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Cheeger estimates of Dirichlet-to-Neumann operators on infinite subgraphs of graphs

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Abstract. In this paper, we study the Dirichlet-to-Neumann operators on infinite subgraphs of graphs. For an infinite subgraph, we prove Cheeger-type estimates for the bottom spectrum of the Dirichlet-to-Neumann operator, and the higher order Cheeger estimates for higher order eigenvalues of the Dirichlet-to-Neumann operator.

1. Introduction

Eigenvalue estimates are of interest in Riemannian geometry and mathematical physics. There are various eigenvalue estimates using geometric quantities. In this paper, we focus on the isoperimetric-type estimate introduced by Cheeger [6], now called the *Cheeger estimate,* which reveals a close relation between the first non-trivial eigenvalue of the Laplace-Beltrami operator on a closed manifold, and the isoperimetric constant called *Cheeger constant.*

Let (M, g) be a compact, connected, smooth Riemannian manifold with smooth boundary ∂M. The Dirichlet-to-Neumann operator Λ, called the *DtN operator* for short, is defined as

$$\Lambda\colon H^{\frac{1}{2}}(\partial M) \to H^{-\frac{1}{2}}(\partial M),$$
$$f \mapsto \Lambda(f) := \frac{\partial u_f}{\partial n},$$

where u_f is the harmonic extension of f to M. The DtN operator Λ is a first order elliptic pseudo-differential operator [37, p. 37]. Since ∂M is compact, the spectrum of Λ is non-negative, discrete and unbounded [2, p. 95]. We refer to [21] for a survey of the spectral properties of the DtN operators. The eigenvalue problem associated to the DtN operator Λ is also known as the *Steklov problem.* For the history of this problem, cf. [28]. There are many results on the Steklov problem on Riemannian manifolds; see, e.g., [9, 15–18, 24, 27].

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For the first non-trivial eigenvalue of the DtN operator, Jammes [27] introduced a type of Cheeger constant,

$$h_J(M) := \inf_{\substack{\Omega\subset M \\ \mathrm{Vol}(\Omega)\le\frac{1}{2}\mathrm{Vol}(M)}} \frac{\mathrm{Area}(\partial\Omega\cap\mathrm{int}(M))}{\mathrm{Area}(\Omega\cap\partial M)},$$

where Area$(\cdot)$ and Vol$(\cdot)$ denote the Riemannian area (i.e., the $(n-1)$-dim Hausdorff measure) of the boundary, and the Riemannian volume, respectively. We call $h_J(M)$ the *Jammes-type Cheeger* constant and the subscript "J" indicates the Jammes-type. Let $\sigma_2(M)$ be the first non-trivial eigenvalue of Λ. The Jammes-type Cheeger estimate [27, Theorem 1] reads as

$$\sigma_2(M) \ge \frac{1}{4}h_N(M)h_J(M), \tag{1.1}$$

where $h_N(M)$ is the Neumann Cheeger constant associated to the Neumann Laplacian on M.

There are many interesting interactions between Riemannian geometry and discrete analysis on graphs. Many methods initiated in Riemannian geometry have been generalized to the discrete setting, and conversely some approaches found on graphs may also be applied to Riemannian geometry. The Cheeger estimate was first generalized to graphs by Dodziuk [13] and Alon and Milman [1] independently. Miclo introduced higher order Cheeger constants and conjectured related higher order Cheeger estimates; see [11, 32]. The conjecture was proved by Lee, Gharan, and Tevisan [29] via random partition methods on graphs. Then Miclo [33] and Funano [20] extended the results to the Riemannian case and found some important applications.

Recently, the authors [26] defined the DtN operator on a finite subgraph of a graph, and proved two Cheeger-type estimates for the first non-trivial eigenvalue of the DtN operator: the Escobar-type Cheeger estimate following [15] and the Jammes-type Cheeger estimate following [27]. Hassannezhad and Miclo [25] proved the Jammes-type Cheeger estimate independently, and generalized it to higher order Cheeger estimates for eigenvalues of the DtN operator in terms of higher order Cheeger–Steklov constants. Their result is as follows. Consider a Markov process on a finite state space U and a proper subset V of U, serving as the boundary, on which the DtN operator is defined. Let σ_k be the k-th eigenvalue of the DtN operator. Hassannezhad and Miclo [25, Theorem A] proved that there exists a universal constant c such that

$$\sigma_k \ge \frac{c}{k^6}\frac{\varsigma_k}{A}, \quad \text{for all } 1\le k\le \sharp V, \tag{1.2}$$

where A is the largest absolute value of the diagonal elements of the irreducible Markov generator and ς_k is the k-th order Cheeger–Steklov constant.

Spectral theory for finite graphs has been extensively studied in the literature; see, e.g., [3, 5, 7, 10]. For infinite graphs, there are also many results on the spectra of the Laplacians; see [4, 14, 19, 34–36]. Infinite graphs have many applications in geometric group theory, probability theory, mathematical physics, etc. In this paper, we study the DtN operators on infinite subgraphs of graphs. By the well-known exhaustion methods, see, e.g., [4], we construct the DtN operator on an infinite subgraph of an infinite graph and prove Cheeger-type estimates for the bottom spectrum and higher order Cheeger estimates for the higher order eigenvalues of the DtN operators following [25, 27], which can be viewed as an extension of the results in [25, 26].

We recall some basic definitions on infinite graphs. Let V be a countable infinite set and μ be a symmetric weight function given by

$$\begin{aligned} \mu\colon V \times V &\to [0, \infty),\\ (x, y) &\mapsto \mu_{xy} = \mu_{yx}. \end{aligned}$$

This induces a graph structure $G = (V, E)$ with the set of vertices V and the set of edges E such that $\{x, y\} \in E$ if and only if $\mu_{xy} > 0$. Two vertices x, y satisfying $\{x, y\} \in E$ are called *neighbors,* denoted by $x \sim y$. We only consider locally finite graphs, i.e. each vertex only has finitely many neighbors. We call the triple (V, E, μ) a *weighted graph.*

For an infinite graph, the exhaustion of the whole graph by finite subsets of vertices is an important concept; see [4]. A sequence of subsets of vertices $\mathcal{W} = \{W_i\}_{i=1}^{\infty}$ is called an *exhaustion* of the infinite graph $G = (V, E)$, denoted by $\mathcal{W} \uparrow V$, if it satisfies

- $W_1 \subset W_2 \subset \cdots \subset W_i \subset \cdots \subset V$,
- $\sharp W_i < \infty$, for all $i = 1, 2, \ldots,$
- $V = \bigcup_{i=1}^{\infty} W_i$.

For any quantity τ defined on finite subgraphs of V that is monotone, i.e. for any finite subgraphs $W \subset W'$ of V, one has $\tau(W) \le \tau(W')$ (or $\tau(W) \ge \tau(W')$). The limit of $\tau(W_i)$ is defined as

$$\tau(V) := \lim_{i\to\infty} \tau(W_i). \tag{1.3}$$

One can check that this limit exists in $\mathbb{R} \cup \{\pm\infty\}$ and does not depend on the choice of the exhaustion.

Given $\Omega_1, \Omega_2 \subset V$, the set of edges between Ω_1 and Ω_2 is denoted by

$$E(\Omega_1, \Omega_2) := \{\{x, y\} \in E\colon x \in \Omega_1, y \in \Omega_2\}.$$

For any subset $\Omega \subset V$, there are two notions of boundary:

1. the edge boundary of Ω is defined as

$$\partial\Omega := E(\Omega, \Omega^c), \quad \text{where } \Omega^c := V \setminus \Omega;$$

2. the vertex boundary of Ω is defined as

$$\delta\Omega := \{x \in \Omega^c : x \sim y \text{ for some } y \in \Omega\}.$$

We write $\bar{\Omega} := \Omega \cup \delta\Omega$. From now on, we only consider the graph structure $(\bar{\Omega}, E(\Omega, \bar{\Omega}), \mu)$, still denoted by $\bar{\Omega}$ for simplicity, where the weight μ is modified such that $\mu_{xy} = 0$ for any $\{x, y\} \notin E(\Omega, \bar{\Omega})$, i.e. the edges between vertices in $\delta\Omega$, $E(\delta\Omega, \delta\Omega)$, are removed. In this paper, we always assume $\bar{\Omega}$ is connected, unless otherwise stated.

In what follows, for $\Omega \subset V$, we regard $\bar{\Omega}$ as the graph Ω with its vertex boundary $\delta\Omega$ defined above, for which we may forget about the ambient graph (V, E, μ). We introduce weights on the vertex set $\bar{\Omega}$ as

$$d(x) = \begin{cases} \sum_{y \in \bar{\Omega}} \mu_{xy}, & x \in \Omega, \\ \sum_{y \in \Omega} \mu_{xy}, & x \in \delta\Omega, \end{cases}$$

which extends to a measure $d(\cdot)$ on subsets of $\bar{\Omega}$ by

$$d(A) := \sum_{x \in A} d(x), \quad \text{for all } A \subset \bar{\Omega}.$$

Given a vertex set F, we denote by $\mathbb{R}^F$ the set of all real-valued functions defined on F. Let $W \subset \bar{\Omega}$ be a finite subset and δW be the vertex boundary of W in $\bar{\Omega}$. For any $f \in \mathbb{R}^{W \cap \delta\Omega}$, let u_f^W be the solution of the following equation:

$$\begin{cases} \Delta u_f^W(x) = 0, & x \in W \cap \Omega, \\ u_f^W(x) = f(x), & x \in W \cap \delta\Omega, \\ u_f^W(x) = 0, & x \in \delta W, \end{cases} \tag{1.4}$$

where the Laplacian Δ is defined in (2.2). For the existence of the solution, see Lemma 2.2. The third condition above stands for the Dirichlet boundary condition on W.

Now, let us define the DtN operator for infinite subgraphs of a graph. Let $\ell_0(\delta\Omega)$ denote the set of functions on $\delta\Omega$ with finite support. For any $f \in \ell_0(\delta\Omega)$, we write $f = f^+ - f^-$, where $f^+ := \max\{f, 0\}$ and $f^- := \max\{-f, 0\}$. For any $\mathcal{W} \uparrow \bar{\Omega}$, let $u_{f^+}^{W_i}$ ($u_{f^-}^{W_i}$ resp.) be the solution of (1.4) with W and f replaced by W_i and f^+ (f^- resp.). Set

$$u_f^{W_i} := u_{f^+}^{W_i} - u_{f^-}^{W_i}.$$

Applying the well-known maximum principle, see, e.g., [22], we have

$$u_{f^\pm}^{W_i} \le u_{f^\pm}^{W_{i+1}} \quad \text{and} \quad |u_{f^\pm}^{W_i}| \le \sup_{x \in \delta\Omega} |f(x)|.$$

Hence, the limit of $u_f^{W_i}$ exists and we set

$$u_f := \lim_{i \to \infty} u_f^{W_i}.$$

For any $f \in \ell_0(\delta\Omega)$, we define

$$\Lambda(f) := \frac{\partial u_f}{\partial n},$$

where the operator $\frac{\partial}{\partial n}$ is defined in (2.3). From Lemma 3.6 and Proposition 3.1, Λ is a bounded symmetric linear operator on $\ell_0(\delta\Omega)$, which can be uniquely extended to a self-adjoint operator on $\ell^2(\delta\Omega)$ (See Section 2 for its definition). We call the extension to $\ell^2(\delta\Omega)$ the *DtN operator on* Ω and still denote it by Λ.

Remark 1.1. Our definition of the DtN operator is slightly different from that of Hassannezhad and Miclo in [25]. In fact, the edges between vertices in $\delta\Omega$, i.e., $E(\delta\Omega, \delta\Omega)$, play no role in our definition, but they matter in Hassannezhad and Miclo's. There are no essential differences up to the fixed boundary information $E(\delta\Omega, \delta\Omega)$. The geometric quantities for bounding the eigenvalues should be modified to fit the definitions, and the results are similar.

By the standard spectral theory [12, (4.5.1) on p. 88], for any $k \ge 1$, the k-th eigenvalue of the DtN operator on Ω is equal to

$$\sigma_k(\Omega) := \inf_{\substack{H \subset \ell_0(\delta\Omega), \\ \dim H = k}} \sup_{0 \ne f \in H} \frac{\langle \Lambda(f), f \rangle_{\delta\Omega}}{\langle f, f \rangle_{\delta\Omega}}, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle_{\delta\Omega}$ is the inner product on $\ell^2(\delta\Omega)$ with respect to the measure $d(\cdot)$ defined in Section 2. For $k = 1$, $\sigma_1(\Omega)$ is called the *bottom spectrum* of Λ. If the number of vertices in $\delta\Omega$ is finite, we will prove that $\sigma_1(\Omega) = 0$ if and only if $\bar{\Omega}$ is recurrent; see Proposition 4.6.

By exhaustion methods, in order to obtain Cheeger-type estimates for $\sigma_k(\Omega)$, $k \ge 1$, we consider finite subsets of $\bar{\Omega}$. Let $W \subset \bar{\Omega}$ be a finite subset. Then the Cheeger-type constants of W are defined as follows.

Definition 1.2. Let $W \subset \bar{\Omega}$ be a finite subset. The Jammes-type Cheeger constant of W in $\bar{\Omega}$ is defined as

$$h_J(W) := \min_{\emptyset \neq A \subset W} \frac{\mu(\partial A)}{d(A \cap \delta\Omega)}.$$

We set $h_J(W) = +\infty$ if $W \cap \delta\Omega = \emptyset$. Similarly, the classical Cheeger constant of W in $\bar{\Omega}$ is defined as

$$h(W) := \min_{\emptyset \neq A \subset W} \frac{\mu(\partial A)}{d(A)}.$$

Remark 1.3. For any finite $W \subset \bar{\Omega}$, one easily shows that

$$h_J(W) \geq h(W). \tag{1.6}$$

The DtN operator on W, denoted by Λ_W, is defined as

$$\Lambda_W : \mathbb{R}^{W \cap \delta\Omega} \to \mathbb{R}^{W \cap \delta\Omega},$$
$$f \mapsto \Lambda_W(f) := \frac{\partial u_f^W}{\partial n},$$

where the operator $\frac{\partial}{\partial n}$ is defined in (2.3) and u_f^W is the solution of (1.4).

Let $\sigma_k(W)$, $1 \leq k \leq \sharp(W \cap \delta\Omega)$, be the k-th eigenvalue of Λ_W. Similar to (1.5), $\sigma_k(W)$ can be characterized as

$$\sigma_k(W) := \min_{\substack{H \subset \mathbb{R}^{W \cap \delta\Omega},\\ \dim H = k}} \max_{0 \neq f \in H} \frac{\langle \Lambda_W(f), f \rangle_{W \cap \delta\Omega}}{\langle f, f \rangle_{W \cap \delta\Omega}}. \tag{1.7}$$

We obtain the following Jammes-type Cheeger estimate for $\sigma_1(W)$, the first non-trivial eigenvalue of Λ_W, which is an analog to (1.1) in the Riemannian case.

Theorem 1.4. *For any finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$, we have*

$$\frac{h(W) \cdot h_J(W)}{2} \leq \sigma_1(W) \leq h_J(W).$$

By Lemma 3.7, $\sigma_k(W)$ is non-increasing when W increases. Moreover, we obtain an approximation relation between $\sigma_k(\Omega)$ and $\{\sigma_k(W_i)\}_{i=1}^{\infty}$ for any $\mathcal{W} \uparrow \bar{\Omega}$.

Proposition 1.5. *For any $\mathcal{W} \uparrow \bar{\Omega}$ and $k \geq 1$,*

$$\lim_{\mathcal{W} \uparrow \bar{\Omega}} \sigma_k(W) = \sigma_k(\Omega).$$

As a corollary, we have the following estimate.

Corollary 1.6. *For any $k \geq 1$,*

$$\sigma_k(\Omega) \geq \lambda_k(\bar{\Omega}),$$

where $\lambda_k(\bar{\Omega})$ are eigenvalues of the normalized Laplacian on the graph $\bar{\Omega} = (\bar{\Omega}, E(\Omega, \bar{\Omega}), \mu)$.

By Definition 1.2, $h_J(W)$ and $h(W)$ are non-increasing when W increases. By (1.3), for any $\mathcal{W} \uparrow \bar{\Omega}$, the corresponding Cheeger constants for Ω can be defined as

$$h_J(\Omega) := \lim_{\mathcal{W}\uparrow\bar{\Omega}} h_J(W), \quad h(\bar{\Omega}) := \lim_{\mathcal{W}\uparrow\bar{\Omega}} h(W).$$

Hence, by Theorem 1.4 and Proposition 1.5, we have the following estimates for the bottom spectrum $\sigma_1(\Omega)$.

Theorem 1.7. *Let $\Omega \subset V$ be an infinite subset. We have*

$$\frac{h(\bar{\Omega}) \cdot h_J(\Omega)}{2} \leq \sigma_1(\Omega) \leq h_J(\Omega).$$

Remark 1.8. Combining (1.6) with Theorem 1.7, we have

$$\sigma_1(\Omega) \geq \frac{1}{2} h^2(\bar{\Omega}).$$

This can be also derived from the Cheeger estimate on $\bar{\Omega}$ [19] and Corollary 1.6 for $k = 1$. In particular, this yields that if $h(\bar{\Omega}) > 0$, then $\sigma_1(\Omega) > 0$.

For any finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$, we denote by $\mathcal{A}(W)$ the collection of all non-empty subsets of W and $\mathcal{A}_k(W)$ the set of all disjoint k-tuples $(A_1, \ldots, A_k)$ such that $A_l \in \mathcal{A}(W)$, for all $l \in [k]$, where $[k] := \{1, \ldots, k\}$, $k \in \mathbb{N}$. Following [25], we define the higher order Cheeger-type constants for the DtN operator on W.

Definition 1.9. The k-th order Cheeger–Steklov constant for the DtN operator on W is defined as

$$h_k(W) := \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(W)} \max_{l\in[k]} h_J(A_l) h(A_l).$$

Similarly, the k-th order Jammes-type Cheeger constant for W is defined as

$$h_J^k(W) := \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(W)} \max_{l\in[k]} \frac{\mu(\partial A_l)}{d(A_l \cap \delta\Omega)}.$$

Following [25], as an intermediary step to obtain a Cheeger-type estimate for higher order eigenvalues of the DtN operators, we prove a higher order Cheeger estimate for the Dirichlet problem on finite graphs; see Theorem 5.11 in the paper. Initiated by [25, Proposition 3], any k-th eigenvalue of the DtN operator on W can be approximated by a sequence of k-th eigenvalues of Dirichlet Laplacians defined in (6.1) and (6.2) with blowing-up weights; see Proposition 6.1. Hence, combining Theorem 5.11 with Proposition 6.1, we obtain the following result.

Theorem 1.10. *Let $\Omega \subset V$ be an infinite subset, and W be a finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$. There exists a universal constant $c > 0$ such that*

$$\frac{c}{k^6} h_k(W) \leq \sigma_k(W) \leq 2h_J^k(W),$$

where $\sigma_k(W)$ is the k-th eigenvalue of the DtN operator on W.

By Definition 1.9, $h_k(W)$ and $h^k_J(W)$ are non-increasing when W increases. Hence, the corresponding Cheeger constants for Ω can be defined as

$$h_k(\Omega) := \lim_{W \uparrow \bar{\Omega}} h_k(W), \quad h^k_J(\Omega) := \lim_{W \uparrow \bar{\Omega}} h^k_J(W).$$

Finally, by exhaustion, the monotonicity of the higher order eigenvalues of the DtN operators, see Lemma 3.7 in the paper, and the convergence of eigenvalues, see Proposition 1.5, we have the following higher order Cheeger-type estimate for the DtN operator on infinite graphs.

Theorem 1.11. *Let $\Omega \subset V$ be an infinite subset. For any $k \in \mathbb{N}$, there exists a universal constant $c > 0$ such that*

$$\frac{c}{k^6} h_k(\Omega) \leq \sigma_k(\Omega) \leq 2h^k_J(\Omega).$$

Remark 1.12. Hassannezhad and Miclo [25, Theorem B] defined the DtN operator on a subset of a probability measure space $(M, \mathcal{M}, \nu)$, endowed with a Markov kernel P leaving ν invariant, and proved the higher order Cheeger estimate. Our result applies to general infinite graphs with possibly infinite total measure, which can be regarded as an extension of Hassannezhad and Miclo's result.

The paper is organized as follows. In Section 2, we recall some facts on graphs. In Section 3, we study the spectra of the DtN operators on infinite subgraphs. In Section 4, we prove the Jammes-type Cheeger estimate for the bottom spectrum of the DtN operators. In Section 5, we obtain higher order Cheeger estimates for the Dirichlet problems. In Section 6, we prove higher order Cheeger estimates for the DtN operators on infinite subgraphs of graphs.

2. Preliminaries

Let (X, ν) be a discrete measure space, i.e., X is a countable discrete space equipped with a measure ν. For any $A \subset X$, we denote by $\ell_0(A)$ the set of finitely supported functions on A. For $p \in [1, \infty]$, the space of ℓ^p summable functions on (X, ν) is defined routinely. Given a function $f \in \mathbb{R}^X$, for $p \in [1, \infty)$, we denote by

$$\|f\|_{\ell^p} = \Big(\sum_{x \in X} |f(x)|^p \nu(x)\Big)^{1/p}$$

the ℓ^p norm of f. For $p = \infty$,

$$\|f\|_{\ell^\infty} = \sup_{x \in X} |f(x)|.$$

Let

$$\ell^p(X,\nu) := \{f \in \mathbb{R}^X : \|f\|_{\ell^p} < \infty\}$$

be the space of ℓ^p summable functions on (X,ν). In our setting, these definitions apply to (Ω, d) and $(\delta\Omega, d)$ for $\Omega \subset V$ in a graph (V, E, μ). The case for $p = 2$ is of particular interest, as we have the Hilbert spaces $\ell^2(\Omega, d)$ and $\ell^2(\delta\Omega, d)$ equipped with standard inner products

$$\langle f, g\rangle_\Omega = \sum_{x\in\Omega} f(x)g(x)d(x), \quad f, g \in \mathbb{R}^\Omega,$$
$$\langle \varphi, \psi\rangle_{\delta\Omega} = \sum_{x\in\delta\Omega} \varphi(x)\psi(x)d(x), \quad \varphi, \psi \in \mathbb{R}^{\delta\Omega}.$$

Given $W \subset \bar{\Omega}$, an associated quadratic form is defined as

$$D_W(f,g) = \sum_{e=\{x,y\}\in E(W,\overline{W})} \mu_{xy}(f(x) - f(y))(g(x) - g(y)), \quad f, g \in \mathbb{R}^{\overline{W}}.$$

The Dirichlet energy of $f \in \mathbb{R}^{\overline{W}}$ can be written as

$$D_W(f) := D_W(f, f). \tag{2.1}$$

For any $f \in \mathbb{R}^{\bar{\Omega}}$, the Laplacian of f is defined as

$$\Delta f(x) := \frac{1}{d(x)} \sum_{y\in V: y\sim x} \mu_{xy}(f(y) - f(x)), \quad x \in \Omega. \tag{2.2}$$

For any $f \in \mathbb{R}^{\bar{\Omega}}$, the outward normal derivative of f at $z \in \delta\Omega$ is defined as

$$\frac{\partial f}{\partial n}(z) := \frac{1}{d(z)} \sum_{x\in\Omega: x\sim z} \mu_{zx}(f(z) - f(x)). \tag{2.3}$$

For any finite subset $W \subset \bar{\Omega}$, the Dirichlet eigenvalue problem on W is defined as

$$\begin{cases} \Delta f(x) = -\lambda f(x), & x \in W, \\ f(x) = 0, & x \in \delta W. \end{cases} \tag{2.4}$$

We denote by $\lambda_{k,D}(W)$ the k-th eigenvalue of the above Dirichlet problem. Note that $\lambda_{k,D}$ is monotone, i.e. $\lambda_{k,D}(W) \geq \lambda_{k,D}(W')$ for $W \subset W'$. For any $\mathcal{W} \uparrow \bar{\Omega}$, the k-th eigenvalue of the normalized Laplacian on the graph $\bar{\Omega} = (\bar{\Omega}, E(\Omega, \bar{\Omega}), \mu)$ is defined as

$$\lambda_k(\bar{\Omega}) := \lim_{i\to\infty} \lambda_{k,D}(W_i). \tag{2.5}$$

We recall the following well-known results for the Laplace operators; see, e.g., [22] for their proofs.

Lemma 2.1 (Green's formula). *For any finite subset $W \subset \bar{\Omega}$ and any $f, g \in \mathbb{R}^{\bar{W}}$, we have*

$$\langle \Delta f, g\rangle_W = -D_W(f, g) + \left\langle \frac{\partial f}{\partial n}, g \right\rangle_{\delta W}, \tag{2.6}$$

where $\frac{\partial f}{\partial n}$ on δW is defined similarly to (2.3) with Ω replaced by W.

Lemma 2.2. *For any $f \in \mathbb{R}^{W \cap \delta\Omega}$, there exists a unique function $u_f^W \in \mathbb{R}^{\bar{W}}$ satisfying (1.4).*

We always denote by u_f^W the unique solution of (1.4) with the Dirichlet boundary condition f in this paper. By Green's formula, Lemma 2.1, we have the following lemma.

Lemma 2.3. *For any finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$ and any $f, g \in \mathbb{R}^{W \cap \delta\Omega}$, we have*

$$D_W(u_f^W, u_g^W) = \left\langle \frac{\partial u_f^W}{\partial n}, g \right\rangle_{W \cap \delta\Omega}.$$

3. DtN operators on infinite graphs

Let $G = (V, E, \mu)$ be an infinite graph, $\Omega \subset V$ is an infinite subset. Let Λ be the DtN operator on Ω defined in the introduction.

Proposition 3.1. *For any $f, g \in \ell_0(\delta\Omega)$,*

$$\langle \Lambda(f), g\rangle = \langle f, \Lambda(g)\rangle.$$

Proof. For sufficiently large i such that $W_i \supset \operatorname{supp}(f) \cup \operatorname{supp}(g)$, by Lemma 2.3,

$$\left\langle \frac{\partial u_f^{W_i}}{\partial n}, g \right\rangle_{\delta\Omega} = \left\langle \frac{\partial u_g^{W_i}}{\partial n}, f \right\rangle_{\delta\Omega}.$$

Since f and g are of finite support, only finitely many summands are involved in the above equation. By passing to the limit,

$$u_f^{W_i} \to u_f, \quad u_g^{W_i} \to u_g, \quad i \to \infty,$$

we prove the proposition. ∎

Let $f \in \ell_0(\delta\Omega)$. For any finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$, set

$$\mathcal{L}(f, W) := \{\phi \in \mathbb{R}^{\bar{\Omega}} \mid \operatorname{supp}(\phi) \subset W \text{ and } \phi|_{W \cap \delta\Omega} = f\}.$$

We define the capacity of W with boundary condition f as

$$\mathrm{Cap}(f, W) := \inf_{\phi \in \mathcal{L}(f,W)} D_W(\phi).$$

Similarly, the capacity of $\bar{\Omega}$ with boundary condition f is defined as

$$\mathrm{Cap}(f) := \inf_{\phi \in \ell_0(\bar{\Omega}), \phi|_{\delta\Omega} = f} D_\Omega(\phi).$$

For any $\mathcal{W} \uparrow \bar{\Omega}$, since $f \in \ell_0(\delta\Omega)$, there exists $M \in \mathbb{N}^+$, such that $\mathrm{supp}(f) \subset W_i \cap \delta\Omega$, for all $i > M$. Note that $\mathcal{L}(f, W_{i_1}) \subset \mathcal{L}(f, W_{i_2})$, for any $i_1, i_2 > M$ and $i_1 < i_2$. Hence, by definition, $\mathrm{Cap}(f, W_i)$ is non-increasing when $i > M$. One can verify that

$$\mathrm{Cap}(f) = \lim_{i \to \infty} \mathrm{Cap}(f, W_i). \tag{3.1}$$

Lemma 3.2. $\mathrm{Cap}(f, W) = D_W(u_f^W)$.

Proof. We have

$$\begin{aligned}
\mathrm{Cap}(f, W) &= \inf_{\phi \in \mathcal{L}(f,W)} D_W(\phi) \\
&= \inf_{\phi \in \mathcal{L}(f,W)} D_{W \cap \Omega}(\phi) + \sum_{x \in W \cap \delta\Omega} \sum_{y \in \Omega \setminus W} \mu_{xy}(f(x) - 0)^2 \\
&= D_{W \cap \Omega}(u_f^W) + \sum_{x \in W \cap \delta\Omega} \sum_{y \in \Omega \setminus W} \mu_{xy} f^2(x) \\
&= D_W(u_f^W).
\end{aligned}$$

The second last equality follows from the fact that the harmonic function u_f^W minimizes the Dirichlet energy among functions with the same boundary condition. ∎

By Lemma 3.2 and 2.3, we have

$$\mathrm{Cap}(f, W) = D_W(u_f^W) = \left\langle \frac{\partial u_f^W}{\partial n}, f \right\rangle_{W \cap \delta\Omega}. \tag{3.2}$$

Proposition 3.3. *For any* $f \in \ell_0(\delta\Omega)$*, we have*

$$\mathrm{Cap}(f) = D_\Omega(u_f) = \langle \Lambda(f), f \rangle.$$

Proof. For any $\mathcal{W} \uparrow \bar{\Omega}$, since $f \in \ell_0(\delta\Omega)$, there exists $M \in \mathbb{N}$, such that $\mathrm{supp}(f) \subset W_i \cap \delta\Omega$, for all $i > M$. By (3.2),

$$\mathrm{Cap}(f, W_i) = D_{W_i}(u_f^{W_i}) = \left\langle \frac{\partial u_f^{W_i}}{\partial n}, f \right\rangle_{W_i \cap \delta\Omega}.$$

Letting $i \to \infty$, by (3.1) we have

$$\mathrm{Cap}(f) = \lim_{i\to\infty} \mathrm{Cap}(f, W_i) = \lim_{i\to\infty} \left\langle \frac{\partial u_f^{W_i}}{\partial n}, f \right\rangle_{W_i \cap \delta\Omega} = \left\langle \frac{\partial u_f}{\partial n}, f \right\rangle_{\delta\Omega}, \tag{3.3}$$

where the last equality follows from that $u_f^{W_i}$ converges to u_f pointwise and $f \in \ell_0(\delta\Omega)$. By Fatou's lemma,

$$D_\Omega(u_f) \le \liminf_{i\to\infty} D_{W_i}(u_f^{W_i}) = \liminf_{i\to\infty} \mathrm{Cap}(f, W_i) = \mathrm{Cap}(f). \tag{3.4}$$

For any $i > M$,

$$D_{W_i}(u_f - u_f^{W_i}) = D_{W_i}(u_f) + D_{W_i}(u_f^{W_i}) - 2D_{W_i}(u_f, u_f^{W_i}). \tag{3.5}$$

By Green's formula, Lemma 2.1, the last term in (3.5) can be written as

$$D_{W_i}(u_f, u_f^{W_i}) = \left\langle \frac{\partial u_f}{\partial n}, u_f^{W_i} = \right\rangle_{W_i \cap \delta\Omega} = \left\langle \frac{\partial u_f}{\partial n}, f \right\rangle_{\delta\Omega} = \mathrm{Cap}(f),$$

where the last equality follows from (3.3). Therefore, (3.5) implies that

$$0 \le D_{W_i}(u_f - u_f^{W_i}) = D_{W_i}(u_f) + \mathrm{Cap}(f, W_i) - 2\,\mathrm{Cap}(f),$$

i.e.

$$D_{W_i}(u_f) \ge 2\,\mathrm{Cap}(f) - \mathrm{Cap}(f, W_i),$$

whence by letting $i \to \infty$, we get

$$D_\Omega(u_f) = \lim_{i\to\infty} D_{W_i}(u_f) \ge 2\,\mathrm{Cap}(f) - \lim_{i\to\infty} \mathrm{Cap}(f, W_i) = \mathrm{Cap}(f). \tag{3.6}$$

Combining (3.4) with (3.6), we have

$$D_\Omega(u_f) = \mathrm{Cap}(f). \tag{3.7}$$

Then the proposition follows from (3.3) and (3.7). ∎

The proof of the above proposition yields the following corollary.

Corollary 3.4. *For any $\mathcal{W} \uparrow \bar{\Omega}$ and $f \in \ell_0(\delta\Omega)$, we have*

$$D_\Omega(u_f) = \lim_{\mathcal{W}\uparrow\bar{\Omega}} D_W(u_f^W).$$

Combining (1.5) with Proposition 3.3, we have the following corollary.

Corollary 3.5. *We have*

$$\begin{aligned}\sigma_k(\Omega) &:= \inf_{\substack{H\subset\ell_0(\delta\Omega),\\ \dim H=k}} \sup_{0\neq f\in H} \frac{\langle\Lambda(f),f\rangle_{\delta\Omega}}{\langle f,f\rangle_{\delta\Omega}}\\ &= \inf_{\substack{H\subset\ell_0(\delta\Omega),\\ \dim H=k}} \sup_{0\neq f\in H} \frac{D_\Omega(u_f)}{\langle f,f\rangle_{\delta\Omega}}\\ &= \inf_{\substack{H\subset\ell_0(\delta\Omega),\\ \dim H=k}} \sup_{0\neq f\in H} \frac{\mathrm{Cap}(f)}{\langle f,f\rangle_{\delta\Omega}}.\end{aligned}$$

In the next lemma, we show that Λ is a bounded operator on $\ell_0(\delta\Omega)$.

Lemma 3.6. *For any $f\in\ell_0(\delta\Omega)$, we have*

$$\|\Lambda(f)\|_{\ell^2(\delta\Omega)}\le\|f\|_{\ell^2(\delta\Omega)},$$

i.e. Λ is a bounded linear operator on $\ell_0(\delta\Omega)$.

Proof. We have

$$\begin{aligned}\left\|\frac{\partial u_f}{\partial n}\right\|^2_{\ell^2(\delta\Omega)} &= \sum_{x\in\delta\Omega}\left|\frac{1}{d(x)}\sum_{y\in\Omega}\mu_{xy}(u_f(x)-u_f(y))\right|^2 d(x)\\ &\le \sum_{x\in\delta\Omega}\sum_{y\in\Omega}\mu_{xy}(u_f(x)-u_f(y))^2\\ &\le D_\Omega(u_f).\end{aligned}$$

For any $f\in\ell_0(\delta\Omega)$ and any finite subset $W\subset\bar{\Omega}$, we denote by

$$\bar{f}_W(x)=\begin{cases} f(x), & x\in W\cap\delta\Omega,\\ 0, & \text{otherwise,}\end{cases}$$

the zero extension of $f|_{W\cap\delta\Omega}$ to $\mathbb{R}^{\bar{\Omega}}$. By Corollary 3.4,

$$\begin{aligned}D_\Omega(u_f) &= \lim_{W\uparrow\bar{\Omega}} D_W(u_f^W)\\ &= \lim_{W\uparrow\bar{\Omega}}\Big(\sum_{e=\{x,y\}\in E(W\cap\Omega,\overline{W\cap\Omega})} + \sum_{x\in W\cap\delta\Omega}\sum_{y\in\Omega\setminus W}\Big)\mu_{xy}(u_f^W(x)-u_f^W(y))^2\\ &\le \lim_{W\uparrow\bar{\Omega}}\Big(D_{W\cap\Omega}(\bar{f}_W)+\sum_{x\in W\cap\delta\Omega}\sum_{y\in\Omega\setminus W}\mu_{xy}f^2(x)\Big)\\ &= \lim_{W\uparrow\bar{\Omega}}\Big(\sum_{x\in W\cap\delta\Omega}f^2(x)\sum_{y\in W\cap\Omega}\mu_{xy}+\sum_{x\in W\cap\delta\Omega}\sum_{y\in\Omega\setminus W}\mu_{xy}f^2(x)\Big)\\ &= \lim_{W\uparrow\bar{\Omega}}\|f\|_{\ell^2(W\cap\delta\Omega)}=\|f\|_{\ell^2(\delta\Omega)},\end{aligned}$$

where the inequality above follows from the fact that the harmonic function u_f^W minimizes the Dirichlet energy among functions with the same boundary condition, in particular compared with $\bar{f}_W$.

Hence, we have

$$\left\| \frac{\partial u_f}{\partial n} \right\|^2_{\ell^2(\delta\Omega)} \leq \| f \|^2_{\ell^2(\delta\Omega)}. \qquad \blacksquare$$

By the boundedness of Λ on $\ell_0(\delta\Omega)$ and the density of $\ell_0(\delta\Omega)$ in $\ell^2(\delta\Omega)$, Λ can be uniquely extended to a bounded self-adjoint operator on $\ell^2(\delta\Omega)$.

For any finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$, we denote by $\sigma_k(W)$ the k-th eigenvalue of the DtN operator Λ_W. By (1.7) and (3.2), $\sigma_k(W)$, $1 \leq k \leq \sharp(W \cap \delta\Omega)$, can be characterized as

$$\begin{aligned}\sigma_k(W) :&= \min_{\substack{H \subset \mathbb{R}^{W \cap \delta\Omega},\\ \dim H = k}} \max_{0 \neq f \in H} \frac{\langle \Lambda_W(f), f\rangle_{W\cap\delta\Omega}}{\langle f, f\rangle_{\delta\Omega}} \\ &= \min_{\substack{H \subset \mathbb{R}^{W \cap \delta\Omega},\\ \dim H = k}} \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f, W)}{\langle f, f\rangle_{W\cap\delta\Omega}} \\ &= \min_{\substack{H \subset \mathbb{R}^{W \cap \delta\Omega},\\ \dim H = k}} \max_{0 \neq f \in H} \frac{D_W(u_f^W)}{\langle f, f\rangle_{\delta\Omega}}. \end{aligned} \tag{3.8}$$

In order to give Cheeger estimates for infinite graphs, we need the following monotonicity result.

Lemma 3.7. *For any $\mathcal{W} \uparrow \bar{\Omega}$,*

$$\sigma_k(W_i) \geq \sigma_k(W_{i+1}), \quad \textit{for all } i = 1, 2, \ldots,$$

where $1 \leq k \leq \sharp(W_i \cap \delta\Omega)$.

Proof. For any i, by (3.8), one can choose $H \subset \mathbb{R}^{W_i \cap \delta\Omega}$, $\dim H = k$, such that

$$\sigma_k(W_i) = \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f, W_i)}{\langle f, f\rangle_{W_i \cap \delta\Omega}}.$$

Then we have

$$\begin{aligned}\sigma_k(W_i) &= \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f, W_i)}{\langle f, f\rangle_{W_i\cap\delta\Omega}} \geq \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f, W_{i+1})}{\langle f, f\rangle_{W_{i+1}\cap\delta\Omega}} \\ &\geq \min_{\substack{H' \subset \mathbb{R}^{W_{i+1} \cap \delta\Omega}\\ \dim H' = k}} \max_{0 \neq f \in H'} \frac{\mathrm{Cap}(f, W_{i+1})}{\langle f, f\rangle_{W_{i+1}\cap\delta\Omega}} = \sigma_k(W_{i+1}). \qquad \blacksquare\end{aligned}$$

Now, we are ready to prove the approximation result in Proposition 1.5.

Proof of Proposition 1.5. For any $i \in \mathbb{N}$, by (3.8), choose $H \subset \mathbb{R}^{W_i \cap \delta\Omega}$, $\dim H = k$, such that

$$\sigma_k(W_i) = \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f, W_i)}{\langle f, f \rangle_{W_i \cap \delta\Omega}}.$$

Then by the monotonicity of $\mathrm{Cap}(f, W_i)$ and the definition of $\mathrm{Cap}(f)$,

$$\begin{aligned}
\sigma_k(W_i) &= \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f, W_i)}{\langle f, f \rangle_{W_i \cap \delta\Omega}} \geq \max_{0 \neq f \in H} \frac{\mathrm{Cap}(f)}{\langle f, f \rangle_{W_i \cap \delta\Omega}} \\
&= \max_{0 \neq f \in H} \frac{D_\Omega(u_f)}{\langle f, f \rangle_{\delta\Omega}} \geq \inf_{\substack{H' \subset \ell_0(\delta\Omega) \\ \dim H' = k}} \sup_{0 \neq f \in H'} \frac{D_\Omega(u_f)}{\langle f, f \rangle_{\delta\Omega}} \\
&= \sigma_k(\Omega),
\end{aligned}$$

The second last equality follows from Proposition 3.3. Hence,

$$\lim_{i \to \infty} \sigma_k(W_i) \geq \sigma_k(\Omega).$$

On the other hand, by Corollary 3.5, for any $\varepsilon > 0$, there exists $H \subset \ell_0(\delta\Omega)$, $\dim H = k$, such that

$$\sigma_k(\Omega) \leq \sup_{0 \neq f \in H} \frac{D_\Omega(u_f)}{\langle f, f \rangle_{\delta\Omega}} = \max_{0 \neq f \in H} \frac{D_\Omega(u_f)}{\langle f, f \rangle_{\delta\Omega}} < \sigma_k(\Omega) + \varepsilon.$$

Since H is finite-dimensional, let $\{h_l\}_{l=1}^k$ be an orthonormal basis of H. We obtain that there exists $K \in \mathbb{N}$, such that

$$\mathrm{supp}(g) \subset W_i \cap \delta\Omega, \quad \text{for all } i > K,\ g \in H.$$

By (3.8), for sufficiently large i, there exists $f_i = \sum_{l=1}^k a_l^i h_l$ satisfying $\sum_l (a_l^i)^2 = 1$ such that

$$\sigma_k(W_i) \leq \max_{0 \neq f \in H} \frac{D_{W_i}(u_f^{W_i})}{\langle f, f \rangle_{\delta\Omega}} = \frac{D_{W_i}(u_{f_i}^{W_i})}{\langle f_i, f_i \rangle_{\delta\Omega}} = \frac{\sum_{1 \leq l, s \leq k} a_l^i a_s^i D_{W_i}(u_{h_l}^{W_i}, u_{h_s}^{W_i})}{\sum_{1 \leq l, s \leq k} a_l^i a_s^i \langle h_l, h_s \rangle_{\delta\Omega}}.$$

By Corollary 3.4, using polarization, we get that for any $1 \leq l, s \leq k$

$$\lim_{i \to \infty} D_{W_i}(u_{h_l}^{W_i}, u_{h_s}^{W_i}) = D_\Omega(u_{h_l}, u_{h_s}).$$

Moreover, by passing to a subsequence, still denoted by a_l^i,

$$\lim_{i \to \infty} a_l^i = b_l, \quad \text{for all } 1 \leq l \leq k,$$

where $\sum_l (b_l)^2 = 1$. Set $h_\infty = \sum_l b_l h_l$. Hence, by $h_\infty \in H, h_\infty \neq 0$,

$$\lim_{i \to \infty} \sigma_k(W_i) \leq \frac{D_\Omega(u_{h_\infty})}{\langle h_\infty, h_\infty \rangle_{\delta\Omega}} < \sigma_k(\Omega) + \varepsilon.$$

Letting $\varepsilon \to 0$, we have

$$\lim_{i\to\infty} \sigma_k(W_i) \le \sigma_k(\Omega).$$

Hence, the proposition follows. ∎

Now, we are ready to prove Corollary 1.6.

Proof of Corollary 1.6. By Proposition 1.5 and (2.5), it suffices to prove that for any $\mathcal{W} \uparrow \bar{\Omega}$,

$$\sigma_k(W) \ge \lambda_{k,D}(W),$$

where $\lambda_{k,D}(W)$ is the k-th eigenvalue of the Dirichlet problem (2.4). Given any linear subset $H \subset \mathbb{R}^{W\cap\delta\Omega}$, we denote

$$\tilde{H} := \operatorname{span}\{u_f^W : f \in H\}.$$

By definition,

$$\begin{aligned}
\sigma_k(W) &= \min_{\substack{H\subset\mathbb{R}^{W\cap\delta\Omega}\\ \dim H=k}} \max_{0\neq f\in H} \frac{\sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}(u_f^W(x)-u_f^W(y))^2}{\sum_{x\in W\cap\delta\Omega} f^2(x)d(x)}\\
&= \min_{\substack{H\subset\mathbb{R}^{W\cap\delta\Omega}\\ \dim H=k}} \max_{0\neq g\in\tilde{H}} \frac{\sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}(g(x)-g(y))^2}{\sum_{x\in W\cap\delta\Omega} g^2(x)d(x)}\\
&\ge \min_{\substack{H'\subset\mathbb{R}^{W}\\ \dim(H')=k}} \max_{0\neq f\in H'} \frac{\sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}(f(x)-f(y))^2}{\sum_{x\in W} f^2(x)d(x)} = \lambda_{k,D}(W).
\end{aligned}$$

This proves the corollary. ∎

Remark 3.8. Let Ω be a finite subset of V. By Corollary 1.6, $\sigma_2(\Omega) \ge \lambda_2(\bar{\Omega})$. Note that for any finite graph with the normalized Laplacian, all eigenvalues are bounded by 2. In particular, for the finite graph $\bar{\Omega}$, $\lambda_2(\bar{\Omega}) \le 2$. This implies that

$$\sigma_2(\Omega) \ge \lambda_2(\bar{\Omega}) \ge \frac{1}{2}\lambda_2^2(\bar{\Omega}),$$

which is stronger than [26, Corollary 1.1].

4. Jammes-type Cheeger estimate for the bottom spectrum

Let (V, E, μ) be an infinite graph, $\Omega \subset V$ be an infinite subgraph and $W \subset \bar{\Omega}$ be a finite subset with $W \cap \delta\Omega \neq \emptyset$. Let $0 \neq f \in \mathbb{R}^{W\cap\delta\Omega}$ be the first eigenfunction associated to the first eigenvalue $\sigma_1(W)$. For convenience, we write f for u_f^W in the following. Without loss of generality, we may assume that f is non-negative, since $D_W(|f|) \leq D_W(f)$. By (3.2), we have

$$\sigma_1(W) = \frac{\sum_{e=\{x,y\}\in E(W,\overline{W})} \mu_{xy}(f(y)-f(x))^2}{\sum_{x\in W\cap\delta\Omega} f^2(x)d(x)}. \tag{4.1}$$

Multiplying both the numerator and denominator of the fraction at the right-hand side of (4.1) by $\sum_{x\in W} f^2(x)d(x)$ and setting

$$\frac{M}{N} := \frac{\sum_{x\in W} f^2(x)d(x)\cdot\sum_{e=\{x,y\}\in E(W,\overline{W})} \mu_{xy}(f(y)-f(x))^2}{\sum_{x\in W} f^2(x)d(x)\cdot\sum_{x\in W\cap\delta\Omega} f^2(x)d(x)},$$

we have

$$\sigma_1(W) = \frac{M}{N}.$$

We need the following lemmas to prove Theorem 1.4; see [26] for the proofs.

Lemma 4.1 ([26, Lemma 5.2]).

$$M \geq \frac{1}{2}\Big(\sum_{e=\{x,y\}\in E(W,\overline{W})} \mu_{xy}|f^2(x)-f^2(y)|\Big)^2.$$

Set $t_0 := \max_{x\in W}\{f(x)\}$. For any $t > 0$, set

$$S_t := f^{-1}([\sqrt{t},+\infty)) = \{x \in W : f^2(x) \geq t\}.$$

By the maximum principle, the maximizer of f cannot be only achieved in $W \cap \Omega$, hence $S_t \cap \delta\Omega \neq \emptyset$ for any $t \in (0, t_0]$. The following is the discrete co-area formula; see [22, Lemma 3.3].

Lemma 4.2 ([26, Lemma 5.3]).

$$\int_0^\infty \mu(\partial S_t)dt = \sum_{e=\{x,y\}\in E(W,\overline{W})} \mu_{xy}|f^2(x)-f^2(y)|.$$

Lemma 4.3 ([26, Lemma 5.4]).

$$\int_0^\infty d(S_t)dt = \sum_{x\in W} f^2(x)d(x).$$

$$\int_0^\infty d(S_t\cap\delta\Omega)dt = \sum_{x\in W\cap\delta\Omega} f^2(x)d(x).$$

Now, we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. For the upper bound estimate, choose $A \subset W$ that achieves $h_J(W)$, i.e.,

$$h_J(W) = \frac{\mu(\partial A)}{d(A \cap \delta\Omega)}.$$

Set $g = \chi_A \in \mathbb{R}^{\overline{W}}$, i.e.,

$$g(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in \overline{W} \setminus A. \end{cases}$$

Then we have

$$\sigma_1(W) \leq \frac{D_W(u^W_{g|_{W\cap\delta\Omega}})}{\langle g, g\rangle_{W\cap\delta\Omega}} \leq \frac{D_W(g)}{\langle g, g\rangle_{W\cap\delta\Omega}} = \frac{\mu(\partial A)}{d(A \cap \delta\Omega)} = h_J(W),$$

where the second inequality due to the fact that harmonic function minimizes the Dirichlet energy among the functions with fixed boundary condition.

For the lower bound estimate, combining Lemma 4.1, Lemma 4.2 with Lemma 4.3, we have

$$\begin{aligned}\sigma_1(W) &\geq \frac{1}{2}\frac{\int_0^\infty \mu(\partial S_t)dt \cdot \int_0^\infty \mu(\partial S_t)dt}{\sum_{x\in W} f^2(x)d(x) \cdot \sum_{x\in W\cap\delta\Omega} f^2(x)d(x)} \\ &\geq \frac{1}{2}\frac{\int_0^\infty h(W)d(S_t)dt \cdot \int_0^\infty h_J(W)d(S_t\cap\delta\Omega)dt}{\sum_{x\in W} f^2(x)d(x) \cdot \sum_{x\in W\cap\delta\Omega} f^2(x)d(x)} \\ &= \frac{h(W)h_J(W)}{2}.\end{aligned}$$

The theorem follows from the above estimates. ∎

Finally, we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. For the upper bound estimate, by Proposition 1.5 and Theorem 1.4,

$$\sigma_1(\Omega) = \lim_{W\uparrow\bar{\Omega}} \sigma_1(W) \leq \lim_{W\uparrow\bar{\Omega}} h_J(W) = h_J(\Omega).$$

Similarly, we have the lower bound estimate

$$\sigma_1(\Omega) = \lim_{W\uparrow\bar{\Omega}} \sigma_1(W) \geq \lim_{W\uparrow\bar{\Omega}} \frac{h(W)\cdot h_J(W)}{2} = \frac{h(\bar{\Omega})\cdot h_J(\Omega)}{2}.$$

Hence, we complete the proof of the theorem. ∎

At the end of this section, we give a necessary and sufficient condition for the positivity of $\sigma_1(\Omega)$ for an infinite subgraph Ω with finite vertex boundary.

Definition 4.4. Let (V, E, μ) be an infinite graph. For any finite subset $F \subset V$, we define

$$\mathrm{Cap}_F(V) := \inf_{\phi \in \ell_0(V), \phi|_F = 1} D_V(\phi).$$

In order to obtain the sufficient condition for the positivity of $\sigma_1(\Omega)$, we need the following criterion for an infinite graph to be recurrent; see, e.g., [38, Theorem 2.12].

Lemma 4.5. *An infinite graph $(\overline{\Omega}, E(\Omega, \overline{\Omega}), \mu)$ is recurrent if and only if we have $\mathrm{Cap}_F(\overline{\Omega}) = 0$ for any finite subset $F \subset \overline{\Omega}$.*

Then we have the following result.

Proposition 4.6. *If Ω is a subgraph of $G = (V, E)$ and $\sharp\delta\Omega < \infty$. Then $\sigma_1(\Omega) = 0$ if and only if $\overline{\Omega} = (\overline{\Omega}, E(\Omega, \overline{\Omega}))$ is recurrent.*

Proof. For the case that Ω is a finite subgraph of G, then the Steklov eigenvalues defined in the introduction reduce to those on finite subgraphs defined in [25, 26]. In this case, $\sigma_1(\Omega) = 0$. Moreover, since $\overline{\Omega}$ is finite, it is recurrent.

Now, we consider the case that Ω is an infinite subgraph of G.

For the "if" part. If $\overline{\Omega}$ is recurrent, then $\mathrm{Cap}_F(\overline{\Omega}) = 0$ for any finite subset $F \subset \overline{\Omega}$. Choosing $F = \delta\Omega$, we have

$$0 = \mathrm{Cap}_{\delta\Omega}(\overline{\Omega}) = \mathrm{Cap}(\chi_{\delta\Omega}).$$

Hence, by Corollary 3.5, $\sigma_1(\Omega) = 0$.

For the "only if" part. If $\sigma_1(\Omega) = 0$, then there exist $\{f_i\}_{i=0}^{\infty} \subset \mathbb{R}^{\delta\Omega}$, $\|f_i\|_{\ell^2(\delta\Omega)} = 1$, such that

$$D_\Omega(u_{f_i}) \to 0, \quad i \to \infty.$$

Hence, there exist a subsequence $\{f_{i_j}\}_{j=1}^{\infty}$ and f_∞ such that

$$f_{i_j} \to f_\infty, \quad j \to \infty$$

and

$$\|f_\infty\|_{\ell^2(\delta\Omega)} = 1.$$

Moreover, by the exhaustion method and the maximum principle on finite set, one can show that

$$u_{f_{i_j}} \to u_{f_\infty}, \ j \to \infty.$$

Then by the lower semi-continuity,

$$D_\Omega(u_{f_\infty}) \le \lim_{j \to \infty} D_\Omega(u_{f_{i_j}}) = 0.$$

Hence, $u_{f_\infty} = \text{const}$. This implies that $\mathrm{Cap}_{\delta\Omega}(\overline{\Omega}) = 0$ and $\overline{\Omega}$ is recurrent. ∎

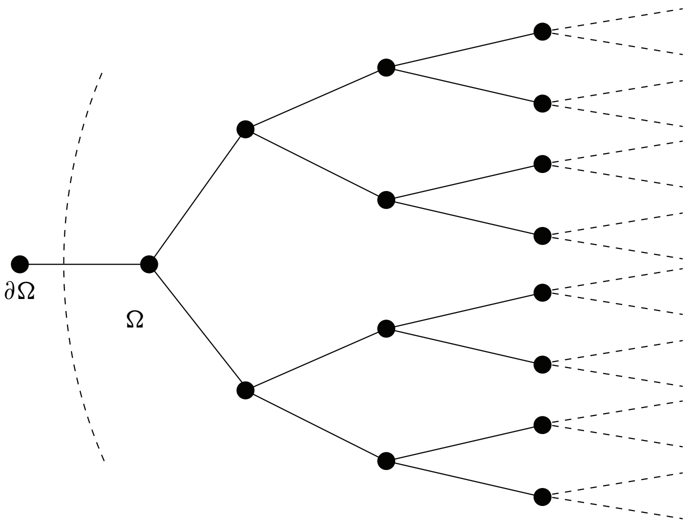

Figure 1

We give an example with positive bottom spectrum for the DtN operator.

Example 4.7. Consider the graph in Figure 1 with unit edge weight. Let Ω be a part of the homogeneous tree with degree three and $\sharp\delta\Omega = 1$. By calculation, $h_J(\Omega) = 1$, $h(\Omega) = \frac{1}{3}$, and $\sigma_1(\Omega) = \frac{1}{2}$. See the related discussions in Remark 1.8.

Remark 4.8. We consider the case that $\bar{\Omega}$ is not connected. Suppose that $\sharp\delta\Omega < \infty$. By the proof of Proposition 4.6, $\sigma_1 = \cdots = \sigma_k = 0 < \sigma_{k+1}$ if and only if the number of the recurrent connected components $\bar{\Omega} = (\bar{\Omega}, E(\Omega, \bar{\Omega}))$ containing $\delta\Omega$ is k. Note that finite connected components are always recurrent.

5. Higher order Cheeger estimates for the Dirichlet eigenvalue problems

Higher order Cheeger estimates for the Laplace operator on finite graphs (without boundary condition) have been proved in [29]. In this section, we prove higher order Cheeger estimates for the eigenvalues of the Dirichlet Laplacian problem on finite graphs with boundary. We remark that the steps of the proof in this section are the same as in the case of finite graphs without boundary.

Let V be a countable set of vertices, and

$$\mu\colon V \times V \to [0,\infty), \quad \{x,y\} \mapsto \mu_{xy} = \mu_{yx}$$

be a symmetric weight function. Instead of the vertex measure $d(\cdot)$ defined in the introduction, we introduce a general measure $\nu(\cdot)$ on V,

$$\begin{aligned}\nu\colon V &\to (0,\infty),\\ x &\mapsto \nu(x).\end{aligned}$$

This induces a (general) weighted graph structure $G=(V,\mu,\nu)$. For any $f\in l^2(V,\nu)$, the Laplacian on (V,μ,ν) is defined as

$$\Delta_\nu^\mu(f)(x) := \frac{1}{\nu(x)}\sum_{y\in V}\mu_{xy}(f(y)-f(x)),\quad \text{for all } x\in V.$$

Let $W\subset V$ be a finite subset. Consider the following Dirichlet eigenvalue problem

$$\begin{cases}\Delta_\nu^\mu(f)(x) = -\lambda f(x), & x\in W,\\ f(x)=0, & x\in\delta W.\end{cases}\tag{5.1}$$

We denote by $\lambda_{k,D}^\nu(W)$ the k-th eigenvalue of the above Dirichlet problem. For any subset $A\subset W$, the associated first Dirichlet eigenvalue can be characterized as

$$\lambda_{1,D}^\nu(A) := \inf\Big\{\frac{D_A(f)}{\langle f,f\rangle_A} : 0\neq f\in\mathbb{R}^{\overline{W}} \text{ and } f(x)=0, \text{ for all } x\in\overline{W}\setminus A\Big\},$$

where $\langle\cdot,\cdot\rangle_A$ is the inner product with respect to the measure $\nu(\cdot)$.

Set $N := \sharp W$. For any $k\in[N]$, define

$$\Gamma_k(W) := \min_{(A_1,\dots,A_k)\in\mathcal{A}_k(W)}\max_{l\in[k]}\lambda_{1,D}^\nu(A_l).\tag{5.2}$$

Let $f_1, f_2,\dots,f_k$ be the first k orthonormal eigenfunctions of the Dirichlet eigenvalue problem (5.1). Consider the following mapping

$$\begin{aligned}F\colon W &\to \mathbb{R}^k,\\ x &\mapsto (f_1(x), f_2(x),\dots,f_k(x)).\end{aligned}\tag{5.3}$$

We denote by $\|\cdot\|$ the Euclidean norm of vectors in $\mathbb{R}^k$. By the Rayleigh quotient characterization of $\lambda_{k,D}^\nu(W)$,

$$\begin{aligned}&\frac{\sum_{e=\{x,y\}\in E(W,\overline{W})}\mu_{xy}\|F(x)-F(y)\|^2}{\sum_{x\in W}\nu(x)\|F(x)\|^2}\\ &= \frac{\sum_{\ell=1}^k\sum_{e=\{x,y\}\in E(W,\overline{W})}\mu_{xy}(f_\ell(x)-f_\ell(y))^2}{\sum_{\ell=1}^k\sum_{x\in W}\nu(x)f_\ell^2(x)}\\ &= \frac{\sum_{\ell=1}^k\lambda_{\ell,D}^\nu(W)\sum_{x\in W}\nu(x)f_\ell^2(x)}{\sum_{\ell=1}^k\sum_{x\in W}\nu(x)f_\ell^2(x)}\le\lambda_{k,D}^\nu(W),\end{aligned}$$

where the second equality follows from the definition of $\lambda^{\nu}_{\ell,D}(W)$ and the inequality follows from the monotonicity of $\{\lambda^{\nu}_{\ell,D}\}_{\ell=1}^{k}$ in ℓ. We denote by $\widetilde{S}_F$ the support set of F, i.e.,

$$\widetilde{S}_F := \{x \in W : F(x) \neq 0\}.$$

Consider the map induced by F

$$\widetilde{F} : \widetilde{S}_F \mapsto \mathbb{S}^{k-1}, \quad x \mapsto \frac{F(x)}{\|F(x)\|}.$$

We define a pseudo-metric $d_{\widetilde{F}}$ on W. For any $x, y \in \widetilde{S}_F$, it is defined as

$$d_{\widetilde{F}}(x, y) := \left\| \frac{F(x)}{\|F(x)\|} - \frac{F(y)}{\|F(y)\|} \right\|.$$

For $x, y \in W$ with $F(x) = F(y) = 0$, set $d_{\widetilde{F}}(x, y) = 0$. Put $d_{\widetilde{F}}(x, y) = \infty$, otherwise.

The goal is to "localize" F on k disjoint subsets T_i to produce functions $\Psi_i : V \to \mathbb{R}^k$ with disjoint support, each with small Rayleigh quotient. For that purpose, we need that $F|_{T_i}$ captures a large fraction of the ℓ^2 mass of F. Then by the variational principle, the Rayleigh quotient of Ψ_i bounds the first Dirichlet eigenvalue $\lambda^{\nu}_{1,D}(\operatorname{supp} \Psi_i)$ and the result follows.

5.1. Spreading lemma and localization lemma

In order to find k disjoint subsets T_i, each with large ℓ^2 mass of F, one needs that the ℓ^2 mass of F is sufficiently well-spread. This follows from the so-called *spreading lemma*. Let F be the map defined in (5.3).

Definition 5.1. For any $r > 0$, $\delta > 0$, F is called *(r, δ)-spreading,* if for any subset $S \subset W$ with $\operatorname{diam}(S \cap \widetilde{S}_F, d_{\widetilde{F}}) \leq r$, one has

$$\sum_{x \in S} \nu(x) \|F(x)\|^2 \leq \delta \sum_{x \in W} \nu(x) \|F(x)\|^2.$$

Following [29, Lemma 3.2], one can prove the following spreading lemma.

Lemma 5.2. *If $0 < r < 1$ and $S \subset W$ is a subset satisfying $\operatorname{diam}(S \cap \widetilde{S}_F, d_{\widetilde{F}}) \leq r$, then we have*

$$\sum_{x \in S} \nu(x) \|F(x)\|^2 \leq \frac{1}{k(1 - r^2)} \sum_{x \in W} \nu(x) \|F(x)\|^2.$$

Remark 5.3. By Definition 5.1, the map F is $(r, \frac{1}{k(1-r^2)})$-spreading.

For the map F and a subset S, we want to localize F near S, i.e. construct a function supported on a small-neighborhood S, which retains the ℓ^2 mass of F on S, and which has controlled Dirichlet energy. This is done by the so-called *localization lemma* using cut-off functions. The ε-neighborhood of $S \subset \widetilde{S}_F$ with respect to $d_{\widetilde{F}}$ is defined as

$$N_\varepsilon(S, d_{\widetilde{F}}) := \{x \in W : d_{\widetilde{F}}(x, S) < \varepsilon\}.$$

For any subset $S \subset W$, we define the cut-off function

$$\theta(x) = \begin{cases} 0, & \text{if } F(x) = 0, \\ \max\Big\{0, 1 - \dfrac{d_{\widetilde{F}}(x, S \cap \widetilde{S}_F)}{\varepsilon}\Big\}, & \text{otherwise.} \end{cases} \tag{5.4}$$

The so-called *localization of F on the subset S* is defined as

$$\Psi := \theta \cdot F : W \mapsto \mathbb{R}^k. \tag{5.5}$$

It is obvious that $\Psi|_S = F|_S$ and $\operatorname{supp}(\Psi) \subset N_\varepsilon(S \cap \widetilde{S}_F, d_{\widetilde{F}})$. One can prove the following localization lemma; see [29, Lemma 3.3].

Lemma 5.4. *For $0 < \varepsilon < 2$, let Ψ be the localization defined in (5.5). Then for any $e = \{x, y\} \in E(W, \overline{W})$, we have*

$$\|\Psi(x) - \Psi(y)\| \le \Big(1 + \frac{2}{\varepsilon}\Big)\|F(x) - F(y)\|. \tag{5.6}$$

5.2. Some results on random partitions

The goal of the section is to partition $(\widetilde{S}_F, d_{\widetilde{F}})$ into well-separated subsets T_i, which retains a large fraction of the ℓ^2 mass of F. This is done by the random partition technique. Random partition theory was firstly developed in theoretical computer science and has many important applications in pure mathematics; see [23, 29–31].

Let (X, d) be a metric space. For any $x \in X$, $r > 0$, we denote by $B(x, r) := \{y \in X : d(y, x) \le r\}$ the ball of radius r centered at x. The metric doubling constant ρ_X of (X, d) is defined as

$$\begin{aligned}\rho_X := \inf\Big\{c \in \mathbb{N} : &\text{ for all } x \in X, r > 0, \\ &\text{there exist } x_1, \dots, x_c \in X, \text{ such that } B(x, r) \subset \bigcup_{i=1}^{c} B\Big(x_i, \frac{r}{2}\Big)\Big\}.\end{aligned}$$

The metric doubling dimension of (X, d) is defined as

$$\dim_d(X) := \log_2 \rho_X.$$

A Borel measure μ on (X, d) is called a *doubling measure* if there exists a finite number C_μ such that for any $x \in X, r > 0$,

$$0 < \mu(B(x,r)) \leq C_\mu \mu\Big(B\Big(x, \frac{r}{2}\Big)\Big) < +\infty.$$

Similarly, the measure doubling dimension is defined as

$$\dim_\mu(X) := \log_2(C_\mu).$$

The two doubling dimensions are related by the following lemma; see [8, p. 67].

Lemma 5.5. *If a metric space (X, d) has a doubling measure μ, then*

$$\dim_d(X) \leq 4 \dim_\mu(X).$$

One can check that $d(x, y) := \|x - y\|$ is a metric on $\mathbb{S}^{k-1}$ and we have the following property.

Proposition 5.6 ([30, Corollary 3.12]). *For the metric space $(\mathbb{S}^{k-1}, d)$, we have*

- $\operatorname{diam}(\mathbb{S}^{k-1}, d) = 2$;
- $\dim_d(\mathbb{S}^{k-1}) \leq 4(k-1)\log_2 \pi$.

A partition of (X, d) is a map $P \colon X \to 2^X$, such that $P(x)$ is the unique set in $\{S_i\}_{i=1}^m$ that contains x, where $S_i \cap S_j = \emptyset$, for all $i \neq j$, and $X = \cup_{i=1}^m S_i$. We denote by $\mathcal{P}(X)$ the collection of partitions of (X, d).

Definition 5.7 (Random partition). Let (X, d) be a finite or countable metric space. Any probability distribution ϖ on $\mathcal{P}(X)$ is called a *random partition of* (X, d).

We denote by $\operatorname{supp}(\varpi) := \{P \in \mathcal{P}(X) \colon \varpi(P) \neq 0\}$ the support set of random partition ϖ. The existence of nice random partition is given in the following theorem; see [23, 30] and [31, Theorem 2.4].

Theorem 5.8. *Let (X, d) be a finite metric subspace of (Y, d). Then for any $r > 0$, $\delta \in (0, 1)$ there exists a random partition ϖ, such that*

- *for any $P \in \operatorname{supp}(\varpi)$, any set S in the partition P, one has $\operatorname{diam}(S) \leq r$;*
- *for any x, one has $\mathbb{P}_\varpi[B(x, \frac{r}{\alpha}) \subset P(x)] \geq 1 - \delta$, where $\alpha = \frac{32 \dim_d(Y)}{\delta}$.*

Remark 5.9. A random partition obtained in the above theorem is called an $(r, \alpha, 1-\delta)$*-padded random partition.*

The following result was proved by [29, Lemma 3.5] and [31, Lemma 6.2]. This yields the well separated subsets T_i, which retains a large fraction of the ℓ^2 mass of F.

Lemma 5.10. *Let $0 < r < 1$, $\alpha > 0$, $k \in \mathbb{N}$. Suppose that F is $(r, \frac{1}{k}(1+\frac{1}{8k}))$-spreading, and there exists an $(r, \alpha, 1-\frac{1}{4k})$-padded random partition on $(\widetilde{S}_F, d_{\widetilde{F}})$, then there exist k non-empty disjoint subsets $T_1, T_2, \ldots, T_k \subset \widetilde{S}_F$ such that*

- $d_{\widetilde{F}}(T_i, T_j) \geq 2\frac{r}{\alpha}$, *for all* $1 \leq i \neq j \leq k$;
- $\sum_{x\in T_i} \nu(x)\|F(x)\|^2 \geq \frac{1}{2k}\sum_{x\in W} \nu(x)\|F(x)\|^2$, *for all* $1 \leq i \leq k$.

5.3. The main result of the section

Combining the previous results, we prove the higher order Cheeger estimates for eigenvalues of the Dirichlet Laplacian on finite graphs with boundary, which is important for our application; see, e.g., [25, Theorem 5].

Theorem 5.11. *Let (V, μ, ν) be a weighted graph and $W \subset V$ be a finite subset. For the Dirichlet eigenvalue problem (5.1) on W, we have*

$$\lambda^{\nu}_{k,D}(W) \geq \frac{c}{k^6}\Gamma_k(W),$$

where $\Gamma_k(W)$ is defined in (5.2).

The proof strategy is as follows. For the map F, by the random partition theorem and the spreading lemma, we have well separated subsets T_i, which retains a large fraction of the ℓ^2 mass of F; see Lemma 5.10. Using cut-off functions on T_i, we obtain localized function Ψ_i supported on the small-neighborhood of T_i with controlled Dirichlet energy; see Lemma 5.4. Then applying the variational principle of $\lambda^{\nu}_{1,D}$ for Ψ_i, we get the desired estimate.

Proof of Theorem 5.11. For the completeness, we give the proof here. Choosing $r = \frac{1}{3\sqrt{k}}$, F is $(r, \frac{1}{k}(1+\frac{1}{8k}))$-spreading by Lemma 5.2. If we further take $\delta = \frac{1}{4k}$, then $\widetilde{S}_F$ has an $(r, \alpha, 1-\frac{1}{4k})$-padded random partition by Theorem 5.8 with

$$\alpha = 128k \dim_d(\mathbb{S}^{k-1}).$$

From Proposition 5.6, we know that $\alpha \leq 128Ck(k-1)$, where $C = 4\log_2 \pi$. Then by Lemma 5.10, we can find k disjoint subsets $T_1, T_2, \ldots, T_k \subset \widetilde{S}_F$, such that

- $d_{\widetilde{F}}(T_i, T_j) \geq 2\frac{r}{\alpha} \geq \frac{2}{3\sqrt{k}}\frac{1}{128Ck(k-1)}$, for all $1 \leq i \neq j \leq k$;
- $\sum_{x\in T_i} \nu(x)\|F(x)\|^2 \geq \frac{1}{2k}\sum_{x\in W} \nu(x)\|F(x)\|^2$, for all $1 \leq i \leq k$.

Let $\{\theta_i\}_{i=1}^k$ be k cut-off functions defined as in (5.4), where S is replaced by T_i and $\varepsilon = \frac{1}{3\sqrt{k}}\frac{1}{128Ck(k-1)}$. Similar to (5.5), we obtain k localizations of F satisfying

$$\begin{aligned} \Psi_i|_{T_i} &= F|_{T_i}, && \text{for all } \leq i \leq k, \\ \operatorname{supp}(\Psi_i) \cap \operatorname{supp}(\Psi_j) &= \emptyset, && \text{for all } 1 \leq i \neq j \leq k. \end{aligned}$$

Applying Lemma 5.4, for any $1 \leq i \leq k$, we have

$$\begin{aligned}
&\frac{\sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}\|\Psi_i(x)-\Psi_i(y)\|^2}{\sum_{x\in \mathrm{supp}(\Psi_i)} \nu(x)\|\Psi_i(x)\|^2} \\
&\leq \frac{(1+\frac{2}{\varepsilon})^2 \sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}\|F(x)-F(y)\|^2}{\frac{1}{2k}\sum_{x\in W} \nu(x)\|F(x)\|^2} \\
&= 2k(1+768C\sqrt{k}k(k-1))^2 \frac{\sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}\|F(x)-F(y)\|^2}{\sum_{x\in W} \nu(x)\|F(x)\|^2} \\
&\leq 2\times (786C)^2 k^6 \lambda^{\nu}_{k,D}(W).
\end{aligned}$$

Write $\Psi_i(x) = (\psi_i^1(x), \psi_i^2(x), \dots, \psi_i^k(x))$. Hence, for any $1 \leq i \leq k$, there exists a coordinate index $a_i \in \{1, 2, \dots, k\}$ such that $\psi_i^{a_i}$ is not identically zero and

$$\frac{\sum_{e=\{x,y\}\in E(W,\bar{W})} \mu_{xy}|\psi_i^{a_i}(x)-\psi_i^{a_i}(y)|^2}{\sum_{x\in W} \nu(x)|\psi_i^{a_i}(x)|^2} \leq ck^6\lambda^{\nu}_{k,D}(W),$$

where $c = 2\times(786C)^2$. Set $A_i := \mathrm{supp}(\psi_i^{a_i})$, for any $1 \leq i \leq k$. Then we have $(A_1, A_2, \dots, A_k) \in \mathcal{A}_k(W)$ and for any $1 \leq i \leq k$,

$$\lambda^{\nu}_{1,D}(A_i) \leq ck^6\lambda^{\nu}_{k,D}(W).$$

Then by the definition of $\Gamma_k(W)$, (5.2), we have

$$\lambda^{\nu}_{k,D}(W) \geq \frac{c}{k^6}\Gamma_k(W). \qquad \blacksquare$$

6. Higher order Cheeger estimate for DtN operators

Let (V, E, μ) be an infinite graph and $\Omega \subset V$ be an infinite subset. For a finite subset $W \subset \bar{\Omega}$ with $W \cap \delta\Omega \neq \emptyset$, let $\sigma_k(W)$ be the k-th eigenvalue of the DtN operator on W.

Following the method proposed in [25], we prove higher order Cheeger estimates for the DtN operators. For any $r > 0$, consider the following measure defined on $\bar{\Omega}$:

$$m_x^{(r)} = \begin{cases} d(x), & x \in \delta\Omega, \\ \frac{1}{r}d(x), & x \in \Omega. \end{cases}$$

For any $f \in \mathbb{R}^{\bar{\Omega}}$, set

$$\Delta^{(r)}(f)(x) := \frac{1}{m_x^{(r)}} \sum_{y\in\bar{\Omega}} \mu_{xy}(f(y)-f(x)), \quad \text{for all } x \in \Omega. \tag{6.1}$$

For $W \subset \bar{\Omega}$, $W \cap \delta\Omega \neq \emptyset$, consider the following Dirichlet problem

$$\begin{cases} -\Delta^{(r)}(f)(x) = \lambda f(x), & x \in W, \\ f(x) = 0, & x \in \delta W. \end{cases} \tag{6.2}$$

We denote by $\lambda_{k,D}^{(r)}(W)$ the k-th eigenvalue of the above Dirichlet problem. Recall that $N = \sharp W$. Set $P := \sharp(W \cap \delta\Omega)$. The following approximation result was proved in [25, Proposition 3].

Proposition 6.1. *For any $k \in [P] := \{1, 2, \ldots, P\}$, we have*

$$\lim_{r\to+\infty} \lambda_{k,D}^{(r)}(W) = \sigma_k(W)$$

and for any $k \in [N] \setminus [P]$,

$$\lim_{r\to+\infty} \lambda_{k,D}^{(r)}(W) = +\infty.$$

Now, we are ready to prove Theorem 1.10.

Proof of Theorem 1.10. For the upper bound estimate, choose $(A_1, \ldots, A_k) \in \mathcal{A}_k(W)$ that achieves $h_J^k(W)$. Consider $H := \mathrm{span}\{\chi_{A_l} : l \in [k]\} \subset \mathbb{R}^W$. Then $\dim H = k$ and $H|_{W\cap\delta\Omega} \subset \mathbb{R}^{W\cap\delta\Omega}$. Note that

$$\frac{D_W(\chi_{A_l})}{\langle \chi_{A_l}, \chi_{A_l}\rangle_{W\cap\delta\Omega}} = \frac{\sum_{e=\{x,y\}\in E(W,\overline{W})} \mu_{xy}(\chi_{A_l}(y) - \chi_{A_l}(x))^2}{d(A_l \cap \delta\Omega)} = \frac{\mu(\partial A_l)}{d(A_l \cap \delta\Omega)}.$$

Hence, by (3.8) and a direct argument

$$\sigma_k(W) \leq 2 \max_{l\in[k]} \frac{D_W(u^W_{\chi_{A_l}})}{\langle \chi_{A_l}, \chi_{A_l}\rangle_{W\cap\delta\Omega}} = 2 \max_{l\in[k]} \frac{\mu(\partial A_l)}{d(A_l \cap \delta\Omega)} = 2h_J^k(W).$$

Next, we prove the lower bound estimate. For any $k \in [P]$,

$$\begin{aligned}
\sigma_k(W) &= \lim_{r\to+\infty} \lambda_{k,D}^{(r)}(W) \geq \lim_{r\to+\infty} \frac{c}{k^6} \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(W)} \max_{l\in[k]} \lambda_{1,D}^{(r)}(A_l) \\
&= \frac{c}{k^6} \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(W)} \max_{l\in[k]} \lim_{r\to+\infty} \lambda_{1,D}^{(r)}(A_l) \\
&= \frac{c}{k^6} \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(W)} \max_{l\in[k]} \sigma_1(A_l) \\
&\geq \frac{c}{k^6} \min_{(A_1,\ldots,A_k)\in\mathcal{A}_k(W)} \max_{l\in[k]} \frac{1}{2} h_J(A_l) h(A_l) \\
&= \frac{c'}{k^6} h_k(W),
\end{aligned}$$

the first inequality follows from Theorem 5.11. ∎

Finally, by Theorem 1.10 and Proposition 1.5, we can prove Theorem 1.11.

Proof of Theorem 1.11. For the upper bound estimate,

$$\sigma_k(\Omega) = \lim_{i\to\infty} \sigma_k(W_i) \le 2 \lim_{i\to\infty} h_J^k(W_i) = 2h_J^k(\Omega).$$

For the lower bound estimate,

$$\sigma_k(\Omega) = \lim_{i\to\infty} \sigma_k(W_i) \ge \lim_{i\to\infty} \frac{c}{k^6} h_k(W_i) = \frac{c}{k^6} h_k(\Omega).$$

This proves the theorem. ■

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