

# On spatial conditioning of the spectrum of discrete Random Schrödinger operators

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**Abstract.** Consider a random Schrödinger-type operator of the form  $H := -H_X + V + \xi$  acting on a general graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $H_X$  is the generator of a Markov process  $X$  on  $\mathcal{G}$ ,  $V$  is a deterministic potential with sufficient growth (so that  $H$  has a purely discrete spectrum), and  $\xi$  is a random noise with at-most-exponential tails. We prove that the eigenvalue point process of  $H$  is number rigid in the sense of Ghosh and Peres (2017); that is, the number of eigenvalues in any bounded domain  $B \subset \mathbb{C}$  is determined by the configuration of eigenvalues outside of  $B$ . Our general setting allows to treat cases where  $X$  could be non-symmetric (hence  $H$  is non-self-adjoint) and  $\xi$  has long-range dependence. Our strategy of proof consists of controlling the variance of the trace of the semigroup  $e^{-tH}$  using the Feynman–Kac formula.

## 1. Introduction

Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a countably infinite connected graph with uniformly bounded degrees and a distinguished vertex  $0 \in \mathcal{V}$ , which we call the *root*. For example,  $\mathcal{G}$  could be the integer lattice  $\mathbb{Z}^d$ , any semiregular tessellation/honeycomb of  $\mathbb{R}^d$  that includes the origin, or a much more general graph.

In this paper, we are interested in the spectral theory of random Schrödinger-type operators of the form

$$Hf(v) = -H_X f(v) + (V(v) + \xi(v))f(v), \quad v \in \mathcal{V}, f: \mathcal{V} \rightarrow \mathbb{R},$$

where we assume that

1.  $H_X$  is the infinitesimal generator of some continuous-time Markov process  $X$  on  $\mathcal{G}$  (which need not be symmetric);
2.  $\xi: \mathcal{V} \rightarrow \mathbb{R}$  is a random noise (which may have long-range dependence);

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*2020 Mathematics Subject Classification.* Primary 60G55; Secondary 47D08, 82B44.

*Keywords.* Random Schrödinger operators, Feynman–Kac formulas, number rigidity, eigenvalue point process, Markov processes.

3.  $V: \mathcal{V} \rightarrow \mathbb{R} \cup \{\infty\}$  is a deterministic potential with sufficient growth at infinity (as measured by the size of  $V(v)$  as  $v$  grows farther away from the root), ensuring that  $H$  has a purely discrete spectrum.

More specifically, we are interested in studying the *spatial conditioning* of the spectrum of  $H$ , i.e., understanding the random configuration of the eigenvalues of  $H$  in some domain  $B \subset \mathbb{C}$  conditional on the configuration of eigenvalues outside of  $B$ . As a first step in this direction, we establish that under general assumptions on  $H_X$ ,  $\xi$ , and  $V$ , the spectrum of  $H$  is *number rigid* in the sense of Ghosh and Peres [25]; that is, the number of eigenvalues of  $H$  in bounded domains  $B \subset \mathbb{C}$  is a measurable function of the configuration of the eigenvalues of  $H$  outside  $B$  (we point to Definition 3.3 for a precise definition). To the best of our knowledge, ours is the first work to study the occurrence of such a phenomenon in the spectrum of random Schrödinger operators acting on discrete spaces.

The spectral theory of differential operators (including non-self-adjoint operators; e.g., [1, 6, 12–14, 18, 27, 32, 35]) is among the most prominent research programs in mathematical physics; see, for instance, [26, 42]. In particular, starting from the pioneering work of Anderson [4], the study of Schrödinger operators perturbed by irregular noise has attracted a lot of attention; we refer to [3, 11] for general introductions to the subject. A particularly active program in this direction is the work on *Anderson localization*, which concerns the appearance of pure point spectrum and eigenfunction decay; see the survey articles [28, 31, 41] for more details.

In contrast to localization and similar questions, in this paper we investigate the *transport of spectral information* from one region to another, whereby observing the configuration of the eigenvalues of  $H$  in some domain  $D \subset \mathbb{C}$  allows to recover nontrivial information about the spectrum in the complement of  $D$ . Such questions of spatial conditioning in general point processes have long been of interest due to their natural applications in mathematics and physics; see, e.g., [5, 29]. In recent years, there has been a renewed interest in such investigations coming from the seminal work of Ghosh and Peres [25] on *rigidity and tolerance*, culminating in a now active field of research (e.g., [7–10, 20–23, 37]; see also [2]). In [34], we studied the occurrence of number rigidity in the spectrum of a class of random Schrödinger operators on one-dimensional continuous space. In this paper, we study a similar problem for discrete random Schrödinger operators.

## 1.1. Organization

In the remainder of this introduction, we provide an outline of our main results and proof strategy, we compare the results in this paper to previous investigations in a similar vein, and we discuss a few natural open questions raised by our work.

In Section 2, we provide a high-level outline of the proof of our main results. We take this opportunity to explain how our technical assumptions arise from our computations. In Section 3, we state our assumptions and main results in full details, namely, Assumptions 3.8 and 3.12 and Theorems 3.16, 3.17, and 3.18. Then, we prove Theorem 3.16 in Section 4, we prove Theorem 3.17 in Sections 5 and 6, and we prove Theorem 3.18 in Section 7.

**1.2. Outline of main results**

Let  $d$  denote the graph distance on  $\mathcal{G}$ . For every  $v \in \mathcal{V}$ , we use  $c_n(v)$ ,  $n \geq 0$ , to denote the coordination sequence of  $v$  in  $\mathcal{G}$ ; that is, for every  $n \in \mathbb{N}$ ,  $c_n(v)$  is the number of vertices  $u \in \mathcal{V}$  such that  $d(u, v) = n$ . Stated informally, our main result is as follows:

**Theorem 1.1** (informal statement). *Suppose that there exists  $d \geq 1$  such that*

$$\sup_{v \in \mathcal{V}} c_n(v) = O(n^{d-1}) \quad \text{as } n \rightarrow \infty. \tag{1.1}$$

*Under mild technical assumptions on the Markov process  $X$  and the noise  $\xi$ , there exists a constant  $d/2 \leq \alpha \leq d$  (which, apart from  $d$ , depends on the the range of the covariance in  $\xi$ ) such that if  $V(v)$  grows faster than  $d(0, v)^\alpha$  as  $d(0, v) \rightarrow \infty$ , then the eigenvalue point process of  $H$  is number rigid.*

See Theorems 3.16 and 3.17 for a formal statement. Our technical assumptions are stated in Assumptions 3.8 and 3.12; roughly speaking, our assumptions are that

1. the jump rates of  $X$  (which may be site-dependent) are uniformly bounded;
2. the tails of  $\xi$  are not worse than exponential.

In particular, our assumptions allow for  $X$  to be non-symmetric (hence, the operator  $H$  need not be self-adjoint) and for  $\xi$  to have a variety of covariance structures, including long-range dependence.

**Remark 1.2.** The constant  $d$  in (1.1), which quantifies the growth rate of the number of vertices, can be thought of as the *dimension* of  $\mathcal{G}$  (or, at least, an upper bound of the dimension). To illustrate this, if  $\mathcal{G}$  is for example  $\mathbb{Z}^d$  or a semiregular tessellation of  $\mathbb{R}^d$ , then it is easy to see that  $cn^{d-1} \leq c_n(v) \leq Cn^{d-1}$  for some  $C, c > 0$ . More generally, the constant  $d$  is closely related to the *intrinsic dimension* of  $\mathcal{G}$ , which is the minimal number  $k$  such that  $\mathcal{G}$  can be embedded in  $\mathbb{Z}^k$ . We refer to, e.g., [33, 36] for more details.

**Remark 1.3.** In Theorem 3.18, we provide concrete examples showing that the growth lower bound of  $d(0, v)^\alpha$  that we impose on  $V$  to get rigidity is the best general sufficient condition that can be obtained with our proof method. The question of whether or not this is actually necessary for rigidity is addressed in Section 1.4.1.

### 1.3. Proof strategy and previous results

Our method to prove number rigidity follows the general scheme introduced by Ghosh and Peres in [25]: Let  $\mathcal{X} = \sum_{k \in \mathbb{N}} \delta_{\lambda_k}$  be a point process on  $\mathbb{C}$ . As per [25, Theorem 6.1], for any bounded set  $B \subset \mathbb{C}$ , if there exists a sequence of functions  $(f_n)_{n \in \mathbb{N}}$  such that, as  $n \rightarrow \infty$ ,

1.  $f_n \rightarrow 1$  uniformly on  $B$ ,
2. the variance of the linear statistics  $\int f_n d\mathcal{X} = \sum_{k \in \mathbb{N}} f_n(\lambda_k)$  vanish,

then  $\mathcal{X}(B)$  is measurable with respect to the configuration of  $\mathcal{X}$  outside of  $B$ .

One of the main difficulties involved with carrying out the above program lies in the computation of upper bounds for the variances of linear statistics  $\text{Var}[\int f d\mathcal{X}]$ . For this reason, much of the previous literature on number rigidity exploits special properties that make the computations more manageable, such as determinantal/Pfaffian or other integrable structure [7, 10, 20, 24, 25], translation invariance and hyperuniformity [19, 23], and finite-dimensional approximations [39].

Among those works, the only result that is related to the spectrum of random Schrödinger operators is the proof of rigidity of the *Airy-2* point process in [7]. Thanks to the work of Edelman, Ramírez, Rider, Sutton, and Virág [16, 38], this implies that the spectrum of the *stochastic Airy operator* with parameter  $\beta = 2$  is number rigid. Given that the method of proof in [7] relies crucially on special algebraic structure only present in that one particular case, however, the result cannot be extended to general Schrödinger operators.

More recently, in [34] we proposed to study number rigidity in the spectrum of random Schrödinger operators using a new *semigroup method*: given that the exponential functions  $e_n(z) := e^{-z/n}$  converge uniformly to 1 on any bounded set as  $n \rightarrow \infty$ , in order to prove number rigidity of any point process, it suffices to prove that  $\text{Var}[\int e_n d\mathcal{X}] \rightarrow 0$  (though the requirement that  $\int e_n d\mathcal{X}$  is finite imposes strong conditions on  $\mathcal{X}$ ). If  $\mathcal{X}$  happens to be the eigenvalue point process of a random Schrödinger operator  $H$ , then  $\int e_n d\mathcal{X}$  is the trace of the operator  $e^{-H/n}$ . Thus, in order to prove the number rigidity of the spectrum of any random Schrödinger operator  $H$ , it suffices to prove that

$$\lim_{t \rightarrow 0} \text{Var}[\text{Tr}[e^{-tH}]] = 0.$$

The reason why this is a particularly attractive strategy to prove number rigidity of general random Schrödinger operators is that, thanks to the Feynman–Kac formula, there exists an explicit probabilistic representation of the semigroup  $(e^{-tH})_{t>0}$  in terms of elementary stochastic processes, making the variance  $\text{Var}[\text{Tr}[e^{-tH}]]$  amenable to computation.

In [34], this strategy was used to prove number rigidity for a class of random Schrödinger operators acting on one-dimensional continuous space (i.e., an interval of the form  $I = (a, b)$  with  $-\infty \leq a < b \leq \infty$ ). In this paper, we apply the same methodology to prove number rigidity for a general class of discrete random Schrödinger operators.

Despite the fact that the general strategy of proof used in the present paper is the same as [34], the differences between the two settings are such that virtually none of the work carried out in [34] can be directly extended to the present paper. For example:

1. Since we consider operators acting on general graphs  $\mathcal{G}$ , the treatment of the geometry of the space on which our operators are defined requires a much more careful analysis than that carried out in [34]. In particular (as per Remark 1.2), in this paper we uncover that the dimension of the space plays an important role in the proof of rigidity using the semigroup method.

2. In [34], we only consider Schrödinger operators whose kinetic energy operator is the standard Laplacian and whose noise is a Gaussian process. As a result, the operators considered therein are all self-adjoint and upper bounds of  $\text{Var}[\text{Tr}[e^{-tH}]]$  can mostly be reduced to the analysis of self-intersection local times of standard Brownian motion. In contrast, in this paper we allow for much more general generators  $H_X$  and noises  $\xi$ . Most notably, the assumptions of this paper allow for non-self-adjoint operators, which increases the technical difficulties involved (e.g., Sections 5 and 6).

#### 1.4. Future directions

Given that our main theorems apply to a very general class of operators, the results of this paper provide substantial evidence of the universality of number rigidity in discrete random Schrödinger operators. That being said, we feel that our results raise a number of interesting follow-up questions. We now discuss three such directions.

**1.4.1. New methods.** It is natural to wonder if the growth condition  $V(v) \gg d(0, v)^\alpha$  that we impose on the potential to get number rigidity is close to optimal. As we show in Theorem 3.18, our main result is optimal in the sense that we can find concrete examples of operators such that

$$\liminf_{t \rightarrow 0} \text{Var}[\text{Tr}[e^{-tH}]] > 0 \tag{1.2}$$

when  $V(v) \asymp d(0, v)^\alpha$ . That being said, the vanishing of the variance of the trace of the semigroup is only a sufficient condition for number rigidity, and, in fact, it was observed in [34, Proposition 2.27] that there exists at least one random Schrödinger operator whose spectrum is known to be number rigid and such that (1.2) holds. For example, the following simple question appears to be outside the scope of the methods used in this paper:

**Problem 1.4.** Suppose that  $X$  is the simple symmetric random walk on  $\mathcal{G} = \mathbb{Z}^d$ , that  $V(v) = d(0, v)^\delta$  for some  $\delta > 0$ , and that  $(\xi(v))_{v \in \mathbb{Z}^d}$  are i.i.d. standard Gaussians (or any other simple distribution). Is the spectrum of  $H$  always number rigid in this case?

More specifically, given that  $c_n(v) \asymp n^{d-1}$  on  $\mathbb{Z}^d$ , our main theorem only implies number rigidity in the above when  $\delta > d/2$ . We expect that solving Problem 1.4 will require developing new methods to study number rigidity in random Schrödinger operators.

**1.4.2. The mechanism of rigidity.** Our main result implies that for every bounded measurable set  $B \subset \mathbb{C}$ , there exists a deterministic function  $\mathcal{N}_B$  such that the identity

$$\begin{aligned} &\text{number of eigenvalues of } H \text{ in } B \\ &= \mathcal{N}_B(\text{configuration of eigenvalues of } H \text{ outside } B) \end{aligned}$$

holds with probability one. That being said, the argument that we use to prove the existence of  $\mathcal{N}_B$  gives little information on its exact form. In other words, the precise nature of the mechanism that makes the number of eigenvalues in  $B$  a deterministic function of the configuration on the outside remains largely unknown. In light of this, an interesting future direction for investigation would be along the following lines:

**Problem 1.5.** Let  $B \subset \mathbb{C}$  be a “simple” bounded subset of the complex plane (e.g., a closed or open ball). Does  $\mathcal{N}_B$  admit an explicit representation?

We point to Remark 6.4 for more details on the construction of  $\mathcal{N}_B$ .

**1.4.3. Spatial conditioning beyond number rigidity.** When the spectrum of  $H$  is number rigid, we know that if we condition  $H$  on having a specific eigenvalue configuration outside of a bounded set  $B$ , then the spectrum of  $H$  inside  $B$  is a point process with a fixed total number of points. It would be interesting to see if more can be learned about the conditional distribution of the eigenvalues in  $B$ . For instance, the following problem (related to the notion of tolerance introduced in [25]) might be a good starting point:

**Problem 1.6.** Suppose that, after conditioning on the outside configuration,  $H$  has  $M \in \mathbb{N}$  random eigenvalues in some bounded set  $B \subset \mathbb{C}$ . Let  $\Lambda \in \mathbb{C}^M$  be the random vector whose components are the random eigenvalues of  $H$  in  $B$  (conditional on the configuration outside  $B$ ), taken in a uniformly random order. What is the support of the probability distribution of  $\Lambda$  on the set  $B^M$ ?

## 2. Proof outline

In this section, we present a sketch of the proof of our main theorem in two simple special cases. We take this opportunity to explain how our technical assumptions arise in our computations. For simplicity of exposition, we assume in this outline that  $\mathcal{G}$  is the integer lattice  $\mathbb{Z}^d$  (i.e.,  $(u, v) \in \mathcal{E}$  if and only if  $\|u - v\|_\infty = 1$ , where  $\|\cdot\|_\infty$  denotes the usual  $\ell^\infty$  norm),  $X$  is the simple symmetric random walk on  $\mathbb{Z}^d$ , and  $\xi$  is a centered stationary Gaussian process with covariance function

$$\gamma(v) := \mathbf{E}[\xi(v)\xi(0)], \quad v \in \mathbb{Z}^d.$$

As alluded to in the introduction (and proved in Section 6), to prove that the eigenvalue point process of  $H$  is number rigid, it suffices to show that the variance of  $\text{Tr}[e^{-tH}]$  vanishes as  $t \rightarrow 0$ . According to the Feynman–Kac formula, we have that

$$\text{Tr}[e^{-tH}] = \sum_{v \in \mathbb{Z}^d} \mathbf{E}_X \left[ \exp \left( \int_0^t V(X(s)) + \xi(X(s)) \, ds \right) \mathbf{1}_{\{X(t)=X(0)\}} \mid X(0) = v \right],$$

where  $\mathbf{E}_X$  means that we are only averaging with respect to the randomness in the path of  $X$ , and we assume that  $X$  is independent of the noise  $\xi$ . In order to ensure that  $e^{-tH}$  is trace class (or even bounded) in the general case, we assume that  $\mathcal{G}$  has uniformly bounded degrees; see Section 6.1 for more details.

Our first step in the analysis of  $\text{Tr}[e^{-tH}]$  is to note that if  $t$  is small, then the probability that there exists some  $0 \leq s \leq t$  such that  $X(s) \neq X(0)$  is close to zero (i.e.,  $1 - e^{-t} \sim t$ ). Thus, by working only with the complement of this event, we have that

$$\text{Tr}[e^{-tH}] \approx \sum_{v \in \mathbb{Z}^d} e^{-tV(v) - t\xi(v)}. \tag{2.1}$$

A rigorous version of this heuristic is carried out in the proof of Lemma 4.6. The latter relies on controlling how far  $X$  can travel from its initial value  $X(0)$  after a small time (e.g., the tail bound (4.22)), which itself depends on the assumptions that the jump rates of  $X$  are uniformly bounded.

Our second step is to identify the leading order asymptotics in the variance of the expression on the right-hand side of (2.1). In the special case where  $\xi$  is a stationary

Gaussian process with covariance  $\gamma$ , an application of Tonelli’s theorem yields

$$\begin{aligned} \text{Var}\left[\sum_{v \in \mathbb{Z}^d} e^{-tV(v)-t\xi(v)}\right] &= \sum_{u,v \in \mathbb{Z}^d} e^{-tV(u)-tV(v)} \text{Cov}[e^{-t\xi(u)}, e^{-t\xi(v)}] \\ &= \sum_{u,v \in \mathbb{Z}^d} e^{-tV(u)-tV(v)} e^{t^2\gamma(0)} (e^{t^2\gamma(u-v)} - 1) \\ &\approx t^2 \sum_{u,v \in \mathbb{Z}^d} e^{-tV(u)-tV(v)} \gamma(u-v), \end{aligned} \tag{2.2}$$

where the last line follows from a Taylor expansion. A bound of this type can be achieved in the general case thanks to our assumption that the tails of  $\xi$  are not worse than exponential. We refer to Proposition 4.2 for the general form of the variance formula. See Lemmas 4.3 and 4.4 for quantitative bounds on the vanishing of the covariance of the exponential random field  $e^{-t\xi}$  as  $t \rightarrow 0$  in terms of the strength of the covariance of  $\xi$ .

Our third and final step is to identify conditions such that the quantity

$$\sum_{u,v \in \mathbb{Z}^d} e^{-tV(u)-tV(v)} \gamma(u-v) \tag{2.3}$$

does not blow up at a faster rate than  $t^{-2}$  as  $t \rightarrow 0$ . As advertised in our informal statement, this depends on the growth rate of the potential  $V$  and the decay rate (if any) of the covariance  $\gamma$  at infinity. To give an illustration of how this is carried out in this paper, we consider the two simplest (and most extreme) cases of covariance structure:

1.  $(\xi(v))_{v \in \mathbb{Z}^d}$  are i.i.d., i.e.,  $\gamma(v) = 0$  whenever  $v \neq 0$ ;
2.  $(\xi(v))_{v \in \mathbb{Z}^d}$  are all equal to each other, i.e.,  $\gamma(v) = \gamma(0)$  for all  $v \in \mathcal{V}$ .

The quantity (2.3) then becomes

$$\sum_{u,v \in \mathbb{Z}^d} e^{-tV(u)-tV(v)} \gamma(u-v) = \begin{cases} \gamma(0) \sum_{v \in \mathbb{Z}^d} e^{-2tV(v)} & \text{i.i.d. case,} \\ \gamma(0) \left( \sum_{v \in \mathbb{Z}^d} e^{-tV(v)} \right)^2 & \text{all equal case.} \end{cases}$$

If we assume that  $V(v) \gg d(0, v)^\alpha$  for some  $\alpha > 0$ , then for any  $\theta > 0$  we have that

$$\sum_{v \in \mathbb{Z}^d} e^{-\theta tV(v)} \ll \sum_{v \in \mathbb{Z}^d} e^{-\theta t d(0,v)^\alpha} = \sum_{n \in \mathbb{N} \cup \{0\}} c_n(0) e^{-\theta t n^\alpha}, \tag{2.4}$$

where we recall that  $c_n(0)$  denotes for every  $n \in \mathbb{N}$  the number of vertices in  $\mathcal{G}$  such that  $d(0, v) = n$ . For the  $d$ -dimensional integer lattice  $\mathbb{Z}^d$ , it is easy to check that



there exists a constant  $C > 0$  such that  $c_n(0) \leq C n^{d-1}$  for every  $n \in \mathbb{N}$ , whence (2.4) yields

$$\sum_{v \in \mathbb{Z}^d} e^{-\theta t V(v)} \ll \sum_{n \in \mathbb{N} \cup \{0\}} n^{d-1} e^{-\theta t n^\alpha} \approx \int_0^\infty x^{d-1} e^{-\theta t x^\alpha} dx = O(t^{-d/\alpha}). \tag{2.5}$$

Summarizing our argument so far in (2.1)–(2.5), we are led to the  $t \rightarrow 0$  asymptotic

$$\text{Var}[\text{Tr}[e^{-tH}]] \ll \begin{cases} t^{2-d/\alpha} & \text{i.i.d. case,} \\ t^{2-2d/\alpha} & \text{all equal case.} \end{cases}$$

Thus, the eigenvalue point process of  $H$  is proved to be number rigid if  $V(v) \gg d(0, v)^{d/2}$  in the i.i.d case and  $V(v) \gg d(0, v)^d$  in the all equal case. If  $\gamma$  has a less extreme decay rate (such as  $\gamma(v) = O(d(0, v)^{-\beta})$  as  $d(0, v) \rightarrow \infty$  for some  $\beta > 0$ ), then the eigenvalue point process of  $H$  is number rigid if  $V(v) \gg d(0, v)^\alpha$  for some  $d/2 \leq \alpha \leq d$ , where the exact value of  $\alpha$  depends on the decay rate of  $\gamma$ . We refer to Theorems 3.16 and 3.17 for the details.

### 3. Main results

#### 3.1. Basic definitions and notations

We begin by introducing basic/standard notations that will be used throughout the paper.

**Notation 3.1** (Function spaces). We use  $\ell^p(\mathcal{V})$  to denote the space of real-valued absolutely  $p$ -summable (or bounded if  $p = \infty$ ) functions on  $\mathcal{V}$ ; we denote the associated norm by  $\|\cdot\|_p$ . We use  $\langle \cdot, \cdot \rangle$  to denote the inner product on  $\ell^2(\mathcal{V})$ . Given a subset  $\mathcal{U} \subset \mathcal{V}$ , we denote

$$\ell^p_{\mathcal{U}}(\mathcal{V}) := \{f \in \ell^p(\mathcal{V}) : f(u) = 0 \text{ for every } u \in \mathcal{U}\}.$$

**Notation 3.2** (Operator theory). Given a linear operator  $T$  on  $\ell^2_{\mathcal{U}}(\mathcal{V})$  (or a dense domain  $D(T) \subset \ell^2_{\mathcal{U}}(\mathcal{V})$ ), we use  $\sigma(T)$  to denote its spectrum, and  $\sigma_p(T) \subset \sigma(T)$  to denote its point spectrum. If  $T$  is bounded, we denote its operator norm by

$$\|T\|_{\text{op}} := \sup_{f \in \ell^2_{\mathcal{U}}(\mathcal{V}), \|f\|_2=1} \|Tf\|_2.$$

We use  $\mathfrak{R}(z, T) := (T - z)^{-1}$  to denote the resolvent of  $T$  for all  $z \in \mathbb{C} \setminus \sigma(T)$ . If  $\lambda$  is an isolated eigenvalue of  $T$ , then we let

$$m_a(\lambda, T) := \dim \left( \text{rg} \left( \frac{1}{2\pi i} \oint_{\Gamma_\lambda} \mathfrak{R}(z, T) dz \right) \right)$$

denote the algebraic multiplicity of  $\lambda$ , where  $\dim$  denotes the dimension of a linear space,  $\text{rg}$  denotes the range of an operator, and  $\Gamma_\lambda$  denotes a Jordan curve that encloses  $\lambda$  and excludes the remainder of the spectrum of  $T$ .

**Definition 3.3** (Rigidity). Let  $\mathcal{X} = \sum_{k \in \mathbb{N}} \delta_{\lambda_k}$  be an infinite point process on  $\mathbb{C}$ . We say that  $\mathcal{X}$  is *real-bounded below* by a random variable  $\omega \in \mathbb{R}$  if  $\Re(\lambda_k) \geq \omega$  almost surely for every  $k \in \mathbb{N}$ . We say that such a point process is *number rigid* if for every Borel set  $B \subset \mathbb{C}$  such that  $B \subset (-\infty, \delta] + i[-\tilde{\delta}, \tilde{\delta}]$  for some  $\delta, \tilde{\delta} > 0$ , the random variable  $\mathcal{X}(B)$  is measurable with respect to the completion (under the law of the point process  $\mathcal{X}$ ) of the sigma algebra generated by the set

$$\{\mathcal{X}(A): A \subset \mathbb{C} \text{ is Borel and } B \cap A = \emptyset\}.$$

**Remark 3.4.** In previous works in the literature, it is most common to define number rigidity as the requirement that  $\mathcal{X}(B)$  is measurable with respect to the configuration in  $\mathbb{C} \setminus B$  for every bounded Borel set  $B$ . This is in part due to the fact that most point processes that have been proved to be number rigid thus far are such that  $\mathcal{X}(B) = \infty$  almost surely whenever  $B$  is unbounded.

That being said, the fact that we are considering the spectrum of Schrödinger operators whose potentials have a strong growth at infinity means that we are considering eigenvalue point processes that are real-bounded below, in which case a more general notion of number rigidity makes sense. We note that a similarly generalized notion of rigidity appeared in the work of Bufetov on the stochastic Airy operator in [7, Proposition 3.2].

### 3.2. Markov process

Next, we introduce the Markov processes on the graph  $\mathcal{G}$  that generate our random operators, as well as some of the notions we need to describe them. We recall that  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  is a countably infinite connected graph with uniformly bounded degrees and a root  $0 \in \mathcal{V}$ .

**Definition 3.5** (Markov process). Let  $\Pi: \mathcal{V} \times \mathcal{V} \rightarrow [0, 1]$  be a matrix such that

1.  $\Pi$  is stochastic, that is, for every  $u \in \mathcal{V}$ ,

$$\sum_{v \in \mathcal{V}} \Pi(u, v) = 1;$$

2.  $\Pi(v, v) = 0$  for all  $v \in \mathcal{V}$ ; and
3. if  $(u, v) \notin \mathcal{E}$ , then  $\Pi(u, v) = \Pi(v, u) = 0$ .

Let  $q: \mathcal{V} \rightarrow (0, \infty)$  be a positive vector and let  $X: [0, \infty) \rightarrow \mathcal{V}$  denote the continuous-time Markov process on  $\mathcal{V}$  defined as follows. If  $X$  is in state  $u \in \mathcal{V}$ , it waits for

a random time with an exponential distribution with rate  $q(u)$ , and then jumps to another state  $v \neq u$  with probability  $\Pi(u, v)$ , independently of the wait time. Once at the new state,  $X$  repeats this procedure independently of all previous jumps.

**Remark 3.6.** We note that condition (3) in the above definition implies that  $X$  is a Markov process on the graph  $\mathcal{G}$ , in the sense that jumps can only occur between vertices that are connected by edges.

**Notation 3.7.** For every  $v \in \mathcal{V}$ , we use  $X^v$  to denote the process  $X$  conditioned on the starting point  $X(0) = v$ . We use  $\mathbf{P}^v$  to denote the law of  $X^v$ , and  $\mathbf{E}^v$  to denote expectation with respect to  $\mathbf{P}^v$ .

We assume throughout that the Markov process  $X$  and the graph  $\mathcal{G}$  satisfy the following.

**Assumption 3.8** (Graph geometry and jump rates). The following two conditions hold.

1. There exists constants  $d \geq 1$  and  $c > 0$  such that

$$\sup_{v \in \mathcal{V}} c_n(v) := \sup_{v \in \mathcal{V}} |\{u \in \mathcal{V} : d(u, v) = n\}| \leq cn^{d-1} \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \quad (3.1)$$

recalling that  $d$  is the graph distance in  $\mathcal{G}$ , that is,  $d(u, v)$  is the length of the shortest path (in terms of number of edges) connecting  $u$  and  $v$ , and with the convention that  $d(v, v) = 0$  for all  $v \in \mathcal{V}$ .

2.  $X$  has uniformly bounded jump rates, that is,

$$\alpha := \sup_{v \in \mathcal{V}} q(v) < \infty.$$

**Remark 3.9.** We note that the assumption (3.1) simultaneously takes care of the requirement that  $\mathcal{G}$  has uniformly bounded degrees (since  $c_1(v) = \text{deg}(v)$ ) and of the asymptotic growth rate (1.1) stated in our informal theorem.

### 3.3. Feynman–Kac kernel

We are now in a position to introduce the central objects of study of this paper, namely, the Feynman–Kac semigroups of the Schrödinger operators we are interested in.

**Notation 3.10** (Local time). For every  $t \geq 0$ , we let  $L_t : \mathcal{V} \rightarrow [0, t]$  denote the local time of  $X$ :

$$L_t(v) := \int_0^t \mathbf{1}_{\{X(s)=v\}} \, ds, \quad v \in \mathcal{V}.$$

**Definition 3.11** (Potential and noise). Let  $V: \mathcal{V} \rightarrow \mathbb{R} \cup \{\infty\}$  be a deterministic function, and let  $\xi: \mathcal{V} \rightarrow \mathbb{R}$  be a random function. We denote the set

$$\mathcal{Z} := \{v \in \mathcal{V}: V(v) = \infty\}. \tag{3.2}$$

Throughout, we make the following assumptions on the noise and potential.

**Assumption 3.12** (Potential growth and noise tails). There exists  $\alpha > 0$  such that

$$\liminf_{d(0,v) \rightarrow \infty} \frac{V(v)}{d(0,v)^\alpha} = \infty. \tag{3.3}$$

Moreover,  $\xi$  satisfies the following conditions:

1.  $\mathbf{E}[\xi(v)] = 0$  for every  $v \in \mathcal{V}$ ;
2. there exists  $m > 0$  such that for every  $p \in \mathbb{N}$ ,

$$\sup_{v \in \mathcal{V}} \mathbf{E}[|\xi(v)|^p] \leq p!m^p. \tag{3.4}$$

In the sequel, it will be useful to characterize noises in terms of the decay rate of their covariances. For this purpose, we make the following definition.

**Definition 3.13** (Covariance decay). We say that  $\xi$  has covariance decay of order (at least)  $\beta > 0$  if there exists a constant  $\mathfrak{C} > 0$  such that

$$|\mathbf{E}[\xi(u)\xi(v)]| \leq \mathfrak{C}(d(u,v) + 1)^{-\beta} \tag{3.5}$$

for every  $u, v \in \mathcal{V}$ , and such that

$$|\mathbf{E}[\xi(u)\xi(v)\xi(w)]| \leq \mathfrak{C} \min_{a,b \in \{u,v,w\}} (d(a,b) + 1)^{-\beta} \tag{3.6}$$

for every  $u, v, w \in \mathcal{V}$ .

**Definition 3.14** (Feynman–Kac kernel). Define the Feynman–Kac kernel

$$K_t(u, v) := \mathbf{E}^u [e^{-(L_t, V + \xi)} \mathbf{1}_{\{X(t)=v\}}], \quad u, v \in \mathcal{V}, \tag{3.7}$$

where we assume that  $X$  is independent of  $\xi$ , and that  $\mathbf{E}^v$  denotes the expectation with respect to the Markov process  $X^v$ , conditional on  $\xi$ . We denote the trace of  $K_t$  by

$$\text{Tr}[K_t] := \sum_{v \in \mathcal{V}} K_t(v, v).$$

**Remark 3.15.** In the above definition, we use the convention that  $e^{-\infty} := 0$  whenever  $V(v) = \infty$ , in particular,  $K_t(u, v) = 0$  whenever  $u \in \mathcal{Z}$  or  $v \in \mathcal{Z}$ .

### 3.4. Main results: variance upper bound and rigidity

We now state our main results. First, we have the following sufficient condition for the vanishing of the variance of the trace of  $K_t$  as  $t \rightarrow 0$ :

**Theorem 3.16.** *Suppose that Assumptions 3.8 and 3.12 hold. In order to have*

$$\lim_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] = 0,$$

*it is sufficient that the constant  $\alpha$  in (3.3) satisfies the following:*

1. *if  $\xi$  has covariance decay of order  $\beta > 0$ , then*

$$\alpha \begin{cases} \geq d/2 & \text{when } \beta > d, \\ > d/2 & \text{when } \beta = d, \\ \geq d - \beta/2 & \text{when } \beta < d; \end{cases} \tag{3.8}$$

2. *otherwise,  $\alpha \geq d$ .*

As a consequence of the above theorem, we have the following result, which states some properties of the infinitesimal generator of  $K_t$ , including number rigidity.

**Theorem 3.17.** *Suppose that Assumptions 3.8 and 3.12 hold, and that we take the constant  $\alpha$  in (3.3) as in Theorem 3.16. The following conditions hold almost surely.*

1. *For every  $t > 0$ ,  $K_t$  is a trace class linear operator on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$ . There exists a random variable  $\omega \leq 0$  such that  $\|K_t\|_{\text{op}} \leq e^{-\omega t}$  for all  $t > 0$ .*
2. *The family of operators  $(K_t)_{t>0}$  is a strongly continuous semigroup on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$ .*
3. *The infinitesimal generator*

$$H := \lim_{t \rightarrow 0} \frac{K_0 - K_t}{t} \tag{3.9}$$

*is closed on some dense domain  $D(H) \subset \ell^2_{\mathbb{Z}}(\mathcal{V})$ , and its action on functions is given by the following matrix:*

$$H(u, v) := \begin{cases} -q(u)\Pi(u, v) & \text{if } u \neq v \text{ and } u, v \notin \mathbb{Z}, \\ q(u) + V(u) + \xi(u) & \text{if } u = v \text{ and } u \notin \mathbb{Z}, \\ 0 & \text{if } u \in \mathbb{Z} \text{ or } v \in \mathbb{Z}. \end{cases} \tag{3.10}$$

*(So, if  $f \in D(H)$ , then  $f(v) = 0$  for every  $v \in \mathbb{Z}$ .)*

*In particular, almost surely,  $H$  has a pure point spectrum without accumulation point, and the eigenvalue point process (counting algebraic multiplicities)*

$$\mathcal{X}_H := \sum_{\lambda \in \sigma(H)} m_a(\lambda, H) \delta_\lambda \tag{3.11}$$

*is real-bounded below by  $\omega$  and number rigid in the sense of Definition 3.3.*

### 3.5. Questions of optimality

In this section, we study the optimality of the growth assumptions we make on  $V$  in Theorem 3.16 by considering three counterexamples.

**Theorem 3.18.** *Suppose that  $X$  is the nearest-neighbor symmetric random walk on the integer lattice  $\mathbb{Z}^d$ , that  $V(v) := d(0, v)^\delta$  for some  $\delta > 0$ , and that  $\xi$  is a centered stationary Gaussian process whose covariance function  $\gamma(v) := \mathbf{E}[\xi(v)\xi(0)]$  is non-negative. If one of the following conditions hold:*

1.  $\delta \leq d/2$  and  $\gamma(v) = \mathbf{1}_{\{v=0\}}$ ;
2.  $\delta \leq d - \beta/2$  for some  $0 < \beta < d$ , and there exists a constant  $\mathfrak{L} > 0$  such that  $\gamma(v) \geq \mathfrak{L}(d(0, v) + 1)^{-\beta}$  for every  $v \in \mathcal{V}$ ;
3.  $\delta \leq d$  and  $\inf_{v \in \mathbb{Z}^d} \gamma(v) > \mathfrak{L}$  for some constant  $\mathfrak{L} > 0$ ;

then we have the variance lower bound

$$\liminf_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] > 0.$$

Thus, given that  $c_n(v) \asymp n^{d-1}$  as  $n \rightarrow \infty$  on  $\mathbb{Z}^d$ , if one is interested in providing a general sufficient condition for number rigidity on graphs using semigroups, then Theorem 3.16 is essentially the optimal result one could hope for.

**Remark 3.19.** An examination of the proof of Theorem 3.18 reveals that similar lower bounds can be proved for more general examples with little effort; we restrict our attention to this elementary setting for simplicity of exposition.

## 4. Proof of Theorem 3.16

Throughout this section, we suppose that Assumptions 3.8 and 3.12 hold. This section is organized as follows. In Section 4.1, we outline the main steps of the proof of Theorem 3.16. That is, we state a number of technical propositions and lemmas, which we then use to prove Theorem 3.16. Then, in Sections 4.2–4.6, we prove the technical results stated Section 4.1, thus wrapping-up the proof of Theorem 3.16.

### 4.1. Proof outline

**4.1.1. Step 1. Variance formula and first bound.** We begin with some notations.

**Notation 4.1.** Let us denote by  $(\Omega_\xi, \mathbf{P}_\xi)$  the probability space on which  $\xi$  is defined. Let  $Y$  be any random element that is independent of  $\xi$ , and let  $F$  be any measurable

function. We denote the random variable

$$\mathbf{E}_\xi[F(\xi, Y)] := \int_{\Omega_\xi} F(x, Y) \, d\mathbf{P}_\xi(x);$$

that is,  $\mathbf{E}_\xi$  is the conditional expectation with respect to  $\xi$ , given  $Y$ . Then, for measurable functions  $F$  and  $G$ , we denote the random variable

$$\text{Cov}_\xi[F(\xi, Y), G(\xi, Y)] := \mathbf{E}_\xi[F(\xi, Y)G(\xi, Y)] - \mathbf{E}_\xi[F(\xi, Y)]\mathbf{E}_\xi[G(\xi, Y)].$$

Our main tool in the proof of Theorem 3.16 is the following variance formula:

**Proposition 4.2.** *For every  $u, v \in \mathcal{V}$ , we let  $X^u$  and  $\tilde{X}^v$  be independent copies of the Markov process  $X$  started from  $u$  and  $v$  respectively. We assume that  $X^u$  and  $\tilde{X}^v$  are independent of the noise  $\xi$ , and we denote their local times by*

$$L_t^u(w) := \int_0^t \mathbf{1}_{\{X^u(s)=w\}} \, ds \quad \text{and} \quad \tilde{L}_t^v(w) := \int_0^t \mathbf{1}_{\{\tilde{X}^v(s)=w\}} \, ds$$

for all  $w \in \mathcal{V}$ . It holds that

$$\text{Var}[\text{Tr}[K_t]] = \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-\langle L_t^u + \tilde{L}_t^v, V \rangle} \text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}] \mathbf{1}_{\{X^u(t)=u, \tilde{X}^v(t)=v\}}].$$

The proof of this proposition, which we provide in Section 4.2 below, is essentially a direct consequence of the definition of  $K_t$  in (3.7). In order to find sufficient conditions for  $\text{Var}[\text{Tr}[K_t]] \rightarrow 0$  as  $t \rightarrow 0$  using this formula, it is convenient to control the contributions coming from  $V$  and  $\xi$  separately. To this end, we use Hölder’s inequality, as well as the elementary fact that  $\mathbf{1}_E \leq 1$  for every event  $E$ , which yields

$$\begin{aligned} & \mathbf{E}[e^{-\langle L_t^u + \tilde{L}_t^v, V \rangle} \text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}] \mathbf{1}_{\{X^u(t)=u, \tilde{X}^v(t)=v\}}] \\ & \leq \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2} \end{aligned}$$

for every fixed  $u, v \in \mathcal{V}$ . Then, by summing both sides of the above inequality over  $u, v \in \mathcal{V}$ , we obtain our first upper bound for the variance:

$$\text{Var}[\text{Tr}[K_t]] \leq \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2}. \tag{4.1}$$

**4.1.2. Step 2. Controlling the contributions from  $\xi$  and  $V$ .** We now state the technical results that we use to control the right-hand side of (4.1). Our first such result is as follows:

**Lemma 4.3.** *Recall the definition of the constant  $\mathfrak{m} > 0$  in (3.4). There exists a constant  $C_1 > 0$  (which only depends on  $\mathfrak{m}$ ) such that for every  $t < 1/C_1$ , one has*

$$\sup_{u,v \in \mathcal{V}} \mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2} \leq C_1 t^2.$$

The proof of Lemma 4.3, which we provide in Section 4.4, follows from estimating expectations of the form  $\mathbf{E}^v[e^{-\theta \langle L_t, \xi \rangle}]$  using our assumption that the tails of  $\xi$  are not worse than exponential (i.e., (3.4)). Next, we have the following result, which provides a tighter decay rate in the case where  $\xi$  has covariance decay:

**Lemma 4.4.** *Suppose that  $\xi$  has covariance decay of order  $\beta$ , as per Definition 3.13. Recall the definitions of the constants  $\mathfrak{q}$ ,  $\mathfrak{m}$ , and  $\mathfrak{C}$  in Assumption 3.8 (3), (3.4), (3.5), and (3.6). There exists a constant  $C_2 > 0$  (which only depends on  $\mathfrak{q}$ ,  $\mathfrak{m}$ ,  $\mathfrak{C}$ , and  $\beta$ ) such that, for every  $t < 1/C_2$  and  $u, v \in \mathcal{V}$ , one has*

$$\mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2} \leq C_2(t^2(d(u, v) + 1)^{-\beta} + t^4).$$

Lemma 4.4 is proved in Section 4.5. The proof of this lemma is rather more subtle than that of Lemma 4.3, and depends on a careful control of how much  $X^u$  and  $\tilde{X}^v$  deviate from their respective starting points  $u$  and  $v$ . We note that the uniform upper bound on the jump rates of  $X$  in Assumption 3.8 (3) is crucial for this lemma.

**Remark 4.5.** The proofs of Lemmas 4.3 and 4.4 both rely on some elementary formulas and estimates of the moment generating functions of the noises and their covariances, which will be stated and proved in Section 4.3.

With Lemmas 4.3 and 4.4 in hand, it now only remains to control the contribution of the potential  $V$  in (4.1). For this, we have the following result:

**Lemma 4.6.** *Recall the definition of  $d \geq 1$  and  $c > 0$  in (3.1). Suppose that we can find some constants  $\kappa, \mu > 0$  such that*

$$V(v) \geq (\kappa d(0, v))^\alpha - \mu, \quad v \in \mathcal{V}. \tag{4.2}$$

*Then, there exists a constant  $C_3 > 0$  (which only depends on  $\alpha, \beta, d$ , and  $c$ ) such that*

$$\limsup_{t \rightarrow 0} t^{2d/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2(L_t^u + \tilde{L}_t^v, V)}]^{1/2} \leq C_3 \kappa^{-2d}; \tag{4.3}$$

$$\limsup_{t \rightarrow 0} t^{(2d-\beta)/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2(L_t^u + \tilde{L}_t^v, V)}]^{1/2} (d(u, v) + 1)^{-\beta} \leq C_3 \kappa^{-2d+\beta} \tag{4.4}$$

*for every  $0 < \beta < d$ ; and*

$$\limsup_{t \rightarrow 0} t^{d/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2(L_t^u + \tilde{L}_t^v, V)}]^{1/2} (d(u, v) + 1)^{-\beta} \leq C_3 \kappa^{-d} \tag{4.5}$$

*for every  $\beta > d$ .*



Lemma 4.6, which is proved in Section 4.6, follows the strategy outlined in (2.4) and (2.5). The first step of the proof of Lemma 4.6 relies on a rigorous implementation of the intuition that, for very small  $t > 0$ , one expects that

$$\mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \approx e^{-tV(u) - tV(v)}. \tag{4.6}$$

This once again relies on controlling how much  $X^u$  and  $\tilde{X}^v$  deviate from their starting points. Once a quantitative version of (4.6) is established, we can then use (4.2), which allows to control  $\mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2}$  in terms of quantities that only depend on the geometry of  $\mathcal{G}$  (more precisely, the graph distance). We then wrap up the proof of the lemma by using the upper bound on the coordination sequences in (3.1), in similar fashion to (2.5).

**4.1.3. Step 3. Conclusion of the proof.** We now combine the technical results stated above to conclude the proof of Theorem 3.16. By applying Lemmas 4.3 and 4.4 to our upper bound (4.1), we get that for every  $t < 1/C_1$ , one has

$$\text{Var}[\text{Tr}[K_t]] \leq C_1 t^2 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2}, \tag{4.7}$$

and if  $\xi$  has covariance decay of order  $\beta > 0$ , then for every  $t < 1/C_2$ , one has

$$\begin{aligned} \text{Var}[\text{Tr}[K_t]] &\leq C_2 t^2 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\beta} \\ &\quad + C_2 t^4 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2}. \end{aligned} \tag{4.8}$$

Thanks to our growth assumption in (3.3), for any choice of  $\kappa > 0$ , we know that there exists a large enough  $\mu > 0$  so that (4.2) holds. We may then complete the proof of Theorem 3.16 by an application of Lemma 4.6. We do this on a case-by-case basis:

Suppose first that  $\xi$  has covariance decay of order  $0 < \beta < d$  and that  $\alpha \geq d - \beta/2 > d/2$ . Then, the fact that  $2 - (2d - \beta)/\alpha \geq 0$  implies by (4.4) that

$$\begin{aligned} &\limsup_{t \rightarrow 0} t^2 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\beta} \\ &= \limsup_{t \rightarrow 0} t^{2 - (2d - \beta)/\alpha} t^{(2d - \beta)/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\beta} \\ &\leq C_3 \kappa^{-2d + \beta}; \end{aligned}$$

and the fact that  $4 - 2d/\alpha > 0$  implies by (4.3) that

$$\begin{aligned} &\limsup_{t \rightarrow 0} t^4 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \\ &= \limsup_{t \rightarrow 0} t^{4 - 2d/\alpha} t^{2d/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} = 0. \end{aligned} \tag{4.9}$$

Combining this with (4.8) implies that

$$\limsup_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] \leq C_2 C_3 \kappa^{-2d+\beta},$$

where we recall that  $C_2, C_3 > 0$  do not depend on  $\kappa$  or  $\mu$ . Since (4.2) holds for any choice of  $\kappa > 0$ , we can take  $\kappa \rightarrow \infty$ , which then yields  $\text{Var}[\text{Tr}[K_t]] \rightarrow 0$  as  $t \rightarrow 0$ .

Next, suppose that  $\xi$  has covariance decay of order  $\beta = d$  and that  $\alpha > d/2$ . We note that this implies that  $\xi$  also has correlation decay of order  $\tilde{\beta}$  for any choice of  $0 < \tilde{\beta} < d$ . Since  $\alpha > d/2$  implies that  $2d - 2\alpha < d$ , we can choose  $\tilde{\beta}$  close enough to  $d$  so that  $2d - 2\alpha < \tilde{\beta}$ , which we can rearrange into  $2 > (2d - \tilde{\beta})/\alpha$ . Thus, (4.4) implies that

$$\begin{aligned} & \limsup_{t \rightarrow 0} t^2 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\beta} \\ &= \limsup_{t \rightarrow 0} t^{2-(2d-\tilde{\beta})/\alpha} t^{(2d-\tilde{\beta})/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\tilde{\beta}} = 0. \end{aligned}$$

Combining this with (4.9), we directly prove that  $\text{Var}[\text{Tr}[K_t]] \rightarrow 0$  as  $t \rightarrow 0$  in this case.

Suppose now that  $\xi$  has covariance decay of order  $\beta > d$  and that  $\alpha \geq d/2$ . Then, the fact that  $2 - d/\alpha \geq 0$  implies by (4.5) that

$$\begin{aligned} & \limsup_{t \rightarrow 0} t^2 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\beta} \\ &= \limsup_{t \rightarrow 0} t^{2-d/\alpha} t^{d/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} (d(u, v) + 1)^{-\beta} \leq C_3 \kappa^{-d}; \end{aligned}$$

and the fact that  $4 - 2d/\alpha \geq 0$  implies by (4.3) that

$$\begin{aligned} & \limsup_{t \rightarrow 0} t^4 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \\ &= \limsup_{t \rightarrow 0} t^{4-2d/\alpha} t^{2d/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \leq C_3 \kappa^{-2d}. \end{aligned} \tag{4.10}$$

Combining this with (4.8) and taking  $\kappa \rightarrow \infty$  then implies that  $\text{Var}[\text{Tr}[K_t]] \rightarrow 0$  as  $t \rightarrow 0$ .

Finally, consider the general case where we simply assume that  $\alpha \geq d$ . Then,  $2 - 2d/\alpha \geq 0$ , and thus (4.3) implies that

$$\begin{aligned} & \limsup_{t \rightarrow 0} t^2 \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \\ &= \limsup_{t \rightarrow 0} t^{2-2d/\alpha} t^{2d/\alpha} \sum_{u,v \in \mathcal{V}} \mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2} \leq C_3 \kappa^{-2d}. \end{aligned}$$

Since the constants  $C_1, C_3 > 0$  are independent of  $\kappa$  and  $\mu$ , combining this with (4.7) and taking  $\kappa \rightarrow \infty$  then implies that  $\text{Var}[\text{Tr}[K_t]] \rightarrow 0$  as  $t \rightarrow 0$  in this case. This then completes the proof of Theorem 3.16.

**4.2. Proof of Proposition 4.2**

Since the random walk  $X$  is assumed independent of  $\xi$ , by applying Fubini’s theorem to the definition of  $K_t$  in (3.7), we have that

$$\mathbf{E}[\text{Tr}[K_t]] = \sum_{v \in \mathcal{V}} \mathbf{E}^v [e^{-\langle L_t, V \rangle} \mathbf{E}_\xi [e^{-\langle L_t, \xi \rangle} \mathbf{1}_{\{X(t)=v\}}]],$$

where we recall the definition of  $\mathbf{E}_\xi$  in Notation 4.1. Taking the square of this expression, we then get once again by Fubini’s theorem that

$$\mathbf{E}[\text{Tr}[K_t]]^2 = \sum_{u, v \in \mathcal{V}} \mathbf{E} [e^{-\langle L_t^u + \tilde{L}_t^v, V \rangle} \mathbf{E}_\xi [e^{-\langle L_t^u, \xi \rangle}] \mathbf{E}_\xi [e^{-\langle \tilde{L}_t^v, \xi \rangle}] \mathbf{1}_{\{X^u(t)=u, \tilde{X}^v(t)=v\}}].$$

Thanks to (3.7), it is easy to check that

$$\text{Tr}[K_t]^2 = \sum_{u, v \in \mathcal{V}} \mathbf{E}_\xi [e^{-\langle L_t^u + \tilde{L}_t^v, V + \xi \rangle} \mathbf{1}_{\{X^u(t)=u, \tilde{X}^v(t)=v\}}].$$

Taking the expectation of this expression using Fubini’s theorem then leads to

$$\mathbf{E}[\text{Tr}[K_t]^2] = \sum_{u, v \in \mathcal{V}} \mathbf{E} [e^{-\langle L_t^u + \tilde{L}_t^v, V \rangle} \mathbf{E}_\xi [e^{-\langle L_t^u + \tilde{L}_t^v, \xi \rangle}] \mathbf{1}_{\{X^u(t)=u, \tilde{X}^v(t)=v\}}].$$

The proof of Proposition 4.2 is then simply a matter of subtracting  $\mathbf{E}[\text{Tr}[K_t]]^2$  from the above expression for  $\mathbf{E}[\text{Tr}[K_t]^2]$ , and using the definition of  $\text{Cov}_\xi$  in Notation 4.1.

**4.3. Auxiliary results on estimates of moment generating functions**

Before discussing the proofs of Lemma 4.3 and Lemma 4.4 in the next two subsections, we list here two simple propositions concerning the tail behaviors of the moment generating functions of the noises and their covariances. The first result is a straightforward consequence of Taylor expansions and Assumption 3.12 on the tails of the noises.

**Proposition 4.7.** *Under Assumption 3.12, for every finitely-supported deterministic functions  $f, g: \mathcal{V} \rightarrow \mathbb{R}$  such that  $\|f + g\|_1, \|f\|_1, \|g\|_1 \leq 1/2m$ , it holds that*

$$|\mathbf{E}[e^{\langle f, \xi \rangle}] - 1| \leq 2m^2 \|f\|_1^2 \tag{4.11}$$

and

$$|\text{Cov}[e^{\langle f, \xi \rangle}, e^{\langle g, \xi \rangle}]| \leq 2m^2 (\|f + g\|_1^2 + \|f\|_1^2 + \|g\|_1^2) + 4m^4 \|f\|_1^2 \|g\|_1^2. \tag{4.12}$$

*Proof.* For every deterministic function  $f: \mathcal{V} \rightarrow \mathbb{R}$ , it follows from a straightforward Taylor expansion of the exponential that

$$\mathbf{E}[e^{\langle f, \xi \rangle}] = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1, \dots, v_p \in \mathcal{V}} \mathbf{E}[\xi(v_1) \cdots \xi(v_p)] f(v_1) \cdots f(v_p), \quad (4.13)$$

with the convention that the term with  $p = 0$  above is equal to one. Firstly, since  $\mathbf{E}[\xi(v)] = 0$  for all  $v$ , the term corresponding to  $p = 1$  in (4.13) is zero. Secondly, thanks to our moment growth assumption  $\mathbf{E}[|\xi(v)|^p] \leq p!m^p$ , for every  $p \geq 2$  we have that

$$\begin{aligned} & \left| \sum_{v_1, \dots, v_p \in \mathcal{V}} \mathbf{E}[\xi(v_1) \cdots \xi(v_p)] f(v_1) \cdots f(v_p) \right| \\ & \leq \sum_{v_1, \dots, v_p \in \mathcal{V}} \mathbf{E}[|\xi(v_1)|^p]^{1/p} \cdots \mathbf{E}[|\xi(v_p)|^p]^{1/p} |f(v_1)| \cdots |f(v_p)| \\ & \leq p!(m \|f\|_1)^p. \end{aligned}$$

Thus, if  $\|f\|_1 \leq 1/2m$ , then we have that

$$|\mathbf{E}[e^{\langle f, \xi \rangle}] - 1| \leq \sum_{p=2}^{\infty} (m \|f\|_1)^p = \frac{(m \|f\|_1)^2}{1 - m \|f\|_1} \leq 2(m \|f\|_1)^2.$$

As for the claim regarding the covariance, for any two random variables  $Y$  and  $Z$ , we have by the triangle inequality that

$$\begin{aligned} |\text{Cov}[Y, Z]| &= |\mathbf{E}[YZ] - \mathbf{E}[Y]\mathbf{E}[Z]| \\ &\leq |\mathbf{E}[YZ] - 1| - |\mathbf{E}[Y] - 1||\mathbf{E}[Z] - 1| + |1 - \mathbf{E}[Y]| + |1 - \mathbf{E}[Z]| \end{aligned}$$

Thus, whenever  $\|f + g\|_1, \|f\|_1, \|g\|_1 \leq 1/2m$ , it follows from (4.11) that

$$|\text{Cov}[e^{\langle f, \xi \rangle}, e^{\langle g, \xi \rangle}]| \leq 2m^2(\|f + g\|_1^2 + \|f\|_1^2 + \|g\|_1^2) + 4m^4\|f\|_1^2\|g\|_1^2,$$

as desired. ■

In cases where we need a more precise control on the covariance, we have the following power series expansion:

**Proposition 4.8.** *Suppose that Assumption 3.12 holds. For any two finitely supported deterministic functions  $f, g: \mathcal{V} \rightarrow \mathbb{R}$ , one has*

$$\text{Cov}[e^{\langle f, \xi \rangle}, e^{\langle g, \xi \rangle}] = \sum_{p=2}^{\infty} \frac{\mathcal{A}_p(f, g)}{p!},$$

where, for every  $p \geq 2$ , we denote

$$\mathcal{A}_p(f, g) := \sum_{v_1, \dots, v_p \in \mathcal{V}} \left( \sum_{m=1}^{p-1} \binom{p}{m} \text{Cov}[\xi(v_1) \cdots \xi(v_m), \xi(v_{m+1}) \cdots \xi(v_p)] \cdot f(v_1) \cdots f(v_m) g(v_{m+1}) \cdots g(v_p) \right). \quad (4.14)$$

*Proof.* Using the same Taylor expansion as in (4.13), we get, on the one hand,

$$\begin{aligned} & \mathbf{E}[e^{(f+g, \xi)}] \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1, \dots, v_p \in \mathcal{V}} \mathbf{E}[\xi(v_1) \cdots \xi(v_p)] (f(v_1) + g(v_1)) \cdots (f(v_p) + g(v_p)) \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1, \dots, v_p \in \mathcal{V}} \sum_{m=0}^p \binom{p}{m} \mathbf{E}[\xi(v_1) \cdots \xi(v_p)] f(v_1) \cdots f(v_m) \cdot g(v_{m+1}) \cdots g(v_p), \end{aligned}$$

and on the other hand

$$\begin{aligned} & \mathbf{E}[e^{(f, \xi)}] \mathbf{E}[e^{(g, \xi)}] \\ &= \sum_{m_1, m_2=0}^{\infty} \frac{1}{m_1! m_2!} \left( \sum_{v_1, \dots, v_{m_1+m_2} \in \mathcal{V}} \mathbf{E}[\xi(v_1) \cdots \xi(v_{m_1})] \mathbf{E}[\xi(v_{m_1+1}) \cdots \xi(v_{m_1+m_2})] \cdot f(v_1) \cdots f(v_{m_1}) g(v_{m_1+1}) \cdots g(v_{m_1+m_2}) \right) \\ &= \sum_{p=0}^{\infty} \sum_{m=0}^p \frac{1}{m!(p-m)!} \left( \sum_{v_1, \dots, v_p \in \mathcal{V}} \mathbf{E}[\xi(v_1) \cdots \xi(v_m)] \mathbf{E}[\xi(v_{m+1}) \cdots \xi(v_p)] \cdot f(v_1) \cdots f(v_m) g(v_{m+1}) \cdots g(v_p) \right) \\ &= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{v_1, \dots, v_p \in \mathcal{V}} \left( \sum_{m=0}^p \binom{p}{m} \mathbf{E}[\xi(v_1) \cdots \xi(v_m)] \mathbf{E}[\xi(v_{m+1}) \cdots \xi(v_p)] \cdot f(v_1) \cdots f(v_m) g(v_{m+1}) \cdots g(v_p) \right). \end{aligned}$$

We then get the result by subtracting these two expressions. ■

#### 4.4. Proof of Lemma 4.3

By definition of local time,  $\|L_t^u\|_1 = \|\tilde{L}_t^v\|_1 = t$ , as well as  $\|L_t^u + \tilde{L}_t^v\|_1 = 2t$ . Thus, by (4.12) in Proposition 4.7, if  $t < 1/4m$ , then we have for any  $u, v \in \mathcal{V}$  that

$$|\text{Cov}_{\xi}[e^{-(L_t^u, \xi)}, e^{-(\tilde{L}_t^v, \xi)}]| \leq 2m^2(4t^2 + t^2 + t^2) + 4m^4 t^4 = 12m^2 t^2 + 4m^4 t^4.$$

Since the right-hand side of this inequality is not random, the result then follows by noting that  $t^4 \leq t^2$  when  $t \leq 1$  and taking  $C_1 := \max\{1, 4m, 12m^2, 4m^4\}$ .

**4.5. Proof of Lemma 4.4**

For every  $u, v \in \mathcal{V}$  and  $t > 0$ , let us denote by

$$\mathfrak{D}_t^{u,v} := \min_{\substack{a,b \in \mathcal{V} \\ L_t^u(a), \tilde{L}_t^v(b) \neq 0}} d(a, b)$$

the distance between the ranges of  $X^u$  and  $\tilde{X}^v$  up to time  $t$ . In Section 4.5.1 below we prove the following crude version of Lemma 4.4: For every  $t < \min\{1, 1/4m\}$  and  $u, v \in \mathcal{V}$ ,

$$|\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]| \leq 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v} + 1)^{-\beta} + 64m^4t^4. \tag{4.15}$$

With this in hand, by Minkowski’s inequality, we have that

$$\mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2} \leq 2\mathfrak{C}t^2\mathbf{E}[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta}]^{1/2} + 64m^4t^4 \tag{4.16}$$

for every  $t < \min\{1, 1/4m\}$  and  $u, v \in \mathcal{V}$ .

Next, we control  $\mathfrak{D}_t^{u,v}$  in terms of  $d(u, v)$ . We do this in two cases. Suppose first that  $d(u, v) < 16$ . In this case, we have the trivial bound

$$\mathbf{E}[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta}]^{1/2} \leq 1 \leq 17^\beta (d(u, v) + 1)^{-\beta},$$

which, when combined with (4.16), yields

$$\mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2} \leq 2 \cdot 17^\beta \mathfrak{C}t^2(d(u, v) + 1)^{-\beta} + 64m^4t^4 \tag{4.17}$$

for every  $t < \min\{1, 1/4m\}$  and  $u, v \in \mathcal{V}$  such that  $d(u, v) < 16$ .

Suppose then that  $d(u, v) \geq 16$ . For any  $u, v \in \mathcal{V}$  and  $t > 0$ , we introduce the event

$$E_t^{u,v} := \left\{ \sup_{0 \leq s \leq t} d(X^u(s), u) \leq \frac{d(u, v)}{4} \text{ and } \sup_{0 \leq \tilde{s} \leq t} d(\tilde{X}^v(\tilde{s}), v) \leq \frac{d(u, v)}{4} \right\}.$$

With this in hand, given that  $(\mathfrak{D}_t^{u,v} + 1)^{-\beta} \leq 1$  and  $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$  for all  $x, y \geq 0$ ,

$$\mathbf{E}[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta}]^{1/2} \leq \mathbf{E}[(\mathfrak{D}_t^{u,v} + 1)^{-2\beta} \mathbf{1}_{E_t^{u,v}}]^{1/2} + \mathbf{P}[(E_t^{u,v})^c]^{1/2}.$$

For any outcome in the event  $E_t^{u,v}$ , we have by the triangle inequality that

$$d(X^u(s), \tilde{X}^v(\tilde{s})) \geq d(u, v) - d(X^u(s), u) - d(\tilde{X}^v(\tilde{s}), v) \geq \frac{d(u, v)}{4}$$

for every  $0 \leq s, \tilde{s} \leq t$ . In particular, this means that  $\mathfrak{D}_t^{u,v} \mathbf{1}_{E_t^{u,v}} \geq d(u, v)/4$ . In Section 4.5.2 below, we prove that if  $t < \min\{4/q, 1/4qe\}$  and  $d(u, v) \geq 16$ , then

$$\mathbf{P}[(E_t^{u,v})^c]^{1/2} \leq \frac{\sqrt{2}q^2e^2t^2}{16}. \tag{4.18}$$

Combining these bounds with (4.16), we are led to

$$\begin{aligned} & \mathbf{E}[\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}]^2]^{1/2} \\ & \leq 2 \cdot 4^\beta \mathfrak{C} t^2 (d(u, v) + 1)^{-\beta} + \left( \frac{\sqrt{2} \alpha^2 e^2 \mathfrak{C}}{8} + 64 m^4 \right) t^4 \end{aligned} \tag{4.19}$$

for all  $t < \min\{1, 1/4m, 4/\alpha, 1/4\alpha e\}$  and  $u, v \in \mathcal{V}$  such that  $d(u, v) \geq 16$ .

With (4.17) and (4.19) in hand, in order to prove Lemma 4.4, it only remains to establish (4.15) and (4.18). We do this in the next two subsections.

**4.5.1. Proof of (4.15).** Our main tool to prove (4.15) consists of the power series expansion proved in Proposition 4.14:

$$\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}] = \sum_{p=2}^{\infty} \frac{\mathcal{A}_p(-L_t^u, -\tilde{L}_t^v)}{p!}, \tag{4.20}$$

where the terms  $\mathcal{A}_p$  are defined in (4.14). Thanks to our moment growth assumptions in (3.4), for every  $p \geq 4$  and  $1 \leq m \leq p - 1$ , we have that

$$\begin{aligned} & |\text{Cov}[\xi(v_1) \cdots \xi(v_m), \xi(v_{m+1}) \cdots \xi(v_p)]| \\ & \leq |\mathbf{E}[\xi(v_1) \cdots \xi(v_p)]| + |\mathbf{E}[\xi(v_1) \cdots \xi(v_m)] \mathbf{E}[\xi(v_{m+1}) \cdots \xi(v_p)]| \\ & \leq \mathbf{E}[|\xi(v_1)|^p]^{1/p} \cdots \mathbf{E}[|\xi(v_p)|^p]^{1/p} \\ & \quad + \mathbf{E}[|\xi(v_1)|^m]^{1/m} \cdots \mathbf{E}[|\xi(v_m)|^m]^{1/m} \\ & \quad \cdot \mathbf{E}[|\xi(v_{m+1})|^{p-m}]^{1/(p-m)} \cdots \mathbf{E}[|\xi(v_p)|^{p-m}]^{1/(p-m)} \\ & \leq p! m^p + m!(p-m)! m^p \leq 2p! m^p. \end{aligned}$$

Therefore, by combining (4.14) with the fact that  $\sum_{m=0}^p \binom{p}{m} = 2^p$ , one has

$$\frac{|\mathcal{A}_p(-L_t^u, -\tilde{L}_t^v)|}{p!} \leq 2m^p \sum_{m=1}^{p-1} \binom{p}{m} \|L_t^u\|_1^m \|\tilde{L}_t^v\|_1^{p-m} \leq 2(2mt)^p.$$

Next, if  $\xi$  has covariance decay of order  $\beta$ , then (3.5) implies that

$$\begin{aligned} |\mathcal{A}_2(-L_t^u, -\tilde{L}_t^v)| & \leq \sum_{w_1, w_2 \in \mathcal{V}} |\text{Cov}[\xi(w_1), \xi(w_2)]| L_t^u(w_1) \tilde{L}_t^v(w_2). \\ & \leq \mathfrak{C} (\mathfrak{D}_t^{u,v} + 1)^{-\beta} \|L_t^u\|_1 \|\tilde{L}_t^v\|_1 \leq \mathfrak{C} t^2 (\mathfrak{D}_t^{u,v} + 1)^{-\beta}. \end{aligned}$$

and similarly (3.6) implies that

$$|\mathcal{A}_3(-L_t^u, -\tilde{L}_t^v)| \leq \mathfrak{C} t^3 (\mathfrak{D}_t^{u,v} + 1)^{-\beta}.$$

At this point if we take  $t < \min\{1, 1/4m\}$ , then  $t^3 \leq t^2$ , and thus it follows from the expansion (4.20) and the estimates above that

$$\begin{aligned} |\text{Cov}_\xi[e^{-(L_t^u, \xi)}, e^{-(\tilde{L}_t^v, \xi)}]| &\leq 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v} + 1)^{-\beta} + 2 \sum_{p=4}^\infty (2mt)^p \\ &= 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v} + 1)^{-\beta} + \frac{32m^4t^4}{1 - 2mt} \\ &\leq 2\mathfrak{C}t^2(\mathfrak{D}_t^{u,v} + 1)^{-\beta} + 64m^4t^4. \end{aligned}$$

**4.5.2. Proof of (4.18).** Let us denote by  $\mathcal{S}_t(X)$  the number of jumps that  $X$  makes in the time interval  $[0, t]$ . For every  $x > 0$  and  $v \in \mathcal{V}$ , it is easy to see that

$$\mathbf{P}^v[\max_{0 \leq s \leq t} d(v, X(s)) \geq x] \leq \mathbf{P}^v[\mathcal{S}_t(X) \geq x]. \tag{4.21}$$

For every  $v \in \mathcal{V}$  and  $t \geq 0$ , the number of jumps  $\mathcal{S}_t(X)$  is stochastically dominated by a Poisson random variable with parameter  $t\mathfrak{q}$ . Therefore, applying the Chernoff bound for the tails of Poisson random variables, we obtain that

$$\sup_{v \in \mathcal{V}} \mathbf{P}^v[\max_{0 \leq s \leq t} d(v, X(s)) \geq x] \leq \sup_{v \in \mathcal{V}} \mathbf{P}^v[\mathcal{S}_t(X) \geq x] \leq e^{-\mathfrak{q}t} \left(\frac{\mathfrak{q}et}{x}\right)^x \tag{4.22}$$

for every  $x > \mathfrak{q}t$ . In order to specialize this to (4.18), we use the parameter  $x := d(u, v)/4$ . If  $t < \min\{4/\mathfrak{q}, 1/4\mathfrak{q}e\}$  and  $d(u, v) \geq 16$ , then we have that  $4\mathfrak{q}et < 1$  and  $x > \mathfrak{q}t$ , and thus it follows by a union bound that

$$\begin{aligned} \mathbf{P}[(E_t^{u,v})^c]^{1/2} &\leq \left(\mathbf{P}^u\left[\mathcal{S}_t(X) \geq \frac{d(u, v)}{4}\right] + \mathbf{P}^v\left[\mathcal{S}_t(X) \geq \frac{d(u, v)}{4}\right]\right)^{1/2} \\ &\leq \sqrt{2}e^{-\mathfrak{q}t/2} \left(\frac{4\mathfrak{q}et}{d(u, v)}\right)^{d(u, v)/8} \leq \frac{\sqrt{2}\mathfrak{q}^2e^2t^2}{16}, \end{aligned}$$

as desired.

**4.6. Proof of Lemma 4.6**

**Notation 4.9.** Throughout this proof, we use  $C > 0$  to denote a constant whose exact value may change from one display to the next. If  $C > 0$  depends on some other parameters, this will be explicitly stated.

**4.6.1. Step 1. General upper bound.** Our first step in this proof is to provide a general upper bound for  $\mathbf{E}[e^{-2(L_t^u + \tilde{L}_t^v, V)}]^{1/2}$  that formalizes the intuition (4.6). To this effect, we claim that if (4.2) holds, then

$$-\langle L_t^u, V \rangle \leq -(\kappa t^{1/\alpha} d(0, u))^{\min\{\alpha, 1\}} + \max_{0 \leq s \leq t} (\kappa t^{1/\alpha} d(u, X^u(s)))^{\min\{\alpha, 1\}} - 1 + \mu t \tag{4.23}$$



for every  $u \in \mathcal{V}$  and  $t > 0$ , and similarly for  $-(\tilde{L}_t^v, V)$ . To see this, we note that

$$\begin{aligned} -\langle L_t^u, V \rangle &\leq -\int_0^t (\kappa d(0, X^u(s)))^\alpha ds + \mu t \\ &= -\int_0^t |\kappa(d(0, u) - d(0, u) + d(0, X^u(s)))|^\alpha ds + \mu t \\ &= -\int_0^1 |\kappa t^{1/\alpha}(d(0, u) - d(0, u) + d(0, X^u(ut)))|^\alpha du + \mu t, \end{aligned} \tag{4.24}$$

where the first line follows directly from (4.2), and the last line follows from a change of variables. For any  $x, y \in \mathbb{R}$ , the triangle inequality implies that

$$|x - y|^\alpha \geq |x - y|^{\min\{\alpha, 1\}} - 1 \geq |x|^{\min\{\alpha, 1\}} - |y|^{\min\{\alpha, 1\}} - 1.$$

Applying this to (4.24) yields

$$-\langle L_t^u, V \rangle \leq -(\kappa t^{1/\alpha} d(0, u))^{\min\{\alpha, 1\}} + \max_{0 \leq s \leq t} |\kappa t^{1/\alpha}(d(0, X^u(s)) - d(0, u))|^{\min\{\alpha, 1\}} - 1 + \mu t.$$

We then obtain (4.23) by combining the fact that  $x \mapsto x^{\min\{\alpha, 1\}}$  is increasing for  $x > 0$  with the reverse triangle inequality  $|d(0, X^u(s)) - d(0, u)| \leq d(u, X^u(s))$ .

With (4.23) in hand, we see that  $\mathbf{E}[e^{-2\langle L_t^u + \tilde{L}_t^v, V \rangle}]^{1/2}$  is bounded above by  $e^{2(\mu t - 1) - (\kappa t^{1/\alpha} d(0, u))^{\min\{\alpha, 1\}} - (\kappa t^{1/\alpha} d(0, v))^{\min\{\alpha, 1\}}}$

$$\cdot \mathbf{E}[\exp(\max_{0 \leq s \leq t} (\kappa t^{1/\alpha} d(u, X^u(s)))^{\min\{\alpha, 1\}} + \max_{0 \leq s \leq t} (\kappa t^{1/\alpha} d(v, \tilde{X}^v(s)))^{\min\{\alpha, 1\}})]^{1/2}. \tag{4.25}$$

On the one hand,  $e^{2(\mu t - 1)} \rightarrow e^{-2}$  as  $t \rightarrow 0$  for any choice of  $\mu > 0$ . On the other hand, thanks to the tail bound (4.22), we know that for every  $\theta, \kappa > 0$ , one has

$$\limsup_{t \rightarrow 0} \sup_{u \in \mathcal{V}} \mathbf{E}[\exp(\theta \max_{0 \leq s \leq t} (\kappa t^{1/\alpha} d(u, X^u(s)))^{\min\{\alpha, 1\}})] = 1,$$

and similarly for  $\tilde{X}$ . Therefore, by a straightforward application of Hölder’s inequality on the second line of (4.25), in order to prove Lemma 4.6, it suffices to prove that there exists a constant  $C > 0$  (which only depends on  $\alpha, \beta, d$ , and  $c$ ) such that

$$\limsup_{t \rightarrow 0} t^{2d/\alpha} \sum_{u, v \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0, u))^{\min\{\alpha, 1\}} - (\kappa t^{1/\alpha} d(0, v))^{\min\{\alpha, 1\}}} \leq C \kappa^{-2d}; \tag{4.26}$$

$$\limsup_{t \rightarrow 0} t^{(2d-\beta)/\alpha} \sum_{u, v \in \mathcal{V}} \frac{e^{-(\kappa t^{1/\alpha} d(0, u))^{\min\{\alpha, 1\}} - (\kappa t^{1/\alpha} d(0, v))^{\min\{\alpha, 1\}}}}{(d(u, v) + 1)^\beta} \leq C \kappa^{-2d+\beta} \tag{4.27}$$

for every  $0 < \beta < d$ ; and

$$\limsup_{t \rightarrow 0} t^{d/\alpha} \sum_{u,v \in \mathcal{V}} \frac{e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha} d(0,v))^{\min\{\alpha,1\}}} }{(d(u,v) + 1)^\beta} \leq C \kappa^{-d} \tag{4.28}$$

for every  $\beta > d$ . We now prove these claims in two steps.

**4.6.2. Step 2. Proof of (4.26).** Recalling the definition and upper bound of the coordination sequences  $c_n(v)$  of  $\mathcal{G}$  in (3.1), we have that

$$\begin{aligned} & \sum_{u,v \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha} d(0,v))^{\min\{\alpha,1\}}} \\ &= \left( \sum_{v \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0,v))^{\min\{\alpha,1\}}} \right)^2 = \left( \sum_{n \in \mathbb{N} \cup \{0\}} c_n(0) e^{-(\kappa t^{1/\alpha} n)^{\min\{\alpha,1\}}} \right)^2 \\ &\leq c^2 \left( \sum_{n \in \mathbb{N} \cup \{0\}} n^{d-1} e^{-(\kappa t^{1/\alpha} n)^{\min\{\alpha,1\}}} \right)^2 \\ &= c^2 t^{(-2d+2)/\alpha} \left( \sum_{n \in t^{1/\alpha} \mathbb{N} \cup \{0\}} n^{d-1} e^{-(\kappa n)^{\min\{\alpha,1\}}} \right)^2. \end{aligned} \tag{4.29}$$

By a Riemann sum, we have that

$$\begin{aligned} \lim_{t \rightarrow \infty} t^{2/\alpha} \left( \sum_{n \in t^{1/\alpha} \mathbb{N} \cup \{0\}} n^{d-1} e^{-(\kappa n)^{\min\{\alpha,1\}}} \right)^2 &= \left( \int_0^\infty x^{d-1} e^{-(\kappa x)^{\min\{\alpha,1\}}} dx \right)^2 \\ &= \frac{\kappa^{-2d} \Gamma\left(\frac{d}{\min\{1,\alpha\}}\right)^2}{\min\{1,\alpha^2\}}. \end{aligned} \tag{4.30}$$

Combining this limit with (4.29) yields (4.26), where, as shown on the right-hand side of (4.30), the constant  $C > 0$  only depends on the parameters  $\alpha$ ,  $d$ , and  $c$ .

**4.6.3. Step 3. Proof of (4.27) and (4.28).** We now conclude the proof of Lemma 4.6 by establishing (4.27) and (4.28). We separate the analysis of the sum on the left-hand sides of (4.27) and (4.28) into two parts, namely, the terms  $u, v \in \mathcal{V}$  such that  $d(u, v) > \kappa^{-1} t^{-1/\alpha}$ , and those such that  $d(u, v) \leq \kappa^{-1} t^{-1/\alpha}$ .

We first consider the terms such that  $d(u, v) > \kappa^{-1} t^{-1/\alpha}$ . For these, we have the sequence of upper bounds

$$\begin{aligned} & \sum_{\substack{u,v \in \mathcal{V} \\ d(u,v) > \kappa^{-1} t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha} d(0,v))^{\min\{\alpha,1\}}} }{(d(u,v) + 1)^\beta} \\ &\leq \sum_{\substack{u,v \in \mathcal{V} \\ d(u,v) > \kappa^{-1} t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha} d(0,v))^{\min\{\alpha,1\}}} }{d(u,v)^\beta} \end{aligned}$$

$$\begin{aligned} &< \kappa^\beta t^{\beta/\alpha} \sum_{\substack{u,v \in \mathcal{V} \\ d(u,v) > \kappa^{-1}t^{-1/\alpha}}} e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha}d(0,v))^{\min\{\alpha,1\}}} \\ &\leq \kappa^\beta t^{\beta/\alpha} \left( \sum_{v \in \mathcal{V}} e^{-(\kappa t^{1/\alpha}d(0,v))^{\min\{\alpha,1\}}} \right)^2. \end{aligned}$$

At this point, by replicating the arguments in Section 4.6.2, we get that there exists a constant  $C > 0$  that only depends on  $\alpha, d$ , and  $c$ , and such that

$$\limsup_{t \rightarrow 0} t^{(2d-\beta)/\alpha} \sum_{\substack{u,v \in \mathcal{V} \\ d(u,v) > \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha}d(0,v))^{\min\{\alpha,1\}}}}{(d(u, v) + 1)^\beta} \leq C\kappa^{-2d+\beta} \tag{4.31}$$

if  $0 < \beta < d$ ; and

$$\lim_{t \rightarrow 0} t^{d/\alpha} \sum_{\substack{u,v \in \mathcal{V} \\ d(u,v) > \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha}d(0,v))^{\min\{\alpha,1\}}}}{(d(u, v) + 1)^\beta} = 0 \tag{4.32}$$

if  $\beta > d$ .

We now consider the terms such that  $d(u, v) \leq \kappa^{-1}t^{-1/\alpha}$ . For those terms, we can reformulate the summands as follows:

$$\begin{aligned} &\sum_{\substack{u,v \in \mathcal{V} \\ d(u,v) \leq \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha}d(0,v))^{\min\{\alpha,1\}}}}{(d(u, v) + 1)^\beta} \\ &= \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}}} \left( \sum_{\substack{v \in \mathcal{V} \\ d(u,v) \leq \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha}d(0,v))^{\min\{\alpha,1\}}}}{(d(u, v) + 1)^\beta} \right) \\ &= \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}}} \left( \sum_{\substack{v \in \mathcal{V} \\ d(u,v) \leq \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha}(d(u,v)+d(0,v)-d(u,v)))^{\min\{\alpha,1\}}}}{(d(u, v) + 1)^\beta} \right). \end{aligned} \tag{4.33}$$

For every every  $u, v \in \mathcal{V}$  such that  $d(u, v) \leq \kappa^{-1}t^{-1/\alpha}$ , the fact that  $d(0, v) \geq 0$  gives the upper bound  $e^{-(\kappa t^{1/\alpha}(d(0,v)-d(u,v)))^{\min\{\alpha,1\}}} \leq e$ . Putting this into the above equation, we then obtain that

$$\begin{aligned} (4.33) &\leq e \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}}} \left( \sum_{\substack{v \in \mathcal{V} \\ d(u,v) \leq \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha}d(u,v))^{\min\{\alpha,1\}}}}{(d(u, v) + 1)^\beta} \right) \\ &\leq e \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha}d(0,u))^{\min\{\alpha,1\}}} \left( \sum_{n=0}^{\kappa^{-1}t^{-1/\alpha}} \frac{c_n(u) e^{-(\kappa t^{1/\alpha}n)^{\min\{\alpha,1\}}}}{(n + 1)^\beta} \right). \end{aligned}$$

Thanks to the uniform bound in (3.1), we then have that

$$\begin{aligned}
 (4.33) &\leq ec \left( \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}}} \right) \left( \sum_{n=0}^{\kappa^{-1}t^{-1/\alpha}} \frac{n^{d-1} e^{-(\kappa t^{1/\alpha} n)^{\min\{\alpha,1\}}}}{(n+1)^\beta} \right) \\
 &\leq e^{1+(\kappa t^{1/\alpha})^{\min\{\alpha,1\}}} c \left( \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}}} \right) \\
 &\quad \cdot \left( \sum_{n \in \mathbb{N} \cup \{0\}} (n+1)^{d-1-\beta} e^{-(\kappa t^{1/\alpha} (n+1))^{\min\{\alpha,1\}}} \right) \\
 &= e^{1+o(1)} c \left( \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}}} \right) \left( \sum_{n \in \mathbb{N}} n^{d-1-\beta} e^{-(\kappa t^{1/\alpha} n)^{\min\{\alpha,1\}}} \right).
 \end{aligned}
 \tag{4.34}$$

We now analyze the two sums on the right-hand side of (4.34). Looking at the first term, the same analysis carried out in Section 4.6.2 implies that

$$\limsup_{t \rightarrow 0} t^{d/\alpha} \sum_{u \in \mathcal{V}} e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}}} \leq C \kappa^{-d}$$

for some  $C$  that only depends on  $\alpha$ ,  $d$ , and  $c$ . Next, the second sum in (4.34) is analyzed differently depending on whether  $0 < \beta < d$  or  $\beta > d$ : On the one hand, if  $\beta < d$ , then by a Riemann sum we have that

$$\begin{aligned}
 \lim_{t \rightarrow 0} t^{(d-\beta)/\alpha} \sum_{n \in \mathbb{N}} n^{d-1-\beta} e^{-(\kappa t^{1/\alpha} n)^{\min\{\alpha,1\}}} &= \lim_{t \rightarrow 0} t^{1/\alpha} \sum_{n \in t^{1/\alpha} \mathbb{N}} n^{d-1-\beta} e^{-(\kappa n)^{\min\{\alpha,1\}}} \\
 &= \int_0^\infty x^{d-1-\beta} e^{-(\kappa x)^{\min\{\alpha,1\}}} dx \\
 &= \frac{\kappa^{-d+\beta} \Gamma\left(\frac{d-\beta}{\min\{\alpha,1\}}\right)}{\min\{\alpha,1\}}.
 \end{aligned}$$

On the other hand, if  $\beta > d$ , then we have by dominated convergence that

$$\lim_{t \rightarrow 0} \sum_{n \in \mathbb{N}} n^{d-1-\beta} e^{-(\kappa t^{1/\alpha} n)^{\min\{\alpha,1\}}} = \sum_{n \in \mathbb{N}} n^{d-1-\beta};$$

we know that the sum on the right-hand side is convergent since  $\beta > d$ .

Putting these two limits back into (4.34), we then get that there exists a constant  $C > 0$  (which only depends on  $\alpha$ ,  $d$ ,  $\beta$ , and  $c$ ) such that

$$\limsup_{t \rightarrow 0} t^{(2d-\beta)/\alpha} \sum_{\substack{u, v \in \mathcal{V} \\ d(u,v) \leq \kappa^{-1}t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha} d(0,u))^{\min\{\alpha,1\}} - (\kappa t^{1/\alpha} d(0,v))^{\min\{\alpha,1\}}}}{(d(u,v) + 1)^\beta} \leq C \kappa^{-2d+\beta}$$

when  $\beta < d$ , and such that

$$\limsup_{t \rightarrow 0} t^{d/\alpha} \sum_{\substack{u, v \in \mathcal{V} \\ d(u, v) \leq \kappa^{-1} t^{-1/\alpha}}} \frac{e^{-(\kappa t^{1/\alpha} d(0, u))^{\min\{\alpha, 1\}} - (\kappa t^{1/\alpha} d(0, v))^{\min\{\alpha, 1\}}}}{(d(u, v) + 1)^\beta} \leq C \kappa^{-d}$$

when  $\beta > d$ . Combining this with (4.31) and (4.32) concludes the proof of (4.27) and (4.28). With this in hand, we have now completed the proof of Lemma 4.6.

### 5. Spectral mapping and multiplicity

A crucial aspect of the proof of Theorem 3.17 is the ability to relate exponential linear statistics of the eigenvalue point process (3.11) to the trace of  $K_t$  via the identities

$$\text{Tr}[K_t] = \sum_{\mu \in \sigma(K_t) \setminus \{0\}} m_\alpha(\mu, K_t) \mu = \sum_{\lambda \in \sigma(H)} m_\alpha(\lambda, H) e^{-t\lambda} \in (0, \infty). \tag{5.1}$$

Though we expect that such a result is known (or at least folklore) in the operator theory community, we were not able to locate any reference that contains all of the precise statements that we need to prove (5.1). (This is especially so since the level of generality in this paper allows for non-self-adjoint operators.) As such, our purpose in this section is to provide a general criterion for an identity of the form (5.1) to hold (as well as a few more properties), which we then use in Section 6 to wrap up the proof of Theorem 3.17.

We begin this section with a definition:

**Definition 5.1.** We say that a linear operator  $T$  on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$  is *finite-dimensional* if there exists a finite set  $\mathcal{U} \subset \mathcal{V}$  such that  $T(u, v) = 0$  whenever  $(u, v) \notin \mathcal{U} \times \mathcal{U}$ . In particular, if we enumerate the set  $\mathcal{U} = \{u_1, \dots, u_{|\mathcal{U}|}\}$ , then  $T$  has the same spectrum as the  $|\mathcal{U}| \times |\mathcal{U}|$  matrix  $M_T$  with entries

$$M_T(i, j) := T(u_i, u_j), \quad 1 \leq i, j \leq |\mathcal{U}|. \tag{5.2}$$

The result that we prove in this section is as follows:

**Proposition 5.2.** *Let  $(T_t)_{t>0}$  be a strongly continuous semigroup of trace class operators on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$  such that  $\|T_t\|_{\text{op}} \leq e^{-\omega t}$  for some  $\omega < 0$ , and let  $G$  be its infinitesimal generator. The following holds:*

1.  $G$  is closed and densely defined on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$ ;
2.  $\sigma(G) = \sigma_p(G)$ , and  $\Re(\lambda) \geq \omega$  for all  $\lambda \in \sigma(G)$ ;
3. for every  $t > 0$ ,  $\sigma(T_t) \setminus \{0\} = \{e^{-t\lambda} : \lambda \in \sigma(G)\}$ .

Moreover, if there exists a sequence of finite-dimensional operators  $(G_n)_{n \in \mathbb{N}}$  such that

$$\lim_{n \rightarrow \infty} \|\Re(z, G_n) - \Re(z, G)\|_{\text{op}} = 0 \tag{5.3}$$

for at least one  $z \in \mathbb{C} \setminus \sigma(G)$  and such that

$$\lim_{n \rightarrow \infty} \|e^{-tG_n} - T_t\|_{\text{op}} = 0, \tag{5.4}$$

then for every  $t > 0$  and  $\mu \in \sigma(T_t) \setminus \{0\}$ ,

$$m_a(\mu, T_t) = \sum_{\lambda \in \sigma(G): e^{-t\lambda} = \mu} m_a(\lambda, G). \tag{5.5}$$

As a direct consequence of the above proposition, we have that

$$\text{Tr}[T_t] = \sum_{\mu \in \sigma(T_t) \setminus \{0\}} m_a(\mu, T_t)\mu = \sum_{\lambda \in \sigma(G)} m_a(\lambda, G)e^{-t\lambda} \in \mathbb{C}$$

for all  $t > 0$ , which is precisely the kind of statement that we are looking for. The remainder of this section is now devoted to the proof of Proposition 5.2.

### 5.1. Step 1. Closed generator and spectral mapping

We begin with the more straightforward aspects of the statement of Proposition 5.2, namely, items (1)–(3). Since  $(T_t)_{t>0}$  is strongly continuous and  $\|T_t\|_{\text{op}} \leq e^{-\omega t}$ , it follows from the Hille–Yosida theorem (e.g., [17, Chapter II, Corollary 3.6]) that  $G$  is closed and densely defined on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$ . Moreover,  $\Re(\lambda) \geq \omega$  for every  $\lambda \in \sigma(G)$ . Given that the  $T_t$  are trace class, we know that  $\sigma(T_t) = \sigma_p(T_t)$  and that

$$\text{Tr}[T_t] = \sum_{\mu \in \sigma(T_t) \setminus \{0\}} m_a(\mu, T_t)\mu \in \mathbb{C}$$

by Lidskii’s theorem (e.g., [40, Sections 3.6 and 3.12]). Next, by the spectral mapping theorem (e.g., [17, Chapter IV, (3.7) and (3.16)]), we know that for every  $t > 0$ ,

$$\{e^{-t\lambda}: \lambda \in \sigma(G)\} \subset \sigma(T_t) \quad \text{and} \quad \{e^{-t\lambda}: \lambda \in \sigma_p(G)\} = \sigma_p(T_t) \setminus \{0\}. \tag{5.6}$$

In particular,  $\sigma(G) = \sigma_p(G)$ , concluding the proof of Proposition 5.2(1)–(3).

### 5.2. Step 2. Multiplicities in finite dimensions

It now remains to prove (5.5). Before we prove this result, we first prove the corresponding statement in finite dimensions, namely:

**Lemma 5.3.** *Let  $T$  be a finite-dimensional linear operator on  $\ell^2_{\mathbb{Z}}(\mathcal{V})$  and  $F: \mathbb{C} \rightarrow \mathbb{C}$  be an analytic function. For every  $\mu \in \sigma(F(T)) = F(\sigma(T))$ , one has*

$$m_a(\mu, F(T)) = \sum_{\lambda \in \sigma(T): F(\lambda) = \mu} m_a(\lambda, T).$$

Applying this to the exponential map and the operators  $e^{-tG_n}$ , we are led to the fact that for every  $n \in \mathbb{N}$ ,  $t > 0$ , and  $\mu \in \sigma(G_n)$  one has

$$m_a(\mu, e^{-tG_n}) = \sum_{\lambda \in \sigma(G_n): e^{-t\lambda} = \mu} m_a(\lambda, G_n). \tag{5.7}$$

*Proof of Lemma 5.3.* It suffices to prove the result with  $T$  replaced by  $M_T$  and  $F(T)$  replaced by  $F(M_T)$ , where  $M_T$  is the matrix defined in (5.2). Let  $M_T = PJP^{-1}$  be the Jordan canonical form of  $M_T$ . That is,  $J$  is the direct sum of the Jordan blocks of  $M_T$ , and in particular the number of times any  $\lambda \in \mathbb{C}$  appears on the diagonal of  $J$  is equal to  $m_a(\lambda, M_T)$ . By the standard analytic functional calculus for matrices, we know that  $F(M_T) = PF(J)P^{-1}$ , where  $F(J)$  is the direct sum of the transformed Jordan blocks of  $M_T$ , wherein any  $k \times k$  Jordan block of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix}$$

is transformed into the upper triangular matrix

$$\begin{bmatrix} F(\lambda) & F'(\lambda) & F''(\lambda)/2 & \dots & F^{(k-1)}(\lambda)/(k-1)! \\ & F(\lambda) & F'(\lambda) & \dots & F^{(k-2)}(\lambda)/(k-2)! \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & F'(\lambda) \\ & & & & F(\lambda) \end{bmatrix}.$$

Given that the characteristic polynomial of  $F(M_T)$  is the same as that of  $F(J)$ , this readily implies the result. ■

### 5.3. Step 3. Passing to the limit

We now complete the proof of Proposition 5.2 by arguing that the identity (5.7) persists in the large  $n$  limit. Thanks to (5.3) and (5.4), we know that we have the convergences  $G_n \rightarrow G$  and  $e^{-tG_n} \rightarrow T_t$  for every  $t > 0$  in the generalized sense of Kato (see [30, Chapter IV, (2.9), (2.20), and p. 206] for a definition of convergence in

the generalized sense, and [30, Chapter IV, Theorems 2.23 a) and (2.25)] for a proof that norm-resolvent and norm convergence implies convergence in the generalized sense). As shown in [30, Chapter IV, Theorem 3.16] (see also [30, Chapter IV, Section 5] for a discussion specific to the context of isolated eigenvalues), convergence in the generalized sense implies the following spectral continuity results:

**Notation 5.4.** In what follows, we use  $B(z, r)$  to denote the closed ball in the complex plane centered at  $z \in \mathbb{Z}$  and with radius  $r > 0$ .

**Corollary 5.5.** For every  $\lambda \in \sigma(G)$ , if  $\varepsilon > 0$  is such that  $\sigma(G) \cap B(\lambda, \varepsilon) = \{\lambda\}$ , then there exists  $N \in \mathbb{N}$  large enough so that

$$\sum_{\tilde{\lambda} \in \sigma(G_n) \cap B(\lambda, \varepsilon)} m_a(\tilde{\lambda}, G_n) = m_a(\lambda, G) \tag{5.8}$$

whenever  $n \geq N$ .

Conversely, for every  $t > 0$  and  $\mu \in \sigma(T_t) \setminus \{0\}$ , if  $\varepsilon > 0$  is such that  $\sigma(T_t) \cap B(\mu, \varepsilon) = \{\mu\}$ , then there exists  $N \in \mathbb{N}$  large enough so that

$$\sum_{\tilde{\mu} \in (e^{-tG_n}) \cap B(\mu, \varepsilon)} m_a(\tilde{\mu}, e^{-tG_n}) = m_a(\mu, T_t) \tag{5.9}$$

whenever  $n \geq N$ .

We are now ready to prove (5.2). We first show that for every  $t > 0$  and  $\mu \in \sigma(T_t) \setminus \{0\}$ , the set  $\{\lambda \in \sigma(G) : e^{-t\lambda} = \mu\}$  is finite. Suppose by contradiction that this is not the case. Then, for any integer  $M > 0$ , we can find at least  $M$  distinct eigenvalues  $\lambda_1, \dots, \lambda_M \in \sigma(G)$  such that  $e^{-t\lambda_i} = \mu$ . By taking a small enough  $\varepsilon > 0$  and large enough  $N \in \mathbb{N}$ , a combination of (5.7) and (5.9) yields

$$m_a(\mu, T_t) = \sum_{\tilde{\mu} \in \sigma(e^{-tG_N}) \cap B(\mu, \varepsilon)} m_a(\tilde{\mu}, e^{-tG_N}) = \sum_{\tilde{\lambda} \in \sigma(G_N) : e^{-t\tilde{\lambda}} \in B(\mu, \varepsilon)} m_a(\tilde{\lambda}, G_N). \tag{5.10}$$

Since  $z \mapsto e^{-tz}$  is continuous, we can take  $\delta > 0$  small enough so that

1. if  $\tilde{\lambda} \in B(\lambda_i, \delta)$  for some  $1 \leq i \leq M$ , then  $e^{-t\tilde{\lambda}} \in B(\mu, \varepsilon)$ ;
2.  $\sigma(G) \cap B(\lambda_i, \delta) = \{\lambda_i\}$  for every  $1 \leq i \leq M$ .

Thus, up to increasing the value of  $N$  if necessary, an application of (5.8) to the right-hand side of (5.10) then gives

$$m_a(\mu, T_t) \geq \sum_{i=1}^M \sum_{\tilde{\lambda} \in \sigma(G_N) \cap B(\lambda_i, \delta)} m_a(\tilde{\lambda}, G_N) = \sum_{i=1}^M m_a(\lambda_i, G) \geq M. \tag{5.11}$$



Since  $M$  was arbitrary, this implies that  $m_a(\mu, T_t) = \infty$ . Since  $T_t$  is trace class this cannot be the case, hence we conclude that  $\{\lambda \in \sigma(G) : e^{-t\lambda} = \mu\}$  is finite.

By repeating the argument leading up to (5.11), but this time letting  $M$  be equal to the number of eigenvalues in the set  $\{\lambda \in \sigma(G) : e^{-t\lambda} = \mu\}$ , we obtain that

$$m_a(\mu, T_t) \geq \sum_{\lambda \in \sigma(G) : e^{-t\lambda} = \mu} m_a(\lambda, G).$$

We now proceed to prove the reverse inequality. Recall that  $\{\lambda \in \sigma(G) : e^{-t\lambda} = \mu\}$  contains finitely many elements. Denote them by  $\lambda_1, \dots, \lambda_M$  for some  $M \in \mathbb{N}$ . Thanks to (5.8), we can find a small enough  $\varepsilon > 0$  and large enough  $N \in \mathbb{N}$  such that

$$\sum_{i=1}^M m_a(\lambda_i, G) = \sum_{\tilde{\lambda} \in \bigcup_{i=1}^M \sigma(G_N) \cap B(\lambda_i, \varepsilon)} m_a(\tilde{\lambda}, G_N) = \sum_{\tilde{\lambda} \in \sigma(G_N) \cap (\bigcup_{i=1}^M B(\lambda_i, \varepsilon))} m_a(\tilde{\lambda}, G_N).$$

Then, by (5.7), one has

$$\sum_{\tilde{\lambda} \in \sigma(G_N) \cap (\bigcup_{i=1}^M B(\lambda_i, \varepsilon))} m_a(\tilde{\lambda}, G_N) = \sum_{\substack{\tilde{\mu} \in \sigma(e^{-tG_N}) \\ \tilde{\mu} \in e^{-t}(\bigcup_{i=1}^M B(\lambda_i, \varepsilon))}} m(\tilde{\mu}, e^{-tG_N}), \tag{5.12}$$

where we use  $e^{-t}(B)$  to denote the image of a set  $B \subset \mathbb{C}$  through the exponential map  $z \mapsto e^{-tz}$ . Since the exponential map is open and  $e^{-t\lambda_i} = \mu$  for all  $1 \leq i \leq M$ , we can find a small enough  $\delta > 0$  such that  $B(\mu, \delta) \subset e^{-t}(\bigcup_{i=1}^M B(\lambda_i, \varepsilon))$  and  $\sigma(T_t) \cap B(\mu, \delta) = \{\mu\}$ . As a result, we get

$$\sum_{i=1}^M m_a(\lambda_i, G) \geq \text{r.h.s. of (5.12)} \geq \sum_{\tilde{\mu} \in \sigma(e^{-tG_N}) \cap B(\mu, \delta)} m_a(\tilde{\mu}, e^{-tG_N}). \tag{5.13}$$

At this point, up to increasing  $N$  if necessary an application of (5.9) then yields

$$\sum_{i=1}^M m_a(\lambda_i, G) \geq \sum_{\tilde{\mu} \in \sigma(e^{-tG_N}) \cap B(\mu, \delta)} m_a(\tilde{\mu}, e^{-tG_N}) = m_a(\mu, T_t),$$

thus concluding the proof of (5.5) and Proposition 5.2.

### 6. Proof of Theorem 3.17

In this section, we prove Theorem 3.17. We suppose throughout that Assumptions 3.8 and 3.12 hold. We begin with a notation:

**Notation 6.1.** Throughout this proof, we denote transition semigroup of  $X$  by

$$\Pi_t(u, v) = \mathbf{P}^u[X(t) = v], \quad t \geq 0, u, v \in \mathcal{V}.$$

**6.1. Step 1. Boundedness**

Our first step in the proof is to show that, almost surely,  $K_t$  is a bounded linear operator on  $\ell^2_{\mathcal{Z}}(\mathcal{V})$  with  $\|K_t\|_{\text{op}} \leq e^{\omega t}$  for every  $t > 0$  for some  $\omega < 0$ . As is typical in Schrödinger semigroup theory, this relies on controlling the minimum of the random potential  $V + \xi$ . To this end, we have the following result:

**Lemma 6.2.** *Define the random variable*

$$\omega_0 := \inf_{v \in \mathcal{V}} (V(v) + \xi(v)). \tag{6.1}$$

$\omega_0 > -\infty$  almost surely.

*Proof.* Thanks to (3.3), it suffices to prove that

$$\liminf_{n \rightarrow \infty} \left( \inf_{v \in \mathcal{V}: d(0,v) \leq n} \frac{\xi(v)}{\log n} \right) > -\infty \text{ almost surely.} \tag{6.2}$$

By a union bound and Markov’s inequality, for every  $\theta, \lambda > 0$ ,

$$\mathbf{P} \left( \inf_{v \in \mathcal{V}: d(0,v) \leq n} \xi(v) \leq -\lambda \right) \leq \sum_{v \in \mathcal{V}: d(0,v) \leq n} e^{-\theta \lambda} \mathbf{E}[e^{-\theta \xi(v)}].$$

On the one hand, thanks to (3.1), we have that

$$|\{v \in \mathcal{V}: d(0, v) \leq n\}| \leq c \sum_{m=1}^n m^{d-1} \leq c + c \int_1^n x^{d-1} dx \leq Cn^d$$

for some constant  $C > 0$ . On the other hand, thanks to the moment bound (3.4), there exists a  $\theta > 0$  small enough so that

$$\sup_{v \in \mathcal{V}} \mathbf{E}[e^{-\theta \xi(v)}] < \infty.$$

Combining these two observations, we conclude that there exists  $\tilde{C}, \theta > 0$  such that

$$\mathbf{P} \left( \inf_{v \in \mathcal{V}: d(0,v) \leq n} \xi(v) \leq -\lambda \right) \leq \tilde{C} n^d e^{-\theta \lambda}, \quad \lambda > 0.$$

If we take  $\lambda = \lambda(n) = c \log n$  for large enough  $c > 0$ , then  $\sum_{n \in \mathbb{N}} \tilde{C} n^d e^{-\theta \lambda(n)} < \infty$ ; hence (6.2) holds by the Borel–Cantelli lemma. ■

As a direct application of Lemma 6.2, we have  $K_t(u, v) \leq e^{-\omega_0 t} \Pi_t(u, v)$  for every  $u, v \in \mathcal{V}$ , where we take  $\omega_0$  as in (6.1). In particular,  $\|K_t\|_{\text{op}} \leq e^{-\omega_0 t} \|\Pi_t\|_{\text{op}}$ . Given that  $\omega_0 > -\infty$  almost surely by Lemma 6.2, it suffices to prove that  $\Pi_t$  is bounded with  $\|\Pi_t\|_{\text{op}} \leq e^{-t\omega_1}$  for some constant  $\omega_1 \leq 0$ . We now prove this.

Note that, for every  $f \in \ell^2(\mathcal{V})$ , we have by Jensen’s inequality that

$$\|\Pi_t f\|_2^2 = \sum_{v \in \mathcal{V}} \mathbf{E}^v [f(X(t))]^2 \leq \sum_{v \in \mathcal{V}} \mathbf{E}^v [f(X(t))^2] = \sum_{u, v \in \mathcal{V}} \Pi_t(v, u) f(u)^2,$$

from which we conclude that

$$\|\Pi_t\|_{\text{op}} \leq \sqrt{\sup_{u \in \mathcal{V}} \sum_{v \in \mathcal{V}} \Pi_t(v, u)}.$$

If we define the matrix

$$H_X(u, v) := \begin{cases} -q(u)\Pi(u, v) & \text{if } u \neq v \\ q(u) & \text{if } u = v \end{cases}, \quad u, v \in \mathcal{V}$$

(i.e., the Markov generator of  $X$ ), then we can write

$$\sum_{v \in \mathcal{V}} \Pi_t(v, u) = \sum_{v \in \mathcal{V}} \sum_{n=0}^{\infty} \frac{(-t)^n H_X^n(v, u)}{n!} \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{v \in \mathcal{V}} |H_X^n(v, u)|.$$

Noting that

$$\sup_{u, v \in \mathcal{V}} |H_X^n(u, v)| \leq \|H_X^n\|_{\text{op}} \leq \|H_X\|_{\text{op}}^n,$$

for every  $u, v \in \mathcal{V}$ , we have the bound

$$|H_X^n(v, u)| \leq \|H_X\|_{\text{op}}^n \mathbf{1}_{\{d(u, v) \leq n\}}.$$

By (3.1), for any  $u \in \mathcal{V}$ , the number of  $v \in \mathcal{V}$  such that  $(u, v)$  is an edge is bounded by  $c$ . Thus, the number of  $v \in \mathcal{V}$  such that  $d(u, v) \leq n$  is crudely bounded by  $c^n$ . Consequently,

$$\|\Pi_t\|_{\text{op}}^2 \leq \sup_{u \in \mathcal{V}} \sum_{v \in \mathcal{V}} \Pi_t(v, u) \leq \sum_{n=0}^{\infty} \frac{(tc \|H_X\|_{\text{op}})^n}{n!} = e^{c \|H_X\|_{\text{op}} t}.$$

Thus, it now suffices to prove that  $\|H_X\|_{\text{op}} < \infty$ .

Recall that, by assumption,  $q := \sup_{u \in \mathcal{V}} q(u) < \infty$ . For every  $f \in \ell^2(\mathcal{V})$ ,

$$\|H_X f\|_2^2 \leq q^2 \sum_{u \in \mathcal{V}} \left( \sum_{v \in \mathcal{V}} \mathbf{1}_{\{(u, v) \in \mathcal{E}\}} f(v) \right)^2 \leq q^2 2^c \sum_{u, v \in \mathcal{V}} \mathbf{1}_{\{(u, v) \in \mathcal{E}\}} f(v)^2,$$

where the last inequality comes from the fact that

$$(x_1 + \dots + x_c)^2 \leq 2^c (x_1^2 + \dots + x_c^2), \quad x_i \in \mathbb{R},$$

and that, by (3.1), for every  $v \in \mathcal{V}$  there are at most  $c$  vertices  $u$  such that  $(u, v) \in \mathcal{E}$ . Using once again this last observation, we have that

$$\sum_{u,v \in \mathcal{V}} \mathbf{1}_{\{(u,v) \in \mathcal{E}\}} f(v)^2 \leq c \|f\|_2^2,$$

from which we conclude that  $\|H_X\|_{\text{op}}^2 \leq c^2 2^c c$ , as desired.

**6.2. Step 2. Continuity of the semigroup**

We now prove the almost-sure strong continuity and semigroup property. Since  $X$  is Markov and local time is additive, the semigroup property is trivial. We now prove strong continuity. Let  $C_{0,z}(\mathcal{V})$  denote the set of functions  $f: \mathcal{V} \rightarrow \mathbb{R}$  that are finitely supported on  $\mathcal{V} \setminus z$ . Since  $C_{0,z}(\mathcal{V})$  is dense in  $\ell_z^2(\mathcal{V})$  and a semigroup of bounded linear operators is strongly continuous if and only if it is weakly continuous (e.g., [17, Chapter I, Theorem 5.8]), it suffices to prove that  $\langle f, K_t g - g \rangle \rightarrow 0$  as  $t \rightarrow 0$  for every  $f, g \in C_{0,z}(\mathcal{V})$ . For every  $g \in C_{0,z}(\mathcal{V})$ , we know that

$$\lim_{t \rightarrow 0} g(X(t)) e^{-\langle L_t, V + \xi \rangle} = g(X(0)) \mathbf{1}_{\{X(0) \neq z\}} = g(X(0)) \quad \text{almost surely.}$$

By the definition of  $\omega_0$ , it follows that  $\langle L_t, V + \xi \rangle \geq \omega_0 t$  which implies that

$$|g(X(t)) e^{-\langle L_t, V + \xi \rangle}| \leq \|g\|_{\ell^\infty} e^{-\omega_0 t}.$$

Since the right-hand side of this inequality is independent of  $X$ , it follows from dominated convergence that

$$\lim_{t \rightarrow 0} K_t g(v) = \lim_{t \rightarrow 0} \mathbf{E}^v [g(X(t)) e^{-\langle L_t, V + \xi \rangle}] = g(v) \quad \text{almost surely}$$

for every  $v \in \mathcal{V}$ . Finally, given that for every  $v \in \mathcal{V}$ , we have

$$|f(v)(K_t g(v) - g(v))| \leq \|f\|_{\ell^\infty} \|g\|_{\ell^\infty} (e^{-\omega_0 t} + 1) \mathbf{1}_{\{f(v) \neq 0\}},$$

which is summable in  $v$  whenever  $f \in C_{0,z}(\mathcal{V})$ , we obtain  $\langle f, K_t g - g \rangle \rightarrow 0$  as  $t \rightarrow 0$  by dominated convergence.

**6.3. Step 3. Trace class**

By the semigroup property, for every  $t > 0$ , we can write  $K_t$  as the product  $K_{t/2} K_{t/2}$ . Thus, given that the product of any two Hilbert–Schmidt operators is trace class (e.g., [40, Theorem 3.7.4]), it suffices to prove that, almost surely,  $K_t$  is Hilbert–Schmidt for all  $t > 0$ , that is,

$$\sum_{u,v \in \mathcal{V}} K_t(u, v)^2 < \infty.$$

By (6.2), there exists finite random variables  $\kappa, \mu > 0$  that only depend on  $\xi$  such that

$$V(v) + \xi(v) \geq (\kappa d(0, v))^\alpha - \mu, \quad v \in \mathcal{V}$$

almost surely. Therefore, it suffices to prove the result with  $K_t$  replaced by the kernel

$$\tilde{K}_t(u, v) := e^{\mu t} \mathbf{E}^u [e^{-\langle L_t, (\kappa d(0, \cdot))^\alpha \rangle} \mathbf{1}_{\{X(t)=v\}}], \quad u, v \in \mathcal{V}.$$

By Jensen’s inequality,

$$\begin{aligned} \sum_{u, v \in \mathcal{V}} \tilde{K}_t(u, v)^2 &\leq e^{2\mu t} \sum_{u, v \in \mathcal{V}} \mathbf{E}^u [e^{-2\langle L_t, (\kappa d(0, \cdot))^\alpha \rangle} \mathbf{1}_{\{X(t)=v\}}] \\ &\leq e^{2\mu t} \sum_{u \in \mathcal{V}} \mathbf{E}^u [e^{-2\langle L_t, (\kappa d(0, \cdot))^\alpha \rangle}]. \end{aligned}$$

At this point, the same argument used in (4.22), (4.24), and (4.25) implies that there exists some finite constant  $C_{\kappa, t} > 0$  (which depends on  $\kappa$  and  $t$ ) such that

$$\sum_{u, v \in \mathcal{V}} \tilde{K}_t(u, v)^2 \leq C_{\kappa, t} e^{2\mu t} \sum_{u \in \mathcal{V}} e^{-2t(\kappa d(0, u))^\alpha}.$$

Then, writing the above sum as

$$\sum_{u \in \mathcal{V}} e^{-2t(\kappa d(0, u))^\alpha} = \sum_{n \in \mathbb{N}} c_n(0) e^{-2t(\kappa n)^\alpha},$$

this is easily seen to be finite for all  $t > 0$  by (3.1).

### 6.4. Step 4. Infinitesimal generator

We now prove the properties of the generator  $H$ , except for number rigidity of its spectrum, which is relegated to the next (and final) step of the proof. That the generator of  $K_t$  is of the form (3.10) follows from the straightforward computation that for every  $u, v \in \mathcal{V} \setminus \mathcal{Z}$ ,

$$\lim_{t \rightarrow 0} \frac{\mathbf{1}_{\{u=v\}} - K_t(u, v)}{t} = H(u, v) \quad \text{almost surely}$$

(indeed, recall that by definition of the process  $X$ ,  $\Pi_t(u, v) = q(u)\Pi(u, v)t + o(t)$  as  $t \rightarrow 0$  whenever  $u \neq v$ , and that  $K_t(u, v) = 0$  if  $u \in \mathcal{Z}$  or  $v \in \mathcal{Z}$ ).

Almost surely,  $(K_t)_{t>0}$  is a strongly continuous semigroup of trace class operators and  $\|K_t\|_{\text{op}} \leq e^{-\omega t}$ . Therefore, by Proposition 5.2 (1)–(3), the following holds almost surely:

1.  $H$  is closed and densely defined on  $\ell^2(\mathcal{V})$ ;
2.  $\sigma(H) = \sigma_p(H)$ , and  $\Re(\lambda) \geq \omega$  for all  $\lambda \in \sigma(H)$ ;

3. for every  $t > 0$ ,  $\sigma(K_t) \setminus \{0\} = \{e^{-t\lambda} : \lambda \in \sigma(H)\}$ .

It now remains to establish the trace identity (5.1), which is crucial in our proof of rigidity. The fact that  $\text{Tr}[K_t]$  is a positive real number follows from the fact that

$$\text{Tr}[K_t] = \sum_{v \in \mathcal{V}} K_t(v, v)$$

and that  $K_t(u, v) \in [0, \infty)$  for all  $u, v \in \mathcal{V}$ . To prove the remainder of (5.1), as per Proposition 5.2, we need to find a sequence of finite-dimensional operators that converge to  $H$  and  $K_t$  in the sense of (5.3) and (5.4).

To this end, for every  $n \in \mathbb{N}$ , let us denote the subset

$$\mathcal{V}_n := \{v \in \mathcal{V} : d(0, v) \leq n\} \subset \mathcal{V}.$$

Given that  $\mathcal{G}$  has uniformly bounded degrees, this must be finite. Thus, the operators

$$H_n(u, v) := H(u, v)\mathbf{1}_{\{(u,v) \in \mathcal{V}_n\}}, \quad u, v \in \mathcal{V}$$

are finite-dimensional in the sense of Definition 5.1. More specifically,  $H_n$  is the restriction of  $H$  to the set  $\mathcal{V}_n$  with Dirichlet boundary on  $\mathcal{V} \setminus \mathcal{V}_n$ . In particular, if for every  $n \in \mathbb{N}$  we denote the hitting time

$$\tau_n := \inf_{t \geq 0} \{t : X(t) \notin \mathcal{V}_n\},$$

then  $e^{-tH_n}$  is the integral operator on  $\ell^2_{\infty}(\mathcal{V})$  with kernel

$$e^{-tH_n}(u, v) = \mathbf{E}^u[e^{-\langle L_t, V + \xi \rangle} \mathbf{1}_{\{X(t)=v\}} \mathbf{1}_{\{\tau_n > t\}}]. \tag{6.3}$$

The proof of (5.1) is now a matter of establishing the following result:

**Lemma 6.3.** *Almost surely, it holds that*

$$\lim_{n \rightarrow \infty} \|\Re(z, H_n) - \Re(z, H)\|_{\text{op}} = 0 \tag{6.4}$$

for every  $z \in \mathbb{C}$  such that  $\Re(z) < \omega$  and

$$\lim_{n \rightarrow \infty} \|e^{-tG_n} - K_t\|_{\text{op}} = 0 \tag{6.5}$$

for every  $t > 0$ .

*Proof.* Given that  $0 \leq e^{-tH_n}(u, v) \leq K_t(u, v)$  for all  $u, v \in \mathcal{V}$ , it is easy to see that  $\|e^{-tH_n}\|_{\text{op}} \leq \|K_t\|_{\text{op}} \leq e^{-\omega t}$  for all  $t > 0$  almost surely. In particular, any  $z \in \mathbb{C}$  such

that  $\Re(z) < \omega$  is in the resolvent set of  $H_n$  and  $H$  for all  $n$ . Consequently, it follows from [17, Chapter II, Theorem 1.10] that

$$\|\Re(z, H_n) - \Re(z, H)\|_{\text{op}} = \left\| \int_0^\infty e^{tz} (e^{-tG_n} - K_t) dt \right\|_{\text{op}} \leq \int_0^\infty e^{tz} \|e^{-tG_n} - K_t\|_{\text{op}} dt,$$

where the last inequality follows from [15, Chapter II, Theorem 4 (ii)]. Given that

$$\int_0^\infty e^{tz} \|e^{-tG_n} - K_t\|_{\text{op}} dt \leq \int_0^\infty e^{tz} (\|e^{-tG_n}\|_{\text{op}} + \|K_t\|_{\text{op}}) dt \leq 2 \int_0^\infty e^{t(z-\omega)} dt < \infty$$

whenever  $\Re(z) < \omega$ , we get that (6.4) is a consequence of (6.5) by an application of the dominated convergence theorem.

Let us then prove (6.5). Since the Hilbert–Schmidt norm dominates the operator norm, it suffices to prove that

$$\sum_{u,v \in \mathcal{V}} (e^{-tG_n}(u, v) - K_t(u, v))^2 = \sum_{u,v \in \mathcal{V}} \mathbf{E}^u [e^{-\langle L_t, V + \xi \rangle} \mathbf{1}_{\{X(t)=v\}} \mathbf{1}_{\{\tau_n \leq t\}}]^2 \tag{6.6}$$

vanishes as  $n \rightarrow \infty$  for all  $t > 0$  almost surely. By Hölder’s inequality, the right-hand side of (6.6) is bounded above by

$$\sum_{u,v \in \mathcal{V}} \mathbf{E}^u [e^{-2\langle L_t, V + \xi \rangle} \mathbf{1}_{\{X(t)=v\}}] \mathbf{P}^u [\tau_n \leq t].$$

By mimicking our proof that  $K_t$  is trace class, we know that

$$\sum_{u,v \in \mathcal{V}} \mathbf{E}^u [e^{-2\langle L_t, V + \xi \rangle} \mathbf{1}_{\{X(t)=v\}}] < \infty$$

for every  $t > 0$  almost surely. Thus, by dominated convergence, it suffices to prove that

$$\lim_{n \rightarrow \infty} \mathbf{P}^u [\tau_n \leq t] = 0$$

for every  $u \in \mathcal{V}$  and  $t > 0$ . Noting that

$$\mathbf{P}^u \left[ \max_{0 \leq s \leq t} d(0, X(s)) > n \right] \leq \mathbf{P}^u \left[ \max_{0 \leq s \leq t} d(u, X(s)) > n - d(0, u) \right]$$

for all  $n \in \mathbb{N}$  by the triangle inequality, this follows directly from the tail bound (4.22). ■

**6.5. Step 5. Rigidity**

It now only remains to prove that the point process (3.11) is number rigid in the sense of Definition 3.3. The proof of this amounts to a minor modification of the argument in [25, Theorem 6.1] (see also [34, Proposition 2.2]).

Let  $B \subset \mathbb{C}$  be a Borel set such that  $B \subset (-\infty, \delta] + i[-\tilde{\delta}, \tilde{\delta}]$  for some  $\delta, \tilde{\delta} > 0$ . Thanks to the trace identity (5.1), almost surely, we can write

$$\mathcal{X}_H(B) = \sum_{\lambda \in \sigma(H) \cap B} m_a(\lambda, H)$$

as the sum of the following three terms:

$$\sum_{\lambda \in \sigma(H)} m_a(\lambda, H) e^{-t\lambda} - \mathbf{E} \left[ \sum_{\lambda \in \sigma(H)} m_a(\lambda, H) e^{-t\lambda} \right] = \text{Tr}[K_t] - \mathbf{E}[\text{Tr}[K_t]], \tag{6.7}$$

$$\sum_{\lambda \in \sigma(H) \cap B} m_a(\lambda, H) (1 - e^{-t\lambda}), \tag{6.8}$$

$$\mathbf{E} \left[ \sum_{\lambda \in \sigma(H)} m_a(\lambda, H) e^{-t\lambda} \right] - \sum_{\lambda \in \sigma(H) \setminus B} m_a(\lambda, H) e^{-t\lambda}. \tag{6.9}$$

Since we choose the exponent  $\alpha$  in the same way as Theorem 3.16, (6.7) converges to zero as  $t \rightarrow 0$  almost surely along a subsequence. Next, we have that (6.8) is bounded above in absolute value by

$$\mathcal{X}_H(B) \sup_{\zeta \in [\omega, \delta] + i[\alpha, \beta]} |1 - e^{-t\zeta}|,$$

where we recall that  $\omega$  is the random lower bound on the real part of the points in  $\mathcal{X}_H$ . Since  $\mathcal{X}_H$  is real-bounded below and  $B \subset (-\infty, \delta] + i[-\tilde{\delta}, \tilde{\delta}]$ ,  $\mathcal{X}_H(B) < \infty$  almost surely. Thus, (6.8) converges to zero almost surely as  $t \rightarrow 0$ . Thus,  $\mathcal{X}_H(B)$  is the almost sure limit of (6.9) as  $t \rightarrow 0$ , along a subsequence. Given that (6.9) is measurable with respect to the configuration of points outside of  $B$  for every  $t$  and that the almost-sure limit of measurable functions is measurable (assuming the sigma algebra is complete), we conclude that  $\mathcal{X}_H(B)$  is measurable with respect to the configuration outside of  $B$ . This then concludes the proof of number rigidity, and thus of Theorem 3.17.

**Remark 6.4.** Referring back to the point raised in Section 1.4.2, we see that the function denoted  $\mathcal{N}_B$  therein satisfies the relation

$$\mathcal{N}_B(\sigma(H) \setminus B) = \lim_{n \rightarrow \infty} \left( \mathbf{E} \left[ \sum_{\lambda \in \sigma(H)} m_a(\lambda, H) e^{-t_n \lambda} \right] - \sum_{\lambda \in \sigma(H) \setminus B} m_a(\lambda, H) e^{-t_n \lambda} \right) \tag{6.10}$$

with probability one, where  $(t_n)_{n \in \mathbb{N}}$  is a sparse enough sequence that vanishes in the large  $n$  limit. In particular, understanding the precise form of  $\mathcal{N}_B$  relies, among other



things, on understanding how the divergences of the two terms inside the limit on the right-hand side of (6.10) somehow cancel out as  $n \rightarrow \infty$ .

### 7. Proof of Theorem 3.18

#### 7.1. Step 1. General lower bound

We begin by providing a lower bound for  $\text{Var}[\text{Tr}[K_t]]$  in the general setting of the statement of Theorem 3.18. This bound will then be shown to remain positive as  $t \rightarrow 0$  in the cases labelled (1)–(3).

Recalling that  $\gamma$  is the positive definite covariance function of  $\xi$ , if we denote the semi-inner-product

$$\langle f, g \rangle_\gamma := \sum_{u, v \in \mathbb{Z}^d} f(u)\gamma(u - v)g(v), \quad f, g: \mathbb{Z}^d \rightarrow \mathbb{R},$$

then our assumption that  $\gamma$  is nonnegative implies that  $\langle f, g \rangle_\gamma \geq 0$  whenever  $f$  and  $g$  are nonnegative. In particular, we have that

$$\text{Cov}_\xi[e^{-\langle L_t^u, \xi \rangle}, e^{-\langle \tilde{L}_t^v, \xi \rangle}] = e^{\frac{1}{2}\langle L_t^u, L_t^u \rangle_\gamma + \frac{1}{2}\langle \tilde{L}_t^v, \tilde{L}_t^v \rangle_\gamma} (e^{\langle L_t^u, \tilde{L}_t^v \rangle_\gamma} - 1) \geq 0. \quad (7.1)$$

For every  $u, v \in \mathbb{Z}^d$  and  $t > 0$ , denote the event  $J_t(u, v) := \{L_t^u = t\mathbf{1}_u \text{ and } \tilde{L}_t^v = t\mathbf{1}_v\}$ . Clearly,  $J_t(u, v) \subset \{X^u(t) = u, \tilde{X}^v(t) = v\}$ , and by independence of  $X^u$  and  $\tilde{X}^v$ ,

$$\inf_{u, v \in \mathbb{Z}^d} \mathbf{P}[J_t(u, v)] = \inf_{v \in \mathbb{Z}^d} \mathbf{P}^v[X(s) = v \text{ for every } s \leq t]^2 \geq e^{-2t}. \quad (7.2)$$

We now combine (7.1) and (7.2) to lower bound the variance of  $\text{Tr}[K_t]$ . By Proposition 4.2, we may write

$$\begin{aligned} \text{Var}[\text{Tr}[K_t]] &\geq \sum_{u, v \in \mathbb{Z}^d} \mathbf{E}[e^{-\langle L_t^u + \tilde{L}_t^v, V \rangle} e^{\frac{1}{2}\langle L_t^u, L_t^u \rangle_\gamma + \frac{1}{2}\langle \tilde{L}_t^v, \tilde{L}_t^v \rangle_\gamma} (e^{\langle L_t^u, \tilde{L}_t^v \rangle_\gamma} - 1) \mathbf{1}_{J_t(u, v)}] \\ &= \sum_{u, v \in \mathbb{Z}^d} e^{-tV(u) - tV(v)} e^{t^2\gamma(0)} (e^{t^2\gamma(u-v)} - 1) \mathbf{P}[J_t(u, v)] \\ &\geq e^{-2t + t^2\gamma(0)} \sum_{u, v \in \mathbb{Z}^d} e^{-tV(u) - tV(v)} (e^{t^2\gamma(u-v)} - 1) \\ &= e^{-2t + t^2\gamma(0)} \sum_{u, v \in \mathbb{Z}^d} e^{-td(0, u)^\delta - td(0, v)^\delta} (e^{t^2\gamma(u-v)} - 1), \end{aligned} \quad (7.3)$$

where the first line comes from (7.1) and the fact that  $\mathbf{E}[Y] \geq \mathbf{E}[Y\mathbf{1}_E]$  for any non-negative random variable  $Y$  and event  $E$ , the second line comes from the definition of the event  $J_t(u, v)$ , the third line comes from (7.2), and the last line comes from the

assumption on  $V$  stated in Theorem 3.18. As  $e^{-2t+t^2\gamma(0)} \rightarrow 1$  as  $t \rightarrow 0$ , we obtain our general lower bound:

$$\liminf_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] \geq \liminf_{t \rightarrow 0} \sum_{u,v \in \mathbb{Z}^d} e^{-td(0,u)^\delta - td(0,v)^\delta} (e^{t^2\gamma(u-v)} - 1). \tag{7.4}$$

We now prove that the right-hand side of (7.4) is positive in cases (1)–(3).

### 7.2. Step 2. Three examples

Suppose first that  $\delta \leq d/2$  and  $\gamma(v) = \mathbf{1}_{\{v=0\}}$ . On the integer lattice  $\mathbb{Z}^d$ , it is easy to see that there exists a constant  $C > 0$  such that  $c_n(0) \geq Cn^{d-1}$ . Therefore, by an application of (7.4), followed by the inequality  $e^x - 1 \geq x$  for all  $x \geq 0$  and a Riemann sum, we have that

$$\begin{aligned} \liminf_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] &\geq \liminf_{t \rightarrow 0} (e^{t^2} - 1) \sum_{v \in \mathbb{Z}^d} e^{-2td(0,v)^\delta} \geq \liminf_{t \rightarrow 0} t^2 \sum_{n \in \mathbb{N} \cup \{0\}} c_n(0) e^{-2tn^\delta} \\ &\geq C \liminf_{t \rightarrow 0} t^{2-d/\delta} t^{1/\delta} \sum_{n \in t^{1/\delta} \mathbb{N} \cup \{0\}} n^{d-1} e^{-2n} \geq C \int_0^\infty x^{d-1} e^{-2x} dx > 0. \end{aligned}$$

Next, suppose that  $\delta \leq d - \beta/2$  and that  $\gamma(v) \geq \mathcal{L}(d(0, v) + 1)^{-\beta}$  for some  $0 < \beta < d$  and  $\mathcal{L} > 0$ . Then, (7.4), the triangle inequality, and the same arguments as in the previous case yield

$$\begin{aligned} \liminf_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] &\geq \liminf_{t \rightarrow 0} \sum_{u,v \in \mathbb{Z}^d} e^{-td(0,u)^\delta - td(0,v)^\delta} (e^{\mathcal{L}t^2(d(u,v)+1)^{-\beta}} - 1) \\ &\geq \mathcal{L} \liminf_{t \rightarrow 0} t^2 \sum_{u,v \in \mathbb{Z}^d} e^{-td(0,u)^\delta - td(0,v)^\delta} (d(0, u) + d(0, v) + 1)^{-\beta} \\ &= \mathcal{L} \liminf_{t \rightarrow 0} t^2 \sum_{m,n \in \mathbb{N} \cup \{0\}} c_m(0)c_n(0) e^{-tm^\delta - tn^\delta} (m + n + 1)^{-\beta} \\ &\geq \mathcal{L} C^2 \liminf_{t \rightarrow 0} t^{2-2(d-1)/\delta + \beta/\delta} \sum_{m,n \in t^{1/\delta} \mathbb{N} \cup \{0\}} (mn)^{d-1} e^{-m^\delta - n^\delta} (m + n + t^\delta)^{-\beta} \\ &= \mathcal{L} C^2 \liminf_{t \rightarrow 0} t^{2-2(d-\beta/2)/\delta} \int_0^\infty \int_0^\infty \frac{(xy)^{d-1}}{(x+y)^\beta} e^{-x^\delta - y^\delta} dx dy > 0. \end{aligned}$$

Finally, suppose that  $\delta \leq d$  and  $\inf_{v \in \mathbb{Z}^d} \gamma(v) > \mathcal{L} > 0$ . In this case we obtain that

$$\liminf_{t \rightarrow 0} \text{Var}[\text{Tr}[K_t]] \geq \liminf_{t \rightarrow 0} (e^{\mathcal{L}t^2} - 1) \sum_{u,v \in \mathbb{Z}^d} e^{-td(0,u)^\delta - td(0,v)^\delta}$$

$$\begin{aligned} &\geq \mathcal{L}C^2 \liminf_{t \rightarrow 0} t^2 \left( \sum_{n \in \mathbb{N}} n^{d-1} e^{-2tn^\delta} \right)^2 \\ &= \mathcal{L}C^2 \liminf_{t \rightarrow 0} t^{2-2d/\delta} \left( \int_0^\infty x^{d-1} e^{-2x} dx \right)^2 > 0, \end{aligned}$$

thus concluding the proof.

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Received 12 January 2021; revised 26 June 2021.

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