2s}{2}t}f$,

$$
\begin{aligned}
\tilde{f} = {} & \Big[\omega_{n+1}^{1-2s}\partial_t^2 + \sum_{j=1}^{n+1}\Omega_j\omega_{n+1}^{1-2s}\Omega_j - \omega_{n+1}^{1-2s}\frac{(n-2s)^2}{4}\Big]\bar{u} \\
& + \omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[\omega_j\omega_k\partial_t^2 + \omega_j\Omega_k\partial_t + \omega_k\Omega_j\partial_t \\
& \qquad + \frac{1}{2}\Omega_j\Omega_k + \frac{1}{2}\Omega_k\Omega_j\Big]\bar{u} \\
& + \omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[(\delta_{jk}-(n+2-2s)\omega_j\omega_k)\partial_t \\
& \qquad - \frac{n+1-2s}{2}\omega_j\Omega_k - \frac{n+1-2s}{2}\omega_k\Omega_j\Big]\bar{u} \\
& + \omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[\frac{(n-2s)^2}{4}\omega_j\omega_k \\
& \qquad - \frac{n-2s}{2}(\delta_{jk}-2\omega_j\omega_k)\Big]\bar{u} \quad \text{in } \mathcal{S}_+^n\times\mathbb{R}.
\end{aligned}
\tag{4.2}
$$

Also,

$$
\lim_{\omega_{n+1}\to 0}\omega_{n+1}^{1-2s}\Omega_{n+1}\bar{u} = \widetilde{V}\bar{u},
$$

where $\widetilde{V} = e^{2st}V$.

Step 2: Conjugation. Next, setting $\bar{v} = \omega_{n+1}^{\frac{1-2s}{2}}e^{\tau\varphi}\bar{u}$, where $\varphi(t) = \phi(e^t\omega) = e^{\alpha t}$, we reach

$$
\omega_{n+1}^{\frac{2s-1}{2}}e^{\tau\varphi}\tilde{f} = L^+\bar{v} = (S - A + (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}))\bar{v} \quad \text{in } \mathcal{S}_+^n\times\mathbb{R}, \tag{4.3}
$$

where

$$
S = \partial_t^2 + \widetilde{\Delta}_\omega + \tau^2|\varphi'|^2 - \tau\varphi'' - \frac{(n-2s)^2}{4},
$$

$$\widetilde{\Delta}_\omega = \sum_{j=1}^{n+1} \omega_{n+1}^{\frac{2s-1}{2}} \Omega_j \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}},$$
$$A = 2\tau\varphi'\partial_t,$$
$$(\mathrm{I}) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \Big[\omega_j \omega_k \partial_t^2 + \omega_j \Omega_k \partial_t + \omega_k \Omega_j \partial_t + \frac{1}{2}\Omega_j \Omega_k + \frac{1}{2}\Omega_k \Omega_j \Big],$$
$$(\mathrm{II}) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \Big[(-2\tau\varphi'\omega_j\omega_k + (\delta_{jk} - (n+1)\omega_j\omega_k))\partial_t - \Big(\tau\varphi' + \frac{n}{2}\Big)(\omega_j \Omega_k + \omega_k \Omega_j) \Big],$$
$$(\mathrm{III}) = \sum_{j,k=1}^{n} (a_{jk} - \delta_{jk}) \big[\omega_j \omega_k (\tau^2 |\varphi'|^2 - \tau\varphi'' + (n+1)\tau\varphi' + C_1) + C_2 \big],$$

for some constants C_1 and C_2. Also,

$$\lim_{\omega_{n+1} \to 0} \omega_{n+1}^{1-2s} \Omega_{n+1} \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} = \widetilde{V} \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} \quad \text{on } \partial\mathcal{S}_+^n \times \mathbb{R}. \tag{4.4}$$

We denote the norm and the scalar product in the bulk and the boundary space by

$$\|\bullet\| := \|\bullet\|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}, \quad \|\bullet\|_0 := \|\bullet\|_{L^2(\partial\mathcal{S}_+^n \times \mathbb{R})},$$
$$\langle \bullet, \bullet \rangle := \langle \bullet, \bullet \rangle_{L^2(\mathcal{S}_+^n \times \mathbb{R})}, \quad \langle \bullet, \bullet \rangle_0 := \langle \bullet, \bullet \rangle_{L^2(\partial\mathcal{S}_+^n \times \mathbb{R})},$$

and we omit the notation "$\lim_{\omega_{n+1} \to 0}$" in $\|\bullet\|_0$ and $\langle \bullet, \bullet \rangle_0$.

Step 3: Showing the ellipticity of $\widetilde{\Delta}_\omega$. We need to prove the ellipticity of $\widetilde{\Delta}_\omega$:

Lemma 4.2. *Suppose* (4.4) *holds, then*

$$\begin{aligned} \|\widetilde{\Delta}_\omega \bar{v}\|^2 \geq c_0 & \sum_{(j,k) \neq (n+1,n+1)} \|\omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \Omega_k \omega_{n+1}^{\frac{2s-1}{2}} \bar{v}\|^2 \\ & - C\Big(\sum_{j=1}^{n+1} \|\omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \bar{v}\|^2 + \|\bar{v}\|^2 \\ & + \|(|\widetilde{V}|^{\frac{1}{2}} + |\nabla_\omega' \widetilde{V}|^{\frac{1}{2}}) \nabla_\omega' \omega_{n+1}^{\frac{2s-1}{2}} \bar{v}\|_0^2 \\ & + \| |\nabla_\omega' \widetilde{V}|^{\frac{1}{2}} \omega_{n+1}^{\frac{2s-1}{2}} \bar{v}\|_0^2 \Big). \end{aligned}$$

Proof. Note that

$$\|\widetilde{\Delta}_\omega \bar{v}\|^2 = \bigg\| \sum_{j=1}^{n} \omega_{n+1}^{\frac{2s-1}{2}} \Omega_j \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} + \omega_{n+1}^{\frac{2s-1}{2}} \Omega_{n+1} \omega_{n+1}^{1-2s} \Omega_{n+1} \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} \bigg\|^2$$

$$\geq \Big\| \sum_{j=1}^{n} \omega_{n+1}^{\frac{2s-1}{2}} \Omega_j \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} \Big\|^2$$
$$+ 2 \sum_{j=1}^{n} \langle \omega_{n+1}^{\frac{2s-1}{2}} \Omega_j \omega_{n+1}^{1-2s} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \bar{v}, \omega_{n+1}^{\frac{2s-1}{2}} \Omega_{n+1} \omega_{n+1}^{1-2s} \Omega_{n+1} \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} \rangle.$$

The integration by parts is given by

$$\int_{\mathbb{R}^{n+1}_+} (\Omega_{n+1} v) u \, dx + \int_{\mathbb{R}^{n+1}_+} v (\Omega_{n+1} u) \, dx = \int_{\mathbb{R}^{n+1}_+} \Omega_{n+1}(uv) \, dx$$
$$= \int_{\mathbb{R}^{n+1}_+} |x| \partial_{n+1}(uv) \, dx - \int_{S^n_+} \int_0^\infty r \omega_{n+1} \partial_t (uv) r^n \, dr \, d\omega$$
$$= - \int_{\mathbb{R}^n \times \{0\}} |x'| uv \, dx' - \int_{\mathbb{R}^{n+1}_+} \omega_{n+1} uv \, dx + (n+1) \int_{S^n_+} \int_0^\infty \omega_{n+1} (uv) r^n \, dr \, d\omega$$
$$= - \int_{\mathbb{R}^n \times \{0\}} |x'| uv \, dx' + n \int_{\mathbb{R}^{n+1}_+} \omega_{n+1} uv \, dx.$$

Similar integration by parts formula holds for Ω_j for $j = 1, \ldots, n$.

Indeed, by (4.1), we know that for $j = 1, \ldots, n$, Ω_j and ω_{n+1} are commute up to some lower order term. So, to estimate the first term, it is suffice to estimate $\| \sum_{j=1}^n \Omega_j^2 \bar{v} \|^2$. Finally, the lower order terms can be easily estimated using integration by parts. ∎

Defining $L^- := S + A + \text{(I)} - \text{(II)} + \text{(III)}$,

$$\mathcal{D} := \|L^+ \bar{v}\|^2 - \|L^- \bar{v}\|^2 \quad \text{and} \quad \mathcal{S} := \| |\varphi'|^{-\frac{1}{2}} L^+ \bar{v} \|^2 + \| |\varphi'|^{-\frac{1}{2}} L^- \bar{v} \|^2.$$

Step 4: Estimating the difference $\mathcal{D}$. Observe that $\mathcal{D} = -4\langle S\bar{v}, A\bar{v} \rangle + R$, where

$$R = 4\langle S\bar{v}, \text{(II)}\,\bar{v} \rangle - 4\langle A\bar{v}, \text{(I)}\,\bar{v} \rangle - 4\langle A\bar{v}, \text{(III)}\,\bar{v} \rangle + 4\langle \text{(I)}\,\bar{v}, \text{(II)}\,\bar{v} \rangle + 4\langle \text{(II)}\,\bar{v}, \text{(III)}\,\bar{v} \rangle.$$

By using (4.1) and integration by parts, we can compute

$$-4\langle S\bar{v}, A\bar{v} \rangle \geq 4\tau \| |\varphi''|^{\frac{1}{2}} \partial_t \bar{v} \|^2 - 4\tau \sum_{j=1}^{n+1} \| |\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \Omega_j \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} \|^2$$
$$+ \frac{119}{10} \tau^3 \| \varphi' |\varphi''|^{\frac{1}{2}} \bar{v} \|^2 - 2\tau \| (|\widetilde{V}|^{\frac{1}{2}} + |\partial_t \widetilde{V}|^{\frac{1}{2}}) |\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{2s-1}{2}} \bar{v} \|_0^2.$$

Since

$$\max_{1\le j,k\le n}|a_{jk}-\delta_{jk}|+\max_{1\le j,k\le n}|\partial_t a_{jk}|+\max_{1\le j,k\le n}|\nabla'_\omega a_{jk}|\le\varepsilon,$$

by using integration by parts, again we reach

$$\begin{aligned}R\ge&-\tau\varepsilon C\||\varphi'|^{\frac{1}{2}}\partial_t\bar{v}\|^2-\tau\varepsilon C\sum_{j=1}^{n+1}\||\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2\\&-\tau^3\varepsilon C\|\varphi'|\varphi''|^{\frac{1}{2}}\bar{v}\|^2-\tau\varepsilon C\|(|\widetilde{V}|^{\frac{1}{2}}+|\partial_t\widetilde{V}|^{\frac{1}{2}}+|\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2.\end{aligned}$$

Hence, for small $\varepsilon>0$ and large τ_0, we reach

$$\begin{aligned}\mathcal{D}\ge&\frac{39}{10}\tau\||\varphi''|^{\frac{1}{2}}\partial_t\bar{v}\|^2-\frac{41}{10}\tau\sum_{j=1}^{n+1}\||\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2+\frac{118}{10}\tau^3\|\varphi'|\varphi''|^{\frac{1}{2}}\bar{v}\|^2\\&-C\tau\|(|\widetilde{V}|^{\frac{1}{2}}+|\partial_t\widetilde{V}|^{\frac{1}{2}}+|\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2.\end{aligned}\tag{4.5}$$

Step 5: Estimating the sum $\mathcal{S}$. Note that

$$\begin{aligned}\mathcal{S}\ge&\,2\||\varphi'|^{-\frac{1}{2}}S\bar{v}\|^2+2\||\varphi'|^{-\frac{1}{2}}A\bar{v}\|^2\\&-C\varepsilon\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}\|^2-C\varepsilon\sum_{j=1}^{n}\||\varphi'|^{-\frac{1}{2}}\partial_t\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2\\&-C\varepsilon\sum_{j,k=1}^{n}\||\varphi'|^{-\frac{1}{2}}\Omega_j\Omega_k\bar{v}\|^2-C\varepsilon\tau^2\||\varphi'|^{\frac{1}{2}}\partial_t\bar{v}\|^2\\&-C\varepsilon\tau^2\sum_{j=1}^{n}\||\varphi'|^{\frac{1}{2}}\Omega_j\bar{v}\|^2-C\varepsilon\tau^4\||\varphi'|^{\frac{3}{2}}\bar{v}\|^2.\end{aligned}$$

Observe that

$$2\||\varphi'|^{-\frac{1}{2}}S\bar{v}\|^2\ge\frac{19}{10}\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}+|\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}+\tau^2|\varphi'|^{\frac{3}{2}}\bar{v}\|^2-C\tau^2\||\varphi''|^{\frac{1}{2}}\bar{v}\|^2.$$

For $\delta\in(0,1)$, write

$$\begin{aligned}&\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}+|\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}+\tau^2|\varphi'|^{\frac{3}{2}}\bar{v}\|^2\\&\quad=\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}\|^2+(1-\delta)\||\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}\|^2\\&\qquad+\delta\||\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}\|^2+\tau^4\||\varphi'|^{\frac{3}{2}}\bar{v}\|^2\\&\qquad+\langle|\varphi'|^{-1}\partial_t^2\bar{v},\widetilde{\Delta}_\omega\bar{v}\rangle+\tau^2\langle\varphi'\partial_t^2\bar{v},\bar{v}\rangle+\tau^2\langle\varphi'\widetilde{\Delta}_\omega\bar{v},\bar{v}\rangle.\end{aligned}$$

By using integration by parts, and apply Lemma 4.2 on the term $\delta\||\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega \bar{v}\|^2$, choose $\delta > 0$ small, and then choose $\varepsilon > 0$ small, we reach

$$\begin{aligned}
\mathcal{S} \geq{}& \frac{19}{10}\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}\|^2 + \frac{19}{10}\sum_{j=1}^{n+1}\||\varphi'|^{-\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\partial_t\bar{v}\|^2 \\
&+ c_1\sum_{(j,k)\neq(n+1,n+1)}\||\varphi'|^{-\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\Omega_k\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2 + \frac{18}{10}\||\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}\|^2 \\
&+ \frac{9}{10}\tau^4\||\varphi'|^{\frac{3}{2}}\bar{v}\|^2 + \frac{39}{10}\tau^2\||\varphi'|^{\frac{1}{2}}\partial_t\bar{v}\|^2 - \frac{11}{10}\tau^2\sum_{j=1}^{n+1}\||\varphi'|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2 \\
&- C\|(|\widetilde{V}|^{\frac{1}{2}} + |\partial_t\widetilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi'|^{-\frac{1}{2}}\partial_t\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2 \\
&- C\|(|\widetilde{V}|^{\frac{1}{2}} + |\partial_t\widetilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi'|^{-\frac{1}{2}}\nabla'_\omega\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2 \\
&- C\|(|\widetilde{V}|^{\frac{1}{2}} + |\partial_t\widetilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi'|^{\frac{1}{2}}\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2. \qquad (4.6)
\end{aligned}$$

Step 6: Combining the difference $\mathcal{D}$ and the sum $\mathcal{S}$. Multiplying (4.5) by τ, and summing with (4.6), we reach

$$\begin{aligned}
(\tau+1)\|L^+\bar{v}\|^2 \geq{}& \tau\mathcal{D} + \mathcal{S} \\
\geq{}& c_1\Big(\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}\|^2 + \sum_{j=1}^{n+1}\||\varphi'|^{-\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\partial_t\bar{v}\|^2 \\
&+ \sum_{(j,k)\neq(n+1,n+1)}\||\varphi'|^{-\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\Omega_k\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2\Big) \\
&+ \frac{39}{5}\tau^2\||\varphi'|^{\frac{1}{2}}\partial_t\bar{v}\|^2 + \frac{208}{10}\tau^4\|\varphi'|\varphi''|^{\frac{1}{2}}\bar{v}\|^2 \\
&- \frac{11}{10}\tau^2\sum_{j=1}^{n+1}\||\varphi'|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2 + \frac{18}{10}\||\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}\|^2 \\
&- C\|(|\widetilde{V}|^{\frac{1}{2}} + |\partial_t\widetilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi'|^{-\frac{1}{2}}\partial_t\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2 \\
&- C\|(|\widetilde{V}|^{\frac{1}{2}} + |\partial_t\widetilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi'|^{-\frac{1}{2}}\nabla'_\omega\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2 \\
&- C\tau^2\|(|\widetilde{V}|^{\frac{1}{2}} + |\partial_t\widetilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\widetilde{V}|^{\frac{1}{2}})|\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2. \qquad (4.7)
\end{aligned}$$

Step 7: Obtaining gradient estimates. Note that

$$\begin{aligned}
&\frac{12}{10}\tau^2\sum_{j=1}^{n+1}\||\varphi'|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2 + \frac{12}{10}\tau^2\langle\widetilde{V}\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}, \varphi'\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\rangle_0 \\
&= -\frac{12}{10}\tau^2\langle\varphi'\bar{v}, \widetilde{\Delta}_\omega\bar{v}\rangle \leq \frac{16}{10}\||\varphi'|^{-\frac{1}{2}}\widetilde{\Delta}_\omega\bar{v}\|^2 + \frac{144}{100}\tau^4\||\varphi'|^{\frac{3}{2}}\bar{v}\|^2. \qquad (4.8)
\end{aligned}$$

Step 8: Conclusion. Summing up (4.7) and (4.8), we reach

$$\begin{aligned}
&\||\varphi'|^{-\frac{1}{2}}\partial_t^2\bar{v}\|^2 + \sum_{j=1}^{n+1}\||\varphi'|^{-\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\partial_t\bar{v}\|^2 \\
&+ \sum_{(j,k)\neq(n+1,n+1)}\||\varphi'|^{-\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\Omega_k\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2 \\
&+ \tau^2\||\varphi'|^{\frac{1}{2}}\partial_t\bar{v}\|^2 + \tau^2\sum_{j=1}^{n+1}\||\varphi'|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\Omega_j\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|^2 + \tau^4\|\varphi'|\varphi''|^{\frac{1}{2}}\bar{v}\|^2 \\
&\leq C\tau\|\tilde{f}\|^2 + C\|(|\tilde{V}|^{\frac{1}{2}} + |\partial_t\tilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\tilde{V}|^{\frac{1}{2}})|\varphi'|^{-\frac{1}{2}}\partial_t\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2 \\
&\quad + C\|(|\tilde{V}|^{\frac{1}{2}} + |\partial_t\tilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\tilde{V}|^{\frac{1}{2}})|\varphi'|^{-\frac{1}{2}}\nabla'_\omega\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2 \\
&\quad + C\tau^2\|(|\tilde{V}|^{\frac{1}{2}} + |\partial_t\tilde{V}|^{\frac{1}{2}} + |\nabla'_\omega\tilde{V}|^{\frac{1}{2}})|\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{2s-1}{2}}\bar{v}\|_0^2. \qquad (4.9)
\end{aligned}$$

Changing back to the Cartesian coordinate, and we obtain our result. ∎

4.2. A Carleman estimate without differentiability assumptions

Imitating the splitting arguments in [31, Theorem 5], we can prove the following Carleman estimate.

Theorem 4.3. *Let $s \in (0,1)$ and let $\tilde{u} \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2s})$ with $\mathrm{supp}(\tilde{u}) \subset \mathbb{R}^{n+1}_+ \setminus B_1^+$ be a solution to*

$$\begin{aligned}
\Big[\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}\sum_{j,k=1}^{n}a_{jk}\partial_j\partial_k\Big]\tilde{u} &= f \quad \text{in } \mathbb{R}^{n+1}_+, \\
\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u} &= V\tilde{u} \quad \text{on } \mathbb{R}^n\times\{0\},
\end{aligned}$$

where $x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+$, $f \in L^2(\mathbb{R}^{n+1}_+, x_{n+1}^{2s-1})$ with compact support in $\mathbb{R}^{n+1}_+$, and $V \in L^\infty(\mathbb{R}^n)$. Assume that

$$\max_{1\leq j,k\leq n}\sup_{|x'|\geq 1}|a_{jk}(x') - \delta_{jk}(x')| + \max_{1\leq j,k\leq n}\sup_{|x'|\geq 1}|x'||\nabla' a_{jk}(x')| \leq \varepsilon$$

for some sufficiently small $\varepsilon > 0$. Let further $\phi(x) = |x|^\alpha$ for $\alpha \geq 1$. Then there exist constants $C = C(n,s,\alpha)$ and $\tau_0 = \tau_0(n,s,\alpha)$ such that

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\alpha}{2}-1}x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau\|e^{\tau\phi}|x|^{\frac{\alpha}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
&\leq C[\|e^{\tau\phi}x_{n+1}^{\frac{2s-1}{2}}|x|f\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau^{2-2s}\|e^{\tau\phi}V|x|^{(1-\alpha)s}\tilde{u}\|_{L^2(\mathbb{R}^n\times\{0\})}]
\end{aligned}$$

for all $\tau \geq \tau_0$.

Proof of Theorem 4.3. We proceed in three steps.

Step 1: Changing the coordinates. As in the proof of Theorem 4.1, firstly, we pass to conformal coordinates. With the notations mentioned before, recall (4.2):

$$\Big[\omega_{n+1}^{1-2s}\partial_t^2+\sum_{j=1}^{n+1}\Omega_j\omega_{n+1}^{1-2s}\Omega_j-\omega_{n+1}^{1-2s}\frac{(n-2s)^2}{4}\Big]\bar{u}+R\bar{u}=\tilde{f},$$

where

$$\begin{aligned}R&=\omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[\omega_j\omega_k\partial_t^2+\omega_j\Omega_k\partial_t+\omega_k\Omega_j\partial_t+\frac{1}{2}\Omega_j\Omega_k+\frac{1}{2}\Omega_k\Omega_j\Big]\\&+\omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[(\delta_{jk}-(n+2-2s)\omega_j\omega_k)\partial_t\\&\qquad\qquad-\frac{n+1-2s}{2}\omega_j\Omega_k-\frac{n+1-2s}{2}\omega_k\Omega_j\Big]\\&+\omega_{n+1}^{1-2s}\sum_{j,k=1}^{n}(a_{jk}-\delta_{jk})\Big[\frac{(n-2s)^2}{4}\omega_j\omega_k-\frac{n-2s}{2}(\delta_{jk}-2\omega_j\omega_k)\Big].\end{aligned}$$

Step 2: Splitting $\bar{u}$ into elliptic and subelliptic parts. We split $\bar{u}$ into two parts $\bar{u}=u_1+u_2$. Here u_1 is a solution to

$$\begin{aligned}&\Big[\omega_{n+1}^{1-2s}\partial_t^2+\sum_{j=1}^{n+1}\Omega_j\omega_{n+1}^{1-2s}\Omega_j\\&\quad-\omega_{n+1}^{1-2s}\frac{(n-2s)^2}{4}-K^2\tau^2|\varphi'|^2\omega_{n+1}^{1-2s}\Big]u_1+Ru_1=\tilde{f}\quad\text{in } \mathcal{S}_+^n\times\mathbb{R},\\&\qquad\lim_{\omega_{n+1}\to 0}\omega_{n+1}^{1-2s}\Omega_{n+1}u_1=\lim_{\omega_{n+1}\to 0}\omega_{n+1}^{1-2s}\Omega_{n+1}\bar{u}\qquad\text{on } \partial\mathcal{S}_+^n\times\mathbb{R}.\end{aligned}\tag{4.10}$$

We remark that existence of unique energy solution to this problem is followed by the Lax-Milgram theorem in $H^1(\mathcal{S}_+^n\times\mathbb{R},\omega_{n+1}^{1-2s})$.

Step 2.1: Obtain an elliptic estimate. Testing $\tau^2e^{2\tau\varphi}|\varphi''|^2u_1$ in (4.10), for $\delta>0$, we reach

$$\begin{aligned}&\tau^2\|e^{\tau\varphi}\varphi''\omega_{n+1}^{\frac{1-2s}{2}}\partial_tu_1\|^2+\tau^2\|e^{\tau\varphi}\varphi''\omega_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_1\|^2\\&+\tau^2\frac{(n-2s)^2}{4}\|e^{\tau\varphi}\varphi''\omega_{n+1}^{\frac{1-2s}{2}}u_1\|^2+K^2\tau^4\|e^{\tau\varphi}\varphi'\varphi''\omega_{n+1}^{\frac{1-2s}{2}}u_1\|^2\\&\quad=-\tau^2\langle e^{\tau\varphi}\omega_{n+1}^{\frac{2s-1}{2}}\tilde{f},e^{\tau\varphi}|\varphi''|^2\omega_{n+1}^{\frac{1-2s}{2}}u_1\rangle\\&\qquad-\langle\tau e^{\tau\varphi}\varphi'''\omega_{n+1}^{\frac{1-2s}{2}}\partial_tu_1,\tau^2e^{\tau\varphi}\varphi'\varphi''\omega_{n+1}^{\frac{1-2s}{2}}u_1\rangle\end{aligned}$$

$$
\begin{aligned}
&+ \tau^2 \langle R u_1, e^{2\tau\varphi} |\varphi''|^2 u_1 \rangle - 2 \langle \tau e^{\tau\varphi} \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} \partial_t u_1, \tau e^{\tau\varphi} \varphi''' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \rangle \\
&- \tau^2 \langle e^{\tau\varphi} \varphi'' e^{\alpha s t} \omega_{n+1}^{1-2s} \Omega_{n+1} u_1, e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \rangle_0 \\
\leq{}& \| e^{\tau\varphi} \omega_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|^2 + \tau^4 \| e^{\tau\varphi} |\varphi''|^2 \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|^2 + \delta \tau^2 \| e^{\tau\varphi} \varphi''' \omega_{n+1}^{\frac{1-2s}{2}} \partial_t u_1 \|^2 \\
&+ C_\delta \tau^4 \| e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|^2 + \delta \tau^2 \| e^{\tau\varphi} \varphi''' \omega_{n+1}^{\frac{1-2s}{2}} \partial_t u_1 \|^2 \\
&+ C_\delta \tau^2 \| e^{\tau\varphi} \varphi''' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|^2 \\
&+ \tau^2 \| e^{\tau\varphi} \varphi'' e^{\alpha s t} \omega_{n+1}^{1-2s} \Omega_{n+1} \bar{u} \|_0 \| e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|_0 \\
&+ \tau^2 | \langle R u_1, e^{2\tau\varphi} |\varphi''|^2 u_1 \rangle |.
\end{aligned}
$$

Firstly, we choose small $\delta > 0$ and small $\varepsilon > 0$, then choose large $K > 1$, so

$$
\begin{aligned}
&\tau^2 \| e^{\tau\varphi} \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} \partial_t u_1 \|^2 + \tau^2 \| e^{\tau\varphi} \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathbb{S}^n} u_1 \|^2 + \tau^4 \| e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|^2 \\
&\quad \leq C \| e^{\tau\varphi} \omega_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|^2 + C \tau^2 \| e^{\tau\varphi} \varphi'' e^{\alpha s t} \omega_{n+1}^{1-2s} \Omega_{n+1} \bar{u} \|_0 \| e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|_0 \\
&\quad \leq C \| e^{\tau\varphi} \omega_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|^2 + C_\eta \tau^{2-2s} \| e^{\tau\varphi} \varphi'' e^{\alpha s t} \omega_{n+1}^{1-2s} \Omega_{n+1} \bar{u} \|_0 \\
&\qquad + \eta \tau^{2+2s} \| e^{\tau\varphi} \varphi' \varphi'' \omega_{n+1}^{\frac{1-2s}{2}} u_1 \|_0 .
\end{aligned} \tag{4.11}
$$

From Proposition A.2, we have

$$
\begin{aligned}
&|\varphi''|^2 e^{2\alpha s t} \int_{\partial \mathbb{S}_+^n} u_1^2 \\
&\quad \leq C \tilde{\tau}^{2-2s} |\varphi''|^2 e^{2\alpha s t} \int_{\mathbb{S}_+^n} \omega_{n+1}^{1-2s} u_1^2 + C \tilde{\tau}^{-2s} |\varphi''|^2 e^{2\alpha s t} \int_{\mathbb{S}_+^n} \omega_{n+1}^{1-2s} |\nabla_{\mathbb{S}^n} u_1|^2 .
\end{aligned}
$$

Choosing $\tilde{\tau} = e^{\alpha t} \tau$, we reach

$$
\begin{aligned}
&|\varphi''|^2 e^{2\alpha s t} \int_{\partial \mathbb{S}_+^n} u_1^2 \\
&\quad \leq C \tau^{2-2s} |\varphi''|^2 e^{2\alpha t} \int_{\mathbb{S}_+^n} \omega_{n+1}^{1-2s} u_1^2 + C \tau^{-2s} |\varphi''|^2 \int_{\mathbb{S}_+^n} \omega_{n+1}^{1-2s} |\nabla_{\mathbb{S}^n} u_1|^2 .
\end{aligned}
$$

Multiplying with $e^{2\tau\varphi}$, using that $\varphi' = \alpha e^{\alpha t}$ and integrating in the radial direction, thus implies

$$
\tau^{2+2s} \| e^{\tau\varphi} |\varphi''| e^{\alpha s t} u_1 \|_0^2 \leq C \tau^4 \| e^{\tau\varphi} \omega_{n+1}^{\frac{1-2s}{2}} \varphi' \varphi'' u_1 \|^2 + C \tau^2 \| e^{\tau\varphi} \omega_{n+1}^{\frac{1-2s}{2}} \varphi'' \nabla_{\mathbb{S}^n} u_1 \|^2 .
$$

Plug the inequality above into (4.11), and choose $\eta > 0$ small, so

$$\begin{aligned}
&\tau^2\|e^{\tau\varphi}\varphi''\omega_{n+1}^{\frac{1-2s}{2}}\partial_t u_1\|^2 + \tau^2\|e^{\tau\varphi}\varphi''\omega_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_1\|^2 \\
&+\tau^4\|e^{\tau\varphi}\varphi'\varphi''\omega_{n+1}^{\frac{1-2s}{2}}u_1\|^2 \\
&\quad\le C\|e^{\tau\varphi}\omega_{n+1}^{\frac{2s-1}{2}}\tilde f\|^2 + C\tau^{2-2s}\|e^{\tau\varphi}\varphi'' e^{\alpha s t}\omega_{n+1}^{1-2s}\Omega_{n+1}\bar u\|_0.
\end{aligned}\tag{4.12}$$

Step 2.2: Obtaining a sub-elliptic estimate. Indeed, u_2 satisfies

$$\begin{aligned}
&\Big[\omega_{n+1}^{1-2s}\partial_t^2 + \sum_{j=1}^{n+1}\Omega_j\omega_{n+1}^{1-2s}\Omega_j - \omega_{n+1}^{1-2s}\frac{(n-2s)^2}{4}\Big]u_2 + Ru_2 \\
&\quad = -K^2\tau^2|\varphi'|^2\theta_{n+1}^{1-2s}u_1 \quad \text{in } \mathcal{S}^n_+\times\mathbb{R}, \\
&\qquad \lim_{\omega_{n+1}\to 0}\omega_{n+1}^{1-2s}\Omega_{n+1}u_2 = 0 \quad \text{on } \partial\mathcal{S}^n_+\times\mathbb{R}.
\end{aligned}$$

To compare with (4.2), we should put

$$\tilde f = -K^2\tau|\varphi'|^2\omega_{n+1}^{1-2s}u_1 \quad \text{and} \quad \widetilde V \equiv 0$$

in (4.9). Omitting the second derivative terms, we obtain

$$\begin{aligned}
&\tau^3\|\varphi'|\varphi''|^{\frac12}\bar v\|^2 + \tau\||\varphi''|^{\frac12}\partial_t\bar v\|^2 + \tau\||\varphi'|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\omega_{n+1}^{\frac{2s-1}{2}}\bar v\|^2 \\
&\quad\le CK^4\tau^4\|e^{\tau\varphi}|\varphi'|^2\omega_{n+1}^{\frac{1-2s}{2}}u_1\|^2,
\end{aligned}$$

that is,

$$\begin{aligned}
&\tau^3\|e^{\tau\varphi}\varphi'|\varphi''|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}u_2\|^2 \\
&+\tau\|e^{\tau\varphi}|\varphi''|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}\partial_t u_2\|^2 \\
&+\tau\|e^{\tau\varphi}|\varphi''|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}u_2\|^2 \\
&\le CK^4\tau^4\|e^{\tau\varphi}|\varphi'|^2\omega_{n+1}^{\frac{1-2s}{2}}u_1\|^2.
\end{aligned}\tag{4.13}$$

Step 3: Conclusion. Summing up (4.12) and (4.13), since $\bar u = u_1 + u_2$, so

$$\begin{aligned}
&\tau^3\|e^{\tau\varphi}\varphi'|\varphi''|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}\bar u\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}\partial_t\bar u\|^2 + \tau\|e^{\tau\varphi}|\varphi''|^{\frac12}\omega_{n+1}^{\frac{1-2s}{2}}\nabla_{\mathcal{S}^n}\bar u\|^2 \\
&\quad\le C[\|e^{\tau\varphi}\omega_{n+1}^{\frac{2s-1}{2}}\tilde f\|^2 + \tau^{2-2s}\|e^{\tau\varphi}\varphi'' e^{\alpha s t}\omega_{n+1}^{1-2s}\Omega_{n+1}\bar u\|_0^2].
\end{aligned}\tag{4.14}$$

Finally, plug in the boundary condition

$$\lim_{\omega_{n+1}\to 0}\omega_{n+1}^{1-2s}\Omega_{n+1}\bar u = \widetilde V\bar u,$$

and switch back to the Cartesian coordinate, we obtain our result. ∎

5. Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1. We proceed in four steps.

Step 1: Applying Carleman estimate. Define $w := \eta_R \tilde{u}$, where η_R is radial,

$$\eta_R(x) = \begin{cases} 1, & 2 \le |x| \le R, \\ 0, & |x| \le 1 \text{ or } |x| \ge 2R, \end{cases} \tag{5.1}$$

and satisfies

$$\begin{aligned} &|\nabla \eta_R| \le C/R, \quad |\nabla^2 \eta_R| \le C/R^2 \quad \text{in } A^+_{R,2R}, \\ &|\nabla \eta_R| \le C, \qquad |\nabla^2 \eta_R| \le C \qquad \text{in } A^+_{1,2}. \end{aligned}$$

Note that

$$\Big[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} \sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k\Big] w = f,$$

where

$$\begin{aligned} f &= x_{n+1}^{1-2s}[(1-2s) x_{n+1}^{-1} \partial_{n+1} \eta_R] \tilde{u} + x_{n+1}^{1-2s}\Big[\partial_{n+1}^2 \eta_R + \sum_{j,k=1}^{n} a_{jk} \partial_j \partial_k \eta_R\Big] \tilde{u} \\ &\quad + 2 x_{n+1}^{1-2s}\Big[(\partial_{n+1}\eta_R)(\partial_{n+1}\tilde{u}) + \sum_{j,k=1}^{n} a_{jk} (\partial_k \eta_R)(\partial_j \tilde{u})\Big] \\ &\quad - x_{n+1}^{1-2s} \sum_{j,k=1}^{n} (\partial_j a_{jk})(\partial_k \tilde{u}) \eta_R. \end{aligned}$$

Since η_R is radial, then $\partial_{n+1}\eta_R = \eta_R' \partial_{n+1}|x| = 0$ on $\mathbb{R}^n \times \{0\}$. Thus,

$$\begin{aligned} \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1} w &= \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \eta_R \partial_{n+1} \tilde{u} \\ &= c_{n,s}^{-1} q \eta_R u = c_{n,s}^{-1} q w \quad \text{on } \mathbb{R}^n \times \{0\}. \end{aligned}$$

Note that w is admissible in the Carleman estimate in Theorem 4.1. For $\beta > 1$, since $|q| \le 1$ and $|x'||\nabla' q| \le 1$, we have

$$\begin{aligned} &\tau^3 \|e^{\tau\phi} |x|^{\frac{3\beta}{2}-1} x_{n+1}^{\frac{1-2s}{2}} w\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \|e^{\tau\phi} |x|^{\frac{\beta}{2}} x_{n+1}^{\frac{1-2s}{2}} \nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\ &+ \tau^{-1} \|e^{\tau\phi} |x|^{-\frac{\beta}{2}+1} x_{n+1}^{\frac{1-2s}{2}} \nabla(\nabla' w)\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\ &\quad \le C\big[\|e^{\tau\phi} x_{n+1}^{\frac{2s-1}{2}} |x| f\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau \|e^{\tau\phi} |x|^{\frac{\beta}{2}} w\|^2_{L^2(\mathbb{R}^n \times \{0\})} \\ &\qquad + \tau^{-1} \|e^{\tau\phi} |x|^{-\frac{\beta}{2}+1} \nabla' w\|^2_{L^2(\mathbb{R}^n \times \{0\})}\big]. \end{aligned} \tag{5.2}$$

Step 2: Estimating the bulk contributions. Since $1 \le \frac{|x|}{R}$ in $A^+_{R,2R}$ and $1 \le |x|$ in $A^+_{1,2}$, then

$$\begin{aligned}
&\|e^{\tau\phi} x_{n+1}^{\frac{2s-1}{2}} |x| f\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\
&\quad \le C\big[R^{-4}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|^2_{L^2(A^+_{R,2R})} + R^{-2}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|^2_{L^2(A^+_{R,2R})} \\
&\qquad + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|^2_{L^2(A^+_{1,2})} + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|^2_{L^2(A^+_{1,2})} \\
&\qquad + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} \nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\big].
\end{aligned}$$

Write $\tilde{\phi}(r) = \phi(x) = r^\beta$ with $r = |x|$, note that

$$\begin{aligned}
&R^{-4}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|^2_{L^2(A^+_{R,2R})} + R^{-2}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|^2_{L^2(A^+_{R,2R})} \\
&\quad \le C\Big[R^{-2} e^{\tau\tilde{\phi}(2R)}\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|^2_{L^2(A^+_{R,2R})} + e^{\tau\tilde{\phi}(2R)}\|x_{n+1}^{\frac{1-2s}{2}} \nabla\tilde{u}\|^2_{L^2(A^+_{R,2R})}\Big].
\end{aligned}$$

Now, we estimate $\|x_{n+1}^{\frac{1-2s}{2}} \nabla\tilde{u}\|^2_{L^2(A^+_{R,2R})}$. Choose ξ_R satisfies

$$\xi_R(x) = \begin{cases} 1, & R \le |x| \le 2R, \\ 0, & |x| \le \frac{R}{2} \text{ or } |x| \ge 2R, \end{cases}$$

with $|\nabla \xi_R| \le C/R$ for $x \in A^+_{\frac{R}{2},R}$ or $x \in A^+_{2R,3R}$. By testing

$$\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1}\tilde{u} + \sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k \tilde{u} = 0$$

by the function $\tilde{u}\xi_R^2$, we reach

$$\|x_{n+1}^{\frac{1-2s}{2}} \nabla\tilde{u}\|^2_{L^2(A^+_{R,2R})} \le C[\|x_{n+1}^{\frac{1-2s}{2}} u\|^2_{L^2(A'_{\frac{R}{2},3R})} + R^{-2}\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|^2_{L^2(A^+_{\frac{R}{2},3R})}].$$

So,

$$\begin{aligned}
&R^{-4}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|^2_{L^2(A^+_{R,2R})} + R^{-2}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|^2_{L^2(A^+_{R,2R})} \\
&\quad \le C e^{\tau\tilde{\phi}(2R)}[\|x_{n+1}^{\frac{1-2s}{2}} u\|^2_{L^2(A'_{\frac{R}{2},3R})} + R^{-2}\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|^2_{L^2(A^+_{R,2R})}].
\end{aligned}$$

Using Proposition 3.1, we have

$$|\tilde{u}(x)| \le C_1 e^{-C_2 R^\alpha} \quad \text{for } x \in A^+_{\frac{R}{2},3R}.$$

So, if we choose $\beta = \alpha - \varepsilon$ for some $\varepsilon \in (0, \alpha - 1)$, then we have

$$\lim_{R\to\infty} (R^{-4}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|_{L^2(A_{R,2R}^+)}^2 + R^{-2}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|_{L^2(A_{R,2R}^+)}^2) = 0.$$

However, (5.2) writes

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1} x_{n+1}^{\frac{1-2s}{2}} w\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}} x_{n+1}^{\frac{1-2s}{2}} \nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&+ \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1} x_{n+1}^{\frac{1-2s}{2}} \nabla(\nabla' w)\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&\quad \le C\big[R^{-4}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|_{L^2(A_{R,2R}^+)}^2 + R^{-2}\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|_{L^2(A_{R,2R}^+)}^2 \\
&\qquad + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|_{L^2(A_{1,2}^+)}^2 + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|_{L^2(A_{1,2}^+)}^2 \\
&\qquad + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} \nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}} w\|_{L^2(\mathbb{R}^n\times\{0\})}^2 \\
&\qquad + \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1}\nabla' w\|_{L^2(\mathbb{R}^n\times\{0\})}^2\big].
\end{aligned}$$

Taking $R \to \infty$ in (5.2) and choosing large τ, we reach

$$\begin{aligned}
&\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1} x_{n+1}^{\frac{1-2s}{2}} w\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}} x_{n+1}^{\frac{1-2s}{2}} \nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&+ \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1} x_{n+1}^{\frac{1-2s}{2}} \nabla(\nabla' w)\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \\
&\quad \le C\big[\|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\tilde{u}\|_{L^2(A_{1,2}^+)}^2 + \|e^{\tau\phi} x_{n+1}^{\frac{1-2s}{2}} |x|\nabla\tilde{u}\|_{L^2(A_{1,2}^+)}^2 \\
&\qquad + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}} w\|_{L^2(\mathbb{R}^n\times\{0\})}^2 + \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1}\nabla' w\|_{L^2(\mathbb{R}^n\times\{0\})}^2\big].
\end{aligned} \tag{5.3}$$

Step 3: Estimating the boundary contributions. Using Proposition A.2, we have

$$\begin{aligned}
\tilde{\tau}|\varphi''|e^{2st}\|v\|_{L^2(\partial\mathcal{S}^n_+)} \le C\big[&\tilde{\tau}^{2-2s}e^{2st}|\varphi''|\|\omega_{n+1}^{\frac{1-2s}{2}} v\|_{L^2(\mathcal{S}^n_+)}^2 \\
&+ \tilde{\tau}^{-2s}e^{2st}|\varphi''|\|\omega_{n+1}^{\frac{1-2s}{2}} \nabla_\omega v\|_{L^2(\mathcal{S}^n_+)}^2\big].
\end{aligned}$$

Setting $e^{2st}\tilde{\tau}^{-2s} = \tau^{-2s}$ (i.e. $\tilde{\tau} = \tau e^t$), our choice of φ gives

$$\tilde{\tau}^{2-2s}e^{2st}|\varphi''| = \tau^{2-2s}e^{2t}|\varphi''| \le \tau^{2-2s}|\varphi'|^2|\varphi''|.$$

Hence, we reach

$$\begin{aligned}
\tau^{2s+1}|\varphi''|\|v\|_{L^2(\partial\mathcal{S}^n_+)}^2 \le C\big[&\tau^3|\varphi''||\varphi'|^2\|\omega_{n+1}^{\frac{1-2s}{2}} v\|_{L^2(\mathcal{S}^n_+)}^2 \\
&+ \tau|\varphi''|\|\omega_{n+1}^{\frac{1-2s}{2}} \nabla_\omega v\|_{L^2(\mathcal{S}^n_+)}^2\big].
\end{aligned}$$

Multiplying the above inequality by $e^{\tau\varphi}$, and then integrating with respect to the radial variable t, we obtain

$$\tau^{2s+1}\|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}}v\|^2_{L^2(\partial S^n_+\times\mathbb{R})} \le C\big[\tau^3\|e^{\tau\varphi}\varphi'|\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}v\|^2_{L^2(S^n_+\times\mathbb{R})} + \tau\|e^{\tau\varphi}|\varphi''|^{\frac{1}{2}}\omega_{n+1}^{\frac{1-2s}{2}}\nabla_\omega v\|^2_{L^2(S^n_+\times\mathbb{R})}\big],$$

that is,

$$\tau^{2s+1}\|e^{\tau\phi}|x|^{\frac{\beta}{2}}w\|^2_{L^2(\mathbb{R}^n\times\{0\})} \le C\big[\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\big].$$

Similarly, we have

$$\begin{aligned}&\tau^{2s-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1}\nabla' w\|^2_{L^2(\mathbb{R}^n\times\{0\})}\\ &\le C\big[\tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla' w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\ &\quad + \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1}x_{n+1}^{\frac{1-2s}{2}}\nabla(\nabla' w)\|^2_{L^2(\mathbb{R}^{n+1}_+)}\big].\end{aligned}$$

So, for large τ, the boundary terms of (5.3) are absorbed, and we reach

$$\begin{aligned}&\tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|^2_{L^2(B_6^+\setminus B_4^+)} + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(B_6^+\setminus B_4^+)}\\ &\le \tau^3\|e^{\tau\phi}|x|^{\frac{3\beta}{2}-1}x_{n+1}^{\frac{1-2s}{2}}w\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \tau\|e^{\tau\phi}|x|^{\frac{\beta}{2}}x_{n+1}^{\frac{1-2s}{2}}\nabla w\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\ &\quad + \tau^{-1}\|e^{\tau\phi}|x|^{-\frac{\beta}{2}+1}x_{n+1}^{\frac{1-2s}{2}}\nabla(\nabla' w)\|^2_{L^2(\mathbb{R}^{n+1}_+)}\\ &\le C[\|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\tilde{u}\|^2_{L^2(A^+_{1,2})} + \|e^{\tau\phi}x_{n+1}^{\frac{1-2s}{2}}|x|\nabla\tilde{u}\|^2_{L^2(A^+_{1,2})}].\end{aligned}$$

Pulling out the exponential weight in the above estimate yields

$$\begin{aligned}&\tau^3 e^{\tau\tilde{\phi}(4)}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(B_6^+\setminus B_4^+)} + \tau e^{\tau\tilde{\phi}(4)}\|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|^2_{L^2(B_6^+\setminus B_4^+)}\\ &\quad\le C[e^{\tau\tilde{\phi}(2)}\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|^2_{L^2(A^+_{1,2})} + e^{\tau\tilde{\phi}(2)}\|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|^2_{L^2(A^+_{1,2})}].\end{aligned}$$

Step 4: Conclusion. Since $\tilde{\phi}(4) \ge \tilde{\phi(2)}$, taking $\tau \to \infty$ will leads a contradiction, unless $\tilde{u} = 0$ in $B_6^+ \setminus B_4^+$. Finally, applying the unique continuation property for classical second order elliptic equations (see e.g. [27, Theorem 1.1]), we conclude that $\tilde{u} \equiv 0$. ■

Following exactly the arguments in [31, Theorem 2], we can obtain Theorem 1.2. For sake of completeness, here we give a sketch of the proof of Theorem 1.2.

Sketch of the proof of Theorem 1.2. Let η_R be the function given in (5.1), and write $\overline{w}(t,\theta) = \bar{u}(t,\theta)\eta_R(e^t\theta) \equiv \tilde{u}(e^t\theta)\eta_R(e^t\theta)$, where (t,θ) is the conformal polar coordinate used in the proof of Carleman estimates (Theorem 4.1 and Theorem 4.3). Plugging $\overline{w}$ into (4.14) (i.e. the Carleman estimate in Theorem 4.3 with conformal polar coordinate) with $\varphi(t) = e^{\beta t}$ (that is, $\phi(x) = |x|^\beta$) with $\frac{4s}{4s-1} < \beta < \alpha$, and taking the limit $R \to \infty$, we obtain [31, equation (49)]:

$$\begin{aligned}
&\tau^3 \| e^{\tau\varphi} |\varphi'||\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \overline{w} \|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}^2 + \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{w} \|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}^2 \\
&+ \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{w} \|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}^2 \\
&\quad \le C(\| e^{\tau\varphi} \omega_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|_{L^2(\mathcal{S}_+^n \times [1,2])}^2 + \tau^{2-2s} \| e^{\tau\varphi} |\varphi''| \tilde{q} e^{-\beta s t} \overline{w} \|_{L^2(\partial\mathcal{S}_+^n \times \mathbb{R})}^2), \qquad (5.4)
\end{aligned}$$

with

$$|\tilde{f}| \le C\omega_{n+1}^{1-2s}(|\partial_t \bar{u}| + |\nabla_{\mathcal{S}^n}\bar{u}| + |\bar{u}|).$$

Using the trace estimate in Proposition A.2 (by replacing τ by $e^{\beta t}\tau$), the boundary term in (5.4) can be absorbed in to the left-hand side of this estimate:

$$\begin{aligned}
&\tau^3 \| e^{\tau\varphi} |\varphi'||\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \overline{w} \|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}^2 \\
&+ \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \partial_t \overline{w} \|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}^2 \\
&+ \tau \| e^{\tau\varphi} |\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \nabla_{\mathcal{S}^n} \overline{w} \|_{L^2(\mathcal{S}_+^n \times \mathbb{R})}^2 \\
&\quad \le C \| e^{\tau\varphi} \omega_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|_{L^2(\mathcal{S}_+^n \times [1,2])}^2. \qquad (5.5)
\end{aligned}$$

The observation $2\beta + 4s - 2\beta s \le \beta + 2\beta s$ is helpful. Pulling out the weight $e^{\tau\varphi}$ in (5.5) leads to

$$e^{\tau\varphi(4)} \tau^3 \| |\varphi'||\varphi''|^{\frac{1}{2}} \omega_{n+1}^{\frac{1-2s}{2}} \bar{u} \|_{L^2(\mathcal{S}_+^n \times [4,6])} \le C e^{\tau\varphi(2)} \| \omega_{n+1}^{\frac{2s-1}{2}} \tilde{f} \|_{L^2(\mathcal{S}_+^n \times [1,2])}^2.$$

Using the monotonicity of φ, and passing to the limit $\tau \to \infty$, we know that $\bar{u} = 0$ in $\mathcal{S}_+^n \times (4,6)$, i.e. $\tilde{u} = 0$ in $B_6^+ \setminus \overline{B_4^+}$. By unique continuation property, we conclude that $\tilde{u} \equiv 0$ in $\mathbb{R}_+^{n+1}$, which conclude the argument. ∎

A. Auxiliary lemmas

A.1. Some interpolation inequalities

The following Hardy inequality can be found in [30, Lemma 4.6]:

Lemma A.1. *If $\alpha \neq \frac{1}{2}$ and if v vanishes for x_{n+1} large, then*

$$\|x_{n+1}^{-\alpha}u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \leq \frac{4}{(2\alpha-1)^2}\|x_{n+1}^{1-\alpha}\partial_{n+1}u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \frac{2}{2\alpha-1}\|\lim_{x_{n+1}\to 0} x_{n+1}^{\frac{1}{2}-\alpha}u\|^2_{L^2(\mathbb{R}^n\times\{0\})}.$$

Proof. Using integration by parts, we have

$$\begin{aligned}\|x_{n+1}^{-\alpha}u\|^2_{L^2(\mathbb{R}^{n+1}_+)} &= \int \partial_{n+1}\Big[\frac{x_{n+1}^{1-2\alpha}}{1-2\alpha}\Big]u^2 \\ &= \frac{2}{2\alpha-1}\int x_{n+1}^{1-2\alpha}u\partial_{n+1}u + \frac{1}{2\alpha-1}\int \lim_{x_{n+1}\to 0} x_{n+1}^{1-2\alpha}u^2 \\ &\leq \frac{1}{2}\frac{4}{(2\alpha-1)^2}\|x_{n+1}^{1-\alpha}\partial_{n+1}u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \frac{1}{2}\|x_{n+1}^{-\alpha}u\|^2_{L^2(\mathbb{R}^{n+1}_+)} \\ &\quad + \frac{1}{2\alpha-1}\|\lim_{x_{n+1}\to 0} x_{n+1}^{\frac{1}{2}-\alpha}u\|^2_{L^2(\mathbb{R}^n\times\{0\})},\end{aligned}$$

which gives our desired result. ∎

We shall use the following interpolation inequality in [10, 29, 31]:

Proposition A.2 (Interpolation inequality I). *Let $s \in (0,1)$ and $u\colon \mathcal{S}^n_+ \to \mathbb{R}$ with $u \in H^1(\mathcal{S}^n_+, \omega_{n+1}^{1-2s})$. Then there exists a constant $C = C(n,s)$ such that*

$$\|u\|_{L^2(\partial\mathcal{S}^n_+)} \leq C[\tau^{1-s}\|\omega_{n+1}^{\frac{1-2s}{2}}u\|_{L^2(\mathcal{S}^n_+)} + \tau^{-s}\|\omega_{n+1}^{\frac{1-2s}{2}}\nabla_\omega u\|_{L^2(\mathcal{S}^n_+)}]$$

for all $\tau > 1$.

The following trace characterization lemma can be found in [30, Lemma 4.4]:

Lemma A.3. *Let $n \geq 1$ and $0 < \tilde{s} < 1$. There is a bounded surjective linear map*

$$T\colon H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{1-2\tilde{s}}) \to H^{\tilde{s}}(\mathbb{R}^n \times \{0\})$$

so that $u(\bullet, x_{n+1}) \to Tu$ in $L^2(\mathbb{R}^n)$ as $x_{n+1} \to 0$.

We need the following interpolation inequality in [30, Proposition 5.11, Step 1]:

Lemma A.4 (Interpolation inequality II (a)). *For any $w \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{2s-1})$ and any $\mu > 0$, the following interpolation inequality holds:*

$$\|w\|_{L^2(\mathbb{R}^n\times\{0\})} \leq C\big[\mu^{1-s}(\|x_{n+1}^{\frac{2s-1}{2}}w\|_{L^2(\mathbb{R}^{n+1}_+)} + \|x_{n+1}^{\frac{2s-1}{2}}\nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}) + \mu^{-s}\|w\|_{H^{-s}(\mathbb{R}^n\times\{0\})}\big].$$

Proof. Let $\langle \bullet \rangle := \sqrt{1 + |\bullet|^2}$. Note that

$$\begin{aligned}\|w\|_{L^2(\mathbb{R}^n\times\{0\})} &= \left[\int_{\mathbb{R}^n\times\{0\}} (\langle\xi\rangle^{2-2s}|\hat{w}|^2)^s (\langle\xi\rangle^{-2s}|\hat{w}|^2)^{1-s}\, d\xi \right]^{\frac{1}{2}} \\ &\le (\mu^{1-s}\|w\|_{H^{1-s}(\mathbb{R}^n\times\{0\})})^s (\mu^{-s}\|w\|_{H^{-s}(\mathbb{R}^n\times\{0\})})^{1-s}\end{aligned}$$

and hence our result follows by Lemma A.3 with $\tilde{s} = 1 - s$. ∎

Slightly modify the proof, we can obtain the following:

Lemma A.5 (Interpolation inequality II (b)). *For any $w \in H^1(\mathbb{R}^{n+1}_+, x_{n+1}^{2s-1})$ and any $\mu > 0$, the following interpolation inequality holds:*

$$\begin{aligned}\|w\|_{L^2(\mathbb{R}^n\times\{0\})} \le C\big[&\mu^{1-s}(\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^{n+1}_+)} \\ &+ \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}) + \mu^{-2s}\|w\|_{H^{-2s}(\mathbb{R}^n\times\{0\})}\big].\end{aligned}$$

Proof. Using Lemma A.3 with $\tilde{s} = 1 - s$, we have

$$\begin{aligned}&\|w\|_{L^2(\mathbb{R}^n\times\{0\})} \\ &\le C\|w\|_{H^{1-s}(\mathbb{R}^n\times\{0\})}^{\frac{2s}{1+s}} \|w\|_{H^{-2s}(\mathbb{R}^n\times\{0\})}^{\frac{1-s}{1+s}} \\ &\le C(\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^{n+1}_+)} + \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^{n+1}_+)})^{\frac{2s}{1+s}} \|w\|_{H^{-2s}(\mathbb{R}^n\times\{0\})}^{\frac{1-s}{1+s}} \\ &\le C[\mu^{1-s}(\|x_{n+1}^{\frac{2s-1}{2}} w\|_{L^2(\mathbb{R}^{n+1}_+)} + \|x_{n+1}^{\frac{2s-1}{2}} \nabla w\|_{L^2(\mathbb{R}^{n+1}_+)}) \\ &\quad + \mu^{-2s}\|w\|_{H^{-2s}(\mathbb{R}^n\times\{0\})}^{\frac{1-s}{1+s}}],\end{aligned}$$

which is our desired result. ∎

A.2. Caccioppoli inequality

We need a generalized the Caccioppoli inequality in [30, Lemma 4.5]:

Lemma A.6. *Let $s \in (0,1)$ and $u \in H^1(B_{2r}^+, x_{n+1}^{1-2s})$ be a solution to*

$$[\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1} + x_{n+1}^{1-2s} P]\tilde{u} = -x_{n+1}^{1-2s} \sum_{j=1}^n \partial_j f_j \quad \text{in } B_{2r}^+.$$

Then there exists a constant $C = C(n, \lambda)$ such that

$$\begin{aligned}\|x_{n+1}^{\frac{1-2s}{2}} \nabla\tilde{u}\|_{L^2(B_r^+)}^2 \le C\big[&r^{-2}\|x_{n+1}^{\frac{1-2s}{2}} \tilde{u}\|_{L^2(B_{2r}^+)}^2 + \sum_{j=1}^n \|x_{n+1}^{\frac{1-2s}{2}} f_j\|_{L^2(B_{2r}^+)}^2 \\ &+ \|\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B'_{2r})}\|u\|_{L^2(B'_{2r})}\big].\end{aligned}$$

Proof. Let $\eta\colon B_{2r}^+ \to \mathbb{R}$ be a smooth, radial cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ on B_r^+, $\operatorname{supp}(\eta) \subset B_{2r}^+$, and $|\nabla \eta| \le C/r$ for some constant C. Note that

$$
\begin{aligned}
&2\sum_{j=1}^{n} \int_{\mathbb{R}_+^{n+1}} (x_{n+1}^{\frac{1-2s}{2}} \eta f_j)(x_{n+1}^{\frac{1-2s}{2}} (\partial_j \eta)\tilde{u}) + \sum_{j=1}^{n} \int_{\mathbb{R}_+^{n+1}} (x_{n+1}^{\frac{1-2s}{2}} \eta f_j)(x_{n+1}^{\frac{1-2s}{2}} \eta \partial_j \tilde{u}) \\
&= -\sum_{j=1}^{n} \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2s} (\partial_j f_j)(\eta^2 \tilde{u}) \\
&= \int_{\mathbb{R}_+^{n+1}} \Big(\partial_{n+1} x_{n+1}^{1-2s} \partial_{n+1}\tilde{u} + x_{n+1}^{1-2s} \sum_{i,j=1}^{n} \partial_i a_{ij} \partial_j \tilde{u}\Big)(\eta^2 \tilde{u}) \\
&= -\int_{\mathbb{R}^n \times \{0\}} \eta^2 \tilde{u} \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1}\tilde{u} - \int_{\mathbb{R}_+^{n+1}} (x_{n+1}^{1-2s} \partial_{n+1}\tilde{u}) \partial_{n+1}(\eta^2 \tilde{u}) \\
&\quad - \int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2s} \sum_{i,j=1}^{n} a_{ij} \partial_j \tilde{u} \partial_i (\eta^2 \tilde{u}) \\
&= -\int_{\mathbb{R}^n \times \{0\}} \eta^2 \tilde{u} \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s} \partial_{n+1}\tilde{u} - 2\int_{\mathbb{R}_+^{n+1}} (x_{n+1}^{1-2s} \partial_{n+1}\tilde{u}) \eta \partial_{n+1} \eta \tilde{u} \\
&\quad - \int_{\mathbb{R}_+^{n+1}} \eta^2 (x_{n+1}^{1-2s} \partial_{n+1}\tilde{u}) \partial_{n+1}\tilde{u} - 2\int_{\mathbb{R}_+^{n+1}} x_{n+1}^{1-2s} \sum_{i,j=1}^{n} a_{ij} (\eta \partial_j \tilde{u})(\partial_i \eta \tilde{u}) \\
&\quad - \int_{\mathbb{R}_+^{n+1}} \eta^2 x_{n+1}^{1-2s} \Big(\sum_{i,j=1}^{n} a_{ij} \partial_j \tilde{u} \partial_i \tilde{u}\Big) \\
&= -\int_{\mathbb{R}^n \times \{0\}} \lim_{x_{n+1}\to 0} \eta^2 \tilde{u} x_{n+1}^{1-2s} \partial_{n+1}\tilde{u} - 2\langle \eta \nabla \tilde{u}, \tilde{u} \nabla \eta \rangle - \|\eta \nabla \tilde{u}\|^2
\end{aligned}
\tag{A.1}
$$

where $\widetilde{A} = \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Here we use the notation

$$
\langle \bullet, \bullet \rangle = \langle \bullet, \bullet \rangle_{L^2(\mathbb{R}_+^n, x_{n+1}^{1-2s} \widetilde{A})} \quad \text{and} \quad \| \bullet \| = \| \bullet \|_{L^2(\mathbb{R}_+^n, x_{n+1}^{1-2s} \widetilde{A})}.
$$

By (1.3), indeed

$$
\|\eta \nabla \tilde{u}\|^2 \ge \lambda \|\eta x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u}\|^2_{L^2(\mathbb{R}_+^{n+1})} \ge \lambda \|x_{n+1}^{\frac{1-2s}{2}} \nabla \tilde{u}\|^2_{L^2(B_r^+)}.
$$

Also, by (1.3), for $\delta > 0$, we have

$$\begin{aligned} 2\langle \eta\nabla\tilde{u}, \tilde{u}\nabla\eta\rangle &\leq \delta\|\eta\nabla\tilde{u}\|^2 + \delta^{-1}\|\tilde{u}\nabla\eta\|^2 \\ &\leq \delta\lambda^{-1}\|\eta x_{n+1}^{\frac{1-2s}{2}}\nabla u\|^2_{L^2(\mathbb{R}^{n+1}_+)} + \delta^{-1}\lambda^{-1}\|\nabla\eta x_{n+1}^{\frac{1-2s}{2}} u\|^2_{L^2(\mathbb{R}^{n+1}_+)}. \end{aligned}$$

Moreover, we have

$$\left| \int_{\mathbb{R}^n\times\{0\}} \lim_{x_{n+1}\to 0} \eta^2\tilde{u}x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\right| \leq \|\lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}\|_{L^2(B'_{2r})}\|\eta^2\tilde{u}\|_{L^2(B'_{2r})}.$$

Plug the inequalities above into (A.1), with small $\delta > 0$, we obtain our desired result. ∎

A.3. L^∞-L^2 type interior inequality

Following the arguments in [35, Proposition 3.1] (see also [18, Proposition 2.6] or [9, Proposition 3.2]), we can obtain the following:

Lemma A.7. *Let $s \in (0,1)$ and $u \in H^1(B^+_{2r}, x_{n+1}^{1-2s})$ be a solution to*

$$\begin{aligned} [\partial_{n+1}x_{n+1}^{1-2s}\partial_{n+1} + x_{n+1}^{1-2s}P]\tilde{u} &= 0 && \text{in } \mathbb{R}^{n+1}_+, \\ \tilde{u} &= u && \text{on } \mathbb{R}^n\times\{0\}, \\ \lim_{x_{n+1}\to 0} x_{n+1}^{1-2s}\partial_{n+1}\tilde{u}(x) &= Vu && \text{on } \mathbb{R}^n\times\{0\}, \end{aligned}$$

with (1.3) and $|V| \leq 1$. Then there exists a constant $C = C(n,\lambda)$ such that

$$\|\tilde{u}\|_{L^\infty(B^+_{1/2})} \leq C[\|x_{n+1}^{\frac{1-2s}{2}}\tilde{u}\|_{L^2(B^+_1)} + \|x_{n+1}^{\frac{1-2s}{2}}\nabla\tilde{u}\|_{L^2(B^+_1)}].$$

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Pu-Zhao Kow
Department of Mathematics and Statistics, P.O. Box 35 (MaD), 40014 University of Jyväskylä, Finland; pu-zhao.pz.kow@jyu.fi