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Spectral convergence of high-dimensional spheres to Gaussian spaces

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Abstract. We prove that the spectral structure on the N-dimensional standard sphere of radius $(N-1)^{1/2}$ compatible with a projection onto the first n-coordinates converges to the spectral structure on the n-dimensional Gaussian space with variance 1 as $N \to \infty$. We also show the analogue for the first Dirichlet eigenvalue problem on a ball in the sphere and that on a half-space in the Gaussian space.

1. Introduction

A curvature-dimension condition $\mathrm{CD}(\kappa, N)$ imposes restriction on the spectra of the weighted Laplacian on a weighted manifold. For example, the Lichnerowicz–Obata-type eigenvalue estimate is known (see [8, Theorems 1.2], [18, Corollary 1.3], [20, Theorem 5.34], and the references therein). Here a *weighted manifold* (M, μ) is a complete smooth n-dimensional Riemannian manifold (M, g) equipped with a measure μ of the form

$$\mu = \exp(-\Psi)\,\mathrm{vol}_M,$$

where $\Psi \in C^\infty(M)$ and vol_M denotes the Riemannian volume measure on (M, g). The *weighted Laplacian* Δ_μ on (M, μ) is defined as

$$\Delta_\mu f := \Delta_M f - g(\nabla_M \Psi, \nabla_M f) \quad \text{for } f \in C^\infty(M),$$

where ∇_M and Δ_M stand for the gradient and the Laplacian on (M, g), respectively, so that the following integration by parts is satisfied

$$\int_M g(\nabla_M f_1, \nabla_M f_2)\,d\mu = -\int_M f_1 \Delta_\mu f_2\,d\mu \quad \text{for } f_1, f_2 \in C_0^\infty(M).$$

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Given $\kappa \in \mathbb{R}$ and $N \in [n,\infty]$, we say that (M,μ) satisfies the *curvature-dimension condition* $\mathrm{CD}(\kappa,N)$ if

$$\mathrm{Ric}_M(v,v) + \mathrm{Hess}_M\Psi(v,v) - \frac{v(\Psi)^2}{N-n} \geq \kappa g(v,v) \quad \text{for } v \in TM,$$

where Ric_M is the Ricci curvature tensor and Hess_M is the Hessian operator on (M,g), respectively. To make sense, we employ the convention that $\frac{1}{\infty} := 0$, $\frac{1}{0} := +\infty$, and $\infty \cdot 0 := 0$. A model space for comparison geometry under the condition $\mathrm{CD}(1,N)$ is the N-dimensional standard sphere of radius $(N-1)^{1/2}$ for $N \in \mathbb{N}$ with $N \geq 2$, and the one-dimensional Gaussian space with variance 1 for $N = \infty$.

For $N \in \mathbb{N}$ and $a > 0$, let $\mathbb{S}^N(a)$ be the N-dimensional standard sphere of radius a. We denote by $\langle\cdot,\cdot\rangle$ the Euclidean inner product and set $|\cdot|_2 := \langle\cdot,\cdot\rangle^{1/2}$. For $n \in \mathbb{N}$ and $\alpha > 0$, we denote by γ_α^n the *n-dimensional Gaussian measure with variance α^2*, that is,

$$d\gamma_\alpha^n(x) = (2\pi\alpha^2)^{-\frac{n}{2}} \exp\Big(-\frac{|x|_2^2}{2\alpha^2}\Big)dx.$$

The weighted manifold $\Gamma_\alpha^n := (\mathbb{R}^n, \gamma_\alpha^n)$ is called the *n-dimensional Gaussian space with variance α^2*. Notice that a weighted manifold of $\mathbb{S}^N(a)$ equipped with its Riemannian volume measure satisfies $\mathrm{CD}(a^{-2}(N-1),N)$ and Γ_α^n satisfies $\mathrm{CD}(\alpha^{-2},\infty)$, respectively. Set

$$S_N := \mathbb{S}^N(\sqrt{N-1}), \quad \gamma^n := \gamma_1^n, \quad \Gamma^n := \Gamma_1^n.$$

Since $\mathrm{CD}(1,\infty)$ can be regarded as the limit of $\mathrm{CD}(1,N)$ as $N \to \infty$, the spectral structure on Γ^n would be derived from the asymptotic behavior of that on S_N as well. For example, Borell [5, Theorem 3.1] and Sudakov and Cirel'son [25, Corollary 1] independently proved the Brunn–Minkowski inequality on Γ^n by using that on S_N. The Brunn–Minkowski inequality determines a domain minimizing the first Dirichlet eigenvalue under the restriction of the volume. The key of the proof is the following asymptotic behavior, so-called Poincaré's theorem (we refer to [9, Section 6] for the history of Poincaré's theorem). Let σ_N be the normalized Riemannian volume measure on S_N to be a probability measure. For $n, N \in \mathbb{N}$ with $n \leq N$, p_n^N denotes the projection from $\mathbb{R}^{N+1} = \mathbb{R}^n \times \mathbb{R}^{N-n+1}$ onto $\mathbb{R}^n$ defined by

$$p_n^N(x,y) := x \quad \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^{N-n+1}.$$

Then the push-forward measure of σ_N by the restriction of p_n^N to S_N satisfies

$$\lim_{N\to\infty} \frac{d\left((p_n^N|_{S_N})_\sharp\sigma_N\right)}{dx}(x) = \frac{d\gamma^n}{dx}(x) \quad \text{for } x \in \mathbb{R}^n$$

and $\{(p_n^N|_{S_N})_\sharp\sigma_N\}_{N\in\mathbb{N}}$ converges to γ^n weakly as $N\to\infty$. Since the weak convergence of probability measures on $\mathbb{R}^n$ is metrizable by the Prokhorov metric d_P, it holds that

$$\lim_{N\to\infty} d_P((p_n^N|_{S_N})_\sharp\sigma_N, \gamma^n) = 0.$$

Let $\iota_N\colon S_N \hookrightarrow \mathbb{R}^{N+1}$ be the inclusion map. In contrast to Poincaré's theorem, Shioya and the author [24, Theorem 1.4] showed that

$$\liminf_{N\to\infty} d_P(\iota_{N\sharp}\sigma_N, \gamma^{N+1}) > 0.$$

This suggests that the asymptotic behavior of the spectral structure on S_N and Γ^{N+1} are different. Indeed, the multiplicity of the first nonzero eigenvalue on both of S_N and Γ^{N+1} are $N+1$, while the multiplicity of the second nonzero eigenvalue on S_N is $N(N+3)/2$ but that on Γ^{N+1} is $(N+1)(N+2)/2$. See [22, Sections 2.1 and 2.2] for instance. Thus, it is more appropriate to compare the spectral structure on Γ^n with the compatible spectral structure on S_N with p_n^N, rather than the spectral structure on S_N itself.

In this paper, we prove the convergence of eigenvalues on $\mathbb{S}^N(a_N)$ to those on Γ_α^n together with the convergence of the composition of p_n^N and compatible eigenfunctions on $\mathbb{S}^N(a_N)$ to eigenfunctions on Γ_α^n as $N\to\infty$ when $\{a_N/\sqrt{N-1}\}_{N\geq n,2}$ converges to α. We also show the analogue for the first Dirichlet eigenvalue problem on a ball in $\mathbb{S}^N(a_N)$ and that on a half-space in Γ_α^n.

We define some notation needed to state our theorems. Let $\mathbb{N}_0$ denote the set of nonnegative integers. Unless specified otherwise in this paper, let

$$n, N\in\mathbb{N} \quad \text{with} \quad n, 2\leq N, \quad k\in\mathbb{N}_0, \quad a,\alpha>0, \quad \theta\in(0,\pi), \quad R\in\mathbb{R}.$$

We shall for convenience denote a sequence $\{c_N\}_{N\geq N_0}$ by $\{c_N\}_N$.

For the rest of this paper, a weighted manifold (M,μ) is either $\mathbb{S}^N(a)$ equipped with its Riemannian volume measure $\mathrm{vol}_{\mathbb{S}^N(a)}$ or $\Gamma_\alpha^n = (\mathbb{R}^n, \gamma_\alpha^n)$. Note that we have $\Delta_{\mathrm{vol}_{\mathbb{S}^N(a)}} = \Delta_{\mathbb{S}^N(a)}$. When it will introduce no confusion, we shall denote $(\mathbb{S}^N(a), \mathrm{vol}_{\mathbb{S}^N(a)})$ simply by $\mathbb{S}^N(a)$.

A real number λ is called a *closed eigenvalue*, or simply *eigenvalue* of $-\Delta_\mu$ on M if there exists a nontrivial solution $\phi\in C^2(M)$ to

$$\Delta_\mu\phi = -\lambda\phi \quad \text{in } M. \tag{1.1}$$

A solution to (1.1) is called an *eigenfunction* of eigenvalue λ. Any constant function on M is an eigenfunction of eigenvalue 0. We denote the list of distinct eigenvalues on M by

$$0=\lambda_0(M,\mu)<\lambda_1(M,\mu)<\lambda_2(M,\mu)<\cdots<\lambda_k(M,\mu)<\cdots\uparrow\infty.$$

Let $E_k(M,\mu)$ be the linear space of solutions to (1.1) for $\lambda = \lambda_k(M,\mu)$. It is known that the linear space $E_k(\mathbb{S}^N(a))$ is spanned by the restriction of homogeneous harmonic polynomials on $\mathbb{R}^{N+1}$ of degree k to $\mathbb{S}^N(a)$ (see [7, Section II.4]). We denote by $\mathbb{P}(n)$ the linear space of polynomials on $\mathbb{R}^n$. Define the linear subspace of $E_k(\mathbb{S}^N(a))$ by

$$E_k^n(\mathbb{S}^N(a)) := \{\Phi \in E_k(\mathbb{S}^N(a)) \mid Q \circ p_n^N = \Phi \text{ on } \mathbb{S}^N(a) \text{ for some } Q \in \mathbb{P}(n)\}.$$

Theorem 1.1. *Let $\{a_N\}_N$ be a sequence of positive real numbers. For $n, N \in \mathbb{N}$ with $n, 2 \leq N$ and $k \in \mathbb{N}_0$,*

$$\dim E_k^n(\mathbb{S}^N(a_N)) = \dim E_k(\Gamma^n) =: d_k(n). \tag{1.2}$$

Moreover, if $\{a_N/\sqrt{N-1}\}_N$ converges to a positive real number α as $N \to \infty$, then

$$\lim_{N\to\infty} \lambda_k(\mathbb{S}^N(a_N)) = \lambda_k(\Gamma_\alpha^n). \tag{1.3}$$

In this case, there exist a set of homogeneous harmonic polynomials $\{P_{N,j}\}_{j=1}^{d_k(n)}$ on $\mathbb{R}^{N+1}$ of degree k and $\{Q_{N,j}\}_{j=1}^{d_k(n)} \subset \mathbb{P}(n)$ satisfying the following three properties:

- *the restriction of $\{P_{N,j}\}_{j=1}^{d_k(n)}$ to $\mathbb{S}^N(a)$ forms a basis of $E_k^n(\mathbb{S}^N(a_N))$.*
- *$Q_{N,j} \circ p_n^N = P_{N,j}$ on $\mathbb{S}^N(a_N)$.*
- *$\{Q_{N,j}\}_N$ converges to some $Q_j \in \mathbb{P}(n)$ uniformly on compact sets and strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$ for each $1 \leq j \leq d_k(n)$ and $\{Q_j\}_{j=1}^{d_k(n)}$ forms a basis of $E_k(\Gamma_\alpha^n)$.*

Next we consider the analogue for the first Dirichlet eigenvalue problem. For $m, i \in \mathbb{N}$ with $i \leq m$, let e_i^m denote the m-tuple consisting of zeros except for a 1 in the ith spot. Let $d_{\mathbb{S}^N(a)}$ be the Riemannian distance function on $\mathbb{S}^N(a)$. We define the open ball $B_{a\theta}^N$ in $\mathbb{S}^N(a)$ and the open half-space $V_{\alpha R}^n$ in $\mathbb{R}^n$ by

$$B_{a\theta}^N := \{z \in \mathbb{S}^N(a) \mid d_{\mathbb{S}^N(a)}(z, ae_1^{N+1}) < a\theta\},$$
$$V_{\alpha R}^n := \{x = (x_i)_{i=1}^n \in \mathbb{R}^n \mid x_1 > \alpha R\},$$

respectively. Let $\Omega = B_{a\theta}^N$ if $M = \mathbb{S}^N(a)$, and $\Omega = V_{\alpha R}^n$ if $M = \Gamma_\alpha^n$.

A real number λ is called the *first Dirichlet eigenvalue* of $-\Delta_\mu$ on Ω if there exists a solution $\phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to

$$\begin{cases} \Delta_\mu \phi = -\lambda\phi & \text{in } \Omega, \\ \phi > 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The first Dirichlet eigenvalue of $-\Delta_\mu$ on Ω, denoted by $\lambda(\Omega, (M, \mu))$, is positive and a solution to (1.4) is uniquely determined up to a positive constant multiple. A solution to (1.4) is called a *first positive Dirichlet eigenfunction* of $-\Delta_\mu$ on Ω.

Let $H_0^1(V_{\alpha R}^n, \gamma_\alpha^n)$ denote the completion of $C_0^\infty(V_{\alpha R}^n)$ with respect to the inner product given by

$$(f_1, f_2)_{H^1(V_{\alpha R}^n, \gamma_\alpha^n)} := \int_{V_{\alpha R}^n} f_1 f_2 d\gamma_\alpha^n + \int_{V_{\alpha R}^n} \langle \nabla_{\mathbb{R}^n} f_1, \nabla_{\mathbb{R}^n} f_2 \rangle d\gamma_\alpha^n$$

for $f_1, f_2 \in C_0^\infty(V_{\alpha R}^n)$.

Theorem 1.2. *Let $\{a_N\}_N$, $\{\theta_N\}_N$ be sequences of real numbers such that $a_N > 0$ and $\theta_N \in (0, \pi)$ for $N \in \mathbb{N}$. Define two functions s_N, w_N on $[-a_N, a_N]$ and a function w_∞ on $\mathbb{R}$ by*

$$s_N(r) := 1 - \frac{r^2}{a_N^2},$$
$$w_N(r) := s_N(r)^{\frac{N}{2}-1} \cdot \left(\int_{-a_N}^{a_N} s_N(\rho)^{\frac{N}{2}-1} d\rho \right)^{-1},$$
$$w_\infty(r) := \frac{1}{\sqrt{2\pi}\alpha} e^{-\frac{r^2}{2\alpha^2}},$$

respectively. Let ϕ_N be the first positive Dirichlet eigenfunction of $-\Delta_{\mathbb{S}^N(a_N)}$ on $B_{a_N \theta_N}^N$ such that

$$\int_{B_{a_N \theta_N}^N} \phi_N(z)^2 d\mathrm{vol}_{\mathbb{S}^N(a_N)}(z) = \mathrm{vol}_{\mathbb{S}^{N-1}(a_N)}(\mathbb{S}^{N-1}(a_N)) \int_{-a_N}^{a_N} s_N(r)^{\frac{N}{2}-1} dr.$$

Then for $n, N \in \mathbb{N}$ with $n, 2 \leq N$, there exists $\psi_N \in H_0^1(V_{\alpha R}^n, \gamma_\alpha^n)$ such that

$$\psi_N \circ p_n^N = \phi_N \cdot \left\{ \left(s_N \sqrt{\frac{w_N}{w_\infty}} \right) \circ p_1^N \right\} \quad \text{on } B_{a_N \theta_N}^N. \tag{1.5}$$

Moreover, if there exist $\alpha > 0$ and $R \in \mathbb{R}$ such that

$$\lim_{N \to \infty} \frac{a_N}{\sqrt{N-1}} = \alpha,$$
$$\lim_{N \to \infty} a_N \cos \theta_N = \alpha R,$$
$$\sup_{N \in \mathbb{N}} \frac{a_N^2 - \alpha^2 (N-2)}{a_N} < \infty,$$
$$a_N \cos \theta_N \geq \alpha R,$$

then

$$\lim_{N\to\infty} \lambda(B^N_{a_N\theta_N}, \mathbb{S}^N(a_N)) = \lambda(V^n_{\alpha R}, \Gamma^n_\alpha).$$

In this case, $\{\psi_N\}_N$ converges to the first positive Dirichlet eigenfunction ψ_∞ of $-\Delta_{\gamma^n_\alpha}$ on $V^n_{\alpha R}$ strongly in $H^1_0(V^n_{\alpha R}, \gamma^n_\alpha)$ and

$$\int_{V^n_{\alpha R}} \psi_\infty(x)^2 d\gamma^n_\alpha(x) = 1.$$

Let us make a few comments on related works. Aside from the difference between the asymptotic behavior of the spectral structure on S_N and Γ^{N+1}, the study of the relation between the limit of S_N as $N \to \infty$ and the infinite-dimensional Gaussian space has a long history, which goes back to Boltzmann and Maxwell around the 1860s in the study of the motion of gas molecules. McKean [21] gave an exposition to explain how this study is fruitful (see also [14], where the classical idea of Lévy [19] and Wiener [27] is explained with examples in physics and control theory). Its mathematical foundations are established in the 1960s. For example, Hida and Nomoto [15] constructed an infinite-dimensional Gaussian space as the projective limit space of S_N and defined a family of functions analogous to homogeneous harmonic polynomials restricted to S_N, which forms a complete orthonormal system in the L^2-spaces on the infinite-dimensional Gaussian space. Umemura and Kôno [26, Section 4] made clear the relation between the Laplacian on S_N and that on the infinite-dimensional Gaussian space and investigated how this relation reflects on their eigenfunctions. Peterson and Sengupta [23, Section 5] analyzed an asymptotic behavior of the Laplacian on S_N and its eigenfunctions from the algebraic viewpoint. Compare Theorem 1.1 with [26, Proposition 5] and [23, Proposition 4.3]. Note that the difference of eigenvalues on S_N and Γ^1 provides an quantitative estimate of the difference between S_N and Γ^1 by [3, Theorem 1.2].

As for the Dirichlet eigenvalue problem, Friedland and Hayman [10, Theorem 2] proved that the positive root $\nu_N(s)$ of the equation

$$\nu(\nu + N - 1) = \lambda(B^N_{\theta_N}, \mathbb{S}^N(1))$$

with

$$\mathrm{vol}_{\mathbb{S}^N(1)}(B^N_{\theta_N}) / \mathrm{vol}_{\mathbb{S}^N(1)}(\mathbb{S}^N(1)) = s \in (0, 1)$$

is nonincreasing in $N \in \mathbb{N}$ hence the limit of $\{\nu_N(s)\}_N$ as $N \to \infty$ exists. This suggests that $\{\lambda(B^N_{\theta_N}, \mathbb{S}^N(1))/N\}_N$ converges to $\lambda(V^1_R, \Gamma^1)$ as $N \to \infty$ (see [6, p. 218]). In general, the spectral convergence with respect to the pointed measured Gromov–Hausdorff topology under the curvature-dimension condition is known (for instance, see [1, 2, 12, 28] and the references therein). With respect to the pointed measured Gromov–Hausdorff topology, although $\{\mathbb{S}^N(1)\}_N$ diverges (see [11, Proposition 1.1]), $\{(p_n^N(S_N), |\cdot|_2, (p_n^N|_{S_N})_\sharp \sigma_N)\}_N$ converges to Γ^n as $N \to \infty$. This with the metric contraction principle (see [22, Proposition 3.4]) suggests

$$\lambda_m(V_{\alpha R}^n, \Gamma_\alpha^n) \geq \lim_{N\to\infty} \lambda_m(B_{a_N\theta_N}^N, \mathbb{S}^N(a_N)),$$

where $m \in \mathbb{N}$ and $\lambda_m(\Omega, (M, \mu))$ stands for the mth Dirichlet eigenvalue of $-\Delta_\mu$ on Ω. In the case $n = 1$, Kazukawa [17, Example 4.18] used a projection from S_N to $\mathbb{R}$ different from p_1^N and discussed the spectral convergence on S_N in the framework of metric measure foliation.

This paper is organized as follows. Section 2 is devoted to recalling some known facts of Eigenvalue problems on spheres and Gaussian spaces. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4, respectively. We discuss the relation between Dirichlet eigenspaces of high-dimensional spheres and those of Gaussian spaces in Section 5.

2. Eigenvalue problems on $\mathbb{S}^N(a)$ and Γ_α^n

Let us briefly recall some known facts of eigenvalue problems on $\mathbb{S}^N(a)$ and Γ_α^n. We refer to [7, Sections II.4 and II.5] and [22, Sections 2.1 and 2.2] for more details.

2.1. Eigenvalue problem on $\mathbb{S}^N(a)$

The kth distinct eigenvalue on $\mathbb{S}^N(a)$ is given by

$$\lambda_k(\mathbb{S}^N(a)) = \frac{k}{a^2}(k + N - 1) \quad \text{with multiplicity} \quad \binom{N+k}{k} - \binom{N+k-2}{k-2}, \tag{2.1}$$

where we adhere to the convention that $\binom{N-2}{-2}, \binom{N-1}{-1} := 0$.

For a first positive Dirichlet eigenfunction ϕ of $-\Delta_{\mathbb{S}^N(a)}$ on $B_{a\theta}^N$, there exists a solution $\varphi \in C^\infty([0, a\theta)) \cap C([0, a\theta])$ to

$$\begin{cases} \varphi''(\vartheta) + (N-1)\dfrac{\cos(\vartheta/a)}{a\sin(\vartheta/a)}\varphi'(\vartheta) = -\lambda(B_{a\theta}^N, \mathbb{S}^N(a))\varphi(\vartheta) & \text{in } \vartheta \in [0, a\theta), \\ \varphi(\vartheta) > 0 & \text{in } \vartheta \in [0, a\theta), \\ \varphi(a\theta) = 0, \end{cases} \tag{D^N}$$

such that $\phi(z) = \varphi(d_{\mathbb{S}^N(a)}(z, ae_1^{N+1}))$ on $\overline{B^N_{a\theta}}$, where

$$\int_{B^N_{a\theta}} \phi(z)^2 d\,\mathrm{vol}_{\mathbb{S}^N(a)}(z) = \mathrm{vol}_{\mathbb{S}^{N-1}(a)}(\mathbb{S}^{N-1}(a)) \int_0^{a\theta} \varphi(\vartheta)^2 \sin^{N-1}(\vartheta/a)\, d\vartheta < \infty \tag{2.2}$$

holds. It follows that

$$\lambda(B^N_{a\theta}, \mathbb{S}^N(a)) = \frac{1}{a^2}\lambda(B^N_\theta, \mathbb{S}^N(1)).$$

It is known that $\phi(z) = \cos(d_{\mathbb{S}^N(a)}(z, ae_1^{N+1})/a)$ is a first positive Dirichlet eigenfunction of $-\Delta_{\mathbb{S}^N(a)}$ on $B^N_{a\pi/2}$ and $\lambda(B^N_{a\pi/2}, \mathbb{S}^N(a)) = N/a^2$. Hence, $\varphi(r) = \cos(r/a)$ solves (D^N) with the case $\theta = \pi/2$. Notice that

$$d_{\mathbb{S}^N(a)}(z, ae_1^{N+1}) = a \cdot \arccos\left(\frac{z_1}{a}\right) \quad \text{on } z = (z_i)_{i=1}^{N+1} \in \mathbb{S}^N(a) \subset \mathbb{R}^{N+1}.$$

2.2. Eigenvalue problem on Γ^n_α

The weighted Laplacian $\Delta_{\gamma^n_\alpha}$ is also called the *Ornstein–Uhlenbeck operator* and is given by

$$\Delta_{\gamma^n_\alpha} f(x) = \Delta_{\mathbb{R}^n} f(x) - \frac{1}{\alpha^2}\langle x, \nabla_{\mathbb{R}^n} f(x)\rangle \quad \text{for } f \in C^2(\mathbb{R}^n) \text{ and } x \in \mathbb{R}^n.$$

For $K = (K_i)_{i=1}^n \in \mathbb{N}_0^n$ and $k \in \mathbb{N}_0$, set

$$|K| := \sum_{i=1}^n K_i, \quad \mathbb{N}_0^n(k) := \{K \in \mathbb{N}_0^n \mid |K| = k\}.$$

The kth distinct eigenvalue on Γ^n_α is given by

$$\lambda_k(\Gamma^n_\alpha) = \frac{k}{\alpha^2} \quad \text{with multiplicity } d_k(n) := \sharp\mathbb{N}_0^n(k) = \binom{n-1+k}{k} \tag{2.3}$$

and $E_k(\Gamma^n_\alpha)$ is spanned by

$$\left\{x = (x_i)_{i=1}^n \mapsto \prod_{i=i}^n H_{K_i}(\alpha^{-1}x_i)\right\}_{K \in \mathbb{N}_0^n(k)},$$

where H_k is the kth order Hermite polynomial of the form

$$H_k(r) := (-1)^k e^{\frac{r^2}{2}} \frac{d^k}{dr^k} e^{-\frac{r^2}{2}}. \tag{2.4}$$

An argument similar to the first Dirichlet eigenvalue problem on a ball in a sphere implies that, for a first Dirichlet eigenfunction ψ of $-\Delta_{\gamma_\alpha^n}$ on $V_{\alpha R}^n$, there exists a first Dirichlet eigenfunction h of $-\Delta_{\gamma_\alpha^1}$ on $V_{\alpha R}^1 = (\alpha R, \infty)$ such that $\psi(x) = h(x_1)$ on $x = (x_i)_{i=1}^n \in \overline{V_{\alpha R}^n}$, where

$$\int_{V_{\alpha R}^n} \psi(x)^2 d\gamma_\alpha^n(x) = \int_{\alpha R}^{\infty} h(r)^2 d\gamma_\alpha^1(r) < \infty.$$

Moreover, $\lambda(V_{\alpha R}^1, \Gamma_\alpha^1) = \lambda(V_{\alpha R}^n, \Gamma_\alpha^n)$ holds.

3. Proof of Theorem 1.1

To prove Theorem 1.1, we analyze the composition of p_n^N and homogeneous harmonic polynomials on $\mathbb{R}^{N+1}$. Given $j \in \mathbb{N}$ and $m \in \mathbb{N}_0$, set

$$\Delta_{\mathbb{R}^n}^j := \Big(\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}\Big)^j, \quad c_j(m) := -\frac{1}{2j(m+2j-1)}, \quad C_j(m) := \prod_{l=1}^j c_l(m),$$

and $\Delta_{\mathbb{R}^n}^0 := \mathrm{id}_{\mathbb{R}^n}, C_0(m) := 1$. For $K = (K_i)_{i=1}^n \in \mathbb{N}_0^n$ and $x = (x_i)_{i=1}^n \in \mathbb{R}^n$, set

$$x^K := \prod_{i=1}^n x_i^{K_i},$$

where by convention $0^0 := 1$. For $t \in \mathbb{R}$, let $[t]$ be the greatest integer less than or equal to t.

Definition 3.1. For $n, N \in \mathbb{N}$ with $n \leq N$, $K \in \mathbb{N}_0^n$ and $a, \alpha > 0$, define

$$P_{N,n,K}(x,y) := \sum_{j=0}^{[|K|/2]} C_j(N-n)|y|_2^{2j}\,\Delta_{\mathbb{R}^n}^j x^K \qquad \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}^{N-n+1},$$

$$Q_{N,n,K;a}(x) := \sum_{j=0}^{[|K|/2]} C_j(N-n)(a^2 - |x|_2^2)^j\,\Delta_{\mathbb{R}^n}^j x^K \quad \text{for } x \in \mathbb{R}^n,$$

$$Q_{n,K;\alpha}(x) := \sum_{j=0}^{[k/2]} (-1)^j \frac{\alpha^{2j}}{2^j j!} \Delta_{\mathbb{R}^n}^j x^K \qquad \text{for } x \in \mathbb{R}^n.$$

We easily check that $P_{N,n,K}$ is a homogeneous polynomial on $\mathbb{R}^{N+1}$of degree $|K|$ and $Q_{N,n,K;a}, Q_{n,K;\alpha} \in \mathbb{P}(n)$. All of

$$\{P_{N,n,K}\}_{K \in \mathbb{N}_0^n(k)}, \quad \{Q_{N,n,K;a}\}_{K \in \mathbb{N}_0^n(k)}, \quad \{Q_{n,K;\alpha}\}_{K \in \mathbb{N}_0^n(k)}$$

are linearly independent. It turns out that

$$Q_{N,n,K;a} \circ p_n^N = P_{N,n,K} \quad \text{on } \mathbb{S}^N(a).$$

Lemma 3.2. *For $n, N \in \mathbb{N}$ with $n, 2 \leq N$ and $k \in \mathbb{N}_0$, let P be a homogeneous harmonic polynomial on $\mathbb{R}^{N+1}$ of degree k. Then $P|_{\mathbb{S}^N(a)} \in E_k^n(\mathbb{S}^N(a))$ if and only if there exists $b_K \in \mathbb{R}$ for each $K \in \mathbb{N}_0^n(k)$ such that P is decomposed as*

$$P = \sum_{K \in \mathbb{N}_0^n(k)} b_K P_{N,n,K} \quad \text{on } \mathbb{R}^{N+1}.$$

Proof. Let P be a homogeneous harmonic polynomial on $\mathbb{R}^{N+1}$ of degree k.

If $P|_{\mathbb{S}^N(a)} \in E_k^n(\mathbb{S}^N(a))$, then P satisfies

$$P(x, y) = P(x, |y|_2 e_1^{N-n+1}) = P(x, -|y|_2 e_1^{N-n+1})$$

for $(x, y) \in \mathbb{S}^N(a) \subset \mathbb{R}^n \times \mathbb{R}^{N-n+1}$. This implies that there exists a homogeneous polynomial Q_{k-2j} on $\mathbb{R}^n$ of degree $k - 2j$ for each $0 \leq j \leq [k/2]$ such that

$$P(x, y) = \sum_{j=0}^{[k/2]} |y|_2^{2j} Q_{k-2j}(x) \quad \text{for } (x, y) \in \mathbb{R}^n \times \mathbb{R}^{N-n+1}$$

(compare with [23, Proposition 3.10]). Since P is harmonic, we find that

$$\begin{aligned}
0 &= \Delta_{\mathbb{R}^{N+1}} P(x, y) \\
&= \sum_{j=0}^{[k/2]} \{(\Delta_{\mathbb{R}^{N-n+1}} |y|_2^{2j}) Q_{k-2j}(x) + |y|_2^{2j} \Delta_{\mathbb{R}^n} Q_{k-2j}(x)\} \\
&= \sum_{j=0}^{[k/2]} \{2j(N - n + 2j - 1)|y|_2^{2(j-1)} Q_{k-2j}(x) + |y|_2^{2j} \Delta_{\mathbb{R}^n} Q_{k-2j}(x)\} \\
&= \sum_{j=1}^{[k/2]} \{\Delta_{\mathbb{R}^n} Q_{k-2(j-1)}(x) - \frac{1}{c_j(N-n)} Q_{k-2j}(x)\}|y|_2^{2(j-1)} \\
&\quad + |y|_2^{2[k/2]} \Delta_{\mathbb{R}^n} Q_{k-2[k/2]}(x) \\
&= \sum_{j=1}^{[k/2]} \{\Delta_{\mathbb{R}^n} Q_{k-2(j-1)}(x) - \frac{1}{c_j(N-n)} Q_{k-2j}(x)\}|y|_2^{2(j-1)},
\end{aligned} \tag{3.1}$$

which implies that

$$\begin{aligned}
Q_{k-2j}(x) &= c_j(N-n)\Delta_{\mathbb{R}^n} Q_{k-2(j-1)}(x) \\
&= \cdots = C_j(N-n)\Delta_{\mathbb{R}^n}^j Q_k(x) \quad \text{for } 1 \leq j \leq [k/2].
\end{aligned}$$

Thus, there exists $b_K \in \mathbb{R}$ for each $K \in \mathbb{N}_0^n(k)$ such that

$$P(x,y) = \sum_{j=0}^{[k/2]} C_j(N-n)|y|_2^{2j} \Delta_{\mathbb{R}^n}^j \Big(\sum_{K \in \mathbb{N}_0^n(k)} b_K x^K \Big) = \sum_{K \in \mathbb{N}_0^n(k)} b_K P_{N,n,K}(x,y).$$

Conversely, we observe from (3.1) that $P_{N,n,K}$ is harmonic for each $K \in \mathbb{N}_0^n(k)$. This complete the proof of the lemma. ∎

Proof of Theorem 1.1. The relation (1.2) follows from Lemma 3.2 and (1.3) follows from (2.1) together with (2.3), respectively.

Fix $K \in \mathbb{N}_0^n(k)$. We prove that $\{Q_{N,n,K;a_N}\}_N$ converges to $Q_{n,K;\alpha}$ uniformly on compact sets and strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$ together with $Q_{n,K;\alpha} \in E_k(\Gamma_\alpha^n)$. For $0 \le j \le [k/2]$, define $q_{N,j} \in \mathbb{P}(n)$ and $q_j \in \mathbb{R}$ by

$$q_{N,j}(x) := C_j(N-n)(a_N^2 - |x|_2^2)^j, \quad q_j := (-1)^j \frac{\alpha^{2j}}{2^j j!},$$

respectively. Then

$$Q_{N,n,K;a_N}(x) = \sum_{j=0}^{[k/2]} q_{N,j}(x) \Delta_{\mathbb{R}^n}^j x^K, \quad Q_{n,K;\alpha}(x) = \sum_{j=0}^{[k/2]} q_j \Delta_{\mathbb{R}^n}^j x^K \quad \text{for } x \in \mathbb{R}^n.$$

Notice that $q_{N,0} \equiv 1$ on $\mathbb{R}^n$ and $q_0 = 1$. For $1 \le j \le [k/2]$ and $x \in \mathbb{R}^n$, we see that

$$q_{N,j}(x) = (-1)^j \prod_{l=1}^{j} \frac{a_N^2 - |x|_2^2}{2l(N-n+2l-1)} \xrightarrow{N\to\infty} (-1)^j \prod_{l=1}^{j} \frac{\alpha^2}{2l} = q_j.$$

Moreover, $\{q_{N,j}\}_N$ converges to q_j uniformly on compact sets as $N \to \infty$, which implies that $\{Q_{N,n,K;a_N}\}_N$ converges to $Q_{n,K;\alpha}$ uniformly on compact sets as $N \to \infty$. We see that $\{q_{N,j}\}_N$ is dominated by $\alpha^{2j}(1+|x|_2^2)^j$ hence $\{Q_{N,n,K;a_N}\}_N$ is dominated by a certain polynomial on $\mathbb{R}^n$. Since any polynomials on $\mathbb{R}^n$ belongs to $L^2(\Gamma_\alpha^n)$, the dominated convergence theorem implies that $\{Q_{N,n,K;a_N}\}_N$ converges to $Q_{n,K;\alpha}$ strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$.

A direct computation gives

$$\begin{aligned} \Delta_{\gamma_\alpha^n} Q_{n,K;\alpha}(x) &= \Delta_{\mathbb{R}^n} Q_{n,K;\alpha}(x) - \frac{1}{\alpha^2}\langle x, \nabla_{\mathbb{R}^n} Q_{n,K;\alpha}(x)\rangle \\ &= \sum_{j=0}^{[k/2]} q_j \Delta_{\mathbb{R}^n}^{j+1} x^K - \sum_{j=0}^{[k/2]} \frac{q_j}{\alpha^2} \langle x, \nabla_{\mathbb{R}^n} \Delta_{\mathbb{R}^n}^j x^K \rangle. \end{aligned}$$

We find that $\Delta_{\mathbb{R}^n}^{[k/2]+1} x^K = 0$. Since $\Delta_{\mathbb{R}^n}^j x^K$ is a linear combination of $\{x^J\}_{J \in \mathbb{N}_0^n(k-2j)}$ and $\langle x, \nabla_{\mathbb{R}^n} x^J \rangle = |J| x^J$ holds for $J \in \mathbb{N}_0^n$, it turns out that

$$\langle x, \nabla_{\mathbb{R}^n} \Delta_{\mathbb{R}^n}^j x^K \rangle = (k-2j) \Delta_{\mathbb{R}^n}^j x^K,$$

and consequently

$$\begin{aligned}\Delta_{\gamma_\alpha^n} Q_{n,K;\alpha}(x) &= \sum_{j=0}^{[k/2]-1} q_j \Delta_{\mathbb{R}^n}^{j+1} x^K - \sum_{j=0}^{[k/2]} \frac{q_j}{\alpha^2}(k-2j)\Delta_{\mathbb{R}^n}^{j} x^K \\ &= \sum_{j=1}^{[k/2]} \left\{ q_{j-1} - \frac{q_j}{\alpha^2}(k-2j) \right\} \Delta_{\mathbb{R}^n}^{j} x^K - \frac{q_0 k}{\alpha^2} \Delta_{\mathbb{R}^n}^{0} x^K \\ &= -\sum_{j=1}^{[k/2]} \frac{q_j k}{\alpha^2} \Delta_{\mathbb{R}^n}^{j} x^K - \frac{q_0 k}{\alpha^2} \Delta_{\mathbb{R}^n}^{0} x^K \\ &= -\frac{k}{\alpha^2} Q_{n,K;\alpha}(x).\end{aligned}$$

Thus, $Q_{n,K;\alpha} \in E_k(\Gamma_\alpha^n)$ and the proof is complete. ∎

Remark 3.3. Notice that $\{Q_{N,n,K;a_N}\}_N$ does not converge to $Q_{n,K;\alpha}$ uniformly on $\mathbb{R}^n$. Indeed, if we take $n = 1, k = 2, I = 2$ and $a_N = N^{1/2}$, then

$$Q_{N,1,2;\sqrt{N}}(x) = x^2 - \frac{N - x^2}{N}, \quad Q_{1,2;1}(x) = x^2 - 1,$$
$$\sup_{x \in \mathbb{R}} |Q_{N,1,2;\sqrt{N}}(x) - Q_{1,2;1}(x)| = \infty.$$

For $(M, \mu) = (\mathbb{S}^N(a), \mathrm{vol}_{\mathbb{S}^N(a)})$ and Γ_α^n, it is well known that all eigenfunctions of $-\Delta_\mu$ on M forms an orthogonal system in $L^2(M, \mu)$. We denote by $(\cdot, \cdot)_{L^2(M,\mu)}$ and $\|\cdot\|_{L^2(M,\mu)}$ the L^2-inner product and L^2-norm on (M, μ), respectively. Let $E^n(\mathbb{S}^N(a))$ be the direct sum of $E_k^n(\mathbb{S}^N(a))$ over $k \in \mathbb{N}_0$ and $E^n(\mathbb{S}^N(a))^\perp$ its orthogonal complement in $L^2(\mathbb{S}^N(a))$. The linear space $E^n(\mathbb{S}^N(a))$ is spanned by

$$\left\{ P_{N,n,K}|_{\mathbb{S}^N(a)} \right\}_{K \in \mathbb{N}_0^n}.$$

Set $D_a^n := \{x \in \mathbb{R}^n \mid |x|_2 < a\}$. We denote by $\mathbb{1}_A$ the indicator function of a set A.

Definition 3.4. Let $\{a_N\}_N$ be a sequence of positive real numbers such that the sequence $\{a_N/\sqrt{N-1}\}_N$ converges to a positive real number α as $N \to \infty$. We define a function $\omega_{a_N,\alpha}$ on $\mathbb{R}^n$ by

$$\omega_{a_N,\alpha}(x) := \left(1 - \frac{|x|_2^2}{a_N^2}\right)^{\frac{N-n-1}{2}} (2\pi\alpha^2)^{\frac{n}{2}} e^{\frac{|x|_2^2}{2\alpha^2}} \mathbb{1}_{D_{a_N}^n}(x).$$

For $F_N \in E^n(\mathbb{S}^N(a_N))$, define a function f_N on $\mathbb{R}^n$ by

$$f_N(x) := F_N(x, \sqrt{a_N^2 - |x|_2^2} e_1^{N-n+1}) \sqrt{\omega_{a_N,\alpha}}.$$

We call f_N the *horizontal part* of F_N.

It is easy to see that the horizontal part of $P_{N,n,K}|_{\mathbb{S}^N(a_N)}$ is $Q_{N,n,K;a_N}\sqrt{\omega_{a_N,\alpha}}$.

Lemma 3.5. *Let $\{a_N\}_N$ be a sequence of positive real numbers such that the sequence $\{a_N/\sqrt{N-1}\}_N$ converges to a positive real number α as $N \to \infty$. Assume $n < N$. For $F_N \in E^n(\mathbb{S}^N(a_N))$ and its horizontal part f_N, it follows that*

$$\|f_N\|^2_{L^2(\Gamma^n_\alpha)} = \frac{\|F_N\|^2_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))}.$$

Proof. Set

$$\Theta := [0,\pi]^{N-1} \times [0, 2\pi]$$

and define

$$\zeta = (\xi,\eta) \colon \Theta \to \mathbb{S}^N(1) \subset \mathbb{R}^n \times \mathbb{R}^{N-n+1}$$

by

$$\zeta(\theta)_i = \begin{cases} \cos\theta_1 & \text{if } i = 1, \\ \Big(\prod_{j=1}^{i-1} \sin\theta_j\Big)\cos\theta_i & \text{if } 2 \le i \le N, \\ \prod_{j=1}^{N} \sin\theta_j & \text{if } i = N+1. \end{cases}$$

Moreover, put

$$f(x) := F_N(x, \sqrt{a_N^2 - |x|_2^2}\, e_1^{N-n+1})$$

for $x \in D^n_{a_N}$. Then the change of variables yields

$$\begin{aligned}
&\|F_N\|^2_{L^2(\mathbb{S}^N(a_N))} \\
&= a_N^N \int_\Theta F_N(a_N\zeta(\theta))^2 \Big(\prod_{i=1}^{N-1} \sin^{N-i}\theta_i\Big) d\theta \\
&= a_N^N \int_\Theta f(a_N\xi(\theta))^2 \Big(\prod_{i=1}^{N-1} \sin^{N-i}\theta_i\Big) d\theta \\
&= 2\pi \Big(\prod_{i=n+1}^{N-1} \int_0^\pi \sin^{N-i}\theta\, d\theta\Big) \cdot a_N^{N-n} \int_{D^n_{a_N}} f(x)^2 \Big(1 - \frac{|x|_2^2}{a_N^2}\Big)^{\frac{N-n-1}{2}} dx \\
&= \mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N)) \int_{\mathbb{R}^n} f_N(x)^2 d\gamma^n_\alpha(x).
\end{aligned}$$

This concludes the proof of the lemma. ∎

As a corollary of Theorem 1.1, we show the L^2-strong convergence of the heat flow and the Mosco convergence of the Cheeger energy. These convergences with respect to the pointed measured Gromov–Hausdorff topology under the curvature-dimension condition are known. For example, see [1, Theorem 1.5.4], [12, Theorems 6.8 and 6.11], [17, Theorem 1.1], and also [2, Theorem 3.4 and Proposition 3.9] and [28, Theorem 3.8]. The results are concerned with the asymptotic behaviors of Laplacians. It should be mentioned that, for each $k \in \mathbb{N}$, Peterson and Sengputa [23, Proposition 5.4] proved the convergence of $\Delta_{\mathbb{S}^N(\sqrt{N-1})}$ to the Hermite operator as $N \to \infty$ on the space of homogeneous polynomials of degree at most k, and that the projection of the Hermite operator onto the first n-coordinates is $\Delta_{\Gamma_1^n}$ (see also [26, Proposition 3]).

Corollary 3.6. *Let $\{a_N\}_N$ be a sequence of positive real numbers such that the sequence $\{a_N/\sqrt{N-1}\}_N$ converges to a positive real number α as $N \to \infty$. Let $U_N\colon [0,\infty) \times \mathbb{S}^N(a_N) \to \mathbb{R}$ denote the solution to the heat equation*

$$\begin{cases} \dfrac{\partial}{\partial t} U = \Delta_{\mathbb{S}^N(a_N)} U & \text{in } (0,\infty) \times \mathbb{S}^N(a_N), \\ U(0,\cdot) = F_N & \text{in } \mathbb{S}^N(a_N), \end{cases}$$

where $F_N \in E^n(\mathbb{S}^N(a_N))$. Then $U_N(t,\cdot) \in E^n(\mathbb{S}^N(a_N))$ for any $t \geq 0$.

Let f_N and $u_N(\cdot,t)$ be the horizontal part of F_N and $U_N(t,\cdot)$, respectively. If $\{f_N\}_N$ converges to f_∞ weakly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$, then $\{u_N(t,\cdot)\}_N$ converges to $u_\infty(t,\cdot)$ strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$ for each $t > 0$ and $\{u_\infty(t,\cdot)\}_{t\geq 0}$ solves the heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u = \Delta_{\gamma_\alpha^n} u & \text{in } (0,\infty) \times \mathbb{R}^n, \\ u(0,\cdot) = f_\infty & \text{in } \mathbb{R}^n. \end{cases} \tag{3.2}$$

Proof. Let $\{\phi_{N,k}\}_{k\in\mathbb{N}}$ be an orthonormal system in $L^2(\mathbb{S}^N(a_N))$ such that each $\phi_{N,k}$ is an eigenfunction of eigenvalue $\lambda_{N,k}$ and either $\phi_{N,k} \in E^n(\mathbb{S}^N(a_N))$ or $\phi_{N,k} \in E^n(\mathbb{S}^N(a_N))^\perp$ holds. It is well known that $U_N(t,z)$ is given by

$$U_N(t,z) = \sum_{k\in\mathbb{N}} e^{-t\lambda_{N,k}} (F_N, \phi_{N,k})_{L^2(\mathbb{S}^N(a_N))} \phi_{N,k}(z).$$

For instance, see [7, Section VI.1]. We deduce from

$$(F_N, \phi_{N,k})_{L^2(\mathbb{S}^N(a_N))} = 0 \quad \text{for } \phi_{N,k} \in E^n(\mathbb{S}^N(a_N))^\perp$$

that $U_N(t,\cdot) \in E^n(\mathbb{S}^N(a_N))$ holds for any $t \geq 0$.

Without loss of generality, we may assume that, for each $\phi_{N,k} \in E^n(\mathbb{S}^N(a_N))$, there exists $K \in \mathbb{N}_0^n$ such that

$$\phi_{N,k} = \frac{P_{N,n,K}|_{\mathbb{S}^N(a_N)}}{\| P_{N,n,K}|_{\mathbb{S}^N(a_N)} \|_{L^2(\mathbb{S}^N(a_N))}}.$$

We shall abbreviate $P_{N,n,K}|_{\mathbb{S}^N(a_N)}$ by $P_{N,n,K}$ when there is no possibility of confusion. We see that

$$f_N = \sum_{K \in \mathbb{N}_0^n} \frac{(F_N, P_{N,n,K})_{L^2(\mathbb{S}^N(a_N))}}{\| P_{N,n,K} \|^2_{L^2(\mathbb{S}^N(a_N))}} Q_{N,n,K;a_N} \sqrt{\omega_{a_N,\alpha}},$$

$$u_N(t,\cdot) = \sum_{K \in \mathbb{N}_0^n} e^{-t\lambda_{|K|}(\mathbb{S}^N(a_N))} \frac{(F_N, P_{N,n,K})_{L^2(\mathbb{S}^N(a_N))}}{\| P_{N,n,K} \|^2_{L^2(\mathbb{S}^N(a_N))}} Q_{N,n,K;a_N} \sqrt{\omega_{a_N,\alpha}}.$$

Similarly, for f_∞ and a solution u to (3.2), it turns out that

$$f_\infty(x) = \sum_{K \in \mathbb{N}_0^n} \frac{(f_\infty, Q_{n,K;\alpha})_{L^2(\Gamma_\alpha^n)}}{\| Q_{n,K;\alpha} \|^2_{L^2(\Gamma_\alpha^n)}} Q_{n,K;\alpha},$$

$$u(t,x) = \sum_{K \in \mathbb{N}_0^n} e^{-t\lambda_{|K|}(\Gamma_\alpha^n)} \frac{(f_\infty, Q_{n,K;\alpha})_{L^2(\Gamma_\alpha^n)}}{\| Q_{n,K;\alpha} \|^2_{L^2(\Gamma_\alpha^n)}} Q_{n,K;\alpha}.$$

For instance, see [4, Theorem 1.4.4]. As well as the proof of Lemma 3.5, we find that

$$\| Q_{N,n,K;a_N}(x) \sqrt{\omega_{a_N,\alpha}} \|^2_{L^2(\Gamma_\alpha^n)} = \frac{\| P_{N,n,K} \|^2_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))},$$

$$(f_N, Q_{N,n,K;a_N} \sqrt{\omega_{a_N,\alpha}})_{L^2(\Gamma_\alpha^n)} = \frac{(F_N, P_{N,n,K})_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))},$$

for $n < N$. It follows from the inequality $1 - \rho \le e^{-\rho}$ on $\rho \in \mathbb{R}$ that

$$\Big(1 - \frac{r^2}{a_N^2}\Big)^{\frac{N-n-1}{2}} \le \exp\Big(-\frac{r^2}{a_N^2} \cdot \frac{N-n-1}{2}\Big) \le \exp\Big(-\frac{r^2}{4\alpha^2}\Big) \quad \text{on } r \in (-a_N, a_N) \tag{3.3}$$

for large enough $N \in \mathbb{N}$. Then

$$\{(2\pi\alpha^2)^{-\frac{n}{4}} Q_{N,n,K;a_N} \sqrt{\omega_{a_N,\alpha}}\}_N$$

is dominated by the product of $\exp(|x|^2/8\alpha^2)$ and a certain polynomial on $\mathbb{R}^n$, where the product belongs to $L^2(\Gamma_\alpha^n)$, hence the sequence converges to $Q_{n,K;\alpha}$ strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$ by the dominated convergence theorem. This with the weak convergence of $\{f_N\}_N$ in $L^2(\Gamma^n_\alpha)$ yields

$$\begin{aligned}
\|Q_{n,K;\alpha}\|^2_{L^2(\Gamma^n_\alpha)} &= \lim_{N\to\infty} \|(2\pi\alpha^2)^{-\frac{n}{4}} Q_{N,n,K;a_N}\sqrt{\omega_{a_N,\alpha}}\|^2_{L^2(\Gamma^n_\alpha)}\\
&= \lim_{N\to\infty} \frac{\|P_{N,n,K}\|^2_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))},\\
(f_\infty, Q_{n,K;\alpha})_{L^2(\Gamma^n_\alpha)} &= \lim_{N\to\infty} (f_N, (2\pi\alpha^2)^{-\frac{n}{4}} Q_{N,n,K;a_N}\sqrt{\omega_{a_N,\alpha}})_{L^2(\Gamma^n_\alpha)}\\
&= (2\pi\alpha^2)^{\frac{n}{4}} \lim_{N\to\infty} \frac{(F_N, P_{N,n,K})_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))},
\end{aligned} \tag{3.4}$$

where we used the Stirling's approximation to have

$$\frac{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))} \xrightarrow{N\to\infty} (2\pi\alpha^2)^{\frac{n}{2}}.$$

The monotonicity of the L^2-energy along the heat flow (see [7, Proposition VI.1.1]) provides

$$\begin{aligned}
\sup_{N\in\mathbb{N}} \|u_N(t,\cdot)\|^2_{L^2(\Gamma^n_\alpha)} &= \sup_{N\in\mathbb{N}} \frac{\|U_N(t,\cdot)\|^2_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))}\\
&\le \sup_{N\in\mathbb{N}} \frac{\|F_N\|^2_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))} = \sup_{N\in\mathbb{N}} \|f_N\|^2_{L^2(\Gamma^n_\alpha)} < \infty.
\end{aligned}$$

Then the Banach–Alaoglu theorem implies that there exists a subsequence of the sequence $\{u_N(t,\cdot)\}_N$, still denoted by $\{u_N(t,\cdot)\}_N$, converging weakly in $L^2(\Gamma^n_\alpha)$. We denote by $u_\infty(t,\cdot)$ the limit. We apply the strong convergence of

$$\{(2\pi\alpha^2)^{-\frac{n}{4}} Q_{N,n,K;a_N}\sqrt{\omega_{a_N,\alpha}}\}_N$$

again to have

$$\begin{aligned}
(u_\infty(t,\cdot), Q_{n,K;\alpha})_{L^2(\Gamma^n_\alpha)} &= \lim_{N\to\infty} (u_N(t,\cdot), (2\pi\alpha^2)^{-\frac{n}{4}} Q_{N,n,K;a_N}\sqrt{\omega_{a_N,\alpha}})_{L^2(\Gamma^n_\alpha)}\\
&=(2\pi\alpha^2)^{\frac{n}{4}} \lim_{N\to\infty} e^{-t\lambda_{|K|}(\mathbb{S}^N(a_N))} \frac{(F_N, P_{N,n,K})_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))}\\
&=e^{-t\lambda_{|K|}(\Gamma^n_\alpha)} (f_\infty, Q_{n,K;\alpha})_{L^2(\Gamma^n_\alpha)}\\
&=(u(t,\cdot), Q_{n,K;\alpha})_{L^2(\Gamma^n_\alpha)},
\end{aligned}$$

which leads to $u_\infty(t,\cdot) = u(t,\cdot)$. Thus, $\{u_N(t,\cdot)\}_N$ converges to $u(t,\cdot)$ weakly in $L^2(\Gamma^n_\alpha)$ as $N\to\infty$ for each $t \ge 0$.

For $N \in \mathbb{N}$ and $k \in \mathbb{N}_0$, set

$$B_{N,k}(t) := \sum_{K \in \mathbb{N}_0^n, |K| \le k} e^{-2t\lambda_{|K|}(\mathbb{S}^N(a_N))} \frac{(F_N, P_{N,n,K})^2_{L^2(\mathbb{S}^N(a_N))}}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N)) \cdot \|P_{N,n,K}\|^2_{L^2(\mathbb{S}^N(a_N))}},$$

$$B_k(t) := \sum_{K \in \mathbb{N}_0^n, |K| \le k} e^{-2t\lambda_{|K|}(\Gamma_\alpha^n)} \frac{(f_\infty, Q_{n,K;\alpha})^2_{L^2(\Gamma_\alpha^n)}}{\|Q_{n,K;\alpha}\|^2_{L^2(\Gamma_\alpha^n)}}.$$

By (3.4) and Theorem 1.1, we see that $B_{N,k}(t) \to B_k(t)$ as $N \to \infty$ and

$$\begin{aligned}\sup_{N \in \mathbb{N}, k \in \mathbb{N}_0} B_{N,k}(t) &\le \sup_{N \in \mathbb{N}} \lim_{k \to \infty} B_{N,k}(t) = \sup_{N \in \mathbb{N}} \|u_N(t,\cdot)\|^2_{L^2(\Gamma_\alpha^n)} \\ &\le \sup_{N \in \mathbb{N}} \|f_N\|^2_{L^2(\Gamma_\alpha^n)} < \infty.\end{aligned}$$

It follows from Dirichlet's test that

$$\begin{aligned}&|\lim_{m \to \infty} B_{N,m}(t) - B_k(t)| - |B_k(t) - B_{N,k}(t)| \\ &\le |\lim_{m \to \infty} B_{N,m}(t) - B_{N,k}(t)| \\ &\le 2 \sup_{m \in \mathbb{N}} \|f_m\|_{L^2(\Gamma_\alpha^n)} e^{-2t\lambda_{k+1}(\mathbb{S}^N(a_N))}.\end{aligned}$$

For $t > 0$, letting $N \to \infty$ first and then $k \to \infty$ leads to

$$\lim_{N \to \infty} \|u_N(t,\cdot)\|^2_{L^2(\Gamma_\alpha^n)} = \lim_{N \to \infty} \lim_{m \to \infty} B_{N,m}(t) = \lim_{k \to \infty} B_k(t) = \|u(t,\cdot)\|^2_{L^2(\Gamma_\alpha^n)},$$

which is the equivalent to the strong convergence of $\{u_N(t,\cdot)\}_N$ to $u(t,\cdot)$ in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$. This completes the proof of the corollary. ∎

As well as $H_0^1(V_{\alpha R}^n, \gamma_\alpha^n)$, we define $H^1(M,\mu)$ as the completion of $C_0^\infty(M)$ with respect to the inner product given by

$$(f_1, f_2)_{H^1(M,\mu)} := \int_M f_1 f_2 d\mu + \int_M g(\nabla_M f_1, \nabla_M f_2) d\mu \quad \text{for } f_1, f_2 \in C_0^\infty(M).$$

For $f \in H^1(M,\mu)$, we write $|\nabla f|_M := g(\nabla_M f, \nabla_M f)^{1/2}$. By [4, Proposition 1.5.4],

$$H^1(\Gamma_\alpha^n) = \left\{ f \in L^2(\Gamma_\alpha^n) \;\middle|\; \sum_{K \in \mathbb{N}_0^n} \lambda_{|K|}(\Gamma_\alpha^n) \frac{(f, Q_{n,K;\alpha})^2_{L^2(\Gamma_\alpha^n)}}{\|Q_{n,K;\alpha}\|^2_{L^2(\Gamma_\alpha^n)}} < \infty \right\}.$$

Similarly, we see that

$$\begin{aligned}&H^1(\mathbb{S}^N(a_N)) \cap E^n(\mathbb{S}^N(a_N)) \\ &= \left\{ F_N \in E^n(\mathbb{S}^N(a_N)) \;\middle|\; \sum_{K \in \mathbb{N}_0^n} \lambda_{|K|}(\mathbb{S}^N(a_N)) \frac{(F_N, P_{N,n,K})^2_{L^2(\mathbb{S}^N(a_N))}}{\|P_{N,n,K}\|^2_{L^2(\mathbb{S}^N(a_N))}} < \infty \right\}.\end{aligned}$$

For $f \in H^1(\Gamma_\alpha^n)$ and $F_N \in H^1(\mathbb{S}^N(a_N)) \cap E^n(\mathbb{S}^N(a_N))$, we find that

$$\int_{\mathbb{R}^n} |\nabla f|_{\mathbb{R}^n}^2 d\gamma_\alpha^n = \sum_{K\in\mathbb{N}_0^n} \lambda_{|K|}(\Gamma_\alpha^n) \frac{(f, Q_{n,K;\alpha})^2_{L^2(\Gamma_\alpha^n)}}{\|Q_{n,K;\alpha}\|^2_{L^2(\Gamma_\alpha^n)}},$$
$$\int_{\mathbb{S}^N(a_N)} |\nabla F_N|^2_{\mathbb{S}^N(a_N)} d\mathrm{vol}_{\mathbb{S}^N(a_N)} = \sum_{K\in\mathbb{N}_0^n} \lambda_{|K|}(\mathbb{S}^N(a_N)) \frac{(F_N, P_{N,n,K})^2_{L^2(\mathbb{S}^N(a_N))}}{\|P_{N,n,K}\|^2_{L^2(\mathbb{S}^N(a_N))}}. \tag{3.5}$$

Corollary 3.7. *Let $\{a_N\}_N$ be a sequence of positive real numbers such that the sequence $\{a_N/\sqrt{N-1}\}_N$ converges to a positive real number α as $N \to \infty$. Define the* Cheeger energy Ch_N *on $H^1(\mathbb{S}^N(a_N)) \cap E^n(\mathbb{S}^N(a_N))$ by*

$$\mathsf{Ch}_N(F_N) := \frac{1}{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N))} \int_{\mathbb{S}^N(a_N)} |\nabla F_N|^2_{\mathbb{S}^N(a_N)} d\mathrm{vol}_{\mathbb{S}^N(a_N)}.$$

For $F_N \in H^1(\mathbb{S}^N(a_N)) \cap E^n(\mathbb{S}^N(a_N))$ and its horizontal part f_N, if $\{f_N\}_N$ converges to f_∞ weakly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$, then

$$\int_{\mathbb{R}^n} |\nabla f_\infty|^2_{\mathbb{R}^n} d\gamma_\alpha^n \le \liminf_{N\to\infty} \mathsf{Ch}_N(F_N). \tag{3.6}$$

Conversely, for $\tilde{f} \in H^1(\Gamma_\alpha^n)$, there exists $\widetilde{F}_N \in H^1(\mathbb{S}^N(a_N)) \cap E^n(\mathbb{S}^N(a_N))$ such that the sequence of the horizontal parts of $\widetilde{F}_N$ converges to $\tilde{f}$ strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$ and

$$\int_{\mathbb{R}^n} |\nabla \tilde{f}|^2_{\mathbb{R}^n} d\gamma_\alpha^n = \lim_{N\to\infty} \mathsf{Ch}_N(\widetilde{F}_N). \tag{3.7}$$

Proof. By Theorem 1.1, $\lambda_{|K|}(\mathbb{S}^N(a_N)) \to \lambda_{|K|}(\Gamma_\alpha^n)$ as $N \to \infty$. Moreover, if $\{f_N\}_N$ converges to f_∞ weakly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$, then (3.4) holds. These and (3.5) with Fatou's lemma provide (3.6).

Conversely, for $\tilde{f} \in H^1(\Gamma_\alpha^n)$, we can choose $\widetilde{F}_N \in H^1(\mathbb{S}^N(a_N)) \cap E^n(\mathbb{S}^N(a))$ as

$$\widetilde{F}_N = \sum_{K\in\mathbb{N}_0^n} \sqrt{\mathrm{vol}_{\mathbb{S}^{N-n}(a_N)}(\mathbb{S}^{N-n}(a_N)) \frac{a_N^2}{|K|+N-1} \cdot \frac{(\tilde{f}, Q_{n,K;\alpha})^2_{L^2(\Gamma_\alpha^n)}}{\alpha^2 \|Q_{n,K;\alpha}\|^2_{L^2(\Gamma_\alpha^n)}}} \cdot \frac{P_{N,n,K}|_{\mathbb{S}^N(a_N)}}{\|P_{N,n,K}\|_{L^2(\mathbb{S}^N(a_N))}}.$$

In this case, the sequence of the horizontal parts of $\widetilde{F}_N$ converges to $\tilde{f}$ strongly in $L^2(\Gamma_\alpha^n)$ as $N \to \infty$ and (3.7) holds. This completes the proof of the corollary. ∎

4. Proof of Theorem 1.2

We begin with two lemmas concerning boundedness. Notice that Stirling's approximation yields

$$\int_{-a_N}^{a_N} s_N(r)^{\frac{N}{2}-1} dr \xrightarrow{N\to\infty} \sqrt{2\pi}\alpha \quad \text{and} \quad w_N(r) \xrightarrow{N\to\infty} w_\infty(r) \quad \text{for each } r \in \mathbb{R}.$$

Lemma 4.1. *Let $\{a_N\}_N$ be a sequence of positive real numbers such that the sequence $\{a_N/\sqrt{N-1}\}_N$ converges to a positive real number α as $N\to\infty$. For $N\in\mathbb{N}$, set*

$$\varpi_N := \sup_{r\in(-a_N,a_N)} \frac{w_N(r)}{w_\infty(r)}, \qquad A_N := \frac{a_N^2 - \alpha^2(N-2)}{a_N}.$$

Then $\{\varpi_N\}_N$ is bounded if and only if $\{A_N\}_N$ is bounded from above.

Proof. For $r \in (-a_N, a_N)$, we compute

$$\frac{d}{dr}\log\frac{w_N(r)}{w_\infty(r)} = -\frac{(N-2)r}{a_N^2 - r^2} + \frac{r}{\alpha^2} = \frac{r}{\alpha^2(a_N^2 - r^2)}\{a_N^2 - \alpha^2(N-2) - r^2\}.$$

In the case of $a_N^2 - \alpha^2(N-2) \le 0$, we see that

$$\varpi_N = \frac{w_N(0)}{w_\infty(0)} = \left(\int_{-a_N}^{a_N} s_N(r)^{\frac{N}{2}-1} dr\right)^{-1} \cdot \sqrt{2\pi}\alpha \xrightarrow{N\to\infty} 1.$$

Thus, if all $N \in \mathbb{N}$ except a finite number satisfy $a_N^2 - \alpha^2(N-2) \le 0$, then $\{\varpi_N\}_N$ is bounded and $\{A_N\}_N$ is bounded from above.

Assume that $a_N^2 - \alpha^2(N-2) > 0$, that is, $A_N > 0$ for infinitely many $N \in \mathbb{N}$. For such N with $N > 2$, we set

$$r_N := \sqrt{a_N^2 - \alpha^2(N-2)} = \sqrt{a_N A_N}.$$

Then we find that $r_N < a_N$ and

$$\varpi_N = \frac{w_N(r_N)}{w_\infty(r_N)} = \frac{w_N(-r_N)}{w_\infty(-r_N)},$$

$$\begin{aligned}\log\frac{w_N(r_N)}{w_\infty(r_N)} &= \log\frac{w_N(0)}{w_\infty(0)} + \left(\frac{N}{2}-1\right)\log\left(1-\frac{r_N^2}{a_N^2}\right) + \frac{r_N^2}{2\alpha^2}\\ &= \log\frac{w_N(0)}{w_\infty(0)} + \left(\frac{N}{2}-1\right)\left\{\log\left(1-\frac{r_N^2}{a_N^2}\right) + \frac{r_N^2}{a_N^2 - r_N^2}\right\}.\end{aligned}$$

Since $f_1(s) := \log(1-s)$ is strictly concave on $(-\infty, 1)$ and $f_1(0) = 0$, $f_1'(0) = -1$, it turns out that

$$\log \frac{w_N(r_N)}{w_\infty(r_N)} - \log \frac{w_N(0)}{w_\infty(0)} = \Big(\frac{N}{2} - 1\Big) f_1\Big(\frac{r_N^2}{a_N^2}\Big) + \frac{r_N^2}{2\alpha^2}$$
$$< -\Big(\frac{N}{2} - 1\Big)\frac{r_N^2}{a_N^2} + \frac{r_N^2}{2\alpha^2} = \frac{A_N^2}{2\alpha^2}.$$

On the other hand, if we set

$$f_2(s) := \log(1-s) + \frac{s}{1-s} \quad \text{for } s \in (-2, 1),$$

then

$$f_2'(s) = \frac{s}{(1-s)^2}, \quad f_2''(s) = \frac{1+s}{(1-s)^3}, \quad f_2'''(s) = \frac{2(2+s)}{(1-s)^4} > 0,$$

consequently,

$$\log \frac{w_N(r_N)}{w_\infty(r_N)} - \log \frac{w_N(0)}{w_\infty(0)} = \Big(\frac{N}{2} - 1\Big) f_2\Big(\frac{r_N^2}{a_N^2}\Big)$$
$$> \Big(\frac{N}{2} - 1\Big)\frac{1}{2}\Big(\frac{r_N^2}{a_N^2}\Big)^2 f_2''(0) = \frac{N-2}{4a_N^2} A_N^2.$$

Thus, $\{\varpi_N\}_N$ is bounded if and only if $\{A_N\}_N$ is bounded from above. This completes the proof of the lemma. ■

Lemma 4.2. *Let $\{a_N\}_N$, $\{\theta_N\}_N$ be sequences of real numbers so that $a_N > 0$ and $\theta_N \in (0, \pi)$ for $N \in \mathbb{N}$. If there exist $\alpha > 0$ and $R \in \mathbb{R}$ such that*

$$\lim_{N\to\infty} \frac{a_N}{\sqrt{N-1}} = \alpha, \quad \lim_{N\to\infty} a_N \cos\theta_N = \alpha R,$$

then

$$\sup_{N\in\mathbb{N}} \lambda(B^N_{a_N\theta_N}, \mathbb{S}^N(a_N)) < \infty.$$

Proof. Assume $n, 2 \le N$. We see that

$$\frac{\mathrm{vol}_{\mathbb{S}^N(a_N)}(B^N_{a_N\theta_N})}{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))} = \frac{\mathrm{vol}_{\mathbb{S}^N(1)}(B^N_{\theta_N})}{\mathrm{vol}_{\mathbb{S}^N(1)}(\mathbb{S}^N(1))} = \frac{\int_0^{\theta_N} \sin^{N-1}\theta d\theta}{\int_0^{\pi} \sin^{N-1}\theta d\theta} = \int_{a_N\cos\theta_N}^{a_N} w_N(r) dr.$$

By an argument similar to (3.3) with Stirling's approximation, we find that

$$w_N(r)\mathbb{1}_{(a_N\cos\theta_N, a_N)}(r) \leq \frac{1}{\sqrt{\pi}\alpha}e^{-\frac{r^2}{4\alpha^2}} \quad \text{on } r \in \mathbb{R}.$$

Then the dominated convergence theorem yields

$$\lim_{N\to\infty} \frac{\mathrm{vol}_{\mathbb{S}^N(a_N)}(B^N_{a_N\theta_N})}{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))} = \gamma^1_\alpha(\alpha R, \infty) \in (0,1).$$

Let $\theta'_N \in (0,\pi)$ satisfy

$$\frac{\mathrm{vol}_{\mathbb{S}^N(a_N)}(B^N_{a_N\theta'_N})}{\mathrm{vol}_{\mathbb{S}^N(a_N)}(\mathbb{S}^N(a_N))} = \frac{\mathrm{vol}_{\mathbb{S}^N(1)}(B^N_{\theta'_N})}{\mathrm{vol}_{\mathbb{S}^N(1)}(\mathbb{S}^N(1))} = \frac{\gamma^1_\alpha(\alpha R, \infty)}{2}.$$

Then, for all $N \in \mathbb{N}$ except a finite number, we see that $\theta_N \geq \theta'_N$ hence

$$\lambda(B^N_{a_N\theta_N}, \mathbb{S}^N(a_N)) \leq \lambda(B^N_{a_N\theta'_N}, \mathbb{S}^N(a_N)) = \frac{1}{a_N^2}\lambda(B^N_{\theta'_N}, \mathbb{S}^N(1)) \tag{4.1}$$

by the domain monotonicity of eigenvalues (see [7, Section I.5]). Since the right-hand side in (4.1) is bounded by the monotonicity due to Friedland and Hayman [10, Theorem 2] as mentioned in the introduction, this concludes the proof of the lemma. ∎

Proof of Theorem 1.2. Set

$$\lambda_N := \lambda(B^N_{a_N\theta_N}, \mathbb{S}^N(a_N)), \quad I_N := (a_N\cos\theta_N, a_N), \quad I := (\alpha R, \infty).$$

Then $I_N \subset I$ for any $N \in \mathbb{N}$ by the assumption. Notice that the density of γ^1_α with respect to the one-dimensional Lebesgue measure is w_∞.

For a nontrivial solution φ_N to (D^N) for $(a,\theta) = (a_N, \theta_N)$, define

$$h_N \in C^\infty(I_N) \cap C(\overline{I_N})$$

by

$$h_N(r) := \varphi_N\Big(a_N \cdot \arccos\Big(\frac{r}{a_N}\Big)\Big).$$

A direct computation provides

$$\begin{cases} L_N h_N = -\lambda_N h_N & \text{in } I_N, \\ h_N > 0 & \text{in } (a_N\cos\theta_N, a_N], \\ h_N(a_N\cos\theta_N) = 0, & \end{cases}$$

where $L_N\colon C^\infty(I_N) \to C^\infty(I_N)$ is defined for $f \in C^\infty(I_N)$ by

$$L_N f(r) := s_N(r) f''(r) - \frac{Nr}{a_N^2} f'(r).$$

We can assume that

$$\int_{I_N} h_N(r)^2 w_N(r) dr = \int_0^{a_N \theta_N} \varphi_N(\theta)^2 \sin^{N-1}\Big(\frac{\theta}{a_N}\Big) d\theta \cdot \Big(\int_{-a_N}^{a_N} s_N(r)^{\frac{N}{2}-1} dr \Big)^{-1} = 1$$

without loss of generality by (2.2). We see that the first positive Dirichlet eigenfunction $\phi_N(z) := \varphi_N(d_{\mathbb{S}^N(a_N)}(z, a_N e_1^{N+1}))$ of $-\Delta_{\mathbb{S}^N(a_N)}$ on $B^N_{a_N \theta_N}$ satisfies

$$\int_{B^N_{a_N\theta_N}} \phi_N(z)^2 d\,\mathrm{vol}_{\mathbb{S}^N(a_N)}(z) = \mathrm{vol}_{\mathbb{S}^{N-1}(a_N)}(\mathbb{S}^{N-1}(a_N)) \int_{-a_N}^{a_N} s_N(r)^{\frac{N}{2}-1} dr.$$

An integration by parts leads to

$$\begin{aligned} \lambda_N = \lambda_N \int_{I_N} h_N(r)^2 w_N(r) dr &= - \int_{I_N} (L_N h_N(r)) h_N(r) w_N(r) dr \\ &= \int_{I_N} h_N'(r)^2 s_N(r) w_N(r) dr. \end{aligned}$$

Thus, we find that

$$\int_I h_N(r)^2 \frac{w_N(r)}{w_\infty(r)} \mathbb{1}_{I_N}(r) d\gamma_\alpha^1(r) = 1,$$
$$\int_I h_N'(r)^2 s_N(r) \frac{w_N(r)}{w_\infty(r)} \mathbb{1}_{I_N}(r) d\gamma_\alpha^1(r) = \lambda_N.$$

Moreover, an integration by parts yields

$$\begin{aligned} &\int_{I_N} r^2 h_N(r)^2 w_N(r) dr \\ &= -\frac{a_N^2}{N} \int_{I_N} r h_N(r)^2 \left(s_N(r) w_N(r)\right)' dr \\ &= \frac{a_N^2}{N} \int_{I_N} h_N(r)^2 s_N(r) w_N(r) dr + \frac{2a_N^2}{N} \int_{I_N} r h_N(r) h_N'(r) s_N(r) w_N(r) dr \end{aligned}$$

$$\leq \frac{a_N^2}{N} + \frac{1}{2}\int_{I_N} r^2 h_N(r)^2 w_N(r)dr + \frac{2a_N^4}{N^2}\int_{I_N} h'_N(r)^2 s_N(r)^2 w_N(r)dr$$
$$\leq \frac{a_N^2}{N}\Big(1 + \frac{2a_N^2\lambda_N}{N}\Big) + \frac{1}{2}\int_{I_N} r^2 h_N(r)^2 w_N(r)dr,$$

where we used Young's inequality in the first inequality. This ensures that

$$\int_I r^2 h_N(r)^2 \frac{w_N(r)}{w_\infty(r)} \mathbb{1}_{I_N}(r)d\gamma_\alpha^1(r) = \int_{I_N} r^2 h_N(r)^2 w_N(r)dr \leq \frac{2a_N^2}{N}\Big(1 + \frac{2a_N^2\lambda_N}{N}\Big).$$

Since $\{\lambda_N\}_N$ is bounded by Lemma 4.2, by the Banach–Alaoglu theorem, there exist a subsequence $\{N(m)\}_m$ of $\{N\}_N$, $h_\infty, \tilde{h}_\infty \in L^2(I, \gamma_\alpha^1)$ and $\lambda \in \mathbb{R}$ such that

$$h_{N(m)}\sqrt{\frac{w_{N(m)}}{w_\infty}}\mathbb{1}_{I_{N(m)}} \to h_\infty,$$
$$r h_{N(m)}(r)\sqrt{\frac{w_{N(m)}(r)}{w_\infty(r)}}\mathbb{1}_{I_{N(m)}} \to r h_\infty,$$
$$h'_{N(m)}\sqrt{s_{N(m)}\frac{w_{N(m)}}{w_\infty}}\mathbb{1}_{I_{N(m)}} \to \tilde{h}_\infty,$$

weakly in $L^2(I, \gamma_\alpha^1)$ and $\lambda_{N(m)} \to \lambda$ as $m \to \infty$. We see that h_∞ is nonnegative almost everywhere in I. For $r \in I_N$, we calculate that

$$\Big(h_N(r)s_N(r)\sqrt{\frac{w_N(r)}{w_\infty(r)}}\Big)' = h'_N(r)s_N(r)\sqrt{\frac{w_N(r)}{w_\infty(r)}} + \frac{r}{2}h_N(r)\sqrt{\frac{w_N(r)}{w_\infty(r)}}\Big(\frac{1}{\alpha^2}s_N(r) - \frac{N+2}{a_N^2}\Big).$$

This with the boundedness of

$$\sup_{N\in\mathbb{N}} \sqrt{s_N(r)} < \infty,$$
$$\sup_{N\in\mathbb{N}}\sup_{r\in I_N}\Big|\Big(\frac{1}{\alpha^2}s_N(r) - \frac{N+2}{a_N^2}\Big)\Big| < \infty$$

implies that

$$\Big(h_{N(m)}s_{N(m)}\sqrt{\frac{w_{N(m)}}{w_\infty}}\mathbb{1}_{I_{N(m)}}\Big)' \to \tilde{h}_\infty \quad \text{weakly in } L^2(I, \gamma_\alpha^1)$$

as $m \to \infty$. By the compact Sobolev embedding on Γ^1_α (see [16, Theorem 3.1] and also [8, Section 6]), we can extract a subsequence, still denoted by $\{N(m)\}_m$, such that

$$h_{N(m)} s_{N(m)} \sqrt{\frac{w_{N(m)}}{w_\infty}} \mathbb{1}_{I_{N(m)}} \to h_\infty \quad \text{weakly in } H^1_0(I, \gamma^1_\alpha) \text{ and strongly in } L^2(I, \gamma^1_\alpha)$$

as $m \to \infty$, where $h'_\infty = \tilde{h}_\infty$. Moreover, we find that

$$h_{N(m)} \sqrt{\frac{w_{N(m)}}{w_\infty}} \mathbb{1}_{I_{N(m)}} (1 - s_{N(m)}) \to 0 \quad \text{strongly in } L^2(I, \gamma^1_\alpha) \text{ as } m \to \infty$$

and hence

$$\int_I h_\infty(r)^2 d\gamma^1_\alpha(r) = 1. \tag{4.2}$$

For $f \in H^1_0(I, \gamma^1_\alpha)$, we observe from Lemma 4.1 that $\{f' \sqrt{s_N w_N / w_\infty} \mathbb{1}_{I_N}\}_N$ converges to f' strongly in $L^2(I, \gamma^1_\alpha)$ as $N \to \infty$ and compute

$$\begin{aligned}
&\int_I h'_\infty(r) f'(r) d\gamma^1_\alpha(r) \\
&\quad = \lim_{m\to\infty} \int_{I_{N(m)}} h'_{N(m)}(r) s_{N(m)}(r) f'(r) w_{N(m)}(r) dr \\
&\quad = - \lim_{m\to\infty} \int_{I_{N(m)}} \left(s_{N(m)}(r) h''_{N(m)}(r) - \frac{Nr}{a^2_{N(m)}} h'_{N(m)}(r) \right) f(r) w_{N(m)}(r) dr \\
&\quad = \lim_{m\to\infty} \int_{I_{N(m)}} \lambda_{N(m)} h_{N(m)} f(r) w_{N(m)}(r) dr \\
&\quad = \lambda \int_I h_\infty(r) f(r) d\gamma^1_\alpha(r),
\end{aligned}$$

which ensures that h_∞ is a weak solution to the Dirichlet eigenvalue problem of $-\Delta_{\gamma^1_\alpha}$ on I. By the elliptic regularity theory (see [13, Theorem 7.10 and Corollary 8.11] for instance), h_∞ is a Dirichlet eigenfunction of $-\Delta_{\gamma^1_\alpha}$ on I of eigenvalue λ. Since h_∞ is nonnegative on I, h_∞ is a first positive Dirichlet eigenfunction and hence $\lambda = \lambda(I, \gamma^1_\alpha) = \lambda(V^n_{\alpha R}, \gamma^n_\alpha)$. Thus, $\{\lambda_N\}_N$ converges to $\lambda(V^n_{\alpha R}, \gamma^n_\alpha)$ as $N \to \infty$. Moreover, it follows from (4.2) that $\{h_N s_N \sqrt{w_N / w_\infty} \mathbb{1}_{I_N}\}_N$ converges to h_∞ strongly in $H^1_0(I, \gamma^1_\alpha)$ as $N \to \infty$.

If we define

$$\psi_N(x) := h_N(x_1) s_N(x_1) \sqrt{\frac{w_N(x_1)}{w_\infty(x_1)}} \mathbb{1}_{I_N}(x_1), \quad \psi_\infty(x) := h_\infty(x_1),$$

for $x = (x_i)_{i=1}^n \in \overline{V_{\alpha R}^n}$, then $\psi_N, \psi_\infty \in H_0^1(V_{\alpha R}^n, \gamma_\alpha^n)$ and $\{\psi_N\}_N$ converges to ψ_∞ strongly in $H_0^1(V_{\alpha R}^n, \gamma_\alpha^n)$. Moreover, ψ_N satisfies (1.5) and ψ_∞ is the first positive Dirichlet eigenfunction ψ_∞ of $-\Delta_{\gamma_\alpha^n}$ on $V_{\alpha R}^n$ satisfying

$$\int_{V_{\alpha R}^n} \psi_\infty(x)^2 d\gamma_\alpha^n(x) = \int_I h_\infty(r)^2 d\gamma_\alpha^1(r) = 1.$$

Thus, the proof is complete. ∎

5. Projection of Dirichlet eigenspace on high-dimensional sphere

We briefly recall some facts of the Dirichlet eigenvalue problem on a ball in a sphere. See [7, Sections II.5 and XII.5] for details.

The Dirichlet eigenvalue problem on $B_{a\theta}^N$ in $\mathbb{S}^N(a)$ is reduced to a Sturm–Liouville problem of the form

$$\begin{cases} \varphi''(\vartheta) + (N-1)\dfrac{\cos(\vartheta/a)}{a\sin(\vartheta/a)}\varphi'(\vartheta) = -\Big(\lambda - \dfrac{\lambda_k(\mathbb{S}^{N-1}(1))}{a^2\sin^2(\vartheta/a)}\Big)\varphi(\vartheta) & \text{in } [0, a\theta), \\ \varphi(a\theta) = 0, \end{cases} \tag{D_k^N}$$

for some $k \in \mathbb{N}_0$. The collection of $\lambda \in \mathbb{R}$ for which there exists a nontrivial solution $\varphi \in C^2([0, a\theta)) \cap C([0, a\theta])$ to (D_k^N) consists of a sequence

$$0 < \lambda_{k,1}(B_{a\theta}^N, \mathbb{S}^N(a)) < \lambda_{k,2}(B_{a\theta}^N, \mathbb{S}^N(a)) < \cdots < \lambda_{k,j}(B_{a\theta}^N, \mathbb{S}^N(a)) < \cdots \uparrow \infty,$$

and $\lambda_{k,j}(B_{a\theta}^N, \mathbb{S}^N(a))$ determines a one-dimensional linear space of solutions for each $j \in \mathbb{N}$. The set of Dirichlet eigenvalues on $B_{a\theta}^N$ is given by

$$\bigcup_{k\in\mathbb{N}_0, j\in\mathbb{N}} \{\lambda_{k,j}(B_{a\theta}^N, \mathbb{S}^N(a))\}.$$

Let (r, θ) denote polar geodesic coordinates about ae_1^{N+1} in $\mathbb{S}^N(a)$, that is,

$$(r(z), \theta(z)) := \left(d_{\mathbb{S}^N(a)}(z, ae_1^{N+1}), \frac{(z_i)_{i=2}^N}{\sqrt{a^2 - z_1^2}}\right)$$

on

$$z = (z_i)_{i=1}^N \in \mathbb{S}^N(a) \setminus \{\pm ae_1^{N+1}\}.$$

Given a solution $\varphi_{N,k,j}$ to (D_k^N) for $\lambda = \lambda_{k,j}(B_{a\theta}^N, \mathbb{S}^N(a))$ and $\Phi \in E_k(\mathbb{S}^{N-1}(1))$, define a function $\phi_{N,k,j}(\Phi; \cdot)$ on $z \in \overline{B_{a\theta}^N} \setminus \{ae_1^{N+1}\}$ by

$$\phi_{N,k,j}(\Phi; z) := \varphi_{N,k,j}(r(z))\Phi(\theta(z)). \tag{5.1}$$

The function $\phi_{N,k,j}(\Phi;\cdot)$ can be extended to $z = ae_1^{N+1}$ smoothly and becomes a Dirichlet eigenfunction on $B^N_{a\theta}$ of eigenvalue $\lambda_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a))$. Let $E_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a))$ denote the linear space of all Dirichlet eigenfunctions $\phi_{N,k,j}(\Phi;\cdot)$ on $B^N_{a\theta}$ of eigenvalue $\lambda_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a))$ given by the form (5.1). Then the linear space of all Dirichlet eigenfunctions on $B^N_{a\theta}$ coincides with

$$\bigoplus_{k\in\mathbb{N}_0, j\in\mathbb{N}} E_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a)).$$

Notice that

$$\dim E_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a)) = \dim E_k(\mathbb{S}^{N-1}(1)).$$

A similar argument implies that the Dirichlet eigenvalue problem on $V^n_{\alpha R}$ in Γ^n_α is reduced to a Sturm–Liouville problem of the form

$$\begin{cases} \Delta_{\gamma^1_\alpha} h = -\Big(\lambda - \dfrac{k}{\alpha^2}\Big)h & \text{in } (\alpha R, \infty), \\ h(\alpha R) = 0, \end{cases} \tag{P$_k$}$$

for some $k \in \mathbb{N}_0$. The collection of $\lambda \in \mathbb{R}$ for which there exists a nontrivial solution $h \in C^2((\alpha R, \infty)) \cap C([\alpha R, \infty))$ to (P$_k$) consists of a sequence

$$0 < \lambda_{k,1}(V^n_{\alpha R}, \Gamma^n_\alpha) < \lambda_{k,2}(V^n_{\alpha R}, \Gamma^n_\alpha) < \cdots < \lambda_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha) < \cdots \uparrow \infty,$$

and $\lambda_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha)$ determines a one-dimensional linear space of solutions for each $j \in \mathbb{N}$. The set of Dirichlet eigenvalues on $V^n_{\alpha R}$ is given by

$$\bigcup_{k\in\mathbb{N}_0, j\in\mathbb{N}} \{\lambda_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha)\}.$$

Given a solution $h_{k,j}$ to (P$_k$) for $\lambda = \lambda_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha)$ and $K = (K_i)^n_{i=2} \in \mathbb{N}_0^{n-1}(k)$, define a function $\psi_{K,j}$ on $x = (x_i)^n_{i=1} \in \overline{V^n_{\alpha R}}$ by

$$\psi_{K,j}(x) := \begin{cases} h_{k,j}(x) & \text{if } n = 1, \\ h_{k,j}(x_1)\displaystyle\prod_{i=2}^n H_{K_i}(\alpha^{-1}x_i) & \text{if } n \geq 2, \end{cases} \tag{5.2}$$

where H_k is the kth order Hermite polynomial given by (2.4). Then $\psi_{K,j}$ is a Dirichlet eigenfunction on $V^n_{\alpha R}$ of eigenvalue $\lambda_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha)$. Let $E_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha)$ denote the linear space of all Dirichlet eigenfunctions $\psi_{K,j}$ on $V^n_{\alpha R}$ of eigenvalue $\lambda_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha)$ given by the form (5.2). Then the linear space of all Dirichlet eigenfunctions on $V^n_{\alpha R}$ coincides with

$$\bigoplus_{k\in\mathbb{N}_0, j\in\mathbb{N}} E_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha).$$

Notice that

$$\dim E_{k,j}(V^n_{\alpha R}, \Gamma^n_\alpha) = d_k(n-1),$$

where we set $d_k(0) := 1$.

Let $\Omega = B^N_{a\theta}$ if $M = \mathbb{S}^N(a)$, and $\Omega = V^n_{\alpha R}$ if $M = \Gamma^n_\alpha$. The first Dirichlet eigenvalue $\lambda(\Omega, (M, \mu))$ is $\lambda_{0,1}(\Omega, (M, \mu))$ and the multiplicity of $\lambda(\Omega, (M, \mu))$ is 1. However, $\lambda_{k,j}(\Omega, (M, \mu)) = \lambda_{k',j'}(\Omega, (M, \mu))$ may happen for distinct pairs $(k, j), (k', j') \in \mathbb{N}_0 \times \mathbb{N}$.

As a counterpart of $E^n_k(\mathbb{S}^N(a))$, we define

$$\begin{aligned} &E^n_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a)) \\ &\quad := \left\{ \phi \in E_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a)) \;\middle|\; \begin{array}{l} \phi = \phi_{N,k,j}(\Phi; \cdot) \text{ defined in (5.1) such that} \\ \phi(x, y) = \phi(x, |y|_2 e_1^{N-n+1}) \text{ on } (x, y) \in B^N_{a\theta} \end{array} \right\}. \end{aligned}$$

By the definition and Lemma 3.2, we immediately find the following.

Proposition 5.1. *Fix $N, j \in \mathbb{N}$ with $2 \le N$, $k \in \mathbb{N}_0$, $a > 0$ and $\theta \in (0, \pi)$. Let $\phi_{N,k,j}(\Phi; \cdot)$ be a Dirichlet eigenfunction on $B^N_{a\theta}$ of eigenvalue $\lambda_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a))$ defined in (5.1).*

The linear space $E^1_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a))$ is nontrivial if and only if $k = 0$, where $E^1_{0,j}(B^N_{a\theta}, \mathbb{S}^N(a))$ is spanned by

$$\{\phi_{N,0,j}(\mathbb{1}_{\mathbb{S}^{N-1}(1)}; \cdot)\}$$

and hence $\dim E^1_{0,j}(B^N_{a\theta}, \mathbb{S}^N(a)) = 1$.

For $n \in \mathbb{N}$ with $2 \le n \le N$, $E^n_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a))$ is spanned by

$$\{\phi_{N,k,j}(P|_{\mathbb{S}^{N-1}(1)}; \cdot)\}_{P \in E^{n-1}_k(\mathbb{S}^{N-1}(1))}.$$

In the sequel, $\dim E^n_{k,j}(B^N_{a\theta}, \mathbb{S}^N(a)) = d_k(n-1)$.

Given $n, N \in \mathbb{N}$ with $2 \le n \le N$ and $K \in \mathbb{N}_0^{n-1}$, define $R_{N,n,K;a} \in \mathbb{P}(n)$ by

$$R_{N,n,K;a}(x_1, x') := \sum_{j=0}^{[|K|/2]} (a^2 - |x|_2^2)^j C_j(N-n) \Delta^j_{\mathbb{R}^{n-1}} x'^K$$

for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then, for $z = (x, y) \in B^N_{a\theta} \setminus \{a e_1^{N+1}\} \subset \mathbb{R}^n \times \mathbb{R}^{N-n+1}$, it turns out that

$$P_{N-1,n-1,K}(\theta(z)) = (a^2 - x_1^2)^{-\frac{|K|}{2}} R_{N,n,K;a}(x)$$

and hence

$$\begin{aligned}
&\phi_{N,|K|,j}(P_{N-1,n-1,K}|_{\mathbb{S}^{N-1}(1)}; z) \\
&\quad = \varphi_{N,|K|,j}(r(z)) P_{N-1,n-1,K}(\theta(z)) \\
&\quad = \varphi_{N,|K|,j}\Big(a \cdot \arccos\Big(\frac{x_1}{a}\Big)\Big) \cdot (a^2 - x_1^2)^{-\frac{|K|}{2}} R_{N,n,K;a}(x).
\end{aligned}$$

To establish a counterpart of Theorem 1.2 for higher Dirichlet eigenvalues and their eigenfunctions, we may need a uniform estimate of $\lambda_{k,j}(B^N_{a_N\theta_N}, \mathbb{S}^N(a_N))$ with respect to $N \in \mathbb{N}$ as well as Lemma 4.2 and a detailed analysis of $\lambda_{k,j}(V^1_{\alpha R}, \Gamma^1_\alpha)$.

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