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L^p-bounds for semigroups generated by non-elliptic quadratic differential operators

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Abstract. In this note, we establish L^p-bounds for the semigroup $e^{-tq^w(x,D)}$, $t \geq 0$, generated by a quadratic differential operator $q^w(x,D)$ on $\mathbb{R}^n$ that is the Weyl quantization of a complex-valued quadratic form q defined on the phase space $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$ and trivial singular space. Specifically, we show that $e^{-tq^w(x,D)}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $t > 0$ whenever $1 \leq p \leq q \leq \infty$, and we prove bounds on $\|e^{-tq^w(x,D)}\|_{L^p \to L^q}$ in both the large $t \gg 1$ and small $0 < t \ll 1$ time regimes. Regarding $L^p \to L^q$ bounds for the evolution semigroup at large times, we show that $\|e^{-tq^w(x,D)}\|_{L^p \to L^q}$ is exponentially decaying as $t \to \infty$, and we determine the precise rate of exponential decay, which is independent of (p,q). At small times $0 < t \ll 1$, we establish bounds on $\|e^{-tq^w(x,D)}\|_{L^p \to L^q}$ for (p,q) with $1 \leq p \leq q \leq \infty$ that are polynomial in t^{-1}.

1. Introduction and statement of results

In this note, we prove L^p-bounds for the solution operator $e^{-tq^w(x,D)}$ of the Schrödinger initial value problem

$$\begin{cases} \partial_t u(t,x) + q^w(x,D)u(t,x) = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u_0 \in L^2(\mathbb{R}^n)$ is the initial data, $q = q(x,\xi)$ is a complex-valued quadratic form on the phase space $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with non-negative real part $\operatorname{Re} q \geq 0$, and $q^w(x,D)$ is the Weyl quantization of $q(x,\xi)$, defined by

$$q^w(x,D)v(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} q\Big(\frac{x+y}{2}, \xi\Big) v(y)\, dy\, d\xi, \quad v \in \mathcal{S}'(\mathbb{R}^n), \tag{1.2}$$

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in the sense of distributions. Operators of the form (1.2) are quadratic differential operators with a simple, explicit expression. This is because the Weyl quantization of a quadratic monomial of the form $x^\alpha \xi^\beta$, where $\alpha, \beta \in \mathbb{N}^n$, $|\alpha+\beta| = 2$, is

$$\frac{x^\alpha D^\beta + D^\beta x^\alpha}{2}, \quad D := \frac{1}{i}\partial. \tag{1.3}$$

The class of evolution equations of the form (1.1) contains a number of familiar examples, such as the free Schrödinger equation where $q(x,\xi) = i|\xi|^2$, $(x,\xi) \in \mathbb{R}^{2n}$, the quantum harmonic oscillator, where $q(x,\xi) = i(|x|^2 + |\xi|^2)$, $(x,\xi) \in \mathbb{R}^{2n}$, the heat equation, where $q(x,\xi) = |\xi|^2$, $(x,\xi) \in \mathbb{R}^{2n}$, and the Kramers–Fokker–Planck equation with a quadratic potential, where $q(x,v,\xi,\eta) = \eta^2 + \frac{1}{4}v^2 + i(v\cdot\xi - ax\cdot\eta)$, for $(x,v,\xi,\eta) \in \mathbb{R}^{4n} = \mathbb{R}^{2n}_{x,v} \times \mathbb{R}^{2n}_{\xi,\eta}$ and $a \in \mathbb{R}\backslash\{0\}$ a constant. From the work [15], it is known that the operator $q^w(x,D)$, regarded as an unbounded operator on $L^2(\mathbb{R}^n)$ equipped with the maximal domain

$$D_{\max} = \{u \in L^2(\mathbb{R}^n) \colon q^w(x,D)u \in L^2(\mathbb{R}^n)\}, \tag{1.4}$$

is maximally accretive and generates a strongly continuous contraction semigroup $G(t) := e^{-tq^w(x,D)}$, $t \geq 0$, on $L^2(\mathbb{R}^n)$. We may regard $G(t)$ as the solution operator for the problem (1.1). Given that a wide range of physical processes may be modeled by equations of the form (1.1), it is of interest to understand the $L^p \to L^q$ mapping properties of the evolution semigroup $G(t)$ and to obtain bounds for the operator norm $\|G(t)\|_{L^p\to L^q}$ at various time regimes. Let us mention that the study of L^p-bounds for semigroups generated by self-adjoint Schrödinger operators has a long and rich tradition in the field of mathematical physics. We refer to [7, 8, 24, 25] for some fundamental results in this area. In particular, L^p-bounds for the propagator $G(t)$ were obtained in [17] in the case when (1.1) is the time evolution of the quantum harmonic oscillator. We also mention that the topic of L^p-bounds for operators with Gaussian kernels is a classical subject. In particular, it is known that the that the $L^p \to L^q$ norm of an operator on $\mathbb{R}^n$ with a Gaussian kernel must be realized by a Gaussian. For more information, see [18].

In this note, we shall be primarily interested in obtaining $L^p \to L^q$ bounds for $G(t)$ in the case when the quadratic form q is non-elliptic. In order to recount the known results in this direction, we pause to recall the notion of the singular space of a complex-valued quadratic form q on $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$. Let $\mathbb{R}^{2n}$ be equipped with the standard symplectic form

$$\sigma((x,\xi),(y,\eta)) = \xi\cdot y - x\cdot\eta, \quad (x,\xi),\ (y,\eta) \in \mathbb{R}^{2n}. \tag{1.5}$$

Suppose $q\colon \mathbb{R}^{2n} \to \mathbb{C}$ is a complex-valued quadratic form with $\operatorname{Re} q \geq 0$ and let $q(\cdot,\cdot)$ denote its symmetric $\mathbb{C}$-bilinear polarization. Because σ is non-degenerate, there is a unique $F \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C})$ such that

$$q((x,\xi),(y,\eta)) = \sigma((x,\xi), F(y,\eta)) \tag{1.6}$$

for all $(x,\xi),(y,\eta) \in \mathbb{R}^{2n}$. This matrix F is called the *Hamilton map* or *Hamilton matrix of q* (see [16, Section 21.5]). Explicitly, the Hamilton matrix of q is given by

$$F = \frac{1}{2}H_q, \tag{1.7}$$

where $H_q = (q'_\xi, -q'_x)$ is the Hamilton vector field of q, viewed as a linear map $\mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Let

$$\mathrm{Re}\, F = \frac{F+\bar{F}}{2}, \quad \mathrm{Im}\, F = \frac{F-\bar{F}}{2i}$$

be the real and imaginary parts of F respectively. The *singular space S of q* is defined as the following finite intersection of kernels:

$$S = \Big(\bigcap_{j=0}^{2n-1} \ker\,[(\mathrm{Re}\, F)(\mathrm{Im}\, F)^j]\Big) \cap \mathbb{R}^{2n}. \tag{1.8}$$

The singular space was first introduced by M. Hitrik and K. Pravda-Starov in [9] where it arose naturally in the study of spectra and semigroup smoothing properties for non-self adjoint quadratic differential operators. The concept of the singular space has since been shown to play a key role in the understanding of hypoelliptic and spectral properties of non-elliptic quadratic differential operators. See for instance [10, 11, 20, 21, 28, 29]. Recent work has also shown that the singular space is vital for the description of the propagation of microlocal singularities for the evolution (1.1). We refer the reader to [2, 3, 5, 22, 23, 30, 31].

Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\mathrm{Re}\, q \geq 0$. Let S be the singular space of q. The quadratic form q is said to be *elliptic* if

$$q(X) = 0,\ X \in \mathbb{R}^{2n} \implies X = 0, \tag{1.9}$$

If (1.9) fails to hold, then we say that q is *non-elliptic*. To the best of our knowledge, there are currently only two general results regarding $L^p \to L^q$ bounds for the semigroup $G(t)$ in the case when q is non-elliptic. First, in [9, Theorem 1.2.3], it was established that $\|G(t)\|_{L^2\to L^2}$ decays exponentially as $t \to \infty$ whenever S is symplectic and distinct from the entire phase space $\mathbb{R}^{2n}$. In other words, if S is symplectic and $S \neq \mathbb{R}^{2n}$, then there are $C, c > 0$ such that

$$\|G(t)\|_{L^2\to L^2} \leq Ce^{-ct}, \quad t \geq 0. \tag{1.10}$$

Thanks to the subsequent work [19], it is also known that if S is trivial, i.e., $S = \{0\}$, then the optimal rate of exponential decay of $\|G(t)\|_{L^2\to L^2}$ is the quantity γ defined below in Theorem 1.1. The second general result concerning $L^p - L^q$ bounds for $G(t)$ is [12, Theorem 1.2], which yields the following $L^2 - L^\infty$ estimate: if $S = \{0\}$, then, for every $s > n/2$, there is $C > 0$ such that

$$\|G(t)\|_{L^2\to L^\infty} \le C t^{-\frac{1}{2}(2k_0+1)(2n+s)}, \quad 0 < t \ll 1, \tag{1.11}$$

where $k_0 \in \{0, 1, \dots, 2n-1\}$ is the smallest non-negative integer such that

$$\bigcap_{j=0}^{k_0} \ker\,[(\operatorname{Re} F)(\operatorname{Im} F)^j] \cap \mathbb{R}^{2n} = \{0\}. \tag{1.12}$$

Our goal in the present work is to prove bounds for the operator norm $\|G(t)\|_{L^p\to L^q}$ with (p, q) more general than $(2, 2)$ and $(2, \infty)$. The main result of this note refines and extends the bounds (1.10) and (1.11) under the assumption that $S = \{0\}$. We recall from [9, Theorem 1.2.2] that when $S = \{0\}$ the spectrum of the quadratic differential operator $q^w(x, D)$ is only composed of eigenvalues of finite algebraic multiplicity with

$$\operatorname{Spec}(q^w(x, D)) = \Big\{ \sum_{\substack{\lambda\in \operatorname{Spec}(F)\\ \operatorname{Im}(\lambda)>0}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \Big\}, \tag{1.13}$$

where r_λ is the dimension of the space of generalized eigenvectors of the Hamilton matrix F of q in $\mathbb{C}^{2n}$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$. In particular, the eigenvalue of $q^w(x, D)$ obtained by setting $k_\lambda = 0$ for all $\lambda \in \operatorname{Spec}(F)$ in (1.13) is

$$\rho = \sum_{\substack{\lambda\in \operatorname{Spec}(F)\\ \operatorname{Im}(\lambda)>0}} -i r_\lambda \lambda. \tag{1.14}$$

We may think of ρ as the "lowest eigenvalue" or "ground state energy" of the operator $q^w(x, D)$.

Theorem 1.1. *Let q, $q^w(x, D)$, $G(t)$, S, and F be as above. Assume that $S = \{0\}$.*

1. *Let $\gamma = \operatorname{Re}(\rho) > 0$. For every $1 \le p \le q \le \infty$ and $\varepsilon > 0$, there are constants $C = C_{\varepsilon,p,q} > 0$ and $c = c_{p,q} > 0$, such that*

$$c e^{-\gamma t} \le \|G(t)\|_{L^p\to L^q} \le C e^{-\gamma t}, \quad t \ge \varepsilon. \tag{1.15}$$

2. *Let $k_0 \in \{0, 1, \dots, 2n-1\}$ be the smallest non-negative integer such that (1.12) holds. There is a time $0 < t_0 \ll 1$ such that for any $1 \le p \le q \le \infty$ we have*

$$c \le \|G(t)\|_{L^p\to L^q} \le C t^{-(2k_0+1)n}, \quad 0 < t \le t_0, \tag{1.16}$$

for some constants $C = C_{p,q} > 0$ and $c = c_{p,q} > 0$.

Remark 1. For any $1 \le p \le q \le \infty$, it is actually true that there is a constant $c = c_{p,q} > 0$ such that

$$ce^{-\gamma t} \le \|G(t)\|_{L^p \to L^q}, \quad 0 \le t < \infty. \tag{1.17}$$

In fact, we have $c \ge \|v\|_{L^q}$, where $v \in \mathcal{S}(\mathbb{R}^n)$ is the L^p-normalized "ground state" for the operator $q^w(x, D)$. For a proof, see the derivation of (4.15) below.

Let us make some general comments regarding Theorem 1.1 First, the bounds (1.15) show that for any $1 \le p \le q \le \infty$ the operator norm $\|G(t)\|_{L^p \to L^q}$ decays exponentially as $t \to \infty$, with γ being the precise rate of decay, independent of (p, q). To prove that γ is the exact rate of exponential decay, one may examine the action of the propagator $G(t)$ on the "ground state" eigenfunction of $q^w(x, D)$ corresponding to the eigenvalue ρ (see Section 4 below). Regarding the short time $0 < t \ll 1$ bounds in Theorem 1.1, it is clear that (1.16) is not sharp for all $1 \le p \le q \le \infty$. For instance, (1.16) fails to reproduce (1.10) when $p = q = 2$. However, one may interpolate (1.16) with the bound $G(t) = \mathcal{O}_{L^2 \to L^2}(1)$ as $t \to 0^+$ to obtain more precise estimates at short times. We also note that when $(p, q) = (2, \infty)$, the bound (1.16) gives $G(t) = \mathcal{O}_{L^2 \to L^\infty}(t^{-(2k_0+1)n})$ as $t \to 0^+$, which is an improvement over (1.11).

Finally, let us briefly touch on the main ideas involved in the proof of Theorem 1.1. In the recent work [31], we showed that if $\mathcal{T}_\varphi$ is a global metaplectic FBI transform on $\mathbb{R}^n$, in the sense of either [32, Chapter 13] or the minicourse [13], then the conjugated propagator $\widetilde{G}(t) := \mathcal{T}_\varphi \circ G(t) \circ \mathcal{T}_\varphi^*$ is, for each $t \ge 0$, a metaplectic Fourier integral operator acting on the Bargmann space $H_{\Phi_0}(\mathbb{C}^n)$, which is the unitary image of $L^2(\mathbb{R}^n)$ under $\mathcal{T}_\varphi$. In particular, we showed that the "Bergman form" ([6, 27]) of $\widetilde{G}(t)$ is given by

$$\begin{aligned}\widetilde{G}(t)u(z) &\\ &= \hat{a}(t) \int_{\mathbb{C}^n} e^{2\Psi_t(z,\bar{w})} u(w) e^{-2\Phi_0(w)} L(dw), \quad z \in \mathbb{C}^n,\ u \in H_{\Phi_0}(\mathbb{C}^n),\ t \ge 0,\end{aligned} \tag{1.18}$$

where $L(dw)$ is the Lebesgue measure on $\mathbb{C}^n$, $\Phi_0(w) := \sup_{y \in \mathbb{R}^n}(-\operatorname{Im}\varphi(w, y))$, $w \in \mathbb{C}^n$, is the strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ associated to φ, Ψ_t is a holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ depending analytically on $t \ge 0$, and $\hat{a} \in C^\omega([0, \infty); \mathbb{C})$ is a non-vanishing amplitude. Moreover, we showed that Ψ_t and $\hat{a}$ are the solutions of an eikonal equation and a transport equation, respectively. In particular, we did not attempt to solve these equations explicitly for Ψ_t and $\hat{a}$. Now, thanks to the work [1], it is known that when the singular space is trivial $S = \{0\}$ it is possible to choose a metaplectic FBI transform $\mathcal{T}_\varphi$ so that conjugated semigroup has the simple form

$$\widetilde{G}(t)u(z) = e^{\frac{i}{2}\operatorname{tr}(M)t}u(e^{itM}z), \quad u \in H_{\Phi_0}(\mathbb{C}^n),\ t \geq 0, \tag{1.19}$$

where $M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a suitable matrix. In the present work, we show that this choice of $\mathcal{T}_\varphi$ leads to equations for Ψ_t and $\hat{a}$ that may be easily solved. One may then show that (1.18) coincides with (1.19), giving an alternative derivation of (1.19). Once the Bergman form of $\widetilde{G}(t)$ is known and a basic estimate for the real part of its phase function is established, the bounds (1.15) and (1.16) follow easily by writing down a formal expression for the Schwartz kernel of the composition $\mathcal{T}_\varphi^* \circ \widetilde{G}(t) \circ \mathcal{T}_\varphi$ using (1.18) and applying Young's integral inequality.

The plan for this note is as follows. In Section 2, we recall how to choose the FBI transform $\mathcal{T}_\varphi$ so that (1.19) holds. In Section 3, we determine the Bergman form (1.18) of $\widetilde{G}(t)$ for $t \geq 0$ and prove some basic estimates. In Section 4, we conclude the proof of Theorem 1.1, as outlined in this introduction.

2. Reduction to a normal form on the FBI transform side

In this section, we follow the approach of [14, 29] for reducing $q^w(x, D)$ to a normal form via a metaplectic FBI transform. We provide additional references where convenient.

Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$ and trivial singular space $S = \{0\}$. Let $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_\zeta$ be equipped with the standard complex symplectic form $\sigma = d\zeta \wedge dz$. Let F be the Hamilton matrix of q introduced in (1.6). From the work [9], it is known that the matrix F has no real eigenvalues. Consequently,

$$\#\{\lambda \in \operatorname{Spec}(F)\colon \operatorname{Im}\lambda > 0\} = \#\{\lambda \in \operatorname{Spec}(F)\colon \operatorname{Im}\lambda < 0\}, \tag{2.1}$$

counting algebraic multiplicities. For $\lambda \in \operatorname{Spec}(F)$, let

$$V_\lambda = \ker((F-\lambda)^{2n}) \subset \mathbb{C}^{2n} \tag{2.2}$$

be the generalized eigenspace of F corresponding to λ. Let us also introduce the stable outgoing and stable incoming manifolds for the quadratic form $-iq$ given by

$$\Lambda^+ = \bigoplus_{\substack{\lambda\in\operatorname{Spec}(F)\\ \operatorname{Im}\lambda>0}} V_\lambda, \quad \Lambda^- = \bigoplus_{\substack{\lambda\in\operatorname{Spec}(F)\\ \operatorname{Im}\lambda<0}} V_\lambda, \tag{2.3}$$

respectively. By [29, Proposition 2.1], Λ^+ is a strictly positive $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n}$ in the sense that Λ^+ is Lagrangian with respect to the complex symplectic form σ and

$$\frac{1}{i}\sigma(Z,\bar{Z}) > 0, \quad Z \in \Lambda^+\backslash\{0\}, \tag{2.4}$$

and Λ^- is a strictly negative $\mathbb{C}$-Lagrangian subspace of $\mathbb{C}^{2n}$ in the sense that Λ^- is Lagrangian for the form σ and (2.4) holds for all $Z \in \Lambda^-\backslash\{0\}$ with "$>$" replaced by "$<$". For background information regarding positive and negative $\mathbb{C}$-Lagrangian subspaces of $\mathbb{C}^{2n}$, we refer to [6, 13]. In particular, from the discussion on [13, pp. 488–489], we know that there exists a holomorphic quadratic form $\varphi = \varphi(z, y)$ on $\mathbb{C}^{2n} = \mathbb{C}^n_z \times \mathbb{C}^n_y$ with

$$\det \varphi''_{zy} \neq 0, \quad \operatorname{Im} \varphi''_{yy} > 0, \tag{2.5}$$

such that the complex linear canonical transformation

$$\kappa_\varphi\colon \mathbb{C}^{2n} \ni (y, -\varphi'_y(z,y)) \mapsto (z, \varphi'_z(z,y)) \in \mathbb{C}^{2n}, \quad (z,y) \in \mathbb{C}^{2n}, \tag{2.6}$$

generated by φ satisfies

$$\kappa_\varphi(\Lambda^+) = \{(z,0)\colon z \in \mathbb{C}^n\}, \quad \kappa_\varphi(\Lambda^-) = \{(0,\zeta)\colon \zeta \in \mathbb{C}^n\}. \tag{2.7}$$

Let

$$\Phi_0(z) = \sup_{y\in\mathbb{R}^n} (-\operatorname{Im} \varphi(z,y)), \quad z \in \mathbb{C}^n, \tag{2.8}$$

be the strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ associated to the phase φ (see [32, Chapter 13] or [13, Section 1.3]), and let

$$\Lambda_{\Phi_0} = \Big\{\Big(z, \frac{2}{i}\Phi'_{0,z}(z)\Big)\colon z \in \mathbb{C}^n\Big\}. \tag{2.9}$$

From either [32, Theorem 13.5] or [13, Proposition 1.3.2], we have

$$\kappa_\varphi(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}, \tag{2.10}$$

and thus Λ_{Φ_0} is I-Lagrangian and R-symplectic for the complex symplectic form σ. Also, the strict positivity of Λ^+ in conjunction with (2.7) gives that the base $\{(z,0)\colon z \in \mathbb{C}^n\}$ is strictly positive relative to Λ_{Φ_0} (see, e.g., [6]). It then follows, as explained in [26, Chapter 11], that the quadratic form Φ_0 is strictly convex.

Let

$$\tilde{q} = q \circ \kappa_\varphi^{-1}, \tag{2.11}$$

regarded as a holomorphic quadratic form on $\mathbb{C}^{2n}$. Since Λ^+ and Λ^- are invariant under F and Lagrangian with respect to σ, we have

$$q(X) = \sigma(X, FX) = 0, \quad X \in \Lambda^+ \cup \Lambda^-. \tag{2.12}$$

From (2.7) and (2.11), it follows that $\tilde{q}$ must be of the form

$$\tilde{q}(z,\zeta) = Mz \cdot \zeta, \quad (z,\zeta) \in \mathbb{C}^{2n}, \tag{2.13}$$

for some $M \in \mathrm{Mat}_{n\times n}(\mathbb{C}^n)$. In particular, the complex Hamilton vector field of $\tilde{q}$ with respect to σ is

$$H_{\tilde{q}} = (Mz, -M^T\zeta), \quad (z,\zeta) \in \mathbb{C}^{2n}. \tag{2.14}$$

The Hamilton map of $\tilde{q}$ is thus given by $\widetilde{F} = \frac{1}{2}H_{\tilde{q}}$, and we have

$$\widetilde{F} = \frac{1}{2}\begin{pmatrix} M & 0 \\ 0 & -M^T \end{pmatrix}. \tag{2.15}$$

As a consequence of (2.11), (1.6), and the invariance of σ under κ_φ, it is true that $\widetilde{F} = \kappa_\varphi \circ F \circ \kappa_\varphi^{-1}$. Since also $\widetilde{F}$ maps $(z,0) \in \kappa_\varphi(\Lambda^+)$ to $\frac{1}{2}(Mz,0) \in \kappa_\varphi(\Lambda^+)$, we have

$$\mathrm{Spec}(M) = \mathrm{Spec}(2F) \cap \{\mathrm{Im}\,\lambda > 0\}, \tag{2.16}$$

with agreement of algebraic multiplicities.

Let $\mathcal{T}_\varphi\colon \mathcal{S}'(\mathbb{R}^n) \to \mathrm{Hol}(\mathbb{C}^n)$ be the metaplectic FBI transform on $\mathbb{R}^n$ associated to φ, given in the sense of distributions by

$$\mathcal{T}_\varphi u(z) = c_\varphi \int_{\mathbb{R}^n} e^{i\varphi(z,y)}u(y)L(dy), \quad u \in \mathcal{S}'(\mathbb{R}^n), \tag{2.17}$$

where

$$c_\varphi = 2^{-n/2}\pi^{-3n/4}(\det \mathrm{Im}\,\varphi''_{yy})^{-1/4}|\det \varphi''_{zy}|. \tag{2.18}$$

By [32, Theorem 13.7], $\mathcal{T}_\varphi$ is unitary $L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n)$, where

$$H_{\Phi_0}(\mathbb{C}^n) := L^2(\mathbb{C}^n, e^{-2\Phi_0(z)}L(dz)) \cap \mathrm{Hol}(\mathbb{C}^n) \tag{2.19}$$

is the Bargmann space associated to the weight Φ_0, equipped with the natural Hilbert space structure inherited from $L^2(\mathbb{C}^n, e^{-2\Phi_0(z)}L(dz))$. Here $L(dz)$ denotes the Lebesgue measure on $\mathbb{C}^n$. Let $\tilde{q}^w(z, D)$ denote the complex Weyl quantization of the symbol $\tilde{q}$ with respect to the weight Φ_0. We recall that $\tilde{q}^w(z, D)$ is defined as an unbounded operator on $H_{\Phi_0}(\mathbb{C}^n)$ that acts on suitable $u \in H_{\Phi_0}(\mathbb{C}^n)$ by

$$\tilde{q}^w(z, D)u(z) = \frac{1}{(2\pi)^n} \iint_{\Gamma_{\Phi_0}(z)} e^{i(z-w)\cdot\zeta} \tilde{q}^w\Big(\frac{z+w}{2}, \zeta\Big) u(w)\, dw \wedge d\zeta, \quad z \in \mathbb{C}^n, \tag{2.20}$$

for the contour of integration

$$\Gamma_{\Phi_0}(z)\colon w \mapsto \zeta = \frac{2}{i}\Phi'_{0,z}\Big(\frac{z+w}{2}\Big), \quad w \in \mathbb{C}^n, \quad z \in \mathbb{C}^n. \tag{2.21}$$

For more information on Weyl quantization in the complex domain, see [32, Chapter 13] or [13, Section 1.4]. By Egorov's theorem (see [32, Theorem 13.9] or [13, Theorem 1.4.2]), we have

$$q^w(x, D) = \mathcal{T}_\varphi^* \circ \tilde{q}^w(z, D) \circ \mathcal{T}_\varphi \tag{2.22}$$

when both sides are viewed as operators acting on the maximal domain of $q^w(x, D)$,

$$D_{\max} = \{u \in L^2(\mathbb{R}^n)\colon q^w(x, D)u \in L^2(\mathbb{R}^n)\}. \tag{2.23}$$

Let

$$\widetilde{D}_{\max} = \{u \in H_{\Phi_0}(\mathbb{C}^n)\colon \tilde{q}^w(z, D)u \in H_{\Phi_0}(\mathbb{C}^n)\} \tag{2.24}$$

be the maximal domain of $\tilde{q}^w(z, D)$, and let us view $\tilde{q}^w(z, D)$ as an unbounded operator on $H_{\Phi_0}(\mathbb{C}^n)$ with the domain $\widetilde{D}_{\max}$. Thanks to (2.22), we have

$$\widetilde{D}_{\max} = \mathcal{T}_\varphi(D_{\max}). \tag{2.25}$$

Let $G(t) = e^{-tq^w(x,D)}$, $t \geq 0$, be the strongly continuous semigroup on $L^2(\mathbb{R}^n)$ generated by $q^w(x, D)$ (see [15]). From (2.22), (2.25), and the unitarity of $\mathcal{T}_\varphi$, it follows that $\tilde{q}^w(z, D)$ generates a strongly continuous semigroup $\widetilde{G}(t) = e^{-t\tilde{q}^w(z,D)}$, $t \geq 0$, on $H_{\Phi_0}(\mathbb{C}^n)$. The semigroups $G(t)$ and $\widetilde{G}(t)$ are related by

$$G(t) = \mathcal{T}_\varphi^* \circ \widetilde{G}(t) \circ \mathcal{T}_\varphi \tag{2.26}$$

for all $t \geq 0$.

We have established the following proposition, which summarizes the discussion in this section.

Proposition 2.1. *Let q be a complex-valued quadratic form on $\mathbb{R}^{2n}$ with non-negative real part $\operatorname{Re} q \geq 0$ and trivial singular space $S = \{0\}$. Let F be the Hamilton matrix of q, and let $q^w(x, D)$ be the Weyl quantization of q, viewed as an unbounded operator* *on* $L^2(\mathbb{R}^n)$ *equipped with its maximal domain* D_{max} *defined in* (2.23). *Let* $G(t) = e^{-tq^w(x,D)}$, $t \geq 0$, *be the strongly continuous semigroup on* $L^2(\mathbb{R}^n)$ *generated by* $q^w(x, D)$.

1. *There exists a holomorphic quadratic form* φ *on* $\mathbb{C}^{2n}$ *satisfying* (2.5) *such that the quadratic form* Φ_0 *defined by* (2.8) *is strictly convex and the complex linear canonical transformation* $\kappa_\varphi\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ *defined implicitly by* (2.6) *has the property that*

$$\tilde{q}(z,\zeta) := (q \circ \kappa_\varphi^{-1})(z,\zeta) = Mz \cdot \zeta, \quad (z,\zeta) \in \mathbb{C}^{2n}, \tag{2.27}$$

   where $M \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ *is such that* $\mathrm{Spec}(M) = \mathrm{Spec}(2F) \cap \{\mathrm{Im}\,\lambda > 0\}$ *with agreement of algebraic multiplicities.*

2. *Let* $\tilde{q}^w(z, D)$ *be the complex Weyl quantization* (2.20) *of* $\tilde{q}$ *with respect to the weight* Φ_0, *realized as an unbounded operator on the Bargmann space* $H_{\Phi_0}(\mathbb{C}^n)$ *introduced in* (2.19) *equipped with the maximal domain* $\widetilde{D}_{max}$ *defined in* (2.24). *The operator* $\tilde{q}^w(z, D)$ *generates a strongly continuous semigroup* $\widetilde{G}(t) = e^{-t\tilde{q}^w(z,D)}$, $t \geq 0$, *on* $H_{\Phi_0}(\mathbb{C}^n)$ *that is unitarily equivalent to* $G(t)$ *for each* $t \geq 0$. *This unitary equivalence is given by the FBI transform* $\mathcal{T}_\varphi$ *introduced in* (2.17), *i.e.,*

$$G(t) = \mathcal{T}_\varphi^* \circ \widetilde{G}(t) \circ \mathcal{T}_\varphi, \quad t \geq 0. \tag{2.28}$$

3. The evolution semigroup on the FBI transform side

We now study the semigroup $\widetilde{G}(t)$, $t \geq 0$. Let Ψ_0 be the polarization of Φ_0, i.e. Ψ_0 is the unique holomorphic quadratic form on $\mathbb{C}^{2n} = \mathbb{C}^n \times \mathbb{C}^n$ such that $\Psi_0(z, \bar{z}) = \Phi_0(z)$ for all $z \in \mathbb{C}^n$. Since

$$\Phi_0(z) = \frac{1}{2}\Phi''_{0,zz}z \cdot z + \Phi''_{0,\bar{z}z}z \cdot \bar{z} + \frac{1}{2}\Phi''_{0,\bar{z}\bar{z}}\bar{z} \cdot \bar{z}, \quad z \in \mathbb{C}^n, \tag{3.1}$$

we see that Ψ_0 is given explicitly by

$$\Psi_0(z,\theta) = \frac{1}{2}\Phi''_{0,zz}z \cdot z + \Phi''_{0,\bar{z}z}z \cdot \theta + \frac{1}{2}\Phi''_{0,\bar{z}\bar{z}}\theta \cdot \theta, \quad (z,\theta) \in \mathbb{C}^{2n}. \tag{3.2}$$

In the work [31], we showed that for every $t \geq 0$ the semigroup $\widetilde{G}(t)$ is a metaplectic Fourier integral operator in the complex domain whose underlying complex canonical transformation is the Hamilton flow $\tilde{\kappa}_t$ of the symbol $\tilde{q}$ at time t/i, i.e.,

$$\tilde{\kappa}_t = \exp\left(\frac{t}{i}H_{\tilde{q}}\right), \quad t \geq 0. \tag{3.3}$$

In view of (2.14), we have

$$\tilde{\kappa}_t(z,\zeta) = (e^{-itM}z, e^{itM^T}\zeta), \quad (z,\zeta) \in \mathbb{C}^{2n},\ t \geq 0. \tag{3.4}$$

For background information regarding metaplectic Fourier integral operators in the complex domain, see [4, Appendix B]. In particular, in the work [6], it was shown that every such metaplectic Fourier integral operator in $\mathbb{C}^n$ possesses a unique "Bergman form." In Section 6 of [31], we proved that the Bergman form of $\widetilde{G}(t)$ is given by

$$\widetilde{G}(t)u(z) = \hat{a}(t) \int_{\mathbb{C}^n} e^{2\Psi_t(z,\bar{w})} u(w) e^{-2\Phi_0(w)} L(dw), \quad z \in \mathbb{C}^n,\ u \in H_{\Phi_0}(\mathbb{C}^n), \tag{3.5}$$

where Ψ_t is a holomorphic quadratic form on $\mathbb{C}^{2n}$, depending analytically on $t \geq 0$, and $\hat{a} \in C^\omega([0,\infty);\mathbb{C})$ is a non-vanishing amplitude. In addition, we showed that Ψ_t, $t \geq 0$, is the unique solution of the eikonal equation

$$\begin{cases} 2\partial_t \Psi_t(z,\theta) + \tilde{q}\Big(z, \dfrac{2}{i}\Psi'_{t,z}(z,\theta)\Big) = 0, & (z,\theta) \in \mathbb{C}^{2n},\ t \geq 0, \\ \Psi_t(z,\theta)|_{t=0} = \Psi_0(z,\theta), & (z,\theta) \in \mathbb{C}^{2n}, \end{cases} \tag{3.6}$$

and $\hat{a}$ is the unique solution of the transport equation

$$\begin{cases} \hat{a}'(t) + \dfrac{1}{2i}\beta(t)\hat{a}(t) = 0, & t \geq 0, \\ \hat{a}(0) = C_{\Phi_0}, \end{cases} \tag{3.7}$$

where

$$\beta(t) = \operatorname{tr}\Big(\tilde{q}''_{\zeta z} + \tilde{q}''_{\zeta\zeta} \cdot \frac{2}{i}\Psi''_{t,zz}\Big), \quad t \geq 0, \tag{3.8}$$

and

$$C_{\Phi_0} = 2^n \pi^{-n} \det \Phi''_{0,z\bar{z}}. \tag{3.9}$$

We note that the initial conditions in (3.6) and (3.7) are chosen so that when $t = 0$ the right-hand side of (3.5) coincides with the orthogonal projector

$$\Pi_{\Phi_0}\colon L^2(\mathbb{C}^n, e^{-2\Phi_0(z)}L(dz)) \to H_{\Phi_0}(\mathbb{C}^n),$$

which has the explicit integral representation

$$\Pi_{\Phi_0}u(z) = C_{\Phi_0} \int_{\mathbb{C}^n} e^{2\Psi_0(z,\bar{w})} u(w) e^{-2\Phi_0(w)} L(dw), \quad u \in L^2(\mathbb{C}^n, e^{-2\Phi(z)}L(dz)). \tag{3.10}$$

In the literature, the operator Π_{Φ_0} is known as the "Bergman projector" associated to the weight Φ_0. For a proof of (3.10), see [32, Theorem 13.6] or [13, Proposition 1.3.4].

Since $\tilde{q}$ has the simple form (2.13), we may determine Ψ_t and $\hat{a}$ by solving (3.6) and (3.7) explicitly. We begin by studying the transport equation (3.7). Thanks to (2.13), we see that

$$\beta(t) = \mathrm{tr}(M), \quad t \geq 0. \tag{3.11}$$

The unique solution of (3.7) is

$$\hat{a}(t) = C_{\Phi_0} e^{\frac{i}{2}\mathrm{tr}(M)t}, \quad t \geq 0. \tag{3.12}$$

Next, we solve (3.6) for Ψ_t. We search for a solution to (3.6) of the form

$$\Psi_t(z,\theta) = \frac{1}{2}A_t z\cdot z + B_t z\cdot\theta + \frac{1}{2}D_t\theta\cdot\theta, \quad (z,\theta)\in\mathbb{C}^{2n},\ t\geq 0, \tag{3.13}$$

where $A_t, B_t, D_t \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ depend smoothly on t and $A_t = A_t^T$ and $D_t = D_t^T$ for all $t \geq 0$. Inserting (3.13) into (3.6) and using (2.13) and (3.2), we see that Ψ_t will be a solution of (3.6) provided A_t, B_t, and D_t satisfy

$$\begin{cases} \partial_t A_t z\cdot z + \dfrac{2}{i}A_t M z\cdot z = 0, & z\in\mathbb{C}^n,\ t\geq 0,\\ A_0 = \Phi''_{0,zz}, \end{cases} \tag{3.14}$$

$$\begin{cases} \partial_t B_t z\cdot\theta + \dfrac{1}{i}B_t M z\cdot\theta = 0, & z,\theta\in\mathbb{C}^n,\ t\geq 0,\\ B_0 = \Phi''_{0,\bar{z}z}, \end{cases} \tag{3.15}$$

and

$$\begin{cases} \partial_t C_t\theta\cdot\theta = 0, & \theta\in\mathbb{C}^n,\ t\geq 0,\\ C_0 = \Phi''_{0,\bar{z}\bar{z}}, \end{cases} \tag{3.16}$$

respectively. The symmetry of A_t implies that

$$2A_t M z\cdot z = (A_t M + M^T A_t)z\cdot z, \quad z\in\mathbb{C}^n,\ t\geq 0. \tag{3.17}$$

Thus, (3.14) holds if and only if

$$\begin{cases} \partial_t A_t + \dfrac{1}{i}A_t M + \dfrac{1}{i}M^T A_t = 0, & t\geq 0,\\ A_0 = \Phi''_{0,zz}. \end{cases} \tag{3.18}$$

The unique solution of (3.18) is

$$A_t = e^{iM^T t}\Phi''_{0,zz}e^{iMt}, \quad t \geq 0. \tag{3.19}$$

By inspection, the solutions of (3.15) and (3.16) are

$$B_t = \Phi''_{0,\bar{z}z}e^{itM}, \quad C_t = \Phi''_{0,\bar{z}\bar{z}}, \quad t \geq 0, \tag{3.20}$$

respectively. Using (3.2), we get

$$\Psi_t(z,\theta) = \Psi_0(e^{itM}z,\theta), \quad (z,\theta) \in \mathbb{C}^{2n},\ t \geq 0. \tag{3.21}$$

From (3.5), (3.10), (3.12), and (3.21), we deduce that

$$\widetilde{G}(t)u(z) = e^{\frac{i}{2}\operatorname{tr}(M)t}u(e^{itM}z), \quad u \in H_{\Phi_0}(\mathbb{C}^n),\ t \geq 0. \tag{3.22}$$

The formula (3.22) for the semigroup $\widetilde{G}(t)$ was obtained by a different method in [1].

For $t \geq 0$, let us define

$$\Phi_t(z) = \Phi_0(e^{itM}z), \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{3.23}$$

Since Φ_0 is strictly convex, Φ_t is a strictly convex quadratic form on $\mathbb{C}^n$ for all $t \geq 0$. In addition, we have $\Phi_t|_{t=0} = \Phi_0$. For $t \geq 0$, let

$$H_{\Phi_t}(\mathbb{C}^n) = L^2(\mathbb{C}^n, e^{-2\Phi_t(z)}L(dz)) \cap \operatorname{Hol}(\mathbb{C}^n) \tag{3.24}$$

be the Bargmann space associated to Φ_t, equipped with the natural Hilbert space structure induced from $L^2(\mathbb{C}^n, e^{-2\Phi_t(z)}L(dz))$. From (3.22), it is clear that $\widetilde{G}(t)$ is bounded $H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_t}(\mathbb{C}^n)$ for every $t \geq 0$, and a direct computation using (3.22), (3.23), and (2.16) gives

$$\|\widetilde{G}(t)u\|_{H_{\Phi_t}(\mathbb{C}^n)} = e^{\gamma t}\|u\|_{H_{\Phi_0}(\mathbb{C}^n)}, \quad u \in H_{\Phi_0}(\mathbb{C}^n),\ t \geq 0, \tag{3.25}$$

where $\gamma > 0$ is as in the statement of Theorem 1.1.

The following proposition summarizes the discussion so far in this section and establishes some basic estimates that will be necessary for the proof of Theorem 1.1 in Section 4.

Proposition 3.1. *Let q, $\tilde{q}$, M, Φ_0, $H_{\Phi_0}(\mathbb{C}^n)$, and $\widetilde{G}(t)$ be as in Proposition 2.1.*

1. *For every $t \geq 0$, we have*

$$\widetilde{G}(t)u(z) = e^{\frac{i}{2}\operatorname{tr}(M)t}u(e^{itM}z), \quad u \in H_{\Phi_0}(\mathbb{C}^n). \tag{3.26}$$

In addition,

$$\|\widetilde{G}(t)u\|_{H_{\Phi_t}(\mathbb{C}^n)} = e^{\gamma t}\|u\|_{H_{\Phi_0}(\mathbb{C}^n)}, \quad t \geq 0, \tag{3.27}$$

where

$$\Phi_t(z) = \Phi_0(e^{itM}z), \quad z \in \mathbb{C}^n,\ t \geq 0, \tag{3.28}$$

the norm $\|\cdot\|_{H_{\Phi_t}(\mathbb{C}^n)}$ *is the norm on the Bargmann space* $H_{\Phi_t}(\mathbb{C}^n)$ *introduced in (3.24), and* $\gamma > 0$ *is as in the statement of Theorem 1.1.*

2. *Let* $R_t = \Phi_0 - \Phi_t$, $t \geq 0$, *and let* $\alpha\colon [0,\infty) \to \mathbb{R}$ *be the continuous function defined by*

$$\alpha(t) = \min_{|z|=1} R_t(z), \tag{3.29}$$

so that

$$R_t(z) \geq \alpha(t)|z|^2, \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{3.30}$$

The function α *has the following properties:*

a. $\alpha(0) = 0$ *and* $\alpha(t) > 0$ *for all* $t > 0$,

b. α *is non-decreasing,*

c. *there is* $0 < t_0 \ll 1$ *and* $c > 0$ *such that*

$$\alpha(t) \geq ct^{2k_0+1}, \quad 0 \leq t \leq t_0, \tag{3.31}$$

where $k_0 \in \{0, 1, \ldots, 2n-1\}$ *is the smallest non-negative integer such that (1.12) holds, and*

d. $\alpha(t) \to \min_{|z|=1} \Phi_0(z) > 0$ *as* $t \to \infty$.

3. *Let* Ψ_0 *be the polarization of* Φ_0 *given by (3.2). For any* $t \geq 0$ *and* $u \in H_{\Phi_0}(\mathbb{C}^n)$, *we have*

$$\widetilde{G}(t)u(z) = C_{\Phi_0} e^{\frac{i}{2}\operatorname{tr}(M)t} \int_{\mathbb{C}^n} e^{2\Psi_t(z,\bar{w})} u(w) e^{-2\Phi_0(w)} L(dw), \quad z \in \mathbb{C}^n, \tag{3.32}$$

where

$$\Psi_t(z,\theta) = \Psi_0(e^{itM}z, \theta), \quad (z,\theta) \in \mathbb{C}^{2n},\ t \geq 0. \tag{3.33}$$

Moreover, there are constants $C, c > 0$, *independent of* t, *such that*

$$\begin{aligned} -C|w - e^{itM}z|^2 &\leq 2\operatorname{Re}\Psi_t(z,\bar{w}) - \Phi_t(z) - \Phi_0(w) \\ &\leq -c|w - e^{itM}z|^2, \quad z, w \in \mathbb{C}^n,\ t \geq 0. \end{aligned} \tag{3.34}$$

Proof. It remains to establish Point 2 and the estimate (3.34). To this end, let

$$R_t(z) = \Phi_0(z) - \Phi_t(z), \quad z \in \mathbb{C}^n,\ t \geq 0, \tag{3.35}$$

and let $\alpha\colon [0,\infty) \to \infty$ be as in (3.29). We will begin by showing that

$$R_t(z) \geq 0, \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{3.36}$$

Let $\tilde{\kappa}_t$, $t \geq 0$, be as in (3.3). A straightforward computation using (2.9), (3.4), and (3.28) gives that

$$\tilde{\kappa}_t(\Lambda_{\Phi_0}) = \Lambda_{\Phi_t} := \Big\{\Big(z, \frac{2}{i}\Phi'_{t,z}(z)\Big) : z \in \mathbb{C}^n\Big\}, \quad t \geq 0. \tag{3.37}$$

From either the discussion in [31, Section 6] or a direct computation, we know that the family $(\Phi_t)_{t\geq 0}$ satisfies the eikonal equation

$$\begin{cases} \partial_t \Phi_t(z) + \operatorname{Re}\tilde{q}\Big(z, \frac{2}{i}\Phi'_{t,z}(z)\Big) = 0, & z \in \mathbb{C}^n,\ t \geq 0, \\ \Phi_t|_{t=0} = \Phi_0 & \text{on } \mathbb{C}^n. \end{cases} \tag{3.38}$$

As a consequence of (3.37), for every $z \in \mathbb{C}^n$ and $t \geq 0$, there is a point $Z \in \Lambda_{\Phi_0}$ such that

$$\Big(z, \frac{2}{i}\Phi'_{t,z}(z)\Big) = \tilde{\kappa}_t(Z). \tag{3.39}$$

Since $\tilde{q}$ is invariant under the flow $\tilde{\kappa}_t$, for every $t \geq 0$ and $z \in \mathbb{C}^n$, there is $Z \in \Lambda_{\Phi_0}$ such that

$$\partial_t \Phi_t(z) = -\operatorname{Re}\tilde{q}(Z). \tag{3.40}$$

Because $\operatorname{Re} q \geq 0$, (2.10) and (2.11) imply that $\operatorname{Re}\tilde{q} \geq 0$ on Λ_{Φ_0}, and we have

$$\partial_t \Phi_t(z) \leq 0, \quad z \in \mathbb{C}^n,\ t \geq 0. \tag{3.41}$$

Thus, for any fixed $z \in \mathbb{C}^n$, the function

$$t \mapsto \Phi_0(z) - \Phi_t(z) \tag{3.42}$$

is non-decreasing. It follows that $R_t \geq 0$ for all $t \geq 0$ and that the function α is non-decreasing.

We next recall from [31, Proposition 6.1] that

$$\Lambda_{\Phi_0} \cap \Lambda_{\Phi_t} = \pi_1(\kappa_\varphi(S)), \quad t > 0, \tag{3.43}$$

where S is the singular space of q, $\kappa_\varphi\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is the complex linear canonical transformation defined by (2.6), and $\pi_1\colon \mathbb{C}^{2n} \to \mathbb{C}^n$ is the projection $\pi_1\colon (z,\zeta) \mapsto z$. Since we assume that $S = \{0\}$, we deduce from (3.43) that

$$\Lambda_{\Phi_0} \cap \Lambda_{\Phi_t} = \{0\}, \quad t > 0. \tag{3.44}$$

Thus, for every $t > 0$ and $z \in \mathbb{C}^n$,

$$\frac{2}{i}\Phi'_{0,z}(z) - \frac{2}{i}\Phi'_{t,z}(z) = 0 \iff z = 0. \tag{3.45}$$

Because R_t is a non-negative quadratic form for each $t \geq 0$, we have

$$\begin{aligned} R_t(z) = 0,\ z \in \mathbb{C}^n,\ t > 0 &\iff \nabla_{\mathrm{Re}\, z, \mathrm{Im}\, z} R_t(z) = 0 \\ &\iff \frac{2}{i}\Phi'_{0,z}(z) - \frac{2}{i}\Phi'_{t,z}(z) = 0. \end{aligned} \tag{3.46}$$

Hence, for any $z \in \mathbb{C}^n$ and $t > 0$,

$$R_t(z) = 0 \iff z = 0. \tag{3.47}$$

Thus, $\alpha(t) > 0$ for all $t > 0$.

To establish (3.31), we recall the main result of [12, Section 2], which states that if the singular space of q is trivial, $S = \{0\}$, then there is a small time $0 < t_0 \ll 1$ and a constant $c > 0$ such that

$$R_t(z) \geq c t^{2k_0+1} |z|^2, \quad z \in \mathbb{C}^n,\ 0 \leq t \leq t_0, \tag{3.48}$$

where $k_0 \in \{0, 1, \dots, 2n-1\}$ is the smallest non-negative integer such that (1.12) holds. It is therefore true that

$$\alpha(t) \geq c t^{2k_0+1}, \quad 0 \leq t \leq t_0. \tag{3.49}$$

To prove the claim regarding the behavior of $\alpha(t)$ as $t \to \infty$, we note that (2.16) implies that $\mathrm{spec}(iM) \subset \{\mathrm{Re}\,\lambda < 0\}$. Thus, there is $c > 0$ such that

$$R_t(z) = \Phi_0(z) + \mathcal{O}(e^{-ct}|z|^2) \quad \text{as } t \to \infty. \tag{3.50}$$

It follows that

$$\alpha(t) \to \min_{|z|=1} \Phi_0(z) \quad \text{as } t \to \infty. \tag{3.51}$$

The proof of Point 2 is complete.

Finally, we prove (3.34). Using (3.1), (3.2), (3.21), and (3.23), we obtain the following identity by elementary algebraic manipulations:

$$\begin{aligned} &2\operatorname{Re}\Psi_t(z,\bar{w}) - \Phi_t(z) - \Phi_0(w) \\ &\quad = -\Phi''_{0,\bar{z}z}(w - e^{iMt}z)\cdot(\overline{w - e^{iMt}z}), \quad z,w\in\mathbb{C}^n,\ t\geq 0. \end{aligned} \tag{3.52}$$

Because Φ_0 is a strictly plurisubharmonic quadratic form, the Levi matrix $\Phi''_{0,\bar{z}z}$ is Hermitian positive-definite. Consequently, there are constants $C, c > 0$, independent of t, such that

$$\begin{aligned} -C|w - e^{itM}z|^2 &\leq 2\operatorname{Re}\Psi_t(z,\bar{w}) - \Phi_t(z) - \Phi_0(w) \\ &\leq -c|w - e^{itM}z|^2, \quad z,w\in\mathbb{C}^n,\ t\geq 0. \end{aligned} \tag{3.53}$$

This proves (3.34). ∎

4. The conclusion of the proof of Theorem 1.1

In view of (2.26), (2.17), (3.5), and (3.12), the Schwartz kernel $K_t(x,y)$ of $G(t)$ is given, formally, by

$$K_t(x,y) = c_\varphi^2 C_{\Phi_0} e^{\frac{i}{2}\operatorname{tr}(M)t} \int\limits_{\mathbb{C}^n}\int\limits_{\mathbb{C}^n} e^{P_t(x,y,z,w)} L(dw)L(dz), \quad (x,y)\in\mathbb{R}^{2n},\ t\geq 0, \tag{4.1}$$

where

$$P_t(x,y,z,w) := -i\overline{\varphi(z,x)} - 2\Phi_0(z) + 2\Psi_t(z,\bar{w}) - 2\Phi_0(w) + i\varphi(w,y), \tag{4.2}$$

for $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$. For $z \in \mathbb{C}^n$, let $r(z) \in \mathbb{R}^n$ be the unique point such that

$$\Phi_0(z) = -\operatorname{Im}\varphi(z, r(z)). \tag{4.3}$$

We note that $r(z)$ is an $\mathbb{R}$-linear function of $z \in \mathbb{C}^n$. Since $\operatorname{Im}\varphi''_{yy} > 0$, there is $c > 0$ such that

$$-\operatorname{Im}\varphi(z,y) - \Phi_0(z) \leq -c|y - r(z)|^2, \quad z\in\mathbb{C}^n,\ y\in\mathbb{R}^n. \tag{4.4}$$

Using (4.4) together with the estimate (3.34), we find that

$$\operatorname{Re} P_t(x,y,z,w) \leq -c|x - r(z)|^2 - R_t(z) - c|w - e^{itM}z|^2 - c|y - r(w)|^2, \tag{4.5}$$

for all $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$, where $R_t(z)$ is as in Proposition 3.1. Let $\alpha\colon [0,\infty) \to \mathbb{R}$ be as in (3.29). Since (3.30) holds, there is $c > 0$ such that

$$\begin{aligned} &\operatorname{Re} P_t(x,y,z,w) \\ &\quad \leq -c|x-r(z)|^2 - \alpha(t)|z|^2 - c|w - e^{itM}z|^2 - c|y - r(w)|^2 \end{aligned} \tag{4.6}$$

for all $x, y \in \mathbb{R}^n$, $z, w \in \mathbb{C}^n$, and $t \geq 0$.

Let γ be as in the statement of Theorem 1.1. Taking the absolute value of (4.1) and using (4.6) and (2.16), we find that there are constants $C, c > 0$ such that

$$\begin{aligned} &|K_t(x,y)| \\ &\quad \leq Ce^{-\gamma t} \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-c|x-r(z)|^2 - \alpha(t)|z|^2 - c|w-\exp(itM)z|^2 - c|y-r(w)|^2} L(dw)L(dz) \end{aligned} \tag{4.7}$$

for every $x, y \in \mathbb{R}^n$ and $t \geq 0$. Let $1 \leq p \leq q \leq \infty$ be given, and let $1 \leq r \leq \infty$ be such that

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}. \tag{4.8}$$

Using Minkowski's integral inequality and the fact that $\alpha(t) > 0$ for every $t > 0$, we get that

$$\begin{aligned} &\|K_t(x,\cdot)\|_{L^r} \\ &\leq Ce^{-\gamma t} \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} e^{-c|x-r(z)|^2 - \alpha(t)|z|^2 - c|w-\exp(itM)z|^2} \|e^{-c|y-r(w)|^2}\|_{L^r_y} L(dw)L(dz) \\ &\leq C\alpha(t)^{-n}e^{-\gamma t}, \quad x \in \mathbb{R}^n,\ t > 0, \end{aligned} \tag{4.9}$$

where $C = C_{p,q} > 0$ depends only on p and q. By similar reasoning, there is $C = C_{p,q} > 0$ such that

$$\|K_t(\cdot,y)\|_{L^r} \leq C\alpha(t)^{-n}e^{-\gamma t}, \quad y \in \mathbb{R}^n,\ t > 0. \tag{4.10}$$

Applying Young's integral inequality with (4.9) and (4.10) gives

$$\|G(t)\|_{L^p \to L^q} \leq C\alpha(t)^{-n}e^{-\gamma t}, \quad t > 0, \tag{4.11}$$

for some $C = C_{p,q} > 0$.

Let $\varepsilon > 0$ be arbitrary. From Proposition 3.1, we know that α is non-decreasing and $\alpha(t) > 0$ for all $t > 0$. Thus,

$$\alpha(t) \geq \alpha(\varepsilon), \quad t \geq \varepsilon. \tag{4.12}$$

In view of (4.11), we may deduce that there is $C = C_{\varepsilon,p,q} > 0$ such that

$$\|G(t)\|_{L^p \to L^q} \leq Ce^{-\gamma t}, \quad t \geq \varepsilon. \tag{4.13}$$

To see that the bound (4.13) is sharp as $t \to \infty$, we recall from [19, Theorem 2.1] that the lowest eigenvalue ρ of $q^w(x, D)$, introduced in (1.14), is simple and that the eigenspace of $q^w(x, D)$ corresponding to ρ is spanned by a "ground state" of the form

$$u_0(x) = e^{-a(x)}, \quad x \in \mathbb{R}^n, \tag{4.14}$$

where a is a complex-valued quadratic form on $\mathbb{R}^n$ with positive-definite real part $\operatorname{Re} a > 0$. Let $v = \|u_0\|^{-1}_{L^p(\mathbb{R}^n)} u_0$. Since $q^w(x, D)v = \rho v$, is is clear that

$$\|e^{-tq^w(x,D)} v\|_{L^q} = e^{-t\gamma} \|v\|_{L^q}, \quad t \geq 0. \tag{4.15}$$

Hence, there is a constant $c = c_{p,q} > 0$ such that

$$\|e^{-tq^w(x,D)}\|_{L^p \to L^q} \geq ce^{-\gamma t}, \quad t \geq 0. \tag{4.16}$$

We conclude that there are constants $C = C_{\varepsilon,p,q} > 0$ and $c = c_{p,q} > 0$ such that (1.15) holds for all $t \geq \varepsilon$.

Finally, we prove the bound (1.16). From (3.31), (4.11), and (4.16), we get that there are constants $C = C_{p,q} > 0$ and $c_{p,q} > 0$ such that

$$c \leq \|G(t)\|_{L^p \to L^q} \leq Ct^{-(2k_0+1)n}, \quad 0 < t \leq t_0. \tag{4.17}$$

The proof of Theorem 1.1 is complete.

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References

[1] A. Aleman and J. Viola, On weak and strong solution operators for evolution equations coming from quadratic operators. *J. Spectr. Theory* **8** (2018), no. 1, 33–121 Zbl 1474.35196 MR 3762128.

[2] P. Alphonse, Quadratic differential equations: partial Gelfand–Shilov smoothing effect and null-controllability. *J. Inst. Math. Jussieu* **20** (2021), no. 6, 1749–1801 Zbl 1476.35074 MR 4332777

[3] P. Alphonse and J. Bernier, Polar decomposition of semigroups generated by non-selfadjoint quadratic differential operators and regularizing effects. 2019, arXiv:1909.03662

[4] E. Caliceti, S. Graffi, M. Hitrik, and J. Sjöstrand, Quadratic $\mathcal{PT}$-symmetric operators with real spectrum and similarity to self-adjoint operators. *J. Phys. A* **45** (2012), no. 44, article no. 444007 20 Zbl 1263.81190 MR 2991874

[5] E. Carypis and P. Wahlberg, Propagation of exponential phase space singularities for Schrödinger equations with quadratic Hamiltonians. *J. Fourier Anal. Appl.* **23** (2017), no. 3, 530–571 Zbl 1377.35228 MR 3649471

[6] L. A. Coburn, M. Hitrik, and J. Sjöstrand, Positivity, complex FIOs, and Toeplitz operators. *Pure Appl. Anal.* **1** (2019), no. 3, 327–357 Zbl 1429.32047 MR 3985088

[7] E. B. Davies and B. Simon, Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59** (1984), no. 2, 335–395 Zbl 0568.47034 MR 766493

[8] E. B. Davies and B. Simon, L^p norms of noncritical Schrödinger semigroups. *J. Funct. Anal.* **102** (1991), no. 1, 95–115 Zbl 0743.47047 MR 1138839

[9] M. Hitrik and K. Pravda-Starov, Spectra and semigroup smoothing for non-elliptic quadratic operators. *Math. Ann.* **344** (2009), no. 4, 801–846 Zbl 1171.47038 MR 2507625

[10] M. Hitrik and K. Pravda-Starov, Semiclassical hypoelliptic estimates for non-selfadjoint operators with double characteristics. *Comm. Partial Differential Equations* **35** (2010), no. 6, 988–1028 Zbl 1247.35015 MR 2753626

[11] M. Hitrik and K. Pravda-Starov, Eigenvalues and subelliptic estimates for non-selfadjoint semiclassical operators with double characteristics. *Ann. Inst. Fourier (Grenoble)* **63** (2013), no. 3, 985–1032 Zbl 1292.35185 MR 3137478

[12] M. Hitrik, K. Pravda-Starov, and J. Viola, From semigroups to subelliptic estimates for quadratic operators. *Trans. Amer. Math. Soc.* **370** (2018), no. 10, 7391–7415 Zbl 1404.35115 MR 3841852

[13] M. Hitrik and J. Sjöstrand, Two minicourses on analytic microlocal analysis. In *Algebraic and analytic microlocal analysis*, pp. 483–540, Springer Proc. Math. Stat. 269, Springer, Cham, 2018 Zbl 1418.32003 MR 3903325

[14] M. Hitrik, J. Sjöstrand, and J. Viola, Resolvent estimates for elliptic quadratic differential operators. *Anal. PDE* **6** (2013), no. 1, 181–196 Zbl 1295.47045 MR 3068543

[15] L. Hörmander, Symplectic classification of quadratic forms, and general Mehler formulas. *Math. Z.* **219** (1995), no. 3, 413–449 Zbl 0829.35150 MR 1339714

[16] L. Hörmander, *The analysis of linear partial differential operators.* III. Classics in Mathematics, Springer, Berlin, 2007 Zbl 1115.35005 MR 2304165

[17] H. Koch and D. Tataru, L^p eigenfunction bounds for the Hermite operator. *Duke Math. J.* **128** (2005), no. 2, 369–392 Zbl 1075.35020 MR 2140267

[18] E. H. Lieb, Gaussian kernels have only Gaussian maximizers. *Invent. Math.* **102** (1990), no. 1, 179–208 Zbl 0726.42005 MR 1069246

[19] M. Ottobre, G. A. Pavliotis, and K. Pravda-Starov, Exponential return to equilibrium for hypoelliptic quadratic systems. *J. Funct. Anal.* **262** (2012), no. 9, 4000–4039 Zbl 1256.47032 MR 2899986

[20] K. Pravda-Starov, Contraction semigroups of elliptic quadratic differential operators. *Math. Z.* **259** (2008), no. 2, 363–391 Zbl 1139.47033 MR 2390087

[21] K. Pravda-Starov, Subelliptic estimates for quadratic differential operators. *Amer. J. Math.* **133** (2011), no. 1, 39–89 Zbl 1257.47058 MR 2752935

[22] K. Pravda-Starov, Generalized Mehler formula for time-dependent non-selfadjoint quadratic operators and propagation of singularities. *Math. Ann.* **372** (2018), no. 3–4, 1335–1382 Zbl 1405.35270 MR 3880300

[23] K. Pravda-Starov, L. Rodino, and P. Wahlberg, Propagation of Gabor singularities for Schrödinger equations with quadratic Hamiltonians. *Math. Nachr.* **291** (2018), no. 1, 128–159 Zbl 1384.35107 MR 3756858

[24] B. Simon, Brownian motion, L^p properties of Schrödinger operators and the localization of binding. *J. Functional Analysis* **35** (1980), no. 2, 215–229 Zbl 0446.47041 MR 561987

[25] B. Simon, Large time behavior of the L^p norm of Schrödinger semigroups. *J. Functional Analysis* **40** (1981), no. 1, 66–83 Zbl 0478.47024 MR 607592

[26] J. Sjöstrand, Singularités analytiques microlocales. *Astérisque* **95** (1982), 1–166 Zbl 0524.35007 MR 699623

[27] J. Sjöstrand, Resolvent estimates for non-selfadjoint operators via semigroups. In *Around the research of Vladimir Maz'ya.* III, pp. 359–384, Int. Math. Ser. (N. Y.) 13, Springer, New York, 2010 Zbl 1198.47068 MR 2664715

[28] J. Viola, Non-elliptic quadratic forms and semiclassical estimates for non-selfadjoint operators. *Int. Math. Res. Not. IMRN* (2013), no. 20, 4615–4671 Zbl 1295.47050 MR 3118871

[29] J. Viola, Spectral projections and resolvent bounds for partially elliptic quadratic differential operators. *J. Pseudo-Differ. Oper. Appl.* **4** (2013), no. 2, 145–221 Zbl 1285.47049 MR 3244980

[30] P. Wahlberg, Propagation of polynomial phase space singularities for Schrödinger equations with quadratic Hamiltonians. *Math. Scand.* **122** (2018), no. 1, 107–140 Zbl 1417.35155 MR 3767155

[31] F. White, Propagation of global analytic singularities for Schrödinger equations with quadratic Hamiltonians. *J. Funct. Anal.* **283** (2022), no. 6, article no. 109569 Zbl 1495.35005 MR 4442595

[32] M. Zworski, *Semiclassical analysis*. Grad. Stud. Math. 138, American Mathematical Society, Providence, RI, 2012 Zbl 1252.58001 MR 2952218 431 p. (2012).

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