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Inertia of Kraus matrices

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Abstract. For positive real numbers r, p_0, and $p_1 < \cdots < p_n$, let K_r be the $n \times n$ Kraus matrix whose (i, j) entry is equal to

$$\frac{1}{p_i - p_j}\Big(\frac{p_i^r - p_0^r}{p_i - p_0} - \frac{p_j^r - p_0^r}{p_j - p_0}\Big).$$

We determine the inertia of this matrix.

1. Introduction

In matrix analysis and operator theory, the notions of matrix monotone functions and matrix convex functions initiated by Löwner [19] and Kraus [18] are quite important. There have been several studies of these two classes of functions, see [1, 3, 8–11, 16] for instance. Let f be a real function defined on an interval I. The function f is said *matrix monotone of order* n if $A \leqq B$ implies $f(A) \leqq f(B)$ for all $n \times n$ Hermitian matrices A, B with eigenvalues in I; it is called *matrix convex of order* n if $f(tA + (1-t)B) \leqq tf(A) + (1-t)f(B)$ for all $n \times n$ Hermitian matrices A, B with eigenvalues in I and for all $t \in [0, 1]$.

Let f be a C^1-function on I. For $\lambda_1, \dots, \lambda_n \in I$, the $n \times n$ matrix

$$L_f(\lambda_1, \dots, \lambda_n) := [\,[\lambda_i, \lambda_j]_f\,]$$

is called a *Loewner matrix associated with* f, where $[\lambda_i, \lambda_j]_f$ is the first divided difference of f; $[\lambda_i, \lambda_j]_f$ is defined as $\frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j}$ if $\lambda_i \neq \lambda_j$, and $f'(\lambda_i)$ if $\lambda_i = \lambda_j$.

Let f be a C^2-function on I. For $\lambda_0, \lambda_1, \dots, \lambda_n$ in I, the $n \times n$ matrix

$$K_f(\lambda_0; \lambda_1, \dots, \lambda_n) := [\,[\lambda_0, \lambda_i, \lambda_j]_f\,] \tag{1.1}$$

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is called a *Kraus matrix associated with* f, where $[\lambda_0, \lambda_i, \lambda_j]_f$ is the second divided difference of f; for distinct $\lambda_0, \lambda_i, \lambda_j$,

$$[\lambda_0, \lambda_i, \lambda_j]_f := \frac{[\lambda_i, \lambda_0]_f - [\lambda_j, \lambda_0]_f}{\lambda_i - \lambda_j} = \frac{1}{\lambda_i - \lambda_j}\Big(\frac{f(\lambda_i) - f(\lambda_0)}{\lambda_i - \lambda_0} - \frac{f(\lambda_j) - f(\lambda_0)}{\lambda_j - \lambda_0}\Big),$$

and this can be extended continuously for any $\lambda_0, \lambda_i, \lambda_j \in I$. To be precise, if $\lambda_1, \dots, \lambda_n$ are distinct and λ_0 is different from them, then the (i, i) entry $[\lambda_0, \lambda_i, \lambda_i]_f$ is

$$\frac{f'(\lambda_i)}{\lambda_i - \lambda_0} - \frac{f(\lambda_i) - f(\lambda_0)}{(\lambda_i - \lambda_0)^2}.$$

If λ_0 coincides with some λ_j, then the (j, j) entry $[\lambda_j, \lambda_j, \lambda_j]_f$ is $f''(\lambda_j)/3!$. We refer to [4, 10, 16, 21] for divided differences.

It is known, thanks to Löwner [19], that for a C^1-function f on I, f is matrix monotone of order n if and only if the Loewner matrix $L_f(\lambda_1, \dots, \lambda_n)$ is positive semidefinite for any $\lambda_1, \dots, \lambda_n \in I$ and, thanks to Kraus and Heinävaara [14, 18], that for a C^2-function f on I, f is matrix convex of order n if and only if the Kraus matrix $K_f(\lambda_0; \lambda_1, \dots, \lambda_n)$ is positive semidefinite for any $\lambda_0, \lambda_1, \dots, \lambda_n \in I$.

Let A be an $n \times n$ Hermitian matrix. The inertia of A is the triple

$$\mathrm{In}(A) := (\pi(A), \zeta(A), \nu(A)),$$

where $\pi(A)$ is the number of positive eigenvalues of A, $\zeta(A)$ is the number of zero eigenvalues of A, and $\nu(A)$ is the number of negative eigenvalues of A.

In [5], Bhatia, Friedland, and Jain settled the conjecture about the inertia of Loewner matrices for the power functions t^r on $(0, \infty)$ which was proposed by Bhatia and Holbrook in [6]. In this article, we study the inertia of Kraus matrices for the power functions. We denote $K_{t^r}(p_0; p_1, \dots, p_n)$ by $K_r(p_0; p_1, \dots, p_n)$, and moreover simply by K_r when $p_0, p_1, \dots, p_n$ are easily inferred from the context:

$$K_r := \Big[\frac{1}{p_i - p_j}\Big(\frac{p_i^r - p_0^r}{p_i - p_0} - \frac{p_j^r - p_0^r}{p_j - p_0}\Big)\Big]. \tag{1.2}$$

Our main theorem is as follows:

Theorem 1.1. *Let r, p_0, and $p_1 < \cdots < p_n$, be positive real numbers. Let K_r be the $n \times n$ Kraus matrix defined in (1.2). Then*

(i) *K_r is singular if and only if $r = 1, \dots, n$;*

(ii) *if r is a positive integer and $r \leqq n + 1$, then*

$$r = 2k \implies \mathrm{In}(K_r) = (k, n + 1 - r, k - 1),$$

and

$$r = 2k - 1 \implies \mathrm{In}(K_r) = (k-1, n+1-r, k-1)$$

for a positive integer k*;*

(iii) *if* r *is not a positive integer and* $r < n$*, then*

$$2(k-1) < r < 2k-1 \implies \mathrm{In}(K_r) = (k-1, 0, n+1-k),$$

and

$$2k-1 < r < 2k \implies \mathrm{In}(K_r) = (n+1-k, 0, k-1)$$

for a positive integer k*;*

(iv) *if* $r > n$*, then* $\mathrm{In}(K_r) = \mathrm{In}(K_{n+1})$.

In Section 2, we give a proof of Theorem 1.1. We note that

$$\mathrm{In}(K_{-r}) = \mathrm{In}(K_{r+1})$$

for $r > 0$, since

$$K_{-r}(p_0; p_1, \dots, p_n) = q_0 D K_{r+1}(q_0; q_1, \dots, q_n) D$$

holds. Here

$$q_i := p_i^{-1}, \quad i = 0, \dots, n$$

and D is the $n \times n$ diagonal matrix $\mathrm{diag}(q_1, \dots, q_n)$; hence, we just consider the case $r > 0$.

In the remainder of this section, we fix our notations and recall several notions. We refer the reader to [4, 17] for matrix analysis.

For an $n \times n$ Hermitian matrix A, all eigenvalues are real numbers and we denote them as

$$\lambda_1(A) \geqq \lambda_2(A) \geqq \cdots \geqq \lambda_n(A).$$

An $n \times n$ Hermitian matrix A is said to be positive semidefinite or simply positive if

$$\langle Ax, x \rangle \geqq 0 \quad \text{for all } x \in \mathbb{C}^n,$$

and positive definite or strictly positive if

$$\langle Ax, x \rangle > 0 \quad \text{for all non-zero } x \in \mathbb{C}^n.$$

For Hermitian matrices A and B, $A \geqq B$ means that $A - B$ is positive semidefinite. Let $\mathcal{H}_1$ be the subspace of $\mathbb{C}^n$ defined as

$$\mathcal{H}_1 := \left\{ x = (x_i) \in \mathbb{C}^n : \sum_{i=1}^n x_i = 0 \right\},$$

which is the kernel space of the $n \times n$ matrix E with all entries 1. An $n \times n$ Hermitian matrix A is said to be *conditionally positive definite* (*cpd* for short) or *almost positive* if

$$\langle Ax, x\rangle \geqq 0 \quad \text{for all } x \in \mathcal{H}_1,$$

and *conditionally negative definite* (*cnd* for short) if $-A$ is cpd. It is known that if A is cpd (resp. cnd), then $\lambda_{n-1}(A) \geqq 0$ (resp. $\lambda_2(A) \leqq 0$). We refer the reader to [2, 10, 16, 21] for properties of these matrices. We also recall our study of the operator/matrix convexity by conditional negative/positive Loewner matrices in [7, 15].

2. Proof

In this section, we give a proof of Theorem 1.1. The following theorem and corollary are obtained similarly to those for Loewner matrices by Bhatia, Friedland, and Jain in [5]. All divided differences are associated with the power function t^r on $(0, \infty)$, so that we simply write them like $[p_0, p_i, p_j]$.

Let $c_1, c_2, \ldots, c_n$ be real numbers, not all of which are zero. Let p_0 and $p_1 < \cdots < p_n$ be positive real numbers. Let us define the continuous function f on $(0, \infty)$ as

$$f(x) = \sum_{j=1}^{n} c_j [p_0, x, p_j] \quad \text{for } x \in (0, \infty). \tag{2.1}$$

Theorem 2.1. *Let r be a positive real number not equal to $1, 2, \ldots, n$. Then the function f defined in (2.1) has at most $n-1$ zeros in $(0, \infty)$.*

Proof. Let $r_1 < r_2 < \cdots < r_m$, and let $a_1, a_2, \ldots, a_m$ be real numbers not all of which are zero. Then, the function

$$g(x) = \sum_{j=1}^{m} a_j x^{r_j} \tag{2.2}$$

has at most $m-1$ zeros in $(0, \infty)$. This is a well known fact: for example, consult [20, p. 46]. For the function f, let

$$g(x) := f(x) \prod_{i=0}^{n} (x - p_i).$$

Then g can be expressed in the form of (2.2) with $m = 2n+1$ and $\{r_1, \ldots, r_{2n+1}\} = \{0, 1, \ldots, n-1, n, r, r+1, \ldots, r+n-1\}$. In fact, we have $g(x) = x^r h_1(x) -$

$xh_2(x) + h_3(x)$, where

$$h_1(x) := \sum_{j=1}^{n} c_j \prod_{i=1, i\neq j}^{n} (x - p_i),$$

$$h_2(x) := \sum_{j=1}^{n} c_j \, [p_j, p_0] \prod_{i=1, i\neq j}^{n} (x - p_i),$$

$$h_3(x) := \sum_{j=1}^{n} c_j \, (p_0[p_j, p_0] - p_0^r) \prod_{i=1, i\neq j}^{n} (x - p_i).$$

Note that

$$[p_0, x, p_j](x - p_j) = [x, p_0] - [p_j, p_0],$$

and

$$[p_0, x, p_j](x - p_j)(x - p_0) = x^r - p_0^r - [p_j, p_0]x + p_0[p_j, p_0].$$

These polynomials $h_1(x), h_2(x)$, and $h_3(x)$ are of degree at most $n - 1$. Since $h_1(p_i) \neq 0$ for some i with $c_i \neq 0$, if $r \neq 1, 2, \ldots, n$, then g is not identically zero, and, by the fact mentioned above, the function g has at most $2n$ zeros in $(0, \infty)$. It is clear that $n + 1$ zeros occur at $x = p_0, p_j$ $(1 \leqq j \leqq n)$, so f has at most $n - 1$ zeros in $(0, \infty)$, and the proof is complete. ∎

Corollary 2.2. *Let r be a positive real number different from $1, 2, \ldots, n$. Then, the $n \times n$ Kraus matrix K_r defined in (1.2) is nonsingular.*

Proof. If the matrix K_r were singular, then there would be a non-zero vector $c = (c_1, \ldots, c_n)$ such that $K_r c = 0$; that is,

$$\sum_{j=1}^{n} c_j [p_0, p_i, p_j] = 0$$

for $i = 1, 2, \ldots, n$. This means that the function $f(x)$ in (2.1) would have n zeros: $x = p_1, \ldots, p_n$. But this contradicts Theorem 2.1. ∎

Proposition 2.3. *Let p_0 and $p_1 < p_2$ be in $(0, \infty)$. If $r > 2$, then the 2×2 Kraus matrix $K_r(p_0; p_1, p_2)$ has a positive eigenvalue and a negative eigenvalue.*

Proof. Let $f(t) = t^r$. Since the function $(p_0, p_1, p_2) \mapsto \det K_r(p_0; p_1, p_2)$ is continuous and the matrix $K_r := K_r(p_0; p_1, p_2)$ is nonsingular by Corollary 2.2, either

$\det K_r > 0$ for any $p_0, p_1 < p_2$ or $\det K_r < 0$ for any $p_0, p_1 < p_2$. Suppose that $\det K_r > 0$. Note that

$$\begin{aligned}
&\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_2] \\ [p_0,p_2,p_1] & [p_0,p_2,p_2] \end{vmatrix} \\
&= \begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_2]-[p_0,p_1,p_1] \\ [p_0,p_2,p_1] & [p_0,p_2,p_2]-[p_0,p_2,p_1] \end{vmatrix} \\
&= \begin{vmatrix} [p_0,p_1,p_1] & (p_2-p_1)[p_0,p_1,p_2,p_1] \\ [p_0,p_2,p_1] & (p_2-p_1)[p_0,p_2,p_2,p_1] \end{vmatrix} \\
&= (p_2-p_1)\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_1,p_2] \\ [p_0,p_2,p_1] & [p_0,p_1,p_2,p_2] \end{vmatrix} \\
&= (p_2-p_1)\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_1,p_2] \\ [p_0,p_2,p_1]-[p_0,p_1,p_1] & [p_0,p_1,p_2,p_2]-[p_0,p_1,p_1,p_2] \end{vmatrix} \\
&= (p_2-p_1)^2\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_1,p_2] \\ [p_0,p_1,p_2,p_1] & [p_0,p_1,p_1,p_2,p_2] \end{vmatrix};
\end{aligned}$$

that is,

$$\det K_r = (p_2-p_1)^2\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_1,p_2] \\ [p_0,p_1,p_1,p_2] & [p_0,p_1,p_1,p_2,p_2] \end{vmatrix}.$$

We refer the reader to [10,12,21] for this computation. It follows from our assumption $\det K_r > 0$ that

$$\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_1,p_2] \\ [p_0,p_1,p_1,p_2] & [p_0,p_1,p_1,p_2,p_2] \end{vmatrix} > 0$$

so that

$$\begin{aligned}
&\lim_{p_1\to p_0, p_2\to p_0}\begin{vmatrix} [p_0,p_1,p_1] & [p_0,p_1,p_1,p_2] \\ [p_0,p_1,p_1,p_2] & [p_0,p_1,p_1,p_2,p_2] \end{vmatrix} \\
&= \begin{vmatrix} f^{(2)}(p_0)/2! & f^{(3)}(p_0)/3! \\ f^{(3)}(p_0)/3! & f^{(4)}(p_0)/4! \end{vmatrix} \geqq 0. \qquad (2.3)
\end{aligned}$$

It is known in [13, Proposition 3.1] for $f(t) = t^r$ on $(0,\infty)$ that

$$\begin{vmatrix} f^{(2)}(t)/2! & f^{(3)}(t)/3! \\ f^{(3)}(t)/3! & f^{(4)}(t)/4! \end{vmatrix} = -\frac{1}{144}r^2(r-1)^2(r-2)(r+1)t^{2r-6},$$

which is negative if $r > 2$. This contradicts (2.3); therefore, $\det K_r < 0$ and we get the conclusion. ■

Corollary 2.4. *Let p_0 and $p_1 < \cdots < p_n$ be in $(0, \infty)$. If $r > 2$, then the $n \times n$ Kraus matrix $K_r(p_0; p_1, \ldots, p_n) = K_r$ admits both a positive eigenvalue and a negative eigenvalue; that is,*

$$\lambda_1(K_r) > 0 > \lambda_n(K_r).$$

Proof. Since

$$\lambda_1(K_r(p_0; p_1, p_2)) > 0 > \lambda_2(K_r(p_0; p_1, p_2))$$

by Proposition 2.3, using Cauchy's interlacing principle, we have the conclusion. ■

Proof of Theorem 1.1 (iv). If the inertia of K_r ($r > 0$) were to change, then one of the eigenvalues of K_r had to change sign, but this contradicts Corollary 2.2. ■

Proof of Theorem 1.1 (iii). For $t > 0$ and $0 < r < 1$ the following formula is well known [4, p. 116]:

$$t^r = \frac{\sin r\pi}{\pi} \int_0^\infty \frac{t}{\lambda + t} \lambda^{r-1}\, d\lambda.$$

We write this as

$$t^r = \int_0^\infty \frac{t}{\lambda + t}\, d\mu(\lambda), \tag{2.4}$$

where μ is a positive measure on $(0, \infty)$. For each $\lambda > 0$ let

$$k_\lambda(t) = \frac{t}{\lambda + t}.$$

Since

$$\frac{1}{p_i - p_j}\Big(\frac{k_\lambda(p_i) - k_\lambda(p_0)}{p_i - p_0} - \frac{k_\lambda(p_j) - k_\lambda(p_0)}{p_j - p_0}\Big) = -\frac{\lambda}{(\lambda + p_0)(\lambda + p_i)(\lambda + p_j)},$$

the Kraus matrix of k_λ is expressed as

$$K_{k_\lambda}(p_0; p_1, \ldots, p_n) = -\frac{\lambda}{\lambda + p_0} D_\lambda E D_\lambda,$$

where E is the matrix with all entries equal to 1: $E = [1] \geqq 0$, and D_λ is the diagonal matrix $\operatorname{diag}(\frac{1}{\lambda+p_1}, \ldots, \frac{1}{\lambda+p_n})$. It follows that $K_{k_\lambda} \leqq 0$; hence, $K_r \leqq 0$. To be precise, by Corollary 2.2 or a direct computation, K_r is negative definite, and

$$\operatorname{In}(K_r) = (0, 0, n).$$

For $1 < r < 2$, we get from (2.4) that

$$t^r = \int_0^\infty \frac{t^2}{\lambda + t}\, d\mu(\lambda).$$

For each $\lambda > 0$ let

$$h_\lambda(t) = \frac{t^2}{\lambda + t}.$$

Since

$$\frac{1}{p_i - p_j}\Big(\frac{h_\lambda(p_i) - h_\lambda(p_0)}{p_i - p_0} - \frac{h_\lambda(p_j) - h_\lambda(p_0)}{p_j - p_0}\Big) = \frac{\lambda^2}{(\lambda + p_0)(\lambda + p_i)(\lambda + p_j)},$$

the Kraus matrix of h_λ is of the form

$$K_{h_\lambda}(p_0; p_1, \dots, p_n) = \frac{\lambda^2}{\lambda + p_0} D_\lambda E D_\lambda \geqq 0,$$

so $K_r \geqq 0$. Moreover, K_r is positive definite by Corollary 2.2, so that

$$\mathrm{In}(K_r) = (n, 0, 0).$$

We pause the proof with a remark. Since $f(t) = t^r$ is operator convex for $1 < r < 2$, that is, matrix convex of any order n, the corresponding Kraus matrix is known to be positive semidefinite for any $p_0, p_1, \dots, p_n$; see [18]. The above argument for K_r is already in [7], and for $2 < r < 3$ or $3 < r < 4$ the functions $g_\lambda(t) := \frac{t^3}{\lambda+t}$ and $f_\lambda(t) := \frac{t^4}{\lambda+t}$ work equally well. Actually, in terms of $D := \mathrm{diag}(p_1, \dots, p_n)$ and D_λ, we see that

$$K_{g_\lambda}(p_0; p_1, \dots, p_n) = E - \frac{\lambda^3}{\lambda + p_0} D_\lambda E D_\lambda,$$

and

$$K_{f_\lambda}(p_0; p_1, \dots, p_n) = DE + ED + p_0 E - \lambda E + \frac{\lambda^4}{\lambda + p_0} D_\lambda E D_\lambda;$$

thus, K_r is cnd for $2 < r < 3$ and cpd for $3 < r < 4$, and we could determine its inertia by [2, Lemma 4.3.5] with Corollary 2.4.

To continue the proof in the general case, we take an alternative approach, following the argument by Bhatia, Friedland, and Jain as in the proof of [5, Theorem 1.1] for Loewner matrices, to determine the inertia of the Kraus matrix for the power function t^r.

Due to the identity

$$\begin{aligned} &p_i\Big(\frac{p_i^{r-2} - p_0^{r-2}}{p_i - p_0} - \frac{p_j^{r-2} - p_0^{r-2}}{p_j - p_0}\Big)p_j \\ &\quad = -(p_i - p_j)\Big(p_i\frac{p_i^{r-2} - p_0^{r-2}}{p_i - p_0} + p_j\frac{p_j^{r-2} - p_0^{r-2}}{p_j - p_0} + p_0^{r-2}\Big) \\ &\qquad + \frac{p_i^r - p_0^r}{p_i - p_0} - \frac{p_j^r - p_0^r}{p_j - p_0}, \end{aligned}$$

the Kraus matrices K_r and K_{r-2} are related as

$$K_r = DK_{r-2}D + D^{\sim}DE + EDD^{\sim} + p_0^{r-2}E,$$

where $E = [1]$, $D = \mathrm{diag}(p_1, \dots, p_n)$ and

$$D^{\sim} := \mathrm{diag}\Big(\frac{p_1^{r-2} - p_0^{r-2}}{p_1 - p_0}, \dots, \frac{p_n^{r-2} - p_0^{r-2}}{p_n - p_0}\Big).$$

Suppose $2 < r < 3$. Then K_r is cnd. In fact, for $x \in \mathcal{H}_1$ or $Ex = 0$,

$$\begin{aligned}\langle K_r x, x\rangle &= \langle DK_{r-2}Dx, x\rangle + \langle D^{\sim}DEx, x\rangle + \langle x, D^{\sim}DEx\rangle + p_0^{r-2}\langle Ex, x\rangle \\ &= \langle DK_{r-2}Dx, x\rangle.\end{aligned}$$

We know that K_{r-2} is negative definite for $0 < r-2 < 1$, so $\langle K_r x, x\rangle \leqq 0$ for $x \in \mathcal{H}_1$ or K_r is cnd; hence, $\lambda_2(K_r) \leqq 0$. Since K_r is nonsingular by Corollary 2.2 and $\lambda_1(K_r) > 0$ by Corollary 2.4, we conclude that

$$\mathrm{In}(K_r) = (1, 0, n-1).$$

Especially, for $n = 2$ and $r > 2$

$$\mathrm{In}(K_r) = (1, 0, 1) = \mathrm{In}(K_3).$$

Let $n > 2$ and suppose $3 < r < 4$. Since K_{r-2} is positive definite for $1 < r-2 < 2$,

$$\langle K_r x, x\rangle = \langle DK_{r-2}Dx, x\rangle \geqq 0$$

for $x \in \mathcal{H}_1$; hence, K_r is cpd, and $\lambda_{n-1}(K_r) \geqq 0$. As K_r is nonsingular by Corollary 2.2 and $\lambda_n(K_r) < 0$ by Corollary 2.4, we have

$$\mathrm{In}(K_r) = (n-1, 0, 1).$$

In particular, for $n = 3$ and $r > 3$

$$\mathrm{In}(K_r) = (2, 0, 1) = \mathrm{In}(K_4).$$

Let us define the subspace $\mathcal{H}_2$ by

$$\begin{aligned}\mathcal{H}_2 &:= \Big\{x = (x_i) \in \mathbb{C}^n \colon \sum_{i=1}^n x_i = 0 = \sum_{i=1}^n p_i x_i\Big\} \\ &= \{x \in \mathbb{C}^n \colon Ex = 0 = EDx\},\end{aligned}$$

where $E = [1]$ and $D = \mathrm{diag}(p_1, \dots, p_n)$, being the orthogonal complement of the span of the vectors $(1, \dots, 1)$ and $(p_1, \dots, p_n)$.

Let $n > 3$ and suppose $4 < r < 5$. For $x \in \mathcal{H}_2$,

$$\langle K_r x, x\rangle = \langle K_{r-2} y, y\rangle,$$

where $y := Dx \in \mathcal{H}_1$. Since K_{r-2} is cnd for $2 < r-2 < 3$, $\langle K_{r-2} y, y\rangle \leqq 0$ or

$$\langle K_r x, x\rangle \leqq 0$$

for $x \in \mathcal{H}_2$. The minmax principle implies that $\lambda_3(K_r) \leqq 0$. We already proved that a 3×3 principal submatrix of K_r has two positive eigenvalues, so $\lambda_2(K_r) > 0$ by Cauchy's interlacing principle. Since K_r is nonsingular by Corollary 2.2, one has

$$\mathrm{In}(K_r) = (2, 0, n-2);$$

especially, for $n = 4$ and $r > 4$

$$\mathrm{In}(K_r) = (2, 0, 2) = \mathrm{In}(K_5).$$

Let $n > 4$ and suppose $5 < r < 6$. For $x \in \mathcal{H}_2$

$$\langle K_r x, x\rangle = \langle K_{r-2} y, y\rangle,$$

where $y := Dx \in \mathcal{H}_1$. Since K_{r-2} is cpd for $3 < r-2 < 4$, $\langle K_{r-2} y, y\rangle \geqq 0$:

$$\langle K_r x, x\rangle \geqq 0$$

for $x \in \mathcal{H}_2$. By the minmax principle and the nonsingularity of K_r $\lambda_{n-2}(K_r) > 0$, and since a 4×4 principal submatrix of K_r has two negative eigenvalues, $\lambda_{n-1}(K_r) < 0$ by Cauchy's interlacing principle. Hence, we have

$$\mathrm{In}(K_r) = (n-2, 0, 2),$$

so for $n = 5$ and $r > 5$

$$\mathrm{In}(K_r) = (3, 0, 2) = \mathrm{In}(K_6).$$

We define the subspace $\mathcal{H}_3$ by

$$\begin{aligned}\mathcal{H}_3 &:= \{x = (x_i) \in \mathbb{C}^n \colon \sum_{i=1}^n x_i = \sum_{i=1}^n p_i x_i = \sum_{i=1}^n p_i^2 x_i = 0\} \\ &= \{x \in \mathbb{C}^n \colon Ex = EDx = ED^2 x = 0\},\end{aligned}$$

which is the orthogonal complement of the span of the vectors $(1, \ldots, 1)$, $(p_1, \ldots, p_n)$, and $(p_1^2, \ldots, p_n^2)$.

Let $n > 5$ and suppose $6 < r < 7$. For $x \in \mathcal{H}_3$, since $Ex = EDx = 0$,

$$\langle K_r x, x\rangle = \langle DK_{r-2}Dx, x\rangle = \langle D^2K_{r-4}D^2x, x\rangle = \langle K_{r-4}y, y\rangle,$$

where $y := D^2x \in \mathcal{H}_1$. Since K_{r-4} is cnd for $2 < r-4 < 3$, $\langle K_{r-4}y, y\rangle \leqq 0$, that is,

$$\langle K_r x, x\rangle \leqq 0 \quad \text{for } x \in \mathcal{H}_3.$$

The minmax principle with the nonsingularity of K_r implies that $\lambda_4(K_r) < 0$. Since a 5×5 principal submatrix of K_r has three positive eigenvalues, $\lambda_3(K_r) > 0$ by Cauchy's interlacing principle. We conclude that

$$\mathrm{In}(K_r) = (3, 0, n-3);$$

in particular, for $n = 6$ and $r > 6$

$$\mathrm{In}(K_r) = (3, 0, 3) = \mathrm{In}(K_7).$$

Let $n > 6$ and suppose $7 < r < 8$. For $x \in \mathcal{H}_3$,

$$\langle K_r x, x\rangle = \langle K_{r-4}y, y\rangle,$$

where $y := D^2x \in \mathcal{H}_1$. Since K_{r-4} is cpd for $3 < r-4 < 4$, $\langle K_{r-4}y, y\rangle \geqq 0$:

$$\langle K_r x, x\rangle \geqq 0$$

for $x \in \mathcal{H}_3$. By the minmax principle and the nonsingularity of K_r, $\lambda_{n-3}(K_r) > 0$, and a 6×6 principal submatrix of K_r has three negative eigenvalues so $\lambda_{n-2}(K_r) < 0$ by Cauchy's interlacing principle; hence,

$$\mathrm{In}(K_r) = (n-3, 0, 3);$$

particularly, for $n = 7$ and $r > 7$,

$$\mathrm{In}(K_r) = (4, 0, 3) = \mathrm{In}(K_8).$$

In this way, we have a proof by induction. Let $n > 2k-3$. If $2(k-1) < r < 2k-1$, then $2 < r - 2(k-2) < 3$. Let us define the subspace $\mathcal{H}_{k-1}$ as the orthogonal complement of the span of the vectors $(1, \dots, 1)$, $(p_1, \dots, p_n)$, $\dots$, $(p_1^{k-2}, \dots, p_n^{k-2})$; that is, $\mathcal{H}_{k-1} := \ker E \cap \ker ED \cap \cdots \cap \ker ED^{k-2}$. For $x \in \mathcal{H}_{k-1}$,

$$\langle K_r x, x\rangle = \langle K_{r-2(k-2)}y, y\rangle,$$

where $y := D^{k-2}x \in \mathcal{H}_1$. Since $K_{r-2(k-2)}$ is cnd for $2 < r - 2(k-2) < 3$,

$$\langle K_{r-2(k-2)}y, y\rangle \leqq 0,$$

or

$$\langle K_r x, x\rangle \leqq 0 \quad \text{for } x \in \mathcal{H}_{k-1}.$$

The minmax principle with the nonsingularity of K_r implies that $\lambda_k(K_r) < 0$. Since a $(2k-3)\times(2k-3)$ principal submatrix of K_r has $(k-1)$ positive eigenvalues by induction, $\lambda_{k-1}(K_r) > 0$ by Cauchy's interlacing principle; hence, we conclude that

$$\mathrm{In}(K_r) = (k-1, 0, n+1-k);$$

in particular, for $n = 2(k-1)$ and $r > n$,

$$\mathrm{In}(K_r) = (k-1, 0, k-1) = \mathrm{In}(K_{n+1}).$$

Similarly, let $n > 2k-2$. If $2k-1 < r < 2k$, then $3 < r - 2(k-2) < 4$ and for $x \in \mathcal{H}_{k-1}$

$$\langle K_r x, x\rangle = \langle K_{r-2(k-2)} y, y\rangle,$$

where

$$y := D^{k-2} x \in \mathcal{H}_1.$$

Since $K_{r-2(k-2)}$ is cpd for $3 < r-2(k-2) < 4$, $\langle K_{r-2(k-2)} y, y\rangle \geqq 0$:

$$\langle K_r x, x\rangle \geqq 0$$

for $x \in \mathcal{H}_{k-1}$, and by the minmax principle, $\lambda_{n-(k-1)}(K_r) \geqq 0$. Since a $2(k-1)$-square principal submatrix of K_r has $(k-1)$ negative eigenvalues by induction, $\lambda_{n-k+2}(K_r) < 0$ by Cauchy's interlacing principle. We conclude by the nonsingularity of K_r that

$$\mathrm{In}(K_r) = (n+1-k, 0, k-1),$$

so that for $n = 2k-1$ and $r > n$

$$\mathrm{In}(K_r) = (k, 0, k-1) = \mathrm{In}(K_{n+1}).$$

The proof of (iii) is complete. ■

Proof of Theorem 1.1 (i) *and Theorem* 1.1 (ii). For $r = 1$, the (i, j) entry of K_1 is

$$\frac{1}{p_i - p_j}\Big(\frac{p_i - p_0}{p_i - p_0} - \frac{p_j - p_0}{p_j - p_0}\Big) = 0,$$

that is, $K_1 = 0$ and $\mathrm{In}(K_1) = (0, n, 0)$.

For $r = 2$,

$$\frac{1}{p_i - p_j}\Big(\frac{p_i^2 - p_0^2}{p_i - p_0} - \frac{p_j^2 - p_0^2}{p_j - p_0}\Big) = \frac{1}{p_i - p_j}\big((p_i + p_0) - (p_j + p_0)\big) = 1,$$

so that $K_2 = E = \begin{bmatrix}1\end{bmatrix}$ and $\mathrm{In}(K_2) = (1, n-1, 0)$.

In general, for a positive integer $r \leqq n+1$, let $\mathcal{H}_r := \ker E \cap \ker ED \cap \cdots \cap \ker ED^{r-1}$. It is easy to see that K_r is of the form

$$K_r = \sum_{k=1}^{r-1} p_0^{k-1} \sum_{l=1}^{r-k} D^{r-k-l} E D^{l-1},$$

so that

$$\ker K_r \supset \ker E \cap \ker ED \cap \ker ED^2 \cap \cdots \cap \ker ED^{r-2} = \mathcal{H}_{r-1},$$

and

$$\dim \ker K_r \geqq \dim \mathcal{H}_{r-1} = n - (r-1) = n + 1 - r.$$

For $r = 3$ and $n \geqq 2$, $\dim\ker K_3 \geqq n-2$. By Corollary 2.4, $\lambda_1(K_3) > 0 > \lambda_n(K_3)$, so that $\mathrm{In}(K_3) = (1, n-2, 1)$ for $n \geqq 2$.

For $r = 4$ and $n \geqq 3$, $\dim\ker K_4 \geqq n-3$. Since a 3×3 principal submatrix of K_4 has the inertia $(2, 0, 1)$ as in the proof of (iii), $\lambda_2(K_4) > 0 > \lambda_n(K_4)$ by Cauchy's interlacing principle; hence, $\mathrm{In}(K_4) = (2, n-3, 1)$ for $n \geqq 3$.

For $r = 5$ and $n \geqq 4$, $\dim\ker K_5 \geqq n-4$. A 4×4 principal submatrix of K_5 has the inertia $(2, 0, 2)$ as in the proof of (iii), we conclude by Cauchy's interlacing principle that $\mathrm{In}(K_5) = (2, n-4, 2)$ for $n \geqq 4$.

We can continue this argument; if $r = 2k$, then a $(r-1)$-square principal submatrix of K_r has the inertia $(k, 0, k-1)$ as in the proof of (iii), so $\mathrm{In}(K_r) = (k, n+1-r, k-1)$ by Cauchy's interlacing principle; if $r = 2k-1$, then a $(r-1)$-square principal submatrix of K_r has the inertia $(k-1, 0, k-1)$ as in the proof of (iii), so $\mathrm{In}(K_r) = (k-1, n+1-r, k-1)$ by Cauchy's interlacing principle.

If $r \in \{1, 2, \ldots, n\}$, then the number of zero eigenvalues is $n+1-r$, which is nonzero; so K_r is singular. The other implication follows from Corollary 2.2; therefore, the proof is complete. ∎

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