J. Spectr. Theory 12 (2022), 1383–1404
DOI 10.4171/JST/428

On inverse problems arising in fractional elasticity

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Abstract. We first formulate an inverse problem for a linear fractional Lamé system. We determine the Lamé parameters from exterior partial measurements of the Dirichlet-to-Neumann map. We further study an inverse obstacle problem as well as an inverse problem for a nonlinear fractional Lamé system. Our arguments are based on the unique continuation property for the fractional operator as well as the associated Runge approximation property.

1. Introduction

The classical Lamé operator $L_{\lambda,\mu}$ for a three-dimensional isotropic elastic body is given by

$$(L_{\lambda,\mu}u)_i := \sum_{j=1}^{3} \partial_i(\lambda u_{j,j}) + \sum_{j=1}^{3} \partial_j(\mu(u_{i,j} + u_{j,i})) \quad (1 \le i \le 3)$$

where u_i denotes the i-th component of the vector-valued displacement function u and $u_{i,j} := \partial_j u_i$. The associated inverse problem has been studied in [8, 21] where the authors considered the Dirichlet problem

$$L_{\lambda,\mu}u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega.$$

They determined the variable Lamé parameters λ, μ from the Dirichlet-to-Neumann (displacement-to-traction) map

$$(\Lambda g)_i := \sum_{j=1}^{3} (\lambda u_{j,j})\nu_i + \sum_{j=1}^{3} \mu(u_{i,j} + u_{j,i})\nu_j \quad (1 \le i \le 3)$$

where ν is the unit outer normal to $\partial\Omega$ under certain assumptions on λ, μ.

In this paper, we study a fractional analogue of $L_{\lambda,\mu}$ and its associated inverse problems.

2020 Mathematics Subject Classification. Primary 35R30; Secondary 35R11.
Keywords. Inverse problems, nonlocal elasticity, unique continuation property, Runge approximation property.

First we recall that the classical elastic model is based on the constitutive relation

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl} e_{kl}$$

where the fourth-order elastic stiffness tensor is

$$C_{ijkl} := \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

the linearized strain tensor is

$$e_{ij} := \frac{1}{2}(u_{i,j} + u_{j,i})$$

and σ_{ij} denotes the stress tensor. Here δ_{ij} is the standard Kronecker delta.

Recently, the theory of nonlocal elasticity has attracted much attention. The integral linear constitutive relation

$$\sigma_{ij} = \sum_{k,l=1}^{3} C_{ijkl}(K * e_{kl})$$

has been introduced to describe complex materials characterized by nonlocality. Then the fractional Taylor series approximation for the Fourier transform of the interaction kernel K, which is given by

$$(\mathscr{F} K)(|\xi|) \approx (\mathscr{F} K)(0) + c_s|\xi|^{2s} \quad (0 < s < 1)$$

leads to the definition of the fractional Lamé operator

$$\mathcal{L} := L_{\lambda,\mu} + (-\Delta)^s L_{\lambda_0,\mu_0}. \tag{1}$$

See [25] and the references there for more background information. In [25], λ, μ are constants proportional to the constants λ_0, μ_0 but in this paper we allow λ, μ to be variable functions.

We consider the exterior Dirichlet problem

$$\mathcal{L}u = 0 \text{ in } \Omega, \quad u = g \text{ in } \Omega_e \tag{2}$$

where Ω is a bounded Lipschitz domain and $\Omega_e := \mathbb{R}^3 \setminus \bar{\Omega}$. Under appropriate assumptions on $\lambda_0, \mu_0, \lambda, \mu$, we can show its well-posedness so we will be able to define the associated Dirichlet-to-Neumann map Λ, which is formally given by

$$\Lambda g := (-\Delta)^s L_{\lambda_0,\mu_0} u_g|_{\Omega_e}. \tag{3}$$

Our goal here is to determine both λ and μ from exterior partial measurements of Λ.

We remark that our problem can be viewed as a variant of the fractional Calderón problem first introduced in [12] where the authors considered the exterior Dirichlet problem

$$((-\Delta)^s + q)u = 0 \text{ in } \Omega, \quad u = g \text{ in } \Omega_e$$

and they proved the fundamental uniqueness theorem that the potential q in Ω can be determined from exterior partial measurements of the map

$$\Lambda_q : g \to (-\Delta)^s u_g|_{\Omega_e}.$$

It has been shown that the knowledge of $\Lambda_q g$ is equivalent to the knowledge of the nonlocal Neumann derivative of u_g (see [12] for more details). Hence, our problem can also be viewed as a nonlocal analogue of the inverse problem for the classical Lamé system.

We mention that inverse problems for fractional operators have been extensively studied so far. See [24] for low regularity and stability results for the fractional Calderón problem. See [11] for reconstruction and single measurement results for the fractional Calderón problem. See [10] for inverse problems for variable coefficients fractional elliptic operators. See [2, 6] for inverse problems for fractional Schrödinger operators with local and non-local perturbations. See [5, 15, 17] for inverse problems for fractional magnetic operators. See [14, 18, 19] for inverse problems for fractional parabolic operators.

The following theorem is our first main result in this paper.

Theorem 1.1. *Let $0 < s < 1$. Suppose the constants λ_0, μ_0 satisfy $\mu_0 > 0$ and $\lambda_0 + \mu_0 \geq 0$. Let $0 \leq \lambda^{(j)}, \mu^{(j)} \in C^1(\bar{\Omega})$ and let $W_j \subset \Omega_e$ be nonempty and open ($j = 1, 2$). Let $\Lambda^{(j)}$ be the Dirichlet-to-Neumann map associated with (2) when λ, μ are replaced by $\lambda^{(j)}, \mu^{(j)}$ in (1). Suppose*

$$\Lambda^{(1)} g|_{W_2} = \Lambda^{(2)} g|_{W_2}$$

for all $g \in C_c^\infty(W_1)$. Then $\lambda^{(1)} = \lambda^{(2)}, \mu^{(1)} = \mu^{(2)}$ in Ω.

We remark that our problem provides an example which suggests that the inverse problem for the fractional operator is more manageable than its classical counterpart.

Recall that to solve the classical inverse problem, we first reduce the Lamé system to a first order system perturbation of the Laplacian. Then we construct complex geometrical optics (CGO) solutions and apply the integral identity to obtain the uniqueness of λ, μ. For some technical reasons, the uniqueness result is only proved provided that μ is close to a constant. The full classical problem remains open. See [8, 22] for details.

Here such a priori knowledge of μ is not required for solving the fractional problem. Instead of constructing CGO solutions, we will use the unique continuation property and the Runge approximation property associated with our fractional operator to prove the strong uniqueness result. This scheme was first introduced in [12] for solving the fractional Calderón problem.

We further study an inverse obstacle problem associated with our fractional operator.

We consider the following obstacle problem

$$\mathcal{L}u = 0 \text{ in } \Omega \setminus \bar{D}, \quad u = 0 \text{ in } D, \quad u = g \text{ in } \Omega_e, \tag{4}$$

where $D \subset \Omega$ is a nonempty open set satisfying that $\Omega \setminus \bar{D}$ is a bounded Lipschitz domain.

As we did for the exterior problem (2), we can similarly show the well-posedness of (4) and define the Dirichlet-to-Neumann map Λ_D by

$$\Lambda_D g := (-\Delta)^s L_{\lambda_0,\mu_0} u_g|_{\Omega_e}. \tag{5}$$

Our next goal is to determine D, λ, μ from the knowledge of Λ_D.

The following theorem is our second main result.

Theorem 1.2. *Let* $0 < s < 1$. *Suppose the constants* λ_0, μ_0 *satisfy* $\mu_0 > 0$ *and* $\lambda_0 + \mu_0 \geq 0$. *Let* $D_j \subset \Omega$ *be nonempty and open such that* $\Omega \setminus \overline{D_j}$ *is a bounded Lipschitz domain,* $0 \leq \lambda^{(j)}, \mu^{(j)} \in C^1(\bar{\Omega} \setminus D_j)$ *and let* $W_j \subset \Omega_e$ *be nonempty and open* ($j = 1, 2$). *Let* $\Lambda^{(j)}_{D_j}$ *be the Dirichlet-to-Neumann map corresponding to* $\lambda^{(j)}, \mu^{(j)}$ *and* D_j. *Suppose*

$$\Lambda^{(1)}_{D_1} g|_{W_2} = \Lambda^{(2)}_{D_2} g|_{W_2}$$

for a nonzero $g \in C^\infty_c(W_1)$. *Then* $D_1 = D_2 =: D$. *Further assume the identity holds for all* $g \in C^\infty_c(W_1)$. *Then* $\lambda^{(1)} = \lambda^{(2)}$ *and* $\mu^{(1)} = \mu^{(2)}$ *in* $\Omega \setminus \bar{D}$.

We also study an inverse problem for a nonlinear fractional Lamé system.

We consider the following nonlinear exterior problem

$$\mathcal{L}u + \mathcal{N}u = 0 \text{ in } \Omega, \quad u = g \text{ in } \Omega_e \tag{6}$$

where the nonlinear operator $\mathcal{N}$ is given by

$$(\mathcal{N}u)_i := \sum_{j=1}^{3} \partial_j N_{ij} u \quad (1 \leq i \leq 3) \tag{7}$$

where

$$\begin{aligned} N_{ij}u := {} & \frac{\lambda + \mathcal{B}}{2} \sum_{m,n} u^2_{m,n} \delta_{ij} + \mathcal{C}\Big(\sum_m u_{m,m}\Big)^2 \delta_{ij} + \frac{\mathcal{B}}{2} \sum_{m,n} u_{m,n} u_{n,m} \delta_{ij} \\ & + \mathcal{B} \sum_m u_{m,m} u_{j,i} + \frac{\mathcal{A}}{4} \sum_m u_{j,m} u_{m,i} + (\lambda + \mathcal{B}) \sum_m u_{m,m} u_{i,j} \\ & + \Big(\mu + \frac{\mathcal{A}}{4}\Big) \sum_m (u_{m,i} u_{m,j} + u_{i,m} u_{j,m} + u_{i,m} u_{m,j}). \end{aligned}$$

This nonlinearity comes from the higher order expansion of the energy density as well as the nonlinear term in the strain tensor

$$e_{ij} := \frac{1}{2}\Big(u_{i,j} + u_{j,i} + \sum_m u_{m,i}u_{m,j}\Big).$$

In fact, $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are the coefficients of terms cubic in e_{ij} in the expansion of the energy density. They are all constants in the original model but in this paper we allow them to be variable functions. See [16, Section 26] for more background information on $\mathscr{N}$.

We remark that our $\mathscr{N}$ is the static version of the nonlinearity considered in [7,26], where the inverse problem for the associated nonlinear elastic wave equation was studied.

Under certain assumptions, we can show the well-posedness of (6) for small $g \in C_c^\infty(\Omega_e)$ and then for such g we can define the associated Dirichlet-to-Neumann map Λ_N formally given by

$$\Lambda_N g := (-\Delta)^s L_{\lambda_0,\mu_0} u_g|_{\Omega_e}. \tag{8}$$

Our last goal is to determine $\lambda, \mu, \mathscr{A}, \mathscr{C}$ from the knowledge of Λ_N.

The following theorem is our third main result.

Theorem 1.3. *Let $\frac{1}{2} \le s < 1$. Suppose the constants λ_0, μ_0 satisfy $\mu_0 > 0$ and $\lambda_0 + \mu_0 \ge 0$ and let $\mathscr{B} \in C^1(\bar{\Omega})$. Let $0 \le \lambda^{(j)}, \mu^{(j)} \in C^1(\bar{\Omega})$, let $\mathscr{A}^{(j)}, \mathscr{C}^{(j)} \in C^1(\bar{\Omega})$ and let $W_j \subset \Omega_e$ be nonempty and open $(j = 1, 2)$. Let $\Lambda_N^{(j)}$ be the Dirichlet-to-Neumann map corresponding to $\lambda^{(j)}, \mu^{(j)}, \mathscr{A}^{(j)}, \mathscr{C}^{(j)}$. Suppose*

$$\Lambda_N^{(1)} g|_{W_2} = \Lambda_N^{(2)} g|_{W_2}$$

for small $g \in C_c^\infty(W_1)$. Then

$$\lambda^{(1)} = \lambda^{(2)}, \quad \mu^{(1)} = \mu^{(2)},$$
$$\mathscr{A}^{(1)} = \mathscr{A}^{(2)}, \quad \mathscr{C}^{(1)} = \mathscr{C}^{(2)}$$

in Ω.

Note that here we only claim that $\mathscr{A}, \mathscr{C}$ can be determined for a fixed $\mathscr{B}$. The question whether we can simultaneously determine $\mathscr{A}, \mathscr{B}, \mathscr{C}$ is still open.

The rest of this paper is organized in the following way. In Section 2, we summarize the background knowledge. In Section 3, we show the well-posedness of the linear exterior problem; We prove the unique continuation property and the Runge approximation property associated with our fractional operator; Then we prove Theorem 1.1 and Theorem 1.2. In Section 4, we show the well-posedness of the nonlinear exterior problem for small exterior data; We combine linearization arguments with the Runge approximation property to prove Theorem 1.3.

2. Preliminaries

Throughout this paper we use the following notations.

- We fix the space dimension $n = 3$ and the fractional power $0 < s < 1$.
- $x = (x_1, x_2, x_3)$ denotes the spatial variable.
- For vector-valued function u, u_i denotes the i-th component of u and $u_{i,j} := \partial_j u_i$.
- Ω denotes a bounded Lipschitz domain and $\Omega_e := \mathbb{R}^3 \setminus \overline{\Omega}$.
- δ_{ij} denotes the standard Kronecker delta.
- $\langle \cdot, \cdot \rangle$ denotes the distributional pairing so formally, $\langle f, g \rangle = \int fg$.

Throughout this paper we refer all function spaces to real-valued function spaces. For convenience, we use the same notation for the scalar-valued function space and the vector-valued one. For instance, $C_c^\infty(\Omega)$ can be either $C_c^\infty(\Omega; \mathbb{R})$ or $C_c^\infty(\Omega; \mathbb{R}^3)$.

2.1. Sobolev spaces

For $r \in \mathbb{R}$, we have the Sobolev space

$$H^r(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \colon \int_{\mathbb{R}^n} (1 + |\xi|^2)^r |\mathcal{F} f(\xi)|^2 d\xi < \infty \right\}$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{S}'(\mathbb{R}^n)$ is the space of temperate distributions.

We have the natural identification

$$H^{-r}(\mathbb{R}^n) = H^r(\mathbb{R}^n)^*.$$

Let U be an open set in $\mathbb{R}^n$. Let F be a closed set in $\mathbb{R}^n$. Then

$$H^r(U) := \{u|_U \colon u \in H^r(\mathbb{R}^n)\}, \quad H^r_F(\mathbb{R}^n) := \{u \in H^r(\mathbb{R}^n) \colon \operatorname{supp} u \subset F\},$$

$$\widetilde{H}^r(U) := \text{the closure of } C_c^\infty(U) \text{ in } H^r(\mathbb{R}^n).$$

Since Ω is a bounded Lipschitz domain, we also have the identifications

$$\widetilde{H}^r(\Omega) = H^r_{\overline{\Omega}}(\mathbb{R}^n), \quad H^{-r}(\Omega) = \widetilde{H}^r(\Omega)^*.$$

Let $0 < s < 1$. It is well known that we have the following continuous embedding

$$\widetilde{H}^s(\Omega) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega).$$

See, for instance, [20, Section 1.5]. It has also been proved that we have the continuous embedding

$$L^{\frac{n}{2s}}(\Omega) \hookrightarrow M(H^s \to H^{-s})$$

where $M(H^s \to H^{-s})$ is the space of pointwise multipliers from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{s,-s} = \sup\{|\langle f, \phi\psi\rangle| : \phi, \psi \in C_c^\infty(\Omega),\ \|\phi\|_{H^s(\mathbb{R}^n)} = \|\psi\|_{H^s(\mathbb{R}^n)} = 1\}.$$

See for instance, [6, Section 2] or [24, Section 2].

2.2. Fractional Laplacian

Let $0 < s < 1$. The fractional Laplacian $(-\Delta)^s$ is formally given by the pointwise definition

$$(-\Delta)^s u(x) := c_{n,s} \lim_{\varepsilon\to 0^+} \int\limits_{\mathbb{R}^n\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

as well as the equivalent Fourier transform definition

$$(-\Delta)^s u(x) := \mathcal{F}^{-1}(|\xi|^{2s}\mathcal{F}u(\xi))(x).$$

It is well known that one of the equivalent forms of the H^s-norm is given by

$$\|f\|_{H^s}^2 := \|f\|_{L^2}^2 + \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2s}}, \quad f \in H^s(\mathbb{R}^n)$$

and we have the following bilinear form formula

$$\langle(-\Delta)^s u, v\rangle = c'_{n,s} \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dxdy, \quad u, v \in H^s(\mathbb{R}^n).$$

It is also well known that one of the equivalent forms of the H^{1+s}-norm is given by

$$\|f\|_{H^{1+s}}^2 := \|f\|_{H^1}^2 + \sum_j \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|\partial_j f(x) - \partial_j f(y)|^2}{|x-y|^{n+2s}}, \quad f \in H^{1+s}(\mathbb{R}^n).$$

By the classical and fractional Poincaré inequalities, we have the following norm equivalence:

$$\begin{aligned} \|f\|_{H^{1+s}}^2 &\sim \sum_j (\|\partial_j f\|_{L^2}^2 + \langle(-\Delta)^s \partial_j f, \partial_j f\rangle) \\ &\sim \sum_j \langle(-\Delta)^s \partial_j f, \partial_j f\rangle), \quad f \in \widetilde{H}^{1+s}(\Omega). \end{aligned}$$

The following unique continuation property of $(-\Delta)^s$ was first proved in [12].

Proposition 2.1. *Suppose $u \in H^r(\mathbb{R}^n)$ for some $r \in \mathbb{R}$. Let $W \subset \mathbb{R}^n$ be open and non-empty. If*

$$(-\Delta)^s u = u = 0 \quad \text{in } W,$$

then $u = 0$ in $\mathbb{R}^n$.

3. Linear fractional elasticity

3.1. Well-posedness

We first study the equation

$$\mathcal{L}u = f \quad \text{in } \Omega. \tag{9}$$

The bilinear form associated with $-\mathcal{L}$ (see (1) in Section 1 for its definition) is

$$\begin{aligned} B[u,v] = &\Big\langle \sum_j (-\Delta)^s \lambda_0 u_{j,j}, \sum_j v_{j,j} \Big\rangle + \int_\Omega \lambda \Big(\sum_j u_{j,j}\Big)\Big(\sum_j v_{j,j}\Big) \\ &+ \frac{1}{2}\sum_{i,j}\Big[\langle (-\Delta)^s \mu_0 (u_{i,j}+u_{j,i}), v_{i,j}+v_{j,i} \rangle \\ &\qquad + \int_\Omega \mu (u_{i,j}+u_{j,i})(v_{i,j}+v_{j,i}) \Big]. \end{aligned} \tag{10}$$

Here λ_0, μ_0 are constants and we assume $\lambda, \mu \in L^\infty(\Omega)$.

We claim that B is bounded over $H^{1+s}(\mathbb{R}^3) \times H^{1+s}(\mathbb{R}^3)$. In fact, note that we have

$$\begin{aligned} &\Big| \Big\langle \sum_j (-\Delta)^s \lambda_0 u_{j,j}, \sum_j v_{j,j} \Big\rangle + \frac{1}{2}\sum_{i,j} \langle (-\Delta)^s \mu_0 (u_{i,j}+u_{j,i}), v_{i,j}+v_{j,i} \rangle \Big| \\ &\quad \le C \sum_{i,j} \|\partial_i u\|_{H^s} \|\partial_j v\|_{H^s} \end{aligned}$$

since the bilinear form $\langle (-\Delta)^s \phi, \psi \rangle$ is bounded over $H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)$. Also note that we have

$$\begin{aligned} &\Big| \int_\Omega \lambda \Big(\sum_j u_{j,j}\Big)\Big(\sum_j v_{j,j}\Big) + \frac{1}{2}\sum_{i,j} \int_\Omega \mu (u_{i,j}+u_{j,i})(v_{i,j}+v_{j,i}) \Big| \\ &\quad \le C' \sum_{i,j} \|\partial_i u\|_{L^2} \|\partial_j v\|_{L^2}. \end{aligned}$$

Hence, we have

$$|B[u,v]| \le C'' \|u\|_{H^{1+s}} \|v\|_{H^{1+s}}.$$

Now, we further assume $\lambda, \mu \ge 0$, $\mu_0 > 0$ and $\lambda_0 + \mu_0 \ge 0$.

We claim that B is coercive over $\widetilde{H}^{1+s}(\Omega) \times \widetilde{H}^{1+s}(\Omega)$ in this case. In fact, note that

$$\langle (-\Delta)^s u_{i,j}, u_{j,i} \rangle = \langle (-\Delta)^s u_{i,i}, u_{j,j} \rangle$$

for $u \in \widetilde{H}^{1+s}(\Omega)$ so we have

$$
\begin{aligned}
B[u,u] &\geq \Big\langle \sum_j (-\Delta)^s \lambda_0 u_{j,j}, \sum_j u_{j,j} \Big\rangle + \frac{1}{2} \sum_{i,j} \langle (-\Delta)^s \mu_0 (u_{i,j} + u_{j,i}), u_{i,j} + u_{j,i} \rangle \\
&= \Big\langle \sum_j (-\Delta)^s \lambda_0 u_{j,j}, \sum_j u_{j,j} \Big\rangle + \sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,j}, u_{i,j} \rangle \\
&\quad + \sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,j}, u_{j,i} \rangle \\
&= \Big\langle \sum_j (-\Delta)^s \lambda_0 u_{j,j}, \sum_j u_{j,j} \Big\rangle + \sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,j}, u_{i,j} \rangle \\
&\quad + \sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,i}, u_{j,j} \rangle \\
&= \Big\langle \sum_j (-\Delta)^s (\lambda_0 + \mu_0) u_{j,j}, \sum_j u_{j,j} \Big\rangle + \sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,j}, u_{i,j} \rangle \\
&\geq \sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,j}, u_{i,j} \rangle.
\end{aligned}
$$

The first inequality holds by the assumption $\lambda, \mu \geq 0$. The last inequality holds by the positivity of the fractional Laplacian and the assumption $\lambda_0 + \mu_0 \geq 0$. Note that for $u \in \widetilde{H}^{1+s}(\Omega)$, we have

$$
\sum_{i,j} \langle (-\Delta)^s \mu_0 u_{i,j}, u_{i,j} \rangle \sim \sum_i \|u_i\|^2_{H^{1+s}} \sim \|u\|^2_{H^{1+s}}
$$

by the assumption $\mu_0 > 0$ and the H^{1+s}-norm equivalence in Section 2.2. Hence, we conclude that B is coercive over $\widetilde{H}^{1+s}(\Omega) \times \widetilde{H}^{1+s}(\Omega)$.

Now, the Lax–Milgram theorem implies that the solution operator $f \to u_f$ associated with (9) is well defined, which is a homeomorphism from $H^{-1-s}(\Omega)$ to $\widetilde{H}^{s+1}(\Omega)$.

From now on, we will always assume $0 \leq \lambda, \mu \in L^\infty(\Omega)$, $\mu_0 > 0$ and $\lambda_0 + \mu_0 \geq 0$.

Let $f := -(-\Delta)^s L_{\lambda_0,\mu_0} g|_\Omega$ in (9). Then we have the well-posedness of the exterior problem (2).

Proposition 3.1. *For each $g \in H^{1+s}(\mathbb{R}^3)$, there exists a unique solution*

$$
u_g \in H^{1+s}(\mathbb{R}^3)
$$

of (2) such that $u_g - g \in \widetilde{H}^{1+s}(\Omega)$. Moreover, the solution operator $g \to u_g$ is bounded on $H^{1+s}(\mathbb{R}^3)$.

3.2. Dirichlet-to-Neumann map and integral identity

Let $X := H^{1+s}(\mathbb{R}^3)/\widetilde{H}^{1+s}(\Omega) = H^{1+s}(\Omega_e)$ and $\tilde{g} :=$ the natural image of $g \in H^{1+s}(\mathbb{R}^3)$ in X.

We define the Dirichlet-to-Neumann map Λ by

$$\langle \Lambda\tilde{g}, \tilde{h}\rangle := -B[u_g, h], \quad g, h \in H^{1+s}(\mathbb{R}^3)$$

where u_g is the solution corresponding to the exterior data g in (2).

Note that if $g_2 - g_1 \in \widetilde{H}^{1+s}(\Omega)$ and $h_2 - h_1 \in \widetilde{H}^{1+s}(\Omega)$, then $u_{g_1} = u_{g_2}$ and

$$B[u_{g_2}, h_2] - B[u_{g_1}, h_1] = B[u_{g_2} - u_{g_1}, h_2] + B[u_{g_1}, h_2 - h_1] = 0$$

so Λ is well defined. For convenience, we will write Λg and $\langle \Lambda g, h\rangle$ instead of $\Lambda\tilde{g}$ and $\langle \Lambda\tilde{g}, \tilde{h}\rangle$.

Also note that for $g, h \in C_c^\infty(\Omega_e)$, by (10) we have

$$\begin{aligned}
\langle \Lambda g, h\rangle &= -\Big\langle \sum_j (-\Delta)^s \lambda_0 (u_g)_{j,j}, \sum_j h_{j,j}\Big\rangle \\
&\quad - \frac{1}{2}\sum_{i,j} \langle (-\Delta)^s \mu_0[(u_g)_{i,j} + (u_g)_{j,i}], h_{i,j} + h_{j,i}\rangle \\
&= -\sum_{i,j} \langle (-\Delta)^s \lambda_0 (u_g)_{j,j}, h_{i,i}\rangle - \sum_{i,j}\langle (-\Delta)^s \mu_0[(u_g)_{i,j} + (u_g)_{j,i}], h_{i,j}\rangle \\
&= \sum_{i,j} \langle \partial_i (-\Delta)^s \lambda_0 (u_g)_{j,j}, h_i\rangle + \sum_{i,j}\langle \partial_j (-\Delta)^s \mu_0[(u_g)_{i,j} + (u_g)_{j,i}], h_i\rangle \\
&= \langle (-\Delta)^s L_{\lambda_0,\mu_0} u_g, h\rangle.
\end{aligned}$$

Hence, the bilinear form definition given here coincides with the one given by (3) for $g \in C_c^\infty(\Omega_e)$.

The symmetry of Λ immediately follows from the symmetry of B. In fact,

$$\langle \Lambda g, h\rangle = -B[u_g, u_h] = -B[u_h, u_g] = \langle \Lambda h, g\rangle$$

so we have the integral identity

$$\begin{aligned}
&\langle \Lambda^{(1)} g^{(1)}, g^{(2)}\rangle - \langle \Lambda^{(2)} g^{(1)}, g^{(2)}\rangle \\
&\quad = -B^{(1)}[u^{(1)}, u^{(2)}] + B^{(2)}[u^{(2)}, u^{(1)}] \\
&\quad = \int_\Omega (\lambda^{(2)} - \lambda^{(1)})\Big(\sum_j u^{(1)}_{j,j}\Big)\Big(\sum_j u^{(2)}_{j,j}\Big) \\
&\qquad + \frac{1}{2}\int_\Omega (\mu^{(2)} - \mu^{(1)}) \sum_{k,j} (u^{(1)}_{k,j} + u^{(1)}_{j,k})(u^{(2)}_{k,j} + u^{(2)}_{j,k}),
\end{aligned} \tag{11}$$

where $\mathcal{L}^{(j)}$,$B^{(j)}$,$\lambda^{(j)}$ correspond to Lamé parameters $\lambda^{(j)}$, $\mu^{(j)}$; $u^{(j)}$ denotes the solution of

$$\mathcal{L}^{(j)}u = 0 \text{ in } \Omega, \quad u = g^{(j)} \text{ in } \Omega_e.$$

We remark that this integral identity has the same form as its classical counterpart (see [8]).

3.3. Unique continuation property and Runge approximation property

Recall that a classical operator L possesses the unique continuation property in a domain U if

$$\mathcal{L}u = 0 \quad \text{in } U, \quad u = 0 \quad \text{in } V$$

where V is a nonempty open subset of U imply that $u = 0$ in U.

It is well known that the classical constant coefficients Lamé operator L_{λ_0,μ_0} possesses the unique continuation property in this sense. (This property even holds true for the general variable coefficients Lamé operator $L_{\lambda,\mu}$. See for instance, the main theorem in [1].)

The following proposition is the unique continuation property of $(-\Delta)^s L_{\lambda_0,\mu_0}$.

Proposition 3.2. *Let $u \in H^{1+s}(\mathbb{R}^3)$. Let W be open. If*

$$(-\Delta)^s L_{\lambda_0,\mu_0} u = u = 0 \quad \textit{in } W,$$

then $u = 0$ in $\mathbb{R}^3$.

Proof. By the unique continuation property of $(-\Delta)^s$ (Proposition 2.1), we have $L_{\lambda_0,\mu_0} u = 0$ in $\mathbb{R}^3$. Then by the unique continuation property of L_{λ_0,μ_0} we have $u = 0$ in $\mathbb{R}^3$. ■

Based on the unique continuation property above, we can prove the following Runge approximation property.

Proposition 3.3. *Let $W \subset \Omega_e$ be nonempty and open. Then*

$$S := \{u_g|_\Omega : g \in C_c^\infty(W)\}$$

is dense in $\widetilde{H}^{1+s}(\Omega)$ where u_g is the solution corresponding to the exterior data g in (2).

Proof. By the Hahn–Banach Theorem, it suffices to show that if $f \in H^{-1-s}(\Omega)$ and $\langle f, w\rangle = 0$ for all $w \in S$, then $f = 0$.

In fact, we can choose $v \in \widetilde{H}^{1+s}(\Omega)$ to be the solution of $\mathcal{L}v = f$ in Ω. For any $g \in C_c^\infty(W)$, note that $u_g - g \in \widetilde{H}^{1+s}(\Omega)$ so by the assumption $\langle f, w\rangle = 0$ for $w \in S$ we have

$$0 = \langle f, u_g - g\rangle = -B[v, u_g - g].$$

Since u_g is the solution of the exterior problem, we have $B[u_g, v] = 0$ so

$$0 = B[v, g] = -\langle (-\Delta)^s L_{\lambda_0,\mu_0} v, g\rangle.$$

This implies $(-\Delta)^s L_{\lambda_0,\mu_0} v = 0$ in W. By the unique continuation property above we have $v = 0$ in $\mathbb{R}^3$. Hence, $f = 0$. ■

3.4. Proof of Theorem 1.1

Now, we are ready to prove Theorem 1.1. The key point is to approximate certain carefully chosen functions by solutions of the linear exterior problems based on the Runge approximation property. This will enable us to exploit the integral identity (11) to determine λ, μ.

Proof. Let $u^{(j)}$ denote the solution of

$$\mathcal{L}^{(j)} u = 0 \text{ in } \Omega, \quad u = g^{(j)} \text{ in } \Omega_e.$$

For any given $\varepsilon > 0$ and $f^{(j)} \in C_c^\infty(\Omega)$, by the Runge approximation property (Proposition 3.3) we can choose $g^{(1)} \in C_c^\infty(W_1)$ such that

$$\|u^{(1)} - f^{(1)}\|_{\widetilde{H}^{1+s}(\Omega)} \leq \varepsilon$$

and for this chosen $g^{(1)}$, again by the Runge approximation property we can choose $g^{(2)} \in C_c^\infty(W_2)$ such that

$$\|u^{(1)}\|_{\widetilde{H}^{1+s}(\Omega)} \|u^{(2)} - f^{(2)}\|_{\widetilde{H}^{1+s}(\Omega)} \leq \varepsilon.$$

(Actually we only need the H^1-norm approximation.) By the assumption

$$\Lambda^{(1)} g|_{W_2} = \Lambda^{(2)} g|_{W_2}$$

for $g \in C_c^\infty(W_1)$ and the integral identity (11) we get

$$\int_\Omega (\lambda^{(2)} - \lambda^{(1)}) \Big(\sum_j u_{j,j}^{(1)}\Big)\Big(\sum_j u_{j,j}^{(2)}\Big) + \frac{1}{2}\int_\Omega (\mu^{(2)} - \mu^{(1)}) \sum_{k,j} (u_{k,j}^{(1)} + u_{j,k}^{(1)})(u_{k,j}^{(2)} + u_{j,k}^{(2)}) = 0.$$

Based on our choice for $g^{(j)}$, we get

$$\Big|\int_\Omega (\lambda^{(2)}-\lambda^{(1)})\Big(\sum_j f^{(1)}_{j,j}\Big)\Big(\sum_j f^{(2)}_{j,j}\Big) + \frac{1}{2}\int_\Omega (\mu^{(2)}-\mu^{(1)}) \sum_{k,j}(f^{(1)}_{k,j}+f^{(1)}_{j,k})(f^{(2)}_{k,j}+f^{(2)}_{j,k})\Big| \le C\varepsilon,$$

where C is a constant depending on $\lambda^{(j)}$, $\mu^{(j)}$ and $f^{(j)}$. Hence, we conclude that

$$\int_\Omega (\lambda^{(2)}-\lambda^{(1)})\Big(\sum_j f^{(1)}_{j,j}\Big)\Big(\sum_j f^{(2)}_{j,j}\Big) + \frac{1}{2}\int_\Omega (\mu^{(2)}-\mu^{(1)}) \sum_{k,j}(f^{(1)}_{k,j}+f^{(1)}_{j,k})(f^{(2)}_{k,j}+f^{(2)}_{j,k}) = 0 \tag{12}$$

since ε is arbitrary.

We will appropriately choose $f^{(j)}$ in (12) to determine λ, μ.

In fact, for any given $\psi \in C_c^\infty(\Omega)$, we can choose ϕ such that $\phi \in C_c^\infty(\Omega)$ and $\phi = x_1$ on $\operatorname{supp}\psi$.

We can show that

$$\int_\Omega (\mu^{(2)}-\mu^{(1)})\partial_j\psi = 0 \quad (1 \le j \le 3).$$

For instance, to obtain the equality above for $j = 2$, we can choose

$$f^{(1)} = (0,\phi,0), \quad f^{(2)} = (\psi,0,0)$$

in (12). We can also show that

$$\int_\Omega (\lambda^{(2)}-\lambda^{(1)})\partial_j\psi = 0 \quad (1 \le j \le 3).$$

For instance, to obtain the equality above for $j = 2$, we can choose

$$f^{(1)} = (\phi,0,0), \quad f^{(2)} = (0,\psi,0)$$

in (12). Hence, we get

$$\nabla(\mu^{(2)}-\mu^{(1)}) = \nabla(\lambda^{(2)}-\lambda^{(1)}) = 0.$$

Now, we show that the constants $c_\mu := \mu^{(2)}-\mu^{(1)}$ and $c_\lambda := \lambda^{(2)}-\lambda^{(1)}$ are zeros.

In fact, we can choose $\psi \in C_c^\infty(\Omega)$ such that $\|\partial_1\psi\|_{L^2} \neq \|\partial_2\psi\|_{L^2}$. Then we can show that

$$(c_\lambda + 2c_\mu)\|\partial_1\psi\|_{L^2}^2 + c_\mu\|\partial_2\psi\|_{L^2}^2 + c_\mu\|\partial_3\psi\|_{L^2}^2 = 0,$$

$$(c_\lambda + 2c_\mu)\|\partial_2\psi\|_{L^2}^2 + c_\mu\|\partial_3\psi\|_{L^2}^2 + c_\mu\|\partial_1\psi\|_{L^2}^2 = 0,$$

$$(c_\lambda + 2c_\mu)\|\partial_3\psi\|_{L^2}^2 + c_\mu\|\partial_1\psi\|_{L^2}^2 + c_\mu\|\partial_2\psi\|_{L^2}^2 = 0.$$

In fact, in order to obtain the first identity, we can choose

$$f^{(1)} = f^{(2)} = (\psi, 0, 0).$$

Then we have

$$\sum_j f_{j,j}^{(1)} = \sum_j f_{j,j}^{(2)} = \partial_1\psi$$

and the five non-vanishing terms in the sum

$$\sum_{k,j}(f_{k,j}^{(1)} + f_{j,k}^{(1)})(f_{k,j}^{(2)} + f_{j,k}^{(2)})$$

are $4|\partial_1\psi|^2$ when $(k, j) = (1, 1)$, $|\partial_2\psi|^2$ when $(k, j) = (1, 2)$ or $(2, 1)$ and $|\partial_3\psi|^2$ when $(k, j) = (1, 3)$ or $(3, 1)$. Now, it is clear that (12) becomes the first identity in this case. Similarly we can choose

$$f^{(1)} = f^{(2)} = (0, \psi, 0) \quad (\text{resp. } f^{(1)} = f^{(2)} = (0, 0, \psi))$$

to obtain the second (resp. third) identity.

On one hand, we sum up the three identities to obtain

$$(c_\lambda + 4c_\mu)(\|\partial_1\psi\|_{L^2}^2 + \|\partial_2\psi\|_{L^2}^2 + \|\partial_3\psi\|_{L^2}^2) = 0.$$

Since at least one of $\partial_1\psi, \partial_2\psi$ is nonzero, we conclude that $c_\lambda + 4c_\mu = 0$.

On the other hand, we combine the first two identities to obtain

$$(c_\lambda + c_\mu)\|\partial_1\psi\|_{L^2}^2 = (c_\lambda + c_\mu)\|\partial_2\psi\|_{L^2}^2,$$

which implies $c_\lambda + c_\mu = 0$. Hence, the only possibility is $c_\lambda = c_\mu = 0$. ∎

Remark. Similar strategy has been applied to solve inverse problems for fractional Schrödinger operators with local perturbations. See [4, 6] for details.

3.5. Proof of Theorem 1.2

We can identically apply the considerations for the exterior problem (2) in previous subsections to the obstacle problem (4). (We just replace Ω by $\Omega \setminus \overline{D}$ in earlier arguments.) Our main task in this subsection is to prove the first part of Theorem 1.2. We will see that this part is an immediate consequence of the unique continuation property (Proposition 3.2). Once we have determined the obstacle D, we can use the same arguments as in the proof of Theorem 1.1 to determine λ, μ.

Proof. For the fixed nonzero $g \in C_c^\infty(W_1)$, let $u^{(j)}$ denote the solution of the obstacle problem

$$\mathcal{L}^{(j)}u = 0 \text{ in } \Omega \setminus \overline{D_j}, \quad u = 0 \text{ in } D_j, \quad u = g \text{ in } \Omega_e.$$

Since we have the assumption

$$\Lambda^{(1)}_{D_1} g|_{W_2} = \Lambda^{(2)}_{D_2} g|_{W_2}$$

and $u^{(1)} = u^{(2)} = g$ in Ω_e, we get

$$(-\Delta)^s L_{\lambda_0,\mu_0}(u^{(1)} - u^{(2)}) = u^{(1)} - u^{(2)} = 0 \quad \text{in } W_2.$$

Then Proposition 3.2 implies that $u^{(1)} = u^{(2)}$ in $\mathbb{R}^3$.

Suppose $D_1 \neq D_2$. Without loss of generality we can assume $V := D_2 \setminus \overline{D}_1$ is nonempty. Note that $u^{(1)} = u^{(2)} = 0$ in V. Then the equation for $u^{(1)}$ in V implies $(-\Delta)^s L_{\lambda_0,\mu_0} u^{(1)} = 0$ in V. But now Proposition 3.2 implies that $u^{(1)} = 0$ in $\mathbb{R}^3$, which contradicts that g is nonzero. ∎

Remark. Similar inverse obstacle problems have been studied for fractional elliptic operators. See [3] for details.

4. Nonlinear fractional elasticity

4.1. Well-posedness and Dirichlet-to-Neumann map

We first study the nonlinear equation

$$\mathcal{L}u + \mathcal{N}u = f \quad \text{in } \Omega \tag{13}$$

See (7) in Section 1 for the definition of $\mathcal{N}$.

From now on, we will always assume $\mathcal{A}, \mathcal{B}, \mathcal{C} \in C^1(\overline{\Omega})$ in (7). Let $\frac{1}{2} \leq s < 1$. Then $\frac{2n}{n-2s} \geq \frac{n}{2s}$ for $n = 3$ so we have the continuous embeddings (see Section 2.1)

$$\widetilde{H}^s(\Omega) \hookrightarrow L^{\frac{2n}{n-2s}}(\Omega) \hookrightarrow L^{\frac{n}{2s}}(\Omega) \hookrightarrow M(H^s \to H^{-s}).$$

Note that for $u \in \widetilde{H}^{s+1}(\Omega)$, each component of Nu is a sum of terms which have the form $\partial_j(a\phi\psi)$ where $a \in C^1(\bar{\Omega})$ and $\phi, \psi \in \widetilde{H}^s(\Omega)$. The embedding $\widetilde{H}^s(\Omega) \hookrightarrow M(H^s \to H^{-s})$ ensures that $\phi\psi \in H^{-s}(\mathbb{R}^3)$ so $Nu \in H^{-1-s}(\mathbb{R}^3)$. Hence, the map F defined by

$$F(f, u) := (\mathcal{L}u + \mathcal{N}u)|_\Omega - f$$

maps from $H^{-1-s}(\Omega) \times \widetilde{H}^{s+1}(\Omega)$ to $H^{-1-s}(\Omega)$.

Note that $F(0, 0) = 0$ and it is easy to verify that the Fréchet derivative

$$D_u F|_{(0,0)}(v) = \mathcal{L}v|_\Omega,$$

which is a homeomorphism from $\widetilde{H}^{s+1}(\Omega)$ to $H^{-1-s}(\Omega)$. By Implicit function theorem (see, for instance, [23, Theorem 10.6]), there exists $\delta > 0$ such that whenever $\|f\|_{H^{-1-s}(\Omega)} \le \delta$, we have both existence and uniqueness of small solutions of (13), and u_f smoothly depends on f.

Let $f := -(-\Delta)^s L_{\lambda_0,\mu_0} g|_\Omega$ in (13). Then we have the well-posedness of (6).

Proposition 4.1. *For each sufficiently small $g \in C_c^\infty(\Omega_e)$, there exists a unique small solution u_g of the exterior problem (6) such that $u_g - g \in \widetilde{H}^{1+s}(\Omega)$ and u_g smoothly depends on g.*

Now, we can conclude that the associated Dirichlet-to-Neumann map Λ_N given by (8) is well defined at least for small $g \in C_c^\infty(\Omega_e)$.

4.2. Proof of Theorem 1.3

We are ready to prove Theorem 1.3. We will first apply the first order linearization and the linear result (Theorem 1.1) to determine λ, μ. Then we will apply the second order linearization and the Runge approximation property (Proposition 3.3) to determine $\mathcal{A}, \mathcal{C}$.

Proof. i. Determine λ, μ. Let $u_{\varepsilon,g}^{(j)}$ be the solution of the exterior problem

$$\mathcal{L}^{(j)}u + \mathcal{N}^{(j)}u = 0 \text{ in } \Omega, \quad u = \varepsilon g \text{ in } \Omega_e \tag{14}$$

for $g \in C_c^\infty(W_1)$ and small ε. Applying $\frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0}$ to (14), we obtain that

$$u_g^{(j)} := \frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0} u_{\varepsilon,g}^{(j)}$$

is the solution of

$$\mathcal{L}^{(j)}u = 0 \text{ in } \Omega, \quad u = g \text{ in } \Omega_e.$$

Since we have the assumption

$$\Lambda_N^{(1)} g = \Lambda_N^{(2)} g \quad \text{in } W_2,$$

i.e.,

$$(-\Delta)^s L_{\lambda_0,\mu_0} u^{(1)}_{\varepsilon,g} = (-\Delta)^s L_{\lambda_0,\mu_0} u^{(2)}_{\varepsilon,g} \quad \text{in } W_2,$$

we can apply $\frac{\partial}{\partial \varepsilon}\big|_{\varepsilon=0}$ to the identity to obtain that

$$(-\Delta)^s L_{\lambda_0,\mu_0} u^{(1)}_{g} = (-\Delta)^s L_{\lambda_0,\mu_0} u^{(2)}_{g} \quad \text{in } W_2,$$

i.e.,

$$\Lambda^{(1)} g = \Lambda^{(2)} g \quad \text{in } W_2.$$

Hence, we conclude that

$$\lambda^{(1)} = \lambda^{(2)} := \lambda,$$
$$\mu^{(1)} = \mu^{(2)} := \mu$$

based on Theorem 1.1.

Now, we use $\mathcal{L}$ to denote both $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$.

ii. Determine $\mathcal{A}, \mathcal{C}$. Let $u^{(j)}_{\varepsilon,g}$ be the solution of the exterior problem

$$\mathcal{L}u + \mathcal{N}^{(j)} u = 0 \text{ in } \Omega, \quad u = \varepsilon_1 g^{(1)} + \varepsilon_2 g^{(2)} \text{ in } \Omega_e \tag{15}$$

for $g^{(j)} \in C^\infty_c(W_1)$ and small ε_j. First note that

$$\frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon_j=0} u^{(1)}_{\varepsilon,g} = \frac{\partial}{\partial \varepsilon_j}\Big|_{\varepsilon_j=0} u^{(2)}_{\varepsilon,g}$$

since both of them are the solution of

$$\mathcal{L}u = 0 \text{ in}\Omega, \quad u = g^{(j)} \text{ in } \Omega_e.$$

Hence, we can denote both of them by $v^{(j)}$. Next we apply $\frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\big|_{\varepsilon_1=\varepsilon_2=0}$ to (15). Then

$$w^{(j)} := \frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}\Big|_{\varepsilon_1=\varepsilon_2=0} u^{(j)}_{\varepsilon,g}$$

satisfies

$$\mathcal{L}w^{(j)} + \widetilde{N}^{(j)}(v^{(1)}, v^{(2)}) = 0 \text{ in } \Omega, \quad w^{(j)} = 0 \text{ in } \Omega_e. \tag{16}$$

Here (based on (7)) we can compute that

$$\widetilde{N}^{(l)}_i = \sum_{j=1}^{3} \partial_j \widetilde{N}^{(l)}_{ij} \quad (1 \le l \le 2;\ 1 \le i \le 3)$$

with

$$
\begin{aligned}
\widetilde{N}_{ij}^{(l)}(v^{(1)}, v^{(2)}) = {} & (\lambda + \mathcal{B}) \sum_{m,n} v_{m,n}^{(1)} v_{m,n}^{(2)} \delta_{ij} + 2\mathcal{C}^{(l)} \Big(\sum_{m} v_{m,m}^{(1)}\Big)\Big(\sum_{m} v_{m,m}^{(2)}\Big)\delta_{ij} \\
& + \mathcal{B} \sum_{m,n} v_{m,n}^{(1)} v_{m,n}^{(2)} \delta_{ij} + \mathcal{B} \sum_{m} (v_{m,m}^{(1)} v_{j,i}^{(2)} + v_{m,m}^{(2)} v_{j,i}^{(1)}) \\
& + \frac{\mathcal{A}^{(l)}}{4} \sum_{m} (v_{j,m}^{(1)} v_{m,i}^{(2)} + v_{j,m}^{(2)} v_{m,i}^{(1)}) \\
& + (\lambda + \mathcal{B}) \sum_{m} (v_{m,m}^{(1)} v_{i,j}^{(2)} + v_{m,m}^{(2)} v_{i,j}^{(1)}) \\
& + \Big(\mu + \frac{\mathcal{A}^{(l)}}{4}\Big) \sum_{m} (v_{m,i}^{(1)} v_{m,j}^{(2)} + v_{m,i}^{(2)} v_{m,j}^{(1)} \\
& \qquad + v_{i,m}^{(1)} v_{j,m}^{(2)} + v_{i,m}^{(2)} v_{j,m}^{(1)} \\
& \qquad + v_{i,m}^{(1)} v_{m,j}^{(2)} + v_{i,m}^{(2)} v_{m,j}^{(1)}).
\end{aligned}
$$

We also apply $\frac{\partial^2}{\partial \varepsilon_1 \partial \varepsilon_2}|_{\varepsilon_1=\varepsilon_2=0}$ to the Dirichlet-to-Neumann map assumption

$$
(-\Delta)^s L_{\lambda_0,\mu_0} u_{\varepsilon,g}^{(1)} = (-\Delta)^s L_{\lambda_0,\mu_0} u_{\varepsilon,g}^{(2)} \quad \text{in } W_2.
$$

Then we get

$$
(-\Delta)^s L_{\lambda_0,\mu_0} w^{(1)} = (-\Delta)^s L_{\lambda_0,\mu_0} w^{(2)} \quad \text{in } W_2.
$$

Since $w^{(1)} = w^{(2)} = 0$ in Ω_e, the unique continuation property (Proposition 3.2) implies that $w^{(1)} = w^{(2)}$ in $\mathbb{R}^3$.

Now, we combine the two equations ($j = 1, 2$) in (16) to obtain

$$
\sum_{j=1}^{3} \partial_j \widetilde{G}_{ij}(v^{(1)}, v^{(2)}) = 0 \quad (1 \le i \le 3) \tag{17}
$$

where

$$
\begin{aligned}
\widetilde{G}_{ij}(v^{(1)}, v^{(2)}) &= \widetilde{N}_{ij}^{(2)}(v^{(1)}, v^{(2)}) - \widetilde{N}_{ij}^{(1)}(v^{(1)}, v^{(2)}) \\
&= \frac{\mathcal{A}^{(2)} - \mathcal{A}^{(1)}}{4} \Big(\sum_{m} (v_{j,m}^{(1)} + v_{m,j}^{(1)})(v_{i,m}^{(2)} + v_{m,i}^{(2)}) \\
&\qquad\qquad + \sum_{m} (v_{j,m}^{(2)} + v_{m,j}^{(2)})(v_{i,m}^{(1)} + v_{m,i}^{(1)}) \Big) \\
&\quad + 2(\mathcal{C}^{(2)} - \mathcal{C}^{(1)}) \Big(\sum_{m} v_{m,m}^{(1)}\Big)\Big(\sum_{m} v_{m,m}^{(2)}\Big)\delta_{ij}.
\end{aligned}
$$

Let both sides of (17) act on $\psi \in C_c^\infty(\Omega)$. Then Proposition 3.3 implies that

$$\sum_{j=1}^{3} \int \widetilde{G}_{ij}(f^{(1)}, f^{(2)})\partial_j \psi = 0 \quad (1 \le i \le 3) \tag{18}$$

for any $f^{(j)} \in C_c^\infty(\Omega)$. We can appropriately choose $f^{(j)}$ to show that

$$\int (\mathscr{A}^{(2)} - \mathscr{A}^{(1)})\partial_j \psi = 0, \quad (1 \le j \le 3).$$

For instance, if we choose $i = 1$, $f^{(1)} = (0, \phi, 0)$ and $f^{(2)} = (\phi, 0, 0)$ where $\phi \in C_c^\infty(\Omega)$ satisfies $\phi = x_1$ on $\operatorname{supp}\psi$ for a chosen $\psi \in C_c^\infty(\Omega)$, a direct computation shows that

$$\widetilde{G}_{11} = 0, \quad \widetilde{G}_{12} = \frac{\mathscr{A}^{(2)} - \mathscr{A}^{(1)}}{2}, \quad \widetilde{G}_{13} = 0,$$

which verifies the equality for $j = 2$. Hence, we conclude that $\mathscr{A}^{(2)} - \mathscr{A}^{(1)} := c_{\mathscr{A}}$ is a constant.

We can also appropriately choose $f^{(j)}$ to show that

$$\int (\mathscr{C}^{(2)} - \mathscr{C}^{(1)})\partial_j \psi = 0 \quad (1 \le j \le 3).$$

For instance, if we choose $i = 1$, $f^{(1)} = (0, \varphi, 0)$ and $f^{(2)} = (\phi, 0, 0)$ where ϕ is defined as before and $\varphi \in C_c^\infty(\Omega)$ satisfies $\varphi = x_2$ on $\operatorname{supp}\psi$ for a chosen $\psi \in C_c^\infty(\Omega)$, a direct computation shows that

$$\widetilde{G}_{11} = 2(\mathscr{C}^{(2)} - \mathscr{C}^{(1)}), \quad \widetilde{G}_{12} = 0, \quad \widetilde{G}_{13} = 0,$$

which verifies the equality for $j = 1$. Hence, we conclude that $\mathscr{C}^{(2)} - \mathscr{C}^{(1)} := c_{\mathscr{C}}$ is a constant.

Now, we show that $c_{\mathscr{A}}$ and $c_{\mathscr{C}}$ are zeros. In fact, we can choose $\psi \in C_c^\infty(\Omega)$ such that $\|\partial_1 \psi\|_{L^2} \ne \|\partial_2 \psi\|_{L^2}$. Then we can appropriately choose $f^{(j)}$ to obtain

$$\begin{aligned}
(4c_{\mathscr{A}} + 4c_{\mathscr{C}})\|\partial_1\psi\|_{L^2}^2 + c_{\mathscr{A}}\|\partial_2\psi\|_{L^2}^2 + c_{\mathscr{A}}\|\partial_3\psi\|_{L^2}^2 = 0,\\
(4c_{\mathscr{A}} + 4c_{\mathscr{C}})\|\partial_2\psi\|_{L^2}^2 + c_{\mathscr{A}}\|\partial_3\psi\|_{L^2}^2 + c_{\mathscr{A}}\|\partial_1\psi\|_{L^2}^2 = 0,\\
(4c_{\mathscr{A}} + 4c_{\mathscr{C}})\|\partial_3\psi\|_{L^2}^2 + c_{\mathscr{A}}\|\partial_1\psi\|_{L^2}^2 + c_{\mathscr{A}}\|\partial_2\psi\|_{L^2}^2 = 0.
\end{aligned}$$

For instance, if we choose $f^{(1)} = (\psi, 0, 0)$ and $f^{(2)} = (\phi, 0, 0)$ where ϕ is defined as before, a direct computation shows that

$$\widetilde{G}_{11} = 2c_{\mathscr{A}}\partial_1\psi + 2c_{\mathscr{C}}\partial_1\psi, \quad \widetilde{G}_{12} = \frac{c_{\mathscr{A}}}{2}\partial_2\psi, \quad \widetilde{G}_{13} = \frac{c_{\mathscr{A}}}{2}\partial_3\psi.$$

Let $i = 1$ in (18). Then we have

$$\int (2c_{\mathcal{A}}\partial_1\psi + 2c_{\mathcal{C}}\partial_1\psi)\partial_1\psi + \int \Big(\frac{c_{\mathcal{A}}}{2}\partial_2\psi\Big)\partial_2\psi + \int \Big(\frac{c_{\mathcal{A}}}{2}\partial_3\psi\Big)\partial_3\psi = 0,$$

which coincides with the first identity.

On one hand, we sum up the three identities to obtain

$$(6c_{\mathcal{A}} + 4c_{\mathcal{C}})(\|\partial_1\psi\|_{L^2}^2 + \|\partial_2\psi\|_{L^2}^2 + \|\partial_3\psi\|_{L^2}^2) = 0.$$

Since at least one of $\partial_1\psi, \partial_2\psi$ is nonzero, we conclude that $6c_{\mathcal{A}} + 4c_{\mathcal{C}} = 0$.

On the other hand, we combine the first two identities to obtain

$$(3c_{\mathcal{A}} + 4c_{\mathcal{C}})\|\partial_1\psi\|_{L^2}^2 = (3c_{\mathcal{A}} + 4c_{\mathcal{C}})\|\partial_2\psi\|_{L^2}^2,$$

which implies $3c_{\mathcal{A}} + 4c_{\mathcal{C}} = 0$. Hence, the only possibility is $c_{\mathcal{A}} = c_{\mathcal{C}} = 0$. ∎

Remark. The multiple-fold linearization procedure performed in the proof has been widely applied in solving inverse problems. For instance, see [9, 13] for this approach for inverse problems for semilinear elliptic operators. Also see [26] for this approach for an inverse problem for a nonlinear elastic wave operator.

Acknowledgement. The author would like to thank Professor Gunther Uhlmann for suggesting the problem and for helpful discussions.

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Received 27 January 2022; revised 23 February 2022.

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