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Generalised norm resolvent convergence: comparison of different concepts

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Abstract. In this paper, we show that the two concepts of generalised norm resolvent convergence introduced by Weidmann and the first named author of this paper are equivalent. We also focus on the convergence speed and provide conditions under which the convergence speed is the same for both concepts. We illustrate the abstract results by a large number of examples.

1. Main results

1.1. Introduction

Convergence of operators is an important topic in many areas of mathematics and applications. For unbounded operators such as Laplacians in Hilbert spaces, one usually studies convergence of their *resolvents*. Often, not only the operators vary with the sequence parameter, but also the spaces in which they are defined, e.g., when one considers Laplace operators on varying domains. We refer to this fact as *generalised* convergence (in contrast to Kato [9, Section IV.2] where *generalised* convergence just means convergence of the resolvents).

Different types of convergence of operators are of interest: convergence in *operator norm* (uniform convergence), *strong* (i.e., pointwise) convergence, and other types. We focus here on operator *norm* convergence of resolvents of (possibly) unbounded self-adjoint operators acting in Hilbert spaces; for results on strong convergence we refer to Section 1.5. Generalisations to non-self-adjoint operators or operators acting in Banach spaces are possible, see Section 1.6 below.

Probably the first abstract approach of convergence of operators acting on varying Banach spaces – a priori not embedded in a common space – is given in [22] (see also the references therein and [23] for a summary, we refer also to [4] for a nice overview). Stummel defines what he calls "discrete convergence of operators" which is a generalisation of *strong* (pointwise) convergence of the resolvents to the

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case where the underlying spaces vary. Stummel's setting includes not only discretisations of partial differential operators (see e.g. [25], also for some older abstract results before [22]), but also the case when the domains of the partial differential operators vary, see Section 1.5 for a more detailed discussion. For an overview on results on domain perturbation up to 2008 we refer to the nice survey [8] and the references therein; for more recent results see [1, 3] and references therein.

Weidmann defines *generalised* norm or strong resolvent convergence implicitly in [27], and explicitly in [26, Section 9.3], having domain perturbations of Laplacians in mind. Weidmann basically embeds all spaces into a large one (called *parent space* here) and considers convergence of the "lifted" resolvents in the parent space. The lifted resolvents are *pseudo-resolvents*, cf. Remark 1.3 (2) below. We introduce Weidmann's concept in Section 1.2. Independently, the first author of this paper defined generalised norm resolvent convergence in [17], based on the concept of *quasi-unitary equivalence* (*QUE* for short), a quantitative generalisation of unitary equivalence, first introduced in [15]. We present the latter *generalised norm resolvent convergence* briefly in Section 1.3.

The main result of this paper (Theorem 1.7) is the equivalence of generalised norm resolvent convergence as introduced by Weidmann and the first author. Let us mention that norm resolvent convergence of operators A_n towards A_∞ as well as both generalisations discussed in this paper imply the convergence of bounded operators of the original operators such as spectral projections and the heat operator in operator norm (in a suitable sense), cf. [26, Satz 9.28] and [17, Section 4.2] and [18, Section 1.3] for precise statements.

An important feature of *norm* resolvent convergence for self-adjoint operators is the *convergence of spectra* in the sense that

$$\lambda_\infty \in \operatorname{spec}(A_\infty) \iff \begin{array}{l} \text{for all } n \in \mathbb{N} \text{ there exists } \lambda_n \in \operatorname{spec}(A_n) \\ \text{such that } \lambda_n \to \lambda_\infty \end{array} \tag{1.1}$$

(cf. [26, Satz 9.24 (a)]). For the weaker notion of *strong* (pointwise) resolvent convergence, this is no longer automatically true, i.e., it may happen that there is an infinite subset $I \subset \mathbb{N}$ such that for each $n \in I$ there exists an element $\lambda_n \in \operatorname{spec}(A_n)$ with $\lambda_n \to \lambda_\infty$ as $n \in I$ and $n \to \infty$, but $\lambda_\infty \notin \operatorname{spec}(A_\infty)$. Such points are called *spectral pollution*.

When considering non-self-adjoint operators, the spectral convergence may fail (cf. e.g. [9, Example IV.3.8]) even for *norm* convergence. The behaviour of the spectrum for general closed (and in particular *non*-self-adjoint) operators under generalised norm and strong resolvent convergence in the sense of Weidmann is discussed in [4–6] and references therein. For example, no spectral pollution occurs under generalised norm resolvent convergence ([5, Theorem 2.4 (i)]).

We will not treat generalised *strong* convergence here, as it is a priori not obvious how to implement it in the setting of quasi-unitary equivalence. Nevertheless, Weidmann's concept allows a straightforward definition, and the equivalence of both concepts in the norm convergence case gives the chance to define generalised strong resolvent convergence also in the QUE-setting. We will treat this in a forthcoming publication.

Many of the above-mentioned concepts are formulated in a variational way, using sesquilinear forms instead of operators. To simplify the presentation, we only use operators here. Note that the QUE-setting was originally formulated for sesquilinear forms in [15].

1.2. Generalised norm resolvent convergence by Weidmann

Weidmann's idea is to compare the operators acting in different Hilbert spaces in a common so-called *parent Hilbert space* where the individual Hilbert spaces are subspaces of. We use the following generalisation here (cf. also Remark 1.3 (3) and Remark 1.3 (4)):

Definition 1.1 (Weidmann's convergence). Let A_n be a self-adjoint bounded or unbounded operator in a Hilbert space $\mathscr{H}_n$ for $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\} = \{1, 2, 3, \dots, \infty\}$. We say that the sequence $(A_n)_{n\in\mathbb{N}}$ *converges to* A_∞ *in generalised norm resolvent sense of Weidmann* (or shortly *Weidmann-converges*), if the following conditions are true.

1. There exist a Hilbert space $\mathscr{H}$, called *parent (Hilbert) space* and for each $n \in \overline{\mathbb{N}}$ an isometry $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$.
2. We have $\delta_n := \|D_n\|_{\mathscr{L}(\mathscr{H})} \to 0$ as $n \to \infty$, where

$$D_n := \iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^* \tag{1.2}$$

   and where

$$R_n := (A_n - z_0)^{-1} \tag{1.3}$$

   for $n \in \overline{\mathbb{N}}$ is the resolvent of A_n at some common resolvent element $z_0 \in \Gamma := \bigcap_{n\in\overline{\mathbb{N}}} \varrho(A_n)$ (we will not stress the dependency of R_n on z_0 in the notation).

For short, we write $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ (with respect to $(\iota_n)_{n\in\overline{\mathbb{N}}}$) and call $(\delta_n)_n$ the *convergence speed*.

Example 1.2 (A motivating example). We treat here a situation appearing often in applications and which is also the basis of Weidmann's consideration: domain perturbations. Assume that X is a measure space with measure μ, and that $\mathscr{H} = \mathsf{L}_2(X)$. We assume that $X_n \subset X$ are measurable subsets of X for $n \in \overline{\mathbb{N}}$, and that the measure on X_n is the restriction of the measure μ to subsets of X_n. To avoid exceptional cases, we assume that $X_n \cap X_\infty$ has positive measure for all $n \in \mathbb{N}$. We set $\mathscr{H}_n = \mathsf{L}_2(X_n)$. Denote the restriction of an equivalence class f of functions from $\mathsf{L}_2(X)$ to X_n by $f \restriction_{X_n}$. Its adjoint is the embedding $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ given by the extension of $f_n \in \mathscr{H}_n$ by 0, denoted as $f_n \oplus 0_{X \setminus X_n}$. We specify the operators A_n acting on (a dense subspace of) $\mathscr{H}_n$ in a moment, but assume here for simplicity that $A_n \geq 0$ so we can choose $z_0 = -1$ as common resolvent point and set $R_n := (A_n + 1)^{-1}$.

A natural candidate for a parent space and isometries are $\mathscr{H} = \mathsf{L}_2(X)$ and

$$\iota_n \colon \mathscr{H}_n = \mathsf{L}_2(X_n) \to \mathscr{H} = \mathsf{L}_2(X), \quad f_n \mapsto f_n \oplus 0_{X \setminus X_n}$$

for $n \in \overline{\mathbb{N}}$. The operator norm estimate in Weidmann's convergence now is equivalent with

$$\begin{aligned}
&\|(\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*) f\|_{\mathscr{H}}^2 \\
&\quad = \int_X |R_n(f \restriction_{X_n}) \oplus 0_{X \setminus X_n} - R_\infty(f \restriction_{X_\infty}) \oplus 0_{X \setminus X_\infty}|^2 \, \mathrm{d}\mu \\
&\quad = \int_{X_n \cap X_\infty} |R_n(f \restriction_{X_n}) - R_\infty(f \restriction_{X_\infty})|^2 \, \mathrm{d}\mu \\
&\qquad + \int_{X_n \setminus X_\infty} |R_n(f \restriction_{X_n})|^2 \, \mathrm{d}\mu + \int_{X_\infty \setminus X_n} \left|R_\infty(f \restriction_{X_\infty})\right|^2 \, \mathrm{d}\mu \\
&\quad \leq \delta_n^2 \|f\|_{\mathsf{L}_2(X)}^2
\end{aligned} \tag{1.4}$$

for all $f \in \mathscr{H} = \mathsf{L}_2(X)$. If A_n is the operator multiplicating with the function $a_n \colon X_n \to [0, \infty)$, then $R_n(f \restriction_{X_n}) = (a_n + 1)^{-1} f \restriction_{X_n}$.

Concrete example A: monotonely decreasing sequence. Let $X = [0, \infty)$ with Lebesgue measure, $X_n = [0, 1] \cup [2^n, \infty)$ and $a_n(x) = x$ for $n \in \mathbb{N}$ and $X_\infty = [0, 1]$, $a_\infty(x) = x$. Here, the action of A_n is the same for all $n \in \overline{\mathbb{N}}$, and $X_\infty \subset X_n$, hence the first and third integral in (1.4) equal 0 and for the second integral we have

$$\int_{X_n \setminus X_\infty} \left|R_n(f \restriction_{X_n})\right|^2 \mathrm{d}\mu = \int_{(2^n, \infty)} \left|\frac{1}{x+1} f(x)\right|^2 \mathrm{d}x \leq \frac{1}{(2^n)^2} \|f\|_{\mathsf{L}_2(X)}^2.$$

In particular, $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with convergence speed $\delta_n = 2^{-n}$.

Concrete example B: monotonely increasing sequence. Let $X = (0, 1]$ with Lebesgue measure, $X_n = [2^{-n}, 1]$ and $a_n(x) = 1/x$ for $n \in \mathbb{N}$ and $X_\infty = (0, 1]$, $a_\infty(x) = 1/x$. Again, the action of A_n is the same for all $n \in \overline{\mathbb{N}}$, and $X_n \subset X_\infty$, hence the first and second integral in (1.4) equal 0 and for the third integral we have

$$\int_{X_\infty \setminus X_n} |R_\infty(f \upharpoonright_{X_\infty})|^2 \,\mathrm{d}\mu = \int_{(0,2^{-n})} \Big|\frac{1}{1/x+1} f(x)\Big|^2 \,\mathrm{d}x \le \frac{1}{(2^n)^2} \|f\|^2_{\mathsf{L}_2(X)}.$$

In particular, $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with convergence speed $\delta_n = 2^{-n}$.

Remark 1.3 (Weidmann's convergence: uniqueness, pseudo-resolvents and isometries versus subspaces). 1. The parent space in (our notion of) Weidmann's convergence is of course not unique.

2. The lifted resolvents $\iota_n R_n(z)\iota_n^*$ in (1.2) for the resolvents

$$R_n(z) = (A_n - z)^{-1}$$

($z \in \Gamma$, $n \in \overline{\mathbb{N}}$) are also called *pseudo-resolvents* (see e.g. [28, Section VIII.4]): a family $(R(z))_{z\in\Gamma}$ of bounded operators $R(z)\colon \mathscr{H} \to \mathscr{H}$ with $z \in \Gamma \subset \mathbb{C}$ is called a family of *pseudo-resolvents* if the (first) resolvent equation

$$R(z) - R(w) = (z - w)R(z)R(w) \tag{1.5}$$

holds for all $z, w \in \Gamma$. One can see, e.g., that $\ker R(z)$ is independent of $z \in \Gamma$. Moreover, $R(z) = (A - z)^{-1}$ for some closed operator A if and only if $\ker R(z) = \{0\}$. (cf. [28, Theorem VIII.4.1]). In particular, Weidmann's generalised resolvent convergence is a rather natural generalisation of the usual resolvent convergence.

3. At this point, it should be noted that the situation in the book of Weidmann [26, Section 9.3] (see also [5,6]) is slightly different. Weidmann assumes that $\mathscr{H}_n$ and $\mathscr{H}_\infty$ are *subspaces* of the common Hilbert space $\mathscr{H}$. Moreover, he uses the notation P_n both for the orthogonal projection onto $\mathscr{H}_n$ as map $\mathscr{H} \to \mathscr{H}$ as well as for the co-isometry (the adjoint of an isometry, denoted in this article by ι_n^*) as map $\mathscr{H} \to \mathscr{H}_n$. Moreover, the inclusion $\mathscr{H}_n \subseteq \mathscr{H}$ (here denoted by $\iota_n\colon \mathscr{H}_n \to \mathscr{H}$) is not given a proper name in [5, 26].

4. Our interpretation of Weidmann's generalised norm resolvent convergence starts with *isometries* $\iota_n\colon \mathscr{H}_n \to \mathscr{H}$ instead of *subspaces* $\mathscr{H}_n \subset \mathscr{H}$. This generalisation is necessary in order to compare the two concepts especially in cases when there is no natural common parent space (see, e.g., Sections 4.2–4.3). But this generalisation allows certain unwanted cases of "convergence": if, e.g., A is a self-adjoint operator in $\mathscr{H}$ and if $U_n\colon \mathscr{H}_n \to \mathscr{H}$ is unitary for each $n \in \overline{\mathbb{N}}$, then $(A_n)_n$ with $A_n := U_n^* A U_n$ *always* Weidmann-converges to A: choose $\iota_n := U_n$ ($n \in \mathbb{N}$) and $\iota_\infty = \mathrm{id}_{\mathscr{H}}$ then

$$\iota_n R_n \iota_n^* = (A - z_0)^{-1} = \iota_\infty R_\infty \iota_\infty^*.$$

One way of avoiding the above-mentioned "unitary mixing" is to use an additional lattice structure on the Hilbert spaces: Assume that the spaces $\mathscr{H}_n$ ($n \in \overline{\mathbb{N}}$) and $\mathscr{H}$

are L_2-spaces. One can then assume that an isometry $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ is *positivity-preserving*, i.e., $f_n \geq 0$ implies $\iota_n f_n \geq 0$ pointwise almost everywhere. This is the case in our motivating example Example 1.2 and in some of our examples in Section 4. A weaker condition is that the corresponding identification operators $J_n = \iota_\infty^* \iota_n$ (see below) are positivity-preserving; we see in Section 4 that these identification operators are *all* positivity-preserving in our examples.

1.3. Generalised norm resolvent convergence based on quasi-unitary equivalence

Independently of Weidmann's concept, the first named author of the present paper developed the notion *quasi-unitary equivalence* of two self-adjoint, unbounded and non-negative operators A_1 and A_2 acting in $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively measuring a sort of "distance". The setting incorporates an identification operator $J \colon \mathscr{H}_1 \to \mathscr{H}_2$ in the norm difference of the resolvents $R_1 = (A_1 + 1)^{-1}$ and $R_2 = (A_2 + 1)^{-1}$, namely we consider $\|JR_1 - R_2 J\|_{\mathfrak{L}(\mathscr{H}_1, \mathscr{H}_2)}$. In order to exclude trivial cases such as $J = 0$ we want that J is "close" to a unitary operator, measured again by a norm estimate:

Definition 1.4 (QUE: quasi-unitary equivalence). Let A_1 and A_2 be two self-adjoint operators acting in $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. For $\delta \geq 0$ we say that A_1 and A_2 are *δ-quasi-unitary equivalent* if there exist a common resolvent element $z_0 \in \varrho(A_1) \cap \varrho(A_2)$ and a bounded operator $J \colon \mathscr{H}_1 \to \mathscr{H}_2$ such that the norm inequalities

$$\|J\|_{\mathfrak{L}(\mathscr{H}_1, \mathscr{H}_2)} \leq 1 + \delta, \tag{1.6a}$$

$$\|(\mathrm{id}_{\mathscr{H}_1} - J^* J) R_1\|_{\mathfrak{L}(\mathscr{H}_1)} \leq \delta, \quad \|(\mathrm{id}_{\mathscr{H}_2} - J J^*) R_2\|_{\mathfrak{L}(\mathscr{H}_2)} \leq \delta, \tag{1.6b}$$

$$\|R_2 J - J R_1\|_{\mathfrak{L}(\mathscr{H}_1, \mathscr{H}_2)} \leq \delta \tag{1.6c}$$

hold. If $z_0 \in \mathbb{C} \setminus \mathbb{R}$, we require that the two estimates in (1.6b) and that (1.6c) also hold with the resolvents $R_j = (A_j - z_0)^{-1}$ replaced by $R_j^* = (A_j - \overline{z}_0)^{-1}$ for $j \in \{1, 2\}$. The operators J and J^* are called *identification operators*, and δ is called *distance bound* or *error*.

In [17] only non-negative operators are considered, hence one can choose $z_0 = -1$. An extension to non-self-adjoint operators is possible, see Remark 1.10.

Next, we want to transfer this concept onto a family of self-adjoint operators to define a convergence. The idea is to check whether every member of the family is quasi-unitary equivalent with the limit operator, and that the sequence of their distance bounds δ_n converges to 0:

Definition 1.5 (QUE-convergence). Let A_n be a self-adjoint bounded or unbounded operator in a Hilbert space $\mathscr{H}_n$ for $n \in \overline{\mathbb{N}}$. We say that the sequence $(A_n)_{n \in \mathbb{N}}$ *converges to A_∞ in generalised norm resolvent sense* (or shortly *QUE-converges*), if there exist $z_0 \in \Gamma = \bigcap_{n\in\overline{\mathbb{N}}} \varrho(A_n)$ and a sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n \to 0$ as $n \to \infty$ such that A_n and A_∞ are δ_n-quasi-unitarily equivalent with common resolvent element z_0. We write for short $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$. We call $(\delta_n)_n$ the *convergence speed*.

In particular, $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ holds (with respect to J_n) if

$$\|J_n\|_{\mathfrak{L}(\mathscr{H}_n,\mathscr{H}_\infty)} \le 1 + \delta_n, \tag{1.7a}$$

$$\|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \le \delta_n, \quad \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \le \delta_n, \tag{1.7b}$$

$$\|R_\infty J_n - J_n R_n\|_{\mathfrak{L}(\mathscr{H}_n,\mathscr{H}_\infty)} \le \delta_n, \tag{1.7c}$$

with $\delta_n \to 0$ as $n \to \infty$ for a family of identification operators $(J_n)_{n\in\mathbb{N}}$ with $J_n\colon \mathscr{H}_n \to \mathscr{H}_\infty$, where $R_n = (A_n - z_0)^{-1}$ is the resolvent in some common resolvent element $z_0 \in \Gamma$. If $z_0 \in \mathbb{C} \setminus \mathbb{R}$, we require that (1.7a)–(1.7c) also hold for z_0^*, i.e., for R_n replaced by R_n^*. Then it is possible to swap the order of the operators by taking adjoints such as

$$\|J_n^* R_\infty - R_n J_n^*\|_{\mathfrak{L}(\mathscr{H}_\infty,\mathscr{H}_n)} = \|R_\infty^* J_n - J_n R_n^*\|_{\mathfrak{L}(\mathscr{H}_n,\mathscr{H}_\infty)} \le \delta_n \to 0 \tag{1.7c$'$}$$

(see also Remark 1.10 for non-self-adjoint operators).

Example 1.6 (Motivating example, continued). We come back to Example 1.2. A natural candidate for the identification operator is

$$J_n\colon \mathscr{H}_n = \mathsf{L}_2(X_n) \to \mathscr{H}_\infty = \mathsf{L}_2(X_\infty), \quad f_n \mapsto (f_n \restriction_{X_n \cap X_\infty}) \oplus 0_{X_\infty \setminus X_n}.$$

As J_n is a non-trivial partial isometry (cf. Section 2.1), we have $\|J_n\| = 1$, hence (1.7a) is trivially fulfilled with $\delta_n = 0$. Moreover, the two estimates in (1.7b) are equivalent with the two estimates

$$\left\|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n f_n\right\|^2_{\mathsf{L}_2(X_n)} = \int_{X_n \setminus X_\infty} |R_n f_n|^2 \,\mathrm{d}\mu \le \delta_n^2 \|f_n\|^2_{\mathsf{L}_2(X_n)}, \tag{1.8a}$$

$$\left\|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty f_\infty\right\|^2_{\mathsf{L}_2(X_\infty)} = \int_{X_\infty \setminus X_n} |R_\infty f_\infty|^2 \,\mathrm{d}\mu \le \delta_n^2 \|f_\infty\|^2_{\mathsf{L}_2(X_\infty)} \tag{1.8b}$$

for all $f_n \in \mathscr{H}_n$ resp. $f_\infty \in \mathscr{H}_\infty$. Finally, (1.7c) is equivalent here with

$$\|(R_\infty J_n - J_n R_n) f_n\|^2_{\mathsf{L}_2(X_\infty)} = \int_{X_n \cap X_\infty} \left|R_\infty (f_n \restriction_{X_n \cap X_\infty}) - R_n f_n\right|^2 \mathrm{d}\mu \le \delta_n^2 \|f_n\|^2_{\mathsf{L}_2(X_n)} \tag{1.8c}$$

for all $f \in \mathscr{H}_\infty = \mathsf{L}_2(X)$. We observe that the three integrals in (1.8a)–(1.8c) can be recovered in (1.4) of Weidmann's convergence.

Concrete example A: monotonely decreasing sequence. Let $X = [0, \infty)$ with Lebesgue measure, $X_n = [0,1] \cup [2^n, \infty)$ and $a_n(x) = x$ for $n \in \mathbb{N}$ and $X_\infty = [0,1]$, $a_\infty(x) = x$. Here, (1.8b) and (1.8c) are trivially valid with $\delta_n = 0$, only (1.8a) is non-trivial, and as in Weidmann's generalised convergence, we can choose $\delta_n = 2^{-n}$. In particular, we have $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with convergence speed $\delta_n = 2^{-n}$.

Concrete example B: monotonely increasing sequence. Let $X = (0, 1]$ with Lebesgue measure, $X_n = [2^{-n}, 1]$ and $a_n(x) = 1/x$ for $n \in \mathbb{N}$ and $X_\infty = (0,1]$, $a_\infty(x) = 1/x$. Here, only (1.8b) is non-trivial and we have (as in Weidmann's case) $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with convergence speed $\delta_n = 2^{-n}$.

1.4. Main result and structure of the paper

The goal of this paper is to relate the two notions of convergence given in Definitions 1.1 and 1.5. As we have seen already in our motivating example, the two notions are indeed equivalent. Here is the main result of our paper:

Theorem 1.7 (Main theorem). *Both notions of resolvent convergence are equivalent. In particular, if A_n is a self-adjoint operator in a Hilbert space $\mathscr{H}_n$ for each $n \in \overline{\mathbb{N}}$, then we have:*

1. *If $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with respect to $(\iota_n)_{n \in \overline{\mathbb{N}}}$, then there are identification operators $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ constructed from ι_n and ι_∞ such that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with respect to $(J_n)_{n \in \mathbb{N}}$.*
2. *If $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with identification operators $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$, then there is a Hilbert space $\mathscr{H}$ and isometries $(\iota_n \colon \mathscr{H}_n \to \mathscr{H})_{n \in \overline{\mathbb{N}}}$ constructed from J_n such that $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with respect to $(\iota_n)_{n \in \overline{\mathbb{N}}}$.*

We will give the proof in four steps.

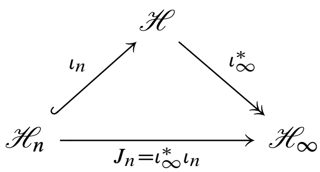

Figure 1. Factorising J_n over $\mathscr{H}$ via an isometry ι_n and a co-isometry ι_∞^*.

• The *first step* in the proof of Theorem 1.7 is showing the rather simple fact that *Weidmann's convergence implies QUE-convergence* in Theorem 2.4. We actually set

$$J_n := \iota_\infty^* \iota_n. \tag{1.9}$$

If the identification operators $J_n\colon \mathscr{H}_n \to \mathscr{H}_\infty$ are of the form (1.9), we say that the isometries $(\iota_n)_{n\in\overline{\mathbb{N}}}$ *factorise* the identification operators $(J_n)_{n\in\mathbb{N}}$ (cf. Figure 1). If the isometries factorise the identification operators J_n, then each J_n necessarily is a *contraction*, i.e., $\|J_n\|_{\mathfrak{L}(\mathscr{H}_n,\mathscr{H}_\infty)} \le 1$ for all $n \in \mathbb{N}$, as a co-isometry and an isometry both have operator norms not greater than 1.

Before showing the first step, we collect some facts about (partial) isometries in Section 2.

• In a *second step*, we show that *QUE-convergence implies Weidmann's convergence* (Theorem 2.8) *assuming that a parent space $\mathscr{H}$ and isometries $\iota_n\colon \mathscr{H}_n \to \mathscr{H}$ factorising the identification operators exist*. If

$$P_n P_\infty = P_\infty P_n \quad \text{for all } n \in \mathbb{N}, \tag{1.10}$$

where $P_n = \iota_n \iota_n^*$ is the orthogonal projection onto the range of ι_n, then the convergence speed has the same order in both cases (Theorem 2.12), while for general identification operators J_n, there is a slight loss in the convergence speed. The condition (1.10) is equivalent with the fact that $P_n P_\infty$ (or $P_\infty P_n$) is an orthogonal projection (cf. [26, Satz. 2.55 (a)]. For *subspaces* $\mathscr{H}_n \subset \mathscr{H}$ ($n \in \overline{\mathbb{N}}$), the condition (1.10) is equivalent with

$$\big(\mathscr{H}_n \ominus (\mathscr{H}_n \cap \mathscr{H}_\infty)\big) \perp \big(\mathscr{H}_\infty \ominus (\mathscr{H}_n \cap \mathscr{H}_\infty)\big)$$

(cf. [26, Aufgabe 2.22, Satz 2.55]) where $\mathscr{H}' \ominus \mathscr{H}''$ is the orthogonal complement of $\mathscr{H}'' \subset \mathscr{H}'$ in $\mathscr{H}'$. In our motivating example, $P_n = \mathbb{1}_{X_n,X}$ (multiplication with the indicator function $\mathbb{1}_{X_n,X}\colon X \to \{0,1\}$), and the commuting condition (1.10) is fulfilled.

Moreover, we characterise whether J_n is a partial isometry in terms of Weidmann's data $(\iota_n\colon \mathscr{H}_n \to \mathscr{H})_{n\in\overline{\mathbb{N}}}$ (cf. Theorem 2.13): namely J_n is a partial isometry if and only if (1.10) holds. The commuting property is therefore an *invariant* among all parent spaces factorising the identification operators, cf. Corollary 2.14.

• As *third step*, we show in Section 3 that *a parent space can always be constructed* from the QUE-data (Theorem 3.1), provided that the identification operators J_n are *contractions* ($\|J_n\| \le 1$). The so-called *defect operators* constructed from the identification operators play a prominent role in the construction of a parent space. As a consequence, QUE-convergence implies Weidmann's convergence without the assumption that a parent space exists (Corollary 3.2). Moreover, we give further equivalent characterisations when the identification operators J_n are partial isometries in terms of the defect operators (Theorem 3.9).

- In a *last step* of the proof of Theorem 1.7, we consider the general case (i.e., that $\|J_n\| > 1$ for some n) in Section 3.4 and Lemma 3.17

Section 4 contains different types of examples, some where a parent space is naturally given, and some where such a parent space is not naturally given. We end this introductory section with a motivating example, more comments on existing literature and some further comments.

Remarks 1.8 (On the main theorem). 1. The main result Theorem 1.7 remains true also for non-self-adjoint operators A_n and A_∞; for the necessary changes see Remark 1.10. For the sake of simplicity, the proofs are written for the self-adjoint case only.

2. The focus in this work lies on the identification operators J_n ($n \in \mathbb{N}$); the operator domains of A_n and A_∞ are rather irrelevant for our analysis here; we only use these operators through their resolvents $R_n := (A_n - z_0)^{-1}$ and $R_\infty := (A_\infty - z_0)^{-1}$ and their adjoints.

3. We present a method in Section 3.1 how to construct a parent space and the corresponding isometries for Weidmann's generalised norm resolvent convergence also in less obvious cases such as thick graphs converging to a metric graph (see Sections 4.2 and 4.3).

Note that a naive choice of a parent space would be

$$\mathscr{H} := \mathscr{H}_\infty \oplus \bigoplus_{n\in\mathbb{N}} \mathscr{H}_n, \quad \iota_\infty f_\infty = (f_\infty, 0, \ldots), \quad \iota_n f_n = (0, \ldots, 0, f_n, 0, \ldots).$$

But, in this case, the isometries are not factorising J_n as $\iota_\infty^* \iota_n = 0$. Moreover, Weidmann's resolvent difference (cf. (2.3)) always fulfils

$$\|D_n f\|^2_{\mathscr{H}} = \|R_n f_n\|^2_{\mathscr{H}_n} + \|R_\infty f_\infty\|^2_{\mathscr{H}_\infty}$$

for $f = (f_\infty, f_1, \ldots) \in \mathscr{H}$, and this expression does not converge to 0. We therefore have to use more elaborated isometries, relating the spaces $\mathscr{H}_n$ and $\mathscr{H}_\infty$ in an appropriate way especially on those subspaces where their resolvents are close to each other.

We see in Remark 2.9 how to characterise Weidmann's convergence using parent spaces factorising the identification operators.

4. If the projection commuting property (1.10) does not hold, then there is in general a loss in the convergence speed when passing from QUE-convergence to Weidmann's generalised norm resolvent convergence. To avoid this loss, it seems to be

better to use a slightly stronger estimate in the QUE-convergence, see Remarks 2.11 and 4.9.

1.5. More comments on existing literature

It is impossible to mention even a small amount of relevant literature concerning resolvent convergence on varying spaces, as it includes, e.g., all types of finite-dimensional approximations of any infinite-dimensional problem. Let us at least comment on a concept weaker than the one considered here, namely the generalised *strong* resolvent convergence:

Generalised strong resolvent convergence and Stummel's discrete convergence. Strong convergence of the resolvents (i.e., the pointwise convergence of the operator resolvents) in Weidmann's setting means that

$$D_n f = \iota_n R_n \iota_n^* f - \iota_\infty R_\infty \iota_\infty^* f$$

converges to 0 in $\mathscr{H}$ for all $f \in \mathscr{H}$. As already mentioned, Stummel [22] introduced an abstract concept of (strong) convergence of operators acting in different Banach spaces (see also [23,25] and references therein). A *discrete approximation* of a Hilbert space $\mathscr{H}_\infty$ by a sequence of Hilbert spaces $\mathscr{H}_n$ in the sense of Stummel is given by a linear map $R\colon \mathscr{H}_\infty \to \prod_{n\in\mathbb{N}} \mathscr{H}_n/\!\sim$ (not to be confused with a resolvent) such that $\|u_n\|_{\mathscr{H}_n} \to \|u_\infty\|_{\mathscr{H}_\infty}$ as $n \to \infty$ for all $u_\infty \in \mathscr{H}_\infty$ and all $[(u_n)_n] \in R(u_\infty)$. Here, $\sim$ is the equivalence relation given by $(u_n)_n \sim (v_n)_n$ if $\|u_n - v_n\|_{\mathscr{H}_n} \to 0$. Stummel then defines the *discrete convergence* $u_n \to u_\infty$ if $[(u_n)_n] = Ru_\infty$ ([22, Section 1.1 (4)]).

- Given Weidmann's setting, $Ru_\infty = [(\iota_n^* \iota_\infty u_\infty)_{n\in\mathbb{N}}]$ defines a discrete approximation provided
$$P_n \to P_\infty \text{ strongly.} \tag{1.11}$$
In particular, $u_n \to u_\infty$ (*discrete convergence* in the sense of Stummel) means that $\|u_n - \iota_n^* \iota_\infty u_\infty\|_{\mathscr{H}_n} \to 0$. If (1.11) holds, the latter is also equivalent with the more natural condition $\|\iota_n u_n - \iota_\infty u_\infty\|_{\mathscr{H}} \to 0$ ("do everything in the parent space").
- Given identification operators $(J_n)_{n\in\mathbb{N}}$ satisfying (1.7a)–(1.7b) then one can show that a discrete approximation is given by $Ru_\infty = [(J_n^* u_\infty)_{n\in\mathbb{N}}]$. In particular, $u_n \to u_\infty$ in the sense of Stummel if and only if $\|u_n - J_n^* u_\infty\|_{\mathscr{H}_n} \to 0$.

A sequence $(S_n)_n$ of bounded operators on $\mathscr{H}_n$ *converges discretely* to a bounded operator S_∞ on $\mathscr{H}_\infty$ (in the sense of Stummel [22, Section 1.2 (2)]) if $u_n \to u_\infty$ implies $S_n u_n \to S_\infty u_\infty$.

- The strong convergence of Weidmann (1.2) is equivalent with Stummel's notion of discrete convergence $S_n \to S_\infty$ provided (1.11) holds.

- If $Ru_\infty := [(J_n^* u_\infty)_n]$, then $S_n \to S_\infty$ in the sense of Stummel is equivalent with the fact that

$$\|u_n - J_n^* u_\infty\|_{\mathscr{H}_n} \to 0 \implies \|S_n u_n - J_n^* S_\infty u_\infty\|_{\mathscr{H}_n} \to 0.$$

  If $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ in the QUE-setting, then

$$\begin{aligned}&\|R_n u_n - J_n^* R_\infty u_\infty\|_{\mathscr{H}_n}\\ &\qquad \le \|R_n(u_n - J_n^* u_\infty)\|_{\mathscr{H}_n} + \|(R_n J_n^* - J_n^* R_\infty)u_\infty)\|_{\mathscr{H}_n} \to 0,\end{aligned}$$

  as $(R_n u)_n$ is *consistent* by Lemma 2.7, i.e., $(\|R_n\|_{\mathscr{L}(\mathscr{H}_n)})_n$ is bounded. In particular, $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ implies the discrete convergence of the resolvents $R_n \to R_\infty$ in the sense of Stummel.

Bögli [4, 5] considers also non-self-adjoint operators and is mainly interested in convergence of spectra and in particular, the non-existence of spectral pollution. She shows also how to upgrade generalised strong resolvent convergence to generalised norm resolvent convergence [5, Theorem 2.7 and Proposition 2.13] under some compactness assumptions on the resolvents. Note that Bögli assumes that the strong convergence of the projections (1.11) holds. If one constructs a parent space according to Section 3 from the QUE-data, and if (1.7b) holds, then (1.11) holds automatically (see Proposition 3.11).

For abstract results on (mostly) strong resolvent convergence in the context of *homogenisation*, we refer to the references presented in [10, Section 1].

Concepts related with the QUE-convergence. Another concept of convergence of operators acting in different Hilbert spaces and related with the QUE-setting is given in [12, Section 2.2–2.7]. Kuwae and Shioya consider families of Hilbert spaces and identification operators between any two members of the family. They define a version of generalised *strong* resolvent convergence. In [12, Theorem 2.4] they prove that their strong resolvent convergence is equivalent with Mosco-convergence of quadratic forms. They apply their results to convergence of families of manifolds (where the limit is not necessarily a manifold any more). Another abstract approach which is applied to so-called *dumbbell domains* is given in [2, Section 4]; here, the authors use a scaled measure on the shrinking thin part of the dumbbell domain.

Domain perturbations. Rauch and Taylor [19] embed the Hilbert spaces $\mathsf{L}_2(X_n)$ and $\mathsf{L}_2(X_\infty)$ into the common Hilbert space $\mathsf{L}_2(\mathbb{R}^d)$ by extending functions by 0 for $X_n, X_\infty \subset \mathbb{R}^d$ as in Example 1.2. Rauch and Taylor show (what Weidmann later called) generalised strong resolvent convergence for Dirichlet and Neumann Laplacians under some "convergence" conditions on X_n and X_∞. From this, Rauch and Taylor conclude in [19, Theorem 1.2 and Theorem 1.5] (although not explicitly stated abstractly) convergence of operator functions and convergence of the rank of spectral projections provided the resolvent is compact. The latter convergence implies convergence of the spectra in the sense of (1.1), and in particular, convergence of the eigenvalues.

In [27, Section 3] Weidmann develops a preliminary version of his generalised strong resolvent convergence based on a monotone convergence theorem for quadratic forms shown by Simon [20]. Weidmann applies his abstract results to domain perturbations ([27, Section 4]). Stollmann [21] generalises results from [19, 20, 27] formulated in the language of Dirichlet forms.

Daners considers pseudo-resolvents (cf. (1.5)) given by $\iota_n R_n(z)\iota_n^*$ as in Weidmann's approach even in Banach spaces, where $R_n(z) = (A_n - z)^{-1}$ is the usual resolvent. Moreover, he shows upper semi-continuity of the spectrum, cf. [8, Section 4] and references therein. Additionally, he gives equivalent conditions under which generalised strong and norm resolvent convergence for Dirichlet Laplacians on $X_n \subset \mathbb{R}^d$ "converging" to $X_\infty \subset \mathbb{R}^d$ holds, cf. [8, Theorem 5.2.4 and Theorem 5.2.6]. For example, the strong resolvent convergence holds provided $\mathsf{H}^1(X_n) \to \mathsf{H}^1(X_\infty)$ in the sense of [14, Section 1], i.e., for any $u \in \mathsf{H}^1(X)$ and $n \in \mathbb{N}$ there exists $u_n \in \mathsf{H}^1(X_n)$ such that $\|u - u_n\|_{\mathsf{H}^1(\mathbb{R}^d)} \to 0$. For some recent results on domain perturbations using a concept close to our identification operators in the QUE-setting we refer to [3] and the references therein.

We plan to address *generalised strong resolvent convergence* in its different appearances in a subsequent publication, see also [4] for a comprehensive overview.

1.6. Some further comments and outlook

Remark 1.9 (Change of common resolvent element). Note that $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ holds for *some* $z_0 \in \Gamma = \bigcap_{n\in\overline{\mathbb{N}}} \varrho(A_n)$ if and only if $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ holds for *any* $z_0 \in \Gamma$ (cf. [26, Satz 9.28]).

Similarly, it can be seen that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ for *some* $z_0 \in \Gamma$ and $\overline{z}_0 \in \Gamma$ if and only if $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ for *any* $z_0 \in \Gamma$. In both cases, the convergence speed changes only by a factor depending on the common resolvent elements.

Remark 1.10 (Extension to non-self-adjoint operators). Both convergence concepts extend to non-self-adjoint (closed) operators A_n ($n \in \overline{\mathbb{N}}$). Weidmann's concept directly applies to closed operators A_n (with resolvents $R_n := (A_n - z_0)^{-1}$ for some $z_0 \in \bigcap_{n\in\overline{\mathbb{N}}} \varrho(A_n)$) as done e.g. in [5–7] (see also references therein); note that

$$\|\iota_n R_n^* \iota_n^* - \iota_\infty R_\infty^* \iota_\infty^*\|_{\mathfrak{L}(\mathscr{H})} = \|\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*\|_{\mathfrak{L}(\mathscr{H})}. \tag{1.12}$$

For the concept of QUE-convergence in the non-self-adjoint case, we assume that (1.7b)–(1.7c) hold for R_n *and* $R_n^* = (A_n^* - \overline{z}_0)^{-1}$ resp. R_∞ *and* $R_\infty^* = (A_\infty^* - \overline{z}_0)^{-1}$.

In particular, our main result Theorem 1.7 remains true with these modifications.

Remark 1.11 (A distance in Weidmann's concept). As in the concept of QUE-convergence, one can also define a distance in Weidmann's concept if there are two operators A_1 and A_2 acting in $\mathscr{H}_1$ and $\mathscr{H}_2$ with resolvents $R_1 = (A_1 - z_0)^{-1}$ and $R_2 = (A_2 - z_0)^{-1}$, respectively. Here, one assumes that there is a parent Hilbert space $\mathscr{H}$ and isometries $\iota_1 : \mathscr{H}_1 \to \mathscr{H}$ and $\iota_2 : \mathscr{H}_2 \to \mathscr{H}$, and the *distance* of R_1 and R_2 is then

$$\|\iota_1 R_1 \iota_1^* - \iota_2 R_2 \iota_2^*\|_{\mathfrak{L}(\mathscr{H})}.$$

Of course, the parent space is not unique, and one could define the distance R_1 and R_2 as the infimum of all parent spaces $\mathscr{H}$ and isometries $\iota_1 : \mathscr{H}_1 \to \mathscr{H}$ and $\iota_2 : \mathscr{H}_2 \to \mathscr{H}$; a similar idea also holds for the QUE-concept as in Definition 1.4.

We will treat such questions in a forthcoming publication, where we also underline some optimality properties of the concrete parent space constructed in Section 3.

Remark 1.12 (Extension to Banach spaces). Many of the concepts extend to operators acting in Banach spaces. Weidmann's convergence in this setting could be understood as

$$\|\iota_n R_n \pi_n - \iota_\infty R_\infty \pi_\infty\|_{\mathfrak{L}(\mathscr{X})} \to 0$$

as $n \to \infty$ for bounded (or resolvents of unbounded) operators $R_n \in \mathfrak{L}(\mathscr{X}_n)$ in Banach spaces $\mathscr{X}_n$ ($n \in \overline{\mathbb{N}}$). Here, $\iota_n : \mathscr{X}_n \to \mathscr{X}$ is an isometry (an injective partial isometry) into another Banach space $\mathscr{X}$ and $\pi_n : \mathscr{X} \to \mathscr{X}_n$ a co-isometry (a surjective partial isometry); partial isometries on Banach spaces are analysed e.g. in [13].

The QUE-convergence could be generalised as follows: Assume that $J_n : \mathscr{X}_n \to \mathscr{X}_\infty$ and $J_n' : \mathscr{X}_\infty \to \mathscr{X}_n$ are operators fulfilling (1.7b)–(1.7c) with $\mathscr{H}_n$ replaced by $\mathscr{X}_n$ and J_n^* replaced by J_n'.

We will treat such convergences and their relation in a forthcoming publication.

2. Relation of the two concepts

2.1. Partial isometries and operator norms

Let us first prove some material needed for our analysis. All our Hilbert spaces here are assumed to be *separable*.

Let $\mathscr{H}$ and $\mathscr{H}_0$ be two Hilbert spaces. An *isometry* is a linear operator $\iota : \mathscr{H}_0 \to \mathscr{H}$ such that $\|\iota f_0\|_{\mathscr{H}} = \|f_0\|_{\mathscr{H}_0}$ for all $f_0 \in \mathscr{H}_0$. Equivalently, ι is an isometry if and only if $\iota^* \iota = \mathrm{id}_{\mathscr{H}_0}$. Moreover, a surjective isometry is unitary, i.e., a Hilbert space isomorphism.

A *partial isometry* is a linear operator $I : \mathscr{H}_0 \to \mathscr{H}$ such that

$$I\restriction_{(\ker I)^\perp} : (\ker I)^\perp \to \mathscr{H} \tag{2.1}$$

is an isometry. We call $(\ker I)^\perp = I^*(\mathscr{H})$ the *initial space* and $I(\mathscr{H}_0) = (\ker I^*)^\perp$ the *final space* of I. Note that $I(\mathscr{H}_0)$ is closed as isometric image of the complete space $(\ker I)^\perp$; a similar argument holds for $I^*(\mathscr{H})$.

Lemma 2.1 (facts about partial isometries). *Let $I\colon \mathscr{H}_0 \to \mathscr{H}$ be an bounded linear operator. Then the following statements are equivalent.*

1. *I is a partial isometry.*
2. *I^* is a partial isometry.*
3. *One of the following equations is true:*

$$I = II^*I \iff I^* = I^*II^* \iff I^*I = (I^*I)^2 \iff II^* = (II^*)^2.$$

The third, resp. fourth, assertion say that I^*I, resp. II^*, are orthogonal projections onto the initial resp. final space of I.

4. *The initial space of I is characterised by*

$$(\ker I)^\perp = I^*(\mathscr{H}) = \{f_0 \in \mathscr{H}_0 \mid \|If_0\|_{\mathscr{H}} = \|f_0\|_{\mathscr{H}_0}\}.$$

5. *The final space of I is characterised by*

$$(\ker I^*)^\perp = I(\mathscr{H}_0) = \{f \in \mathscr{H} \mid \|I^*f\|_{\mathscr{H}_0} = \|f\|_{\mathscr{H}}\}.$$

Proof. (1) $\implies$ (2). Let $g \in (\ker I^*)^\perp = I(\mathscr{H}_0)$, then $g = If_0$ for a $f_0 \in \mathscr{H}_0$. If $f_0 \in \ker I$, then $I^*g = I^*If_0 = 0$ and $g \in (\ker I^*) \cap (\ker I^*)^\perp = \{0\}$. If on the other hand $f_0 \in (\ker I)^\perp$, then $\|I^*g\|_{\mathscr{H}} = \|I^*If_0\|_{\mathscr{H}_0} = \|f_0\|_{\mathscr{H}_0} = \|If_0\|_{\mathscr{H}} = \|g\|_{\mathscr{H}}$. Therefore, I^* is a partial isometry.

(2) $\implies$ (1). This follows from the first implication and the fact, that $(I^*)^* = I$.

(1) $\implies$ (3) (first equation). If $f_0 \in \ker I$, it is clear that: $If_0 = 0 = II^*If_0$. If $f_0 \in (\ker I)^\perp$, we have $II^*If_0 = I\,\mathrm{id}_{\mathscr{H}_0} f_0$.

(3) (first equation) $\implies$ (3) (third equation) is clear by multiplication from the left with I^*.

(3) (third equation) $\implies$ (2). Let $f_0 \in \mathscr{H}_0$ then

$$\|If_0\|^2_{\mathscr{H}} = \langle f_0, I^*If\rangle_{\mathscr{H}_0} = \langle f_0, (I^*I)^2 f_0\rangle_{\mathscr{H}_0} = \|I^*If_0\|^2_{\mathscr{H}_0},$$

and therefore I^* is an isometry on $I(\mathscr{H}_0) = (\ker(I^*))^\perp$.

(2) $\implies$ (3) (second equation) $\implies$ (fourth equation) $\implies$ (1) can be shown analogously.

(1) $\implies$ (4) "$\subseteq$". Let $f_0 \in (\ker I)^\perp$. Then $\|If_0\|_{\mathscr{H}} = \|f_0\|_{\mathscr{H}_0}$ by (2.1).

(1) $\implies$ (4) "$\supseteq$". Let $f_0 \in \mathscr{H}_0$ with $\|If_0\|_{\mathscr{H}} = \|f_0\|_{\mathscr{H}_0}$. We have

$$\begin{aligned}\|f_0 - I^*If_0\|^2_{\mathscr{H}_0} &= \|(\mathrm{id}_{\mathscr{H}_0} - I^*I)f_0\|^2_{\mathscr{H}_0} = \langle(\mathrm{id}_{\mathscr{H}_0} - I^*I)f_0, (\mathrm{id}_{\mathscr{H}_0} - I^*I)f_0\rangle_{\mathscr{H}_0}\\ &= \langle f_0, (\mathrm{id}_{\mathscr{H}_0} - I^*I)f_0\rangle_{\mathscr{H}_0} = \|f_0\|^2_{\mathscr{H}_0} - \|If_0\|^2_{\mathscr{H}} = 0\end{aligned}$$

Thus, $f_0 \in I^*(\mathscr{H}_0)$.

(4) $\implies$ (1) is given directly by (2.1). Analogously we get (2) $\iff$ (5). In summary all statements characterise the notion of a partial isometry. ■

An isometry is hence a partial isometry with maximal initial space, or equivalently, an injective partial isometry.

A *co-isometry* $\pi\colon \mathscr{H} \to \mathscr{H}_0$ is the adjoint of an isometry, i.e., π^* is an isometry. Equivalently, π is a co-isometry if and only if $\pi\pi^* = \mathrm{id}_{\mathscr{H}}$. A co-isometry is a partial isometry with maximal final space, or equivalently, a surjective partial isometry.

Let us now provide some simple facts about isometries and co-isometries.

Lemma 2.2 ((Co-)isometries and operator norms). *Let $A \in \mathfrak{L}(\mathscr{H}_1, \mathscr{H}_2)$ and let $\iota_j\colon \mathscr{H}_j \to \widetilde{\mathscr{H}_j}$ be isometries for $j \in \{1, 2\}$. Then we have*

$$\|A\|_{\mathfrak{L}(\mathscr{H}_1,\mathscr{H}_2)} = \|\iota_2 A\|_{\mathfrak{L}(\mathscr{H}_1,\widetilde{\mathscr{H}}_2)} = \|A^*\iota_2^*\|_{\mathfrak{L}(\widetilde{\mathscr{H}}_2,\mathscr{H}_1)} = \|\iota_1 A^*\iota_2^*\|_{\mathfrak{L}(\widetilde{\mathscr{H}}_2,\widetilde{\mathscr{H}}_1)}$$

Moreover, if $\mathscr{H}_1 = \mathscr{H}_2$ and if $A = A^*$, then all the above equalities hold with A^* replaced by A.

Proof. The first equality follows from

$$\|A\|_{\mathfrak{L}(\mathscr{H}_1,\mathscr{H}_2)} = \sup_{f_1 \in \mathscr{H}_1 \setminus \{0\}} \frac{\|Af_1\|_{\mathscr{H}_2}}{\|f_1\|_{\mathscr{H}_1}} = \sup_{f_1 \in \mathscr{H}_1 \setminus \{0\}} \frac{\|\iota_2 Af_1\|_{\widetilde{\mathscr{H}}_2}}{\|f_1\|_{\mathscr{H}_1}}.$$

For the second note that

$$\|A^*\iota_2^*\|_{\mathfrak{L}(\widetilde{\mathscr{H}}_2,\mathscr{H}_1)} = \|\iota_2 A\|_{\mathfrak{L}(\mathscr{H}_1,\widetilde{\mathscr{H}}_2)} = \|A\|_{\mathfrak{L}(\mathscr{H}_1,\mathscr{H}_2)}$$

by taking the adjoint and using the first equality appropriately. The last equality is a consequence of the first two. ■

Note that when the isometry is on the right or the co-isometry is on the left, we have only the trivial estimate

$$\|\iota_2^* A\iota_1\|_{\mathfrak{L}(\widetilde{\mathscr{H}}_1,\widetilde{\mathscr{H}}_2)} \le \|A\|_{\mathfrak{L}(\mathscr{H}_1,\mathscr{H}_2)} \tag{2.2}$$

and the inequality can be strict (e.g. if $A \neq 0$, $\mathscr{H}_1 = \{0\}$ and $\iota_1 = 0$), see also Remark 2.6.

2.2. Weidmann's convergence implies QUE-convergence

Given a parent space $\mathscr{H}$ with corresponding isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ and resolvents R_n acting in $\mathscr{H}_n$ ($n \in \overline{\mathbb{N}}$), we abbreviate their difference as used in Weidmann's convergence by

$$D_n \colon \mathscr{H} \to \mathscr{H}, \quad D_n := \iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*. \tag{2.3}$$

Moreover, we denote by

$$P_n := \iota_n \iota_n^* \colon \mathscr{H} \to \mathscr{H} \quad \text{and} \quad P_\infty := \iota_\infty \iota_\infty^* \colon \mathscr{H} \to \mathscr{H} \tag{2.4}$$

the orthogonal projections in $\mathscr{H}$ onto $\iota_n(\mathscr{H}_n)$, resp. $\iota_\infty(\mathscr{H}_\infty)$. As ι_n and ι_∞ are isometries, we clearly have

$$\iota_n^* \iota_n = \mathrm{id}_{\mathscr{H}_n} \quad \text{and} \quad \iota_\infty^* \iota_\infty = \mathrm{id}_{\mathscr{H}_\infty}. \tag{2.5}$$

We denote by $P_n^\perp := \mathrm{id}_{\mathscr{H}} - P_n$ and $P_\infty^\perp := \mathrm{id}_{\mathscr{H}} - P_\infty$ the corresponding complementary orthogonal projections. We collect some obvious but useful equalities:

Lemma 2.3. *Let $\mathscr{H}$ be a parent space with corresponding isometries factorising the identification operators $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$. Then we have*

$$P_n \iota_n = \iota_n, \quad \iota_n^* P_n = \iota_n^*, \quad P_\infty \iota_\infty = \iota_\infty, \quad \iota_\infty^* P_\infty = \iota_\infty^*, \tag{2.6a}$$

$$P_n^\perp \iota_n = 0, \quad \iota_n^* P_n^\perp = 0, \quad P_\infty^\perp \iota_\infty = 0, \quad \iota_\infty^* P_\infty^\perp = 0, \tag{2.6b}$$

$$D_n = P_\infty D_n P_n + P_\infty^\perp D_n P_n + P_\infty D_n P_n^\perp. \tag{2.6c}$$

Proof. (2.6a) and (2.6b) follow directly from (2.5). (2.6c) is a direct consequence of $P_n^\perp D_n P_\infty^\perp = 0$ as $P_\infty^\perp \iota_\infty = 0$ and $\iota_n^* P_n^\perp = 0$. ∎

Let us now assume that we are in the situation of Weidmann's convergence (Definition 1.1). In this case, we have the following result:

Theorem 2.4 (Weidmann's convergence implies QUE-convergence). *Let A_n be a self-adjoint operator in a Hilbert space $\mathscr{H}_n$ for $n \in \overline{\mathbb{N}}$ such that $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with convergence speed $(\delta_n)_n$. Then $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with the same speed $(\delta_n)_n$.*

Proof. Let $\mathscr{H}$ be the parent space with corresponding isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ as in Weidmann's convergence. As identification operators $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ we define $J_n := \iota_\infty^* \iota_n$ for each $n \in \mathbb{N}$. Clearly, the operator norm of J_n is not greater than 1, hence (1.7a) is satisfied with $\delta_n = 0$. Moreover, we want to express $(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n$ appearing in (1.7b) in terms of D_n. We have

$$\iota_n^* P_\infty^\perp D_n \iota_n = \iota_n^* (\mathrm{id}_{\mathscr{H}} - \iota_\infty^* \iota_\infty) \iota_n R_n \iota_n^* \iota_n = (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n.$$

using (2.6a) for the first equality and (2.5) for the second. Similarly, we can express the second operator in (1.7b) as

$$-\iota_\infty^* P_n^\perp D_n \iota_\infty = \iota_\infty^* P_n^\perp \iota_\infty R_\infty \iota_\infty^* \iota_\infty = (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty.$$

Finally, for (1.7c) we have

$$\iota_\infty^* D_n \iota_n = J_n R_n - R_\infty J_n.$$

As all operators appearing in (1.7) can be factorised into D_n and operators with norm not greater than 1, we obtain the estimates required in (1.7a)–(1.7c) with respect to the common resolvent element z_0. Similarly, replacing R_n by R_n^* and using (1.12), we see that (1.7a)–(1.7c) also hold for R_n replaced by R_n^* for all $n \in \overline{\mathbb{N}}$. ■

2.3. QUE-convergence implies Weidmann's convergence if a parent space exists

To prove Weidmann's conditions using quasi-unitary equivalence, we face two main difficulties:

- Can we always construct a parent space $\mathscr{H}$ factorising *given* identification operators $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$?
- If such a decomposition exists, is it possible to use QUE-convergence such that the operator norm of D_n from Weidmann's concept is small?

In this subsection, we *assume* that the first question is answered affirmative, i.e., we assume that a parent space exists. We then answer the second question affirmatively in this subsection in Theorem 2.8.

Let us first express the operator norm of D_n (resp. its three summands in (2.6c)) in term of expressions from QUE-convergence:

Lemma 2.5. *For the norms of the three summands of D_n in (2.6c) we have*

$$\|P_\infty D_n P_n\|_{\mathfrak{L}(\mathscr{H})} = \|J_n R_n - R_\infty J_n\|_{\mathfrak{L}(\mathscr{H}_n, \mathscr{H}_\infty)}, \tag{2.7a}$$

$$\|P_\infty^\perp D_n P_n\|_{\mathfrak{L}(\mathscr{H})} = (\|R_n^* (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)})^{1/2}, \tag{2.7b}$$

$$\|P_\infty D_n P_n^\perp\|_{\mathfrak{L}(\mathscr{H})} = (\|R_\infty^* (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)})^{1/2}. \tag{2.7c}$$

Proof. To prove (2.7a) we calculate

$$P_\infty D_n P_n = \iota_\infty \iota_\infty^* (\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*) \iota_n \iota_n^* = \iota_\infty (J_n R_n - R_\infty J_n) \iota_n^*. \tag{2.8}$$

Using Lemma 2.2 we obtain the claimed norm equality (2.7a). For (2.7b), we first calculate

$$P_\infty^\perp D_n P_n = (\mathrm{id}_{\mathscr{H}} - \iota_\infty \iota_\infty^*) \iota_n R_n \iota_n^* = (\iota_n - \iota_\infty J_n) R_n \iota_n^*$$

using again (2.6b). Moreover, we have

$$\begin{aligned}\|P_\infty^\perp D_n P_n f\|_{\mathscr{H}}^2 &= \langle \iota_n R_n^*(\iota_n^* - J_n^*\iota_\infty)(\iota_n - \iota_\infty J_n) R_n \iota_n^* f, f\rangle_{\mathscr{H}} \\ &= \langle \iota_n R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n \iota_n^* f, f\rangle_{\mathscr{H}} \end{aligned} \tag{2.9}$$

for $f \in \mathscr{H}$, where we used

$$(\iota_n - \iota_\infty J_n)^*(\iota_n - \iota_\infty J_n) = \iota_n^*\iota_n - J_n^*\iota_\infty^*\iota_n - \iota_n^*\iota_\infty J_n + J_n^*\iota_\infty^*\iota_\infty J_n = \mathrm{id}_{\mathscr{H}_n} - J_n^* J_n$$

for the last step. Taking the supremum over $f \in \mathscr{H}$ with $\|f\|_{\mathscr{H}} = 1$ we obtain

$$\|P_\infty^\perp D_n P_n\|^2 = \|\iota_n R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n \iota_n^*\| = \|R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|$$

using (2.9) and the fact that $\iota_n R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n \iota_n^*$ is self-adjoint (first equality) and again Lemma 2.2 (second equality), we arrive at the operator norm equality (2.7b).

Similarly, we show (2.7c). ■

Remark 2.6 (A problem with the (co-)isometry being at the "wrong" side). Note that the second and third summand (2.7b)–(2.7c) of D_n, we do not have a norm equality in terms of the one from QUE-convergence *without* square root. A direct calculation shows that we also have

$$(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n = \iota_n^*(\mathrm{id}_{\mathscr{H}} - \iota_\infty\iota_\infty^*)\iota_n R_n = \iota_n^* P_\infty^\perp \iota_n R_n \iota_n^* \iota_n = \iota_n^* P_\infty^\perp D_n P_n \iota_n$$

using Lemma 2.3. Taking the operator norm yields

$$\|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} = \|\iota_n^* P_\infty^\perp D_n P_n \iota_n\|_{\mathfrak{L}(\mathscr{H}_n)} \le \|P_\infty^\perp D_n P_n\|_{\mathfrak{L}(\mathscr{H})}$$

("≤"holds, but not the needed "≥"). Actually, the isometry and the co-isometry are on the "wrong" side of the operator $P_\infty^\perp D_n P_n$, see Lemma 2.2.

Similarly, for the third term (2.7c) we have

$$\begin{aligned}(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty &= \iota_\infty^*(\mathrm{id}_{\mathscr{H}} - \iota_n\iota_n^*)\iota_\infty R_\infty \\ &= \iota_\infty^* P_n^\perp \iota_\infty R_\infty \iota_\infty^* \iota_\infty \\ &= -\iota_\infty^* P_n^\perp D_n P_\infty \iota_\infty,\end{aligned}$$

hence we again only have the "wrong" estimate

$$\|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} = \|\iota_\infty^* P_n^\perp D_n P_\infty \iota_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \le \|P_n^\perp D_n P_\infty\|_{\mathfrak{L}(\mathscr{H})}.$$

Before proving that QUE-convergence implies Weidmann's convergence, we need another technical result:

Lemma 2.7. *Assume that* $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ *converges in generalised norm resolvent sense in the sense of Definition 1.5 with identification operators* J_n *fulfilling* $\|J_n\| \le 1$ *and with convergence speed* $(\delta_n)_n$. *Then we have*

$$\|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \le \|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} + 2\delta_n. \tag{2.10}$$

Proof. We have

$$\begin{aligned}\|R_n\| &\le \|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)R_n\| + \|J_n^*(J_n R_n - R_\infty J_n)\| + \|J_n^* R_\infty J_n\| \\ &\le 2\delta_n + \|R_\infty\|.\end{aligned}$$

∎

We now prove our next main result:

Theorem 2.8 (QUE-convergence implies Weidmann's if a parent space exists). *Let* A_n *be a self-adjoint operator in a Hilbert space* $\mathscr{H}_n$ *for* $n \in \overline{\mathbb{N}}$. *If* $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ *and if a parent space exists that factorises the identification operators (i.e. (1.9) holds), then* $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ *Weidmann-converges with convergence speed*

$$\tilde{\delta}_n := \delta_n^{1/2} \cdot (2(\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} + 2\delta_n)^{1/2} + \delta_n^{1/2}) \in \mathrm{O}(\delta_n^{1/2}). \tag{2.11}$$

Proof. Assume that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with convergence speed δ_n. We estimate the norms of the three summands of D_n (cf. (2.6c)) given in Lemma 2.5 and obtain

$$\|P_n D_n P_\infty\| = \|J_n R_n^* - R_\infty^* J_n\| \le \delta_n$$

for the first summand. For the norm of the second summand (2.7b)–(2.7c) we have

$$\begin{aligned}\|P_\infty^\perp D_n P_n\|_{\mathfrak{L}(\mathscr{H})} &\le (\|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)R_n\|_{\mathfrak{L}(\mathscr{H}_n)})^{1/2} \\ &\le (\|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \cdot \delta_n)^{1/2} \\ &\le ((\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} + 2\delta_n) \cdot \delta_n)^{1/2}\end{aligned} \tag{2.12a}$$

using also Lemma 2.7. For the third summand, we have

$$\begin{aligned}\|P_\infty D_n P_n^\perp\|_{\mathfrak{L}(\mathscr{H})} &\le (\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*)R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)})^{1/2} \\ &\le (\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \cdot \delta_n)^{1/2}.\end{aligned} \tag{2.12b}$$

In particular, we obtain the following norm estimate used in Weidmann's generalised norm resolvent convergence

$$\begin{aligned}\|D_n\|_{\mathfrak{L}(\mathscr{H})} &\le \|P_\infty D_n P_n\|_{\mathfrak{L}(\mathscr{H})} + \|P_\infty^\perp D_n P_n\|_{\mathfrak{L}(\mathscr{H})} + \|P_\infty D_n P_n^\perp\|_{\mathfrak{L}(\mathscr{H})} \\ &\le \delta_n + ((\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} + 2\delta_n)\delta_n)^{1/2} + (\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)}\delta_n)^{1/2} \\ &\le \delta_n^{1/2} \cdot (\delta_n^{1/2} + 2(\|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} + 2\delta_n)^{1/2}).\end{aligned}$$

∎

Remark 2.9 (Characterisation of Weidmann's convergence in a parent space). We can interpret Theorems 2.4 and 2.8 in the following way: the convergence $\|D_n\| \to 0$ (i.e., Weidmann's generalised norm resolvent convergence) is equivalent with $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ among all parent spaces factorising the identification operators.

Remark 2.10 (Defect operators). If the operator $\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n$ is an orthogonal projection (i.e. if $\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n = (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^2$), then we actually have

$$\|P_\infty^\perp D_n P_n\|^2_{\mathfrak{L}(\mathscr{H})} = \|R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^2 R_n\|_{\mathfrak{L}(\mathscr{H}_n)} = \|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|^2_{\mathfrak{L}(\mathscr{H}_n)},$$

hence there is no square root needed in. The so-called *defect operator*

$$W_n = (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^{1/2}$$

plays an important role in Section 3.1 in the construction of a parent space. We also give equivalent characterisations when W_n is an orthogonal projection in Theorem 3.9. A similar remark holds for $(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*)$.

Remark 2.11 (a modified version of QUE-convergence). We see that we loose the convergence speed in the two "bad" estimates (2.12a)–(2.12b). Probably it is more appropriate to change slightly the definition of QUE-convergence and require that

$$\begin{aligned} \|R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} &\le \delta_n^2 \to 0, \\ \|R_\infty^*(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} &\le \delta_n^2 \to 0 \end{aligned} \tag{1.7b$'$}$$

holds in Definition 1.5. Then $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ (with the modified definition) with convergence speed δ_n would lead to $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with convergence speed $3\delta_n$, as one can directly see from (2.6c) and Lemma 2.5. If $\|R_n\| \le 1$ then (1.7b′) implies (1.7b).

Note that the first estimate in (1.7b′) is equivalent with

$$\|f_n\|^2_{\mathscr{H}_n} - \|J_n f_n\|^2_{\mathscr{H}_\infty} \le \delta_n^2 \|(A_n + 1) f_n\|^2_{\mathscr{H}_n} \tag{2.13}$$

for all $f_n \in \mathrm{dom}\, A_n$. Note that we always have $0 \le \|f_n\|^2_{\mathscr{H}_n} - \|J_n f_n\|^2_{\mathscr{H}_\infty}$ as J_n is a contraction. The original first estimate in (1.7b) is equivalent with

$$\|f_n - J_n^* J_n f_n\|^2_{\mathscr{H}_n} \le \delta_n^2 \|(A_n + 1) f_n\|^2_{\mathscr{H}_n} \tag{2.14}$$

for all $f_n \in \mathrm{dom}\, A_n$. Note that (2.13) implies (2.14) (if $\|R_n\| \le 1$, and with a slightly different δ_n in the general case, see Lemma 2.7). A similar remark holds for the second estimate in (1.7b′).

We have already seen in [11, Section 3.2] that the stronger estimate (2.13) is also more appropriate than the weaker one (2.14) when showing spectral convergence (see [11, Theorem 3.5]).

2.4. QUE-convergence implies Weidmann's convergence: better convergence speed

As we have seen, a loss in the speed of convergence occurs, when passing from QUE- to Weidmann's convergence in Theorem 2.8. Under a commutator condition on the projections, the convergence speed in Weidmann's convergence is the same as in QUE-convergence:

Theorem 2.12 (QUE-convergence implies Weidmann's: better convergence speed). *Let A_n be a self-adjoint operator in a Hilbert space $\mathscr{H}_n$ for $n \in \overline{\mathbb{N}}$ such that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ QUE-converges with convergence speed $(\delta_n)_n$. If furthermore a parent space exists that factorises the identification operators (i.e. (1.9) holds), and if the projections $P_n = \iota_n \iota_n^*$ and $P_\infty = \iota_\infty \iota_\infty^*$ commute (i.e., (1.10) holds) then $(A_n)_n$ converges in generalised norm resolvent in the sense of Weidmann with convergence speed $(3\delta_n)_n$.*

Proof. We again use the decomposition of D_n in three terms as in (2.6c). The first term causes no problems (see (2.7a) and (2.8) and can be estimated by δ_n.

For the second and third summand in (2.6c) we need the assumption $P_n P_\infty = P_\infty P_n$. Here, we have

$$\begin{aligned}
P_\infty^\perp D_n P_n &= (\mathrm{id}_{\mathscr{H}} - P_\infty)\iota_n R_n \iota_n^* \\
&= (\mathrm{id}_{\mathscr{H}} - P_\infty P_n)\iota_n R_n \iota_n^* \\
&= (\mathrm{id}_{\mathscr{H}} - P_n P_\infty)\iota_n R_n \iota_n^* \\
&= \iota_n (\mathrm{id}_{\mathscr{H}_n} - \iota_n^* \iota_\infty \iota_\infty^* \iota_n) R_n \iota_n^* \\
&= \iota_n (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n \iota_n^*
\end{aligned}$$

using $\iota_n = P_n \iota_n$ for the second equality and where we used that P_n and P_∞ commute for the third equality. In particular, we have again an isometry on the left and a co-isometry on the right, hence the operator norm equality

$$\|P_\infty^\perp D_n P_n\| = \|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|$$

(see Lemma 2.2). Similarly, we have

$$P_\infty D_n P_n^\perp = -\iota_\infty R_\infty P_n^\perp = -\iota_\infty R_\infty (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*)\iota_\infty^*$$

and

$$\|P_\infty D_n P_n^\perp\| = \|R_\infty (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*)\| = \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty^*\|.$$

In particular, we can estimate $\|D_n\|$ by the three terms above and hence have $\|D_n\| \le 3\delta_n$. ∎

We now give equivalent characterisations for the commuting condition $P_n P_\infty = P_\infty P_n$:

Theorem 2.13 (Equivalent characterisation of partial isometries). *Assume that the operator $J_n\colon \mathscr{H}_n \to \mathscr{H}_\infty$ is a contraction then the following assertions are equivalent.*

1. *J_n is a partial isometry.*
2. *For one parent space (for all parent spaces) with corresponding isometries ι_n and ι_∞ factorising J_n we have*
$$\iota_n(\mathscr{H}_n) \cap \iota_\infty(\mathscr{H}_\infty) = \iota_n(\mathscr{H}_n'),$$
where $\mathscr{H}_n' = (\ker J_n)^\perp = J_n^*(\mathscr{H}_\infty)$.
3. *For one parent space (for all parent spaces) with corresponding isometries ι_n and ι_∞ factorising J_n we have*
$$\iota_n(\mathscr{H}_n) \cap \iota_\infty(\mathscr{H}_\infty) = \iota_\infty(\mathscr{H}_\infty'),$$
where $\mathscr{H}_\infty' = (\ker J_n^*)^\perp = J_n(\mathscr{H}_n)$.
4. *For one parent space (for all parent spaces) with corresponding isometries ι_n and ι_∞ factorising J_n and with orthogonal projections P_n and P_∞ we have $P_n P_\infty = P_\infty P_n$.*

Proof. As (1) is formulated without reference to the parent space, the other assertions are true for one parent space resp. for all parent spaces factorising J_n.

(1) $\implies$ (2) "$\subseteq$". Let $f = \iota_n f_n = \iota_\infty f_\infty$. We have to show that we can actually choose $f_n' \in \mathscr{H}_n'$ such that $\iota_n f_n' = \iota_n f_n = f$. Let $f_n' = J_n^* f_\infty \in J_n^*(\mathscr{H}_\infty) = (\ker J_n)^\perp = \mathscr{H}_n'$, then
$$\iota_n f_n' = \iota_n J_n^* f_\infty = \iota_n \iota_n^* \iota_\infty f_\infty = \iota_n \iota_n^* \iota_n f_n = \iota_n f_n = f,$$
i.e., $f \in \iota_n(\mathscr{H}_n')$.

(1) $\implies$ (2) "$\supseteq$". Let $f = \iota_n f_n' \in \iota_n(\mathscr{H}_n') \subset \iota_n(\mathscr{H}_n)$. We have to show that $f \in \iota_\infty(\mathscr{H}_\infty)$. As the latter space is the final space of the isometry ι_∞, we use the characterisation Lemma 2.1 (5): We have
$$\|\iota_\infty^* f\|_{\mathscr{H}_\infty} = \|\iota_\infty^* \iota_n f_n'\|_{\mathscr{H}_\infty} = \|J_n f_n'\|_{\mathscr{H}_\infty} = \|f_n'\|_{\mathscr{H}_n} = \|\iota_n f_n'\|_{\mathscr{H}} = \|f\|_{\mathscr{H}},$$
where we used for the third equation that f_n' is in the initial space $\mathscr{H}_n' = (\ker J_n)^\perp$ of the partial isometry J_n. In particular, we have shown that $f \in \iota_\infty(\mathscr{H}_\infty)$.

(1) $\implies$ (3). The proof is literally the same as the one for (1) $\implies$ (2), just interchange $(\cdot)_\infty$ and J_n^* with $(\cdot)_n$ and J_n, respectively.

(2) $\implies$ (4). We first observe that $\iota_n(\mathscr{H}_n'') \perp \iota_\infty(\mathscr{H}_\infty)$, where $\mathscr{H}_n'' = (\mathscr{H}_n')^\perp = \ker J_n$: Let $f = \iota_n f_n''$ with $J_n f_n'' = 0$ and $g = \iota_\infty g_\infty \in \iota_\infty(\mathscr{H}_\infty)$. Then
$$\langle f, g\rangle_{\mathscr{H}} = \langle \iota_n f_n'', \iota_\infty g_\infty\rangle_{\mathscr{H}} = \langle \iota_\infty^* \iota_n f_n'', g_\infty\rangle_{\mathscr{H}_\infty} = \langle J_n f_n'', g_\infty\rangle_{\mathscr{H}_\infty} = 0.$$

Now, we have

$$\iota_n(\mathscr{H}_n) = \iota_n(\mathscr{H}_n') \oplus \iota_n(\mathscr{H}_n'') = (\iota_n(\mathscr{H}_n) \cap \iota_\infty(\mathscr{H}_\infty)) \oplus \iota_n(\mathscr{H}_n'')$$

and

$$\iota_\infty(\mathscr{H}_\infty) = (\iota_n(\mathscr{H}_n) \cap \iota_\infty(\mathscr{H}_\infty)) \oplus \widehat{\mathscr{H}}^\perp,$$

where $\widehat{\mathscr{H}}^\perp$ is defined by the last line. As $\widehat{\mathscr{H}}^\perp \subset \iota_\infty(\mathscr{H}_\infty)$ we have $\widehat{\mathscr{H}}^\perp \perp \iota_n(\mathscr{H}_n'')$. From [26, Aufgabe 2.22] we conclude that $P_n P_\infty = P_\infty P_n$.

(3) $\implies$ (4). Again, the proof is literally the same as the one for (2) $\implies$ (4), just interchange $(\cdot)_\infty$ and J_n^* with $(\cdot)_n$ and J_n, respectively.

(4) $\implies$ (1). We have

$$J_n J_n^* J_n = \iota_\infty^* \iota_n \iota_n^* \iota_\infty \iota_\infty^* \iota_n = \iota_\infty^* P_n P_\infty \iota_n = \iota_\infty^* P_\infty P_n \iota_n = \iota_\infty^* \iota_n = J_n.$$

By Lemma 2.1, J_n is a partial isometry. ∎

One consequence of Theorem 2.13 is that the commuting property of the projections is an invariant, i.e., only depending on the quasi-unitary setting given by $(J_n : \mathscr{H}_n \to \mathscr{H}_\infty)_{n \in \mathbb{N}}$:

Corollary 2.14 (Projection commuting is invariant). *If there is a parent space with isometries factorising J_n such that $P_n P_\infty = P_\infty P_n$ then this is true for* all *parent spaces.*

We immediately conclude from Theorems 2.12 and 2.13:

Corollary 2.15 (J_n is a partial isometry). *Assume that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ QUE-converges with identification operators J_n and convergence speed $(\delta_n)_n$ and that a parent space with isometries factorising J_n exists.*

If J_n are partial isometries for all $n \in \mathbb{N}$ then $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with speed $(3\delta_n)_n$.

Two special cases often appearing in applications deserve to be mentioned. Although the claims follow from the fact that a (co-)isometry is a partial isometry, we give explicit proofs here (as they are rather simple):

Corollary 2.16 (J_n is an isometry). *Assume that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ QUE-converges with speed $(\delta_n)_n$. If the identification operators J_n are isometries for all $n \in \mathbb{N}$ then $\mathscr{H} = \mathscr{H}_\infty$ is a parent space factorising J_n and $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with speed $(2\delta_n)_n$.*

Remark. Note that if J_n is an isometry, then $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ just means that

$$\|J_n R_n J_n^* - R_\infty\|_{\mathfrak{L}(\mathscr{H})} \to 0.$$

Proof. If J_n is an isometry, then clearly $\mathscr{H} = \mathscr{H}_\infty$ is a parent space with isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ and $\iota_n = J_n$. Moreover, $\iota_\infty = \mathrm{id}_{\mathscr{H}_\infty}$ is unitary and we can add ι_∞ on the right and ι_∞^* on the left-hand side. In particular, we have

$$\begin{aligned} \|D_n\|_{\mathfrak{L}(\mathscr{H})} &= \|\iota_\infty^*(\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*)\iota_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \\ &= \|J_n R_n J_n^* - R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \\ &\le \|J_n(R_n J_n^* - J_n^* R_\infty)\|_{\mathfrak{L}(\mathscr{H}_\infty)} + \|(J_n J_n^* - \mathrm{id}_{\mathscr{H}_\infty}) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \\ &\le 2\delta_n. \end{aligned}$$ ∎

Corollary 2.17 (J_n is a co-isometry). *Assume that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ QUE-converges with speed $(\delta_n)_n$ and that a parent space factorising J_n exists. If J_n is a co-isometry for each $n \in \mathbb{N}$ then $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ with speed $(2\delta_n)_n$. Moreover, $\iota_\infty(\mathscr{H}_\infty) \subset \iota_n(\mathscr{H}_n)$ for all $n \in \mathbb{N}$.*

Proof. If J_n^* is an isometry then $\mathscr{H}'_\infty = \mathscr{H}_\infty$ and $\iota_\infty(\mathscr{H}_\infty) \subset \iota_n(\mathscr{H}_n)$ by using Theorem 2.13 (3), and therefore ι_n can be added on the left and ι_n^* on the right-hand side. In particular, we have

$$\begin{aligned} \|D_n\|_{\mathfrak{L}(\mathscr{H})} &= \|\iota_n^*(\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*)\iota_n\|_{\mathfrak{L}(\mathscr{H}_n)} \\ &= \|R_n - J_n^* R_\infty J_n\|_{\mathfrak{L}(\mathscr{H}_\infty)} \\ &\le \|(\mathrm{id} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} + \|J_n^*(J_n R_n - R_\infty J_n)\|_{\mathfrak{L}(\mathscr{H}_n)} \le 2\delta_n. \end{aligned}$$ ∎

3. From QUE-convergence to Weidmann's convergence: the general case

3.1. Defect operators and existence of a parent space

The main result in this subsection is to prove the existence of a parent space $\mathscr{H}$ with isometries

$$\iota_n \colon \mathscr{H}_n \to \mathscr{H} \quad (n \in \overline{\mathbb{N}})$$

factorising the identification operators $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$. As $J_n = \iota_\infty^* \iota_n$, we necessarily have

$$\|J_n\|_{\mathfrak{L}(\mathscr{H}_n, \mathscr{H}_\infty)} \le 1 \quad \text{for all } n \in \mathbb{N}, \tag{3.1}$$

hence we assume (3.1) throughout this subsection.

Our main result in this subsection is as follows:

Theorem 3.1 (a parent space exists). *If $J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ are contractions ($\|J_n\| \le 1$) for all $n \in \mathbb{N}$, then there is a parent space $\mathscr{H}$ and isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ factorising the identification operators J_n, i.e., $J_n = \iota_\infty^* \iota_n$.*

Together with Theorem 2.8 we immediately conclude:

Corollary 3.2 (QUE-convergence implies Weidmann's one, the case of contractions). *Let A_n be a self-adjoint operator in a Hilbert space $\mathscr{H}_n$ for $n \in \overline{\mathbb{N}}$. If $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with identification operators J_n fulfilling $\|J_n\| \le 1$ and with convergence speed $(\delta_n)_n$. Then $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ Weidmann-converges with convergence speed $\tilde{\delta}_n \in \mathrm{O}(\delta_n^{1/2})$ given in* (2.11).

The proof of Theorem 3.1 follows from the statement in Lemma 3.6 below. Before, we need some preparations and start with an important ingredient in the isometry ι_n already used in [24, Section I.3]:

Lemma 3.3. *Assume* (3.1).

1. *The so-called* defect operators
$$W_n := (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^{1/2} \quad \textit{and} \quad W_{\infty,n} := (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*)^{1/2} \tag{3.2}$$
are well defined and self-adjoint with spectrum in $[0, 1]$. *Moreover, the spectra of W_n and $W_{\infty,n}$ agree including multiplicity, except for the value* 1*, i.e.,*
$$\mathrm{spec}(W_n) \setminus \{1\} = \mathrm{spec}(W_{\infty,n}) \setminus \{1\}.$$
2. *We have*
$$J_n W_n = W_{\infty,n} J_n \quad \textit{and} \quad W_n J_n^* = J_n^* W_{\infty,n}. \tag{3.3}$$
3. *We have*
$$\|J_n f_n\|^2_{\mathscr{H}_\infty} + \|W_n f_n\|^2_{\mathscr{H}_n} = \|f_n\|^2_{\mathscr{H}_n} \tag{3.4a}$$
and
$$\|J_n^* f_\infty\|^2_{\mathscr{H}_n} + \|W_{\infty,n} f_\infty\|^2_{\mathscr{H}_\infty} = \|f_\infty\|^2_{\mathscr{H}_\infty}. \tag{3.4b}$$

Proof. (1) As $\|J_n\| \le 1$ we have $0 \le \mathrm{id}_{\mathscr{H}_n} - J_n^* J_n$, hence the square root W_n is well defined. Moreover, we have $\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n \le \mathrm{id}_{\mathscr{H}_n}$, hence the spectrum of $\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n$ and therefore of W_n lies in $[0, 1]$. The claim for $W_{\infty,n}$ follows similarly. The assertion on the spectra of W_n and $W_{\infty,n}$ follows from the fact that $\mathrm{id}_{\mathscr{H}_n} - W_n^2 = J_n^* J_n$ and $\mathrm{id}_{\mathscr{H}_\infty} - W_{\infty,n}^2 = J_n J_n^*$ and the fact that the spectra of $J_n^* J_n$ and $J_n J_n^*$ agree except for the value 0 (see e.g. [16, Section 1.2])

(2) For (3.3), we have
$$J_n (W_n)^2 = J_n (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) = J_n - J_n J_n^* J_n = (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) J_n = (W_{\infty,n})^2 J_n,$$
and this can be extended to
$$J_n p((W_n)^2) = p((W_{\infty,n})^2) J_n$$

for any polynomial p defined on $\mathbb{R}$. As we can approximate $t \mapsto \sqrt{t}$ uniformly on $\operatorname{spec}(W_n)$ resp. $\operatorname{spec}(W_{\infty,n}) \subset [0,1]$ by a polynomial, and as uniform convergence turns into operator convergence for the operator functions, we conclude $J_n W_n = W_{\infty,n} J_n$. The second equality in (3.3) follows by taking adjoints.

(3) We have

$$\begin{aligned}\|J_n f_n\|^2_{\mathscr{H}_\infty} + \|W_n f_n\|^2_{\mathscr{H}_n} &= \langle f_n, J_n^* J_n f_n\rangle_{\mathscr{H}_n} + \langle f_n, (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) f_n\rangle_{\mathscr{H}_n} \\ &= \|f_n\|^2_{\mathscr{H}_n}\end{aligned}$$

and similarly for (3.4b). ∎

Using the definition of the defect operators, we conclude from Lemma 2.5:

Corollary 3.4. *If $\mathscr{H}$ is a parent space with corresponding isometries factorising J_n and orthogonal projections P_n, we have*

$$\begin{aligned}\|P_\infty^\perp D_n P_n\|^2_{\mathfrak{L}(\mathscr{H})} &= \|W_n R_n\|^2_{\mathfrak{L}(\mathscr{H}_n)} \\ &= \|R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \\ &\le \|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned}\|P_\infty D_n P_n^\perp\|^2_{\mathfrak{L}(\mathscr{H})} &= \|W_{\infty,n} R_\infty\|^2_{\mathfrak{L}(\mathscr{H}_\infty)} \\ &= \|R_\infty^*(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \\ &\le \|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)}. \end{aligned} \tag{3.5b}$$

Proof. We have

$$\|W_n R_n\|^2_{\mathfrak{L}(\mathscr{H}_n)} = \|R_n^* W_n^2 R_n\|_{\mathfrak{L}(\mathscr{H}_n)} = \|R_n^*(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)},$$

hence the first equality follows from (2.7b). A similar argument holds for (3.5b). ∎

Let us now define the parent space:

Definition 3.5 (Parent space associated with identification operators). Let

$$(J_n \colon \mathscr{H}_n \to \mathscr{H}_\infty)_{n \in \mathbb{N}}$$

be a sequence of contractions. We call

$$\mathscr{H} = \mathscr{H}_\infty \oplus \bigoplus_{n=1}^\infty \mathscr{H}_n, \quad f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H} \tag{3.6a}$$

with

$$\|f\|^2_{\mathscr{H}} = \|f_\infty\|^2_{\mathscr{H}_\infty} + \sum_{n=1}^{\infty} \|f_n\|^2_{\mathscr{H}_n} < \infty \tag{3.6b}$$

the *associated parent space*. Moreover, we set

$$\iota_n : \mathscr{H}_n \to \mathscr{H}, \quad \iota_n f_n = (J_n f_n, 0, \dots, 0, W_n f_n, 0, \dots) \tag{3.7a}$$

for $n \in \mathbb{N}$, where the entry $W_n f_n$ is at the n-th component. Finally, we set

$$\iota_\infty : \mathscr{H}_\infty \to \mathscr{H}, \quad \iota_\infty f_\infty = (f_\infty, 0, \dots). \tag{3.7b}$$

We call $(\iota_n)_{n\in\overline{\mathbb{N}}}$ the *associated isometries*.

The maps ι_n are actually isometries factorising J_n:

Lemma 3.6 (Proof of Theorem 3.1). 1. *ι_n and ι_∞ are isometries.*

2. *The adjoints $\iota_n^* : \mathscr{H} \to \mathscr{H}_n$ and $\iota_\infty^* : \mathscr{H} \to \mathscr{H}_\infty$ act as*

$$\iota_n^* g = J_n^* g_\infty + W_n g_n \quad \textit{and} \quad \iota_\infty^* g = g_\infty,$$

respectively, where $g = (g_\infty, g_1, g_2, \dots) \in \mathscr{H}$.

3. *We have $J_n = \iota_\infty^* \iota_n$ and $J_n^* = \iota_n^* \iota_\infty$.*

Proof. (1) ι_n is an isometry, as

$$\begin{aligned}\|\iota_n f_n\|^2_{\mathscr{H}} &= \|J_n f_n\|^2_{\mathscr{H}_\infty} + \sum_{k=1}^{\infty} \|(\iota_n f_n)_k\|^2_{\mathscr{H}_k} \\ &= \|J_n f_n\|^2_{\mathscr{H}_\infty} + \|W_n f_n\|^2_{\mathscr{H}_n} = \|f_n\|^2_{\mathscr{H}_n}\end{aligned}$$

using (3.4a). Moreover, that ι_∞ is an isometry is obvious.

(2) We have

$$\begin{aligned}\langle \iota_n f_n, g\rangle_{\mathscr{H}} &= \langle J_n f_n, g_\infty\rangle_{\mathscr{H}_\infty} + \sum_{k=1}^{\infty} \langle (\iota_n f_n)_k, g_k\rangle_{\mathscr{H}_k} \\ &= \langle J_n f_n, g_\infty\rangle_{\mathscr{H}_\infty} + \langle W_n f_n, g_n\rangle_{\mathscr{H}_n} \\ &= \langle f_n, J_n^* g_\infty + W_n g_n\rangle_{\mathscr{H}_n}\end{aligned}$$

for $f_n \in \mathscr{H}_n$ and $g \in \mathscr{H}$, as $W_n^* = W_n$. The claim $\iota_\infty^* g = g_\infty$ is obvious.

(3) Obviously, we have $\iota_\infty^* \iota_n f_n = (\iota_n f_n)_\infty = J_n f_n$, and the second claim follows by taking adjoints. ∎

Remark 3.7. We can express Weidmann's resolvent difference here as

$$\begin{aligned} D_n f &= \iota_n R_n \iota_n^* f - \iota_\infty R_\infty \iota_\infty^* f \\ &= \big((J_n R_n J_n^* - R_\infty) f_\infty + J_n R_n W_n f_n, 0, \dots, \\ &\qquad 0, W_n R_n (J_n^* f_\infty + W_n f_n), 0, \dots\big) \end{aligned} \tag{3.8}$$

for all $f = (f_\infty, f_1, \dots, 0, f_n, 0, \dots) \in \mathscr{H}$. Moreover, a direct proof of Corollary 3.2 is possible using Corollary 3.4 and the QUE-convergence.

3.2. Another equivalent characterisation of commuting projections

For further purposes, let us now calculate the projections $P_n = \iota_n \iota_n^*$ and $P_\infty = \iota_\infty \iota_\infty^*$ in this concrete situation: From Lemma 3.6 (2) we conclude

$$P_n f = \big(J_n J_n^* f_\infty + J_n W_n f_n, 0, \dots, 0, W_n J_n^* f_\infty + (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) f_n, 0, \dots\big) \tag{3.9a}$$

where the entry with W_n is at the n-th position. Moreover,

$$P_\infty f = (f_\infty, 0, \dots), \tag{3.9b}$$

where, as usual, $f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H}$.

We consider now the commutator $P_n P_\infty - P_\infty P_n$ needed in Theorem 2.12 in the concrete case here:

Lemma 3.8. *We have*

$$(P_n P_\infty - P_\infty P_n) f = (-J_n W_n f_n, 0, \dots, 0, W_n J_n^* f_\infty, 0, \dots) \tag{3.10a}$$

$$= (-W_{\infty,n} J_n f_n, 0, \dots, 0, J_n^* W_{\infty,n} f_\infty, 0, \dots) \tag{3.10b}$$

for all $f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H}$.

Proof. We have

$$P_n P_\infty f = P_n (f_\infty, 0, 0, \dots) = (J_n J_n^* f_\infty, 0, \dots, 0, W_n J_n^* f_\infty, 0, \dots)$$

and

$$P_\infty P_n f = (P_n f)_\infty = (J_n J_n^* f_\infty + J_n W_n f_n, 0, \dots);$$

hence (3.10a) follows. Equality (3.10b) is a consequence of (3.3). ∎

We now continue with Theorem 2.13 within the setting of the concrete parent space constructed above:

Theorem 3.9 (Equivalent characterisation of partial isometries). *Let $n \in \mathbb{N}$, then the following assertions are equivalent:*

1. *the identification operator J_n is a partial isometry;*
2. *the defect operator W_n is an orthogonal projection (onto* $\ker J_n$*);*
3. $\operatorname{spec}(W_n) \subset \{0, 1\}$*;*
4. $J_n W_n = 0$*;*
5. *the identification operator J_n^* is a partial isometry;*
6. *the defect operator $W_{\infty,n}$ is an orthogonal projection (onto* $\ker J_n^*$*);*
7. $\operatorname{spec}(W_{\infty,n}) \subset \{0, 1\}$*;*
8. $W_{\infty,n} J_n = 0$*;*
9. *the projections commute, i.e., $P_n P_\infty = P_\infty P_n$.*

Proof. (1) $\implies$ (2). We have

$$W_n^4 = (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^2 = \mathrm{id}_{\mathscr{H}_n} - 2J_n^* J_n + J_n^* J_n J_n^* J_n = \mathrm{id}_{\mathscr{H}_n} - J_n^* J_n = W_n^2$$

as $J_n = J_n J_n^* J_n$. In particular, W_n^2 is idempotent and (clearly) self-adjoint, hence an orthogonal projection with

$$W_n^2 f_n = f_n \iff J_n^* J_n f_n = 0 \iff f_n \in \ker J_n,$$

i.e., onto $\ker J_n$. For an orthogonal projection P, we also have $P^{1/2} = P$, and hence W_n itself is an orthogonal projection.

(2) $\iff$ (3) and (6) $\iff$ (7) follow from the fact that a self-adjoint operator P is an orthogonal projection if and only if $\operatorname{spec}(P) \subset \{0, 1\}$.

(2) $\implies$ (4) is obvious.

(1) $\iff$ (5) follows from Lemma 2.1 (1) $\iff$ (2).

(5) $\implies$ (6) $\implies$ (8) follow as above (use $(W_{\infty,n} J_n)^* = J_n^* W_{\infty,n}$).

(4) or (8) $\implies$ (9) follows from Lemma 3.8.

(9) $\implies$ (1) has already been proven in Theorem 2.13 (4) $\implies$ (1). ∎

In the special case of (co-)isometries we have:

Corollary 3.10 ((co-)isometries). *J_n is an isometry if and only if $W_n = 0$; similarly, J_n is a co-isometry if and only if $W_{\infty,n} = 0$.*

Let us finally state another property of our parent space constructed here:

Proposition 3.11. *Assume that* $\|J_n\| \le 1$ *for all* $n \in \mathbb{N}$ *and that* (1.7b) *holds; then there is a parent space such that* $P_n \xrightarrow{s} P_\infty$ *strongly, i.e.,* $\|P_n f - P_\infty f\|_{\mathscr{H}} \to 0$ *for all* $f \in \mathscr{H}$.

Proof. Let $\mathscr{H}$ be the parent space as constructed in Section 3.1. For $f \in \mathscr{H}$ with $f = (f_\infty, f_1, f_2, \dots)$ such that $f_n \neq 0$ only for a finite number of elements, say $f_n = 0$ for all $n > n_0$ and some $n_0 \in \mathbb{N}$. Moreover, we assume that $f_\infty = R_\infty g_\infty$ for some $g_\infty \in \mathscr{H}_\infty$. Then we have

$$\begin{aligned}\|(P_n - P_\infty) f\|^2_{\mathscr{H}} &= \|-(f_\infty - J_n J_n^* f_\infty) + J_n W_n f_n\|^2_{\mathscr{H}_\infty} \\ &\quad + \|(f_n - J_n^* J_n f_n) + J_n^* W_{\infty,n} f_\infty\|^2_{\mathscr{H}_n} \\ &= \|f_\infty - J_n J_n^* f_\infty\|^2_{\mathscr{H}_\infty} + \|J_n^* W_{\infty,n} f_\infty\|^2_{\mathscr{H}_n} \\ &\le \|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) R_\infty g_\infty\|^2_{\mathscr{H}_\infty} + \|W_{\infty,n} R_\infty g_\infty\|^2_{\mathscr{H}_\infty} \qquad (3.11)\end{aligned}$$

for $n > n_0$ by (3.3) and (3.9a)–(3.9b). Now,

$$\|W_{\infty,n} R_\infty\|^2_{\mathfrak{L}(\mathscr{H}_\infty)} \le \|R_\infty\|_{\mathfrak{L}(\mathscr{H}_\infty)} \delta_n \to 0$$

as $n \to \infty$ using (3.5b) and (1.7b). In particular, we conclude that

$$\|(P_n - P_\infty) f\|^2_{\mathscr{H}} \to 0 \quad \text{as } n \to \infty.$$

Since the set of f of the above form are dense in $\mathscr{H}$, the result follows. ∎

A simple consequence is the following:

Corollary 3.12. *The commutator of* P_n *and* P_∞ *converges strongly to* 0.

Proof. We have

$$\begin{aligned}\|(P_n P_\infty - P_\infty P_n) f\|_{\mathscr{H}} &= \|(P_n P_\infty - P_n + P_\infty - P_\infty P_n + P_n - P_\infty) f\|_{\mathscr{H}} \\ &\le \|(P_n - P_\infty) f\|_{\mathscr{H}} + \|(P_\infty - P_n) f\|_{\mathscr{H}} \\ &\quad + \|(P_\infty - P_n) f\|_{\mathscr{H}}\end{aligned}$$

for all $f \in \mathscr{H}$, and the result follows from Proposition 3.11. ∎

Remarks 3.13 (convergence of the projections). We have seen the relevance of $P_n \xrightarrow{s} P_\infty$ already in Stummel's concept discussed near (1.11).

1 (Strong convergence not an invariant). Note that $P_n \xrightarrow{s} P_\infty$ is not an invariant of a parent space, i.e., it may be true in one parent space, but not in another one. In Proposition 3.11 we have shown $P_n \xrightarrow{s} P_\infty$ for the concrete parent space of Section 3.1, but it may not be true in others (see Example 4.1).

2 (No norm convergence of the projections). We cannot expect $\|P_n - P_\infty\| \to 0$ in operator norm: for $f = (f_\infty, 0, \dots)$ we conclude

$$\|(P_n - P_\infty)f\|^2_{\mathscr{H}} \geq \|W^2_{\infty,n} f_\infty\|^2_{\mathscr{H}_\infty}$$

from (3.11). If J_n is a partial isometry then $W_{\infty,n}$ is an orthogonal projection onto $\ker J_n^*$, and $\ker J_n^* \neq \{0\}$ if J_n is not a co-isometry. In particular, $\|P_n - P_\infty\| \geq \|W_{\infty,n}\| = 1$ in the latter case. If J_n is a co-isometry (and not an isometry), then W_n is an orthogonal projection onto $\ker J_n \neq \{0\}$, and we have

$$\|(P_n - P_\infty)f\|^2_{\mathscr{H}} \geq \|W_n^2 f_n\|^2_{\mathscr{H}_\infty}$$

for $f = (0, 0, \dots, f_n, 0, \dots) \in \mathscr{H}$ again by (3.11). As before, we have $\|P_n - P_\infty\| \geq \|W_n\| = 1$. In particular, we have shown that if J_n is a partial isometry (and not unitary), then $\|P_n - P_\infty\| = 1$. If J_n is unitary then $P_n = P_\infty$.

Nevertheless, the commutator $P_n P_\infty - P_\infty P_n$ (if not already 0) might converge to 0 in operator norm in some examples as we will see in Section 4.3.

3.3. A minimal parent space

Definition 3.14 (minimal parent space). Given a parent space $\mathscr{H}$ with isometries $\iota_n \colon \mathscr{H}_n \to \mathscr{H}$ ($n \in \overline{\mathbb{N}}$) we call

$$\mathscr{H}_{\min} = \overline{\operatorname{lin}} \bigcup_{n \in \overline{\mathbb{N}}} \iota_n(\mathscr{H}_n) \tag{3.12}$$

the *minimal parent space* associated with $(\iota_n \colon \mathscr{H}_n \to \mathscr{H})_{n \in \overline{\mathbb{N}}}$. Here, $\overline{\operatorname{lin}}\, M$ denotes the closure of the linear span of $M \subset \mathscr{H}$.

The notion *minimal* parent space reflects the fact that with $\mathscr{H}_{\min}$ as parent space instead of $\mathscr{H}$, the maps $\tilde{\iota}_n \colon \mathscr{H}_n \to \mathscr{H}_{\min}$ are still isometries for each $n \in \overline{\mathbb{N}}$. Moreover, if $f \in \mathscr{H}_{\min}^{\perp} = \mathscr{H} \ominus \mathscr{H}_{\min}$, then $f \in (\iota_n(\mathscr{H}_n))^{\perp} = \ker \iota_n^*$, i.e., $\iota_n^* f = 0$ for all $n \in \overline{\mathbb{N}}$. In particular, whatever happens outside $\mathscr{H}_{\min}$ is not relevant for any objects involving ι_n and ι_n^* and $n \in \overline{\mathbb{N}}$.

Let us now see how the *minimal* parent space in the concrete construction of Definition 3.5 looks like:

Lemma 3.15. *Assume that $W_n(\mathscr{H}_n)$ is closed.*[1] *then we have*

$$\mathscr{H}_{\min} = \mathscr{H}_\infty \oplus \bigoplus_{n \in \mathbb{N}} W_n(\mathscr{H}_n),$$

[1]This is equivalent with the fact that 0 is isolated in $\operatorname{spec}(W_n)$.

i.e., we can replace $\mathscr{H}_n$ by the orthogonal complement in $\mathscr{H}_n$ of the space where J_n is an isometry. In particular, if J_n is an isometry, then $\operatorname{ran} W_n = \{0\}$.

Proof. "$\subseteq$". clearly $\iota_\infty(\mathscr{H}_\infty) \subset \mathscr{H}_{\min}$, and as the n-th component of $\iota_n f_n$ is $W_n f_n$, hence an element of $\overline{\operatorname{ran} W_n}$.

"$\supseteq$". Let $f = (f_\infty, 0, \dots)$, then $f = \iota_\infty f_\infty \in \mathscr{H}_{\min}$. Moreover, if $f = (0, \dots, 0, f_n, 0, \dots) \in \mathscr{H}_\infty \oplus \bigoplus_{n\in\mathbb{N}} W_n(\mathscr{H}_n)$ with $f_n = W_n g_n$ and $g_n \in \mathscr{H}_n$, then $f = \iota_n g_n - \iota_\infty J_n g_n \in \mathscr{H}_{\min}$. ∎

If J_n is a partial isometry, we conclude from Section 3.2:

Proposition 3.16. *Assume that J_n is a partial isometry for each $n \in \mathbb{N}$. Then*

$$\mathscr{H}_{\min} = \mathscr{H}_\infty \oplus \bigoplus_{n\in\mathbb{N}} \ker J_n,$$
$$\iota_n f_n = (J_n f_n, 0, \dots, 0, f_n'', 0, \dots),$$
$$\iota_\infty f_\infty = (f_\infty, 0, \dots),$$

where $f_n'' = f_n - J_n^* J_n f_n$ is the orthogonal projection of f_n onto $\ker J_n$.

Proof. If J_n is a partial isometry, then $W_n = \mathrm{id}_{\mathscr{H}_n} - J_n^* J_n$ is the orthogonal projection onto $\ker J_n$ by Theorem 3.9 (2). In particular, $W_n(\mathscr{H}_n) = \ker J_n$ is closed and Lemma 3.15 applies. ∎

3.4. A change of identification operators

As a final step in the proof of our main result Theorem 1.7, we want to get rid of the condition $\|J_n\| \le 1$ used in Corollary 3.2 and also in Section 3.1. Therefore, we rescale the identification operators so that their norm is less than or equal to 1, but the convergence is maintained.

Lemma 3.17. *Assume that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with convergence speed $(\delta_n)_n$ and with identification operators J_n not necessarily being contractions ($\|J_n\| > 1$ may happen). Then $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with identification operators*

$$\hat{J}_n = \begin{cases} J_n & \text{if } \|J_n\| \le 1, \\ \dfrac{1}{\|J_n\|} J_n & \text{if } \|J_n\| > 1, \end{cases}$$

and convergence speed of the same order.

Proof. For the proof we have to face the four inequalities given in Definition 1.4: The first inequality is clear by $\|\hat{J}_n\| \le \|J_n\| \le 1$. For the second condition we obtain for the interesting case $\|J_n\| > 1$:

$$\begin{aligned}
&\|(\mathrm{id}_{\mathscr{H}_n} - \hat{J}_n^* \hat{J}_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \\
&\quad = \left\| \left(\mathrm{id}_{\mathscr{H}_n} - \frac{1}{\|J_n\|^2} J_n^* J_n - J_n^* J_n + J_n^* J_n \right) R_n \right\|_{\mathfrak{L}(\mathscr{H}_n)} \\
&\quad \le \|(\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n) R_n\|_{\mathfrak{L}(\mathscr{H}_n)} + \left(1 - \frac{1}{\|J_n\|^2}\right) \cdot \|J_n^* J_n\|_{\mathfrak{L}(\mathscr{H}_n)} \|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \\
&\quad \le \delta_n + (\|J_n\|^2 - 1) \|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \\
&\quad \le \delta_n + \delta_n (2 + \delta_n) \|R_n\|_{\mathfrak{L}(\mathscr{H}_n)} \to 0
\end{aligned}$$

using also Lemma 2.7. The third inequality can be shown similarly. For the last condition we estimate

$$\begin{aligned}
\|R_\infty \hat{J}_n - \hat{J}_n R_n\|_{\mathfrak{L}(\mathscr{H}_n, \mathscr{H}_\infty)} &\le \frac{\|R_\infty J_n - J_n R_n\|_{\mathfrak{L}(\mathscr{H}_n, \mathscr{H}_\infty)}}{\|J_n\|} \\
&\le \|R_\infty J_n - J_n R_n\|_{\mathfrak{L}(\mathscr{H}_n, \mathscr{H}_\infty)}
\end{aligned}$$

as $\|J_n\| > 1$ in the interesting case. ■

Therefore, it is always possible to modify the identification operators J_n such that Corollary 3.2 can be applied, keeping the convergence speed of the same order. Hence, the equivalence of both concepts is proofed completely.

4. Examples

In all examples, one encounters convergence of Laplace-like operators on varying spaces X_n with limit space X_∞, therefore we assume here that X_n is a measure space with measure μ_n for each $n \in \overline{\mathbb{N}}$. In particular, we have

$$\mathscr{H}_n := \mathsf{L}_2(X_n) \quad \text{for } n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}.$$

Note that Weidmann's convergence *always* implies the QUE-convergence with the same convergence speed by Theorem 2.4 for *any* parent space and isometries factorising the identification operators (not only the parent space associated with the identification operators of Definition 3.5).

4.1. Examples with natural common parent space

For the first example class, assume that there is a common measure space X with measure μ, that $X_n \subset X$ is a measurable subset and that μ_n is the corresponding restriction of μ to X_n for $n \in \overline{\mathbb{N}}$. In particular, we can embed elements of $\mathscr{H}_n = \mathsf{L}_2(X_n)$ naturally into $\mathscr{H} := \mathsf{L}_2(X)$, by extending them by 0. Denote this extension of an element $f_n \in \mathsf{L}_2(X_n)$ to $\mathsf{L}_2(X)$ by $f_n \oplus 0_{X \setminus X_n}$.

Weidmann's convergence. We then have the natural isometries

$$\iota_n : \mathscr{H}_n \to \mathscr{H}, \quad \iota_n f_n = f_n \oplus 0_{X \setminus X_n} \quad (n \in \overline{\mathbb{N}}).$$

Identifying f_n with $\iota_n f_n$, we are actually in the situation originally considered by Weidmann, namely that $\mathscr{H}_n$ is a *subspace* of the parent space $\mathscr{H}$. The adjoint ι_n^* is the restriction of an equivalent class of functions f from $\mathsf{L}_2(X)$ to X_n, denoted by $\iota_n^* f = f \restriction_{X_n}$. Moreover, the norm of the lifted resolvent difference is given in (1.4).

The corresponding orthogonal projections are

$$P_n f = \iota_n \iota_n^* f = f \restriction_{X_n} \oplus 0_{X \setminus X_n} \quad \text{and} \quad P_\infty f = \iota_\infty \iota_\infty^* f = f \restriction_{X_\infty} \oplus 0_{X \setminus X_\infty}.$$

In particular, the projections can be written as $P_n = \mathbb{1}_{X_n, X}$ and $P_\infty = \mathbb{1}_{X_\infty, X}$ where $\mathbb{1}_{M,X} : X \to \{0, 1\}$ denotes the indicator function ($\mathbb{1}_{M,X}(x) = 1$ if $x \in M \cap X$ and $\mathbb{1}_{M,X}(x) = 0$ if $x \in X \setminus M$) as well as for the corresponding multiplication operator in $\mathsf{L}_2(X)$. Moreover, the projections commute as we have

$$P_n P_\infty = \mathbb{1}_{X_n \cap X_\infty, X} = P_\infty P_n \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

Let us present another example with multiplication operators as in our motivating example Example 1.2 showing the possibility that the limit space suddenly shrinks:

Example 4.1 (Suddenly shrinking limit space, non-converging projections). Let

$$X = X_n = [0, 1/2] \cup [2, \infty) \quad \text{and} \quad X_\infty = [0, 1/2]$$

with Lebesgue measure. Moreover, let A_n be the multiplication operator multiplying with the function $a_n(x) = x^n$ and $A_\infty = 0$. Here, Weidmann's generalised norm resolvent convergence holds, as

$$\begin{aligned}
&\|(\iota_n R_n \iota_n^* - \iota_\infty R_\infty \iota_\infty^*) f\|^2_{\mathsf{L}_2(X)} \\
&= \int_{X_\infty} \Big| \Big(\frac{1}{1 + x^n} - 1 \Big) f(x) \Big|^2 \, dx + \int_{X \setminus X_\infty} \Big| \frac{1}{1 + x^n} f(x) \Big|^2 \, dx \\
&= \int_{[0,1/2]} \Big| \frac{x^n}{1 + x^n} f(x) \Big|^2 \, dx + \int_{[2,\infty)} \Big| \frac{1}{1 + x^n} f(x) \Big|^2 \, dx \\
&\le \frac{1}{(2^n)^2} \int_{[0,1/2] \cup [2,\infty)} |f(x)|^2 \, dx = \frac{1}{(2^n)^2} \|f\|^2_{\mathsf{L}_2(X)}.
\end{aligned}$$

Note that the limit space suddenly shrinks, and $P_n = \mathrm{id}_{\mathscr{H}}$ does not converge to $P_\infty = \mathbb{1}_{[0,1/2]}$ in any topology (cf. Remark 3.13 (1)).

QUE-convergence. The identification operators obtained from Weidmann's isometries are

$$J_n f_n = \iota_\infty^* \iota_n f_n = (f_n \oplus 0_{X \setminus X_n}) \restriction_{X_\infty}$$

and

$$J_n^* f_\infty = \iota_n^* \iota_\infty f_\infty = (f_\infty \oplus 0_{X \setminus X_\infty}) \restriction_{X_n}.$$

From Theorems 2.4 and 2.12 together with (4.1), we conclude the following characterisation of norm resolvent convergence in this case for multiplication operators. Actually, a direct proof follows easily from the motivating examples Examples 1.2 and 1.6 and especially from (1.4):

Proposition 4.2. *If A_n is the multiplication operator with the measurable (not necessarily bounded) function $a_n \colon X_n \to \mathbb{R}$ then the following are equivalent:*

1. $A_n \xrightarrow{\text{W-gnrc}} A_\infty$ *with convergence speed of order* $\delta_n \to 0$*;*
2. $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ *with convergence speed of order* $\delta'_n \to 0$*;*
3. *We have*

$$\|(a_\infty - \mathrm{i})^{-1}\|_{\mathsf{L}_\infty(X_\infty \setminus X_n)} \le \delta'_n, \quad \|(a_n - \mathrm{i})^{-1}\|_{\mathsf{L}_\infty(X_n \setminus X_\infty)} \le \delta'_n$$

and

$$\|(a_n - \mathrm{i})^{-1} - (a_\infty - \mathrm{i})^{-1}\|_{\mathsf{L}_\infty(X_n \cap X_\infty)} \le \delta'_n$$

and $\delta_n \to 0$.

For (1) $\implies$ ((2) or (3)) *we can choose* $\delta'_n = \delta_n$*; for* ((2) or (3)) $\implies$ (1) *we can choose* $\delta_n = 3\delta'_n$.

Remark 4.3 (Abstract minimal parent space not always optimal). Let us now calculate the abstract (minimal) parent space associated with the identification operators J_n (see Definition 3.5 and Lemma 3.15). The defect operators are

$$W_n = (\mathrm{id}_{\mathscr{H}_n} - J_n^* J_n)^{1/2} = \mathbb{1}_{X_n \setminus X_\infty, X_n} \colon \mathsf{L}_2(X_n) \to \mathsf{L}_2(X_n)$$

and

$$W_{\infty,n} = (\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*)^{1/2} = \mathbb{1}_{X_\infty \setminus X_n, X_\infty} \colon \mathsf{L}_2(X_\infty) \to \mathsf{L}_2(X_\infty).$$

Since $\operatorname{ran} W_n = \mathbb{1}_{X_n \setminus X_\infty, X_n}(\mathsf{L}_2(X_n)) = \mathsf{L}_2(X_n \setminus X_\infty)$, the minimal parent space $\mathscr{H}_{\min}$ is

$$\mathscr{H}_{\min} = \mathsf{L}_2(X_\infty) \oplus \bigoplus_{n \in \mathbb{N}} \mathsf{L}_2(X_n \setminus X_\infty) \tag{4.2}$$

by Lemma 3.15. Moreover, the corresponding isometries $\iota_{n,\min}\colon \mathscr{H}_n \to \mathscr{H}_{\min}$ ($n \in \overline{\mathbb{N}}$) are

$$\iota_{n,\min} f_n = ((f_n \oplus 0_{X \setminus X_n})\restriction_{X_\infty}, 0, \dots, 0, f_n \restriction_{X_n \setminus X_\infty} 0, \dots) \tag{4.3a}$$

and

$$\iota_{\infty,\min} f_\infty = (f_\infty, 0, \dots). \tag{4.3b}$$

We can see here that the abstract minimal parent space is not always as simple as the obvious choice $\mathscr{H} = \mathsf{L}_2(X)$.

Nevertheless, this abstract construction (including the artificially added sequence of spaces $\mathsf{L}_2(X_n \setminus X_\infty)$) leads to the strong convergence of the corresponding projections, i.e.,

$$P_{n,\min} f = (\mathbb{1}_{X_n \cap X_\infty, X_\infty} f_\infty, 0, \dots, \mathbb{1}_{X_n \setminus X_\infty, X_n} f_n, 0, \dots) \to P_{\infty,\min} f = (f_\infty, 0, \dots)$$

for all $f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H}_{\min}$.

Example 4.4 (Example 4.1 revisited). The QUE-convergence holds here with

$$J_n\colon \mathscr{H}_n = \mathsf{L}_2([0,1/2] \cup [2,\infty)) \to \mathscr{H}_\infty = \mathsf{L}_2([0,1/2]), \quad J_n f_n = f_n \restriction_{[0,1/2]}.$$

Then $J_n^* f_\infty = f_\infty \oplus 0_{[2,\infty)}$, $J_n^* J_n = \mathbb{1}_{[0,1/2]}$, and $J_n J_n^* = \mathrm{id}_{\mathscr{H}_\infty}$ (for simplicity, we just write $\mathbb{1}_{[0,1/2]}$ instead of the above introduced $\mathbb{1}_{[0,1/2],X_\infty}$). The defect operators are

$$W_n = \mathbb{1}_{[2,\infty)} \quad \text{and} \quad W_{\infty,n} = 0.$$

The minimal Hilbert space is here

$$\mathscr{H}_{\min} = \mathsf{L}_2([0,1/2]) \oplus \bigoplus_{n \in \mathbb{N}} \mathsf{L}_2([2,\infty]).$$

Here, the strong convergence $P_{n,\min} \xrightarrow{\mathrm{s}} P_{\infty,\min}$ (cf. Proposition 3.11) holds, as we have

$$P_{n,\min} f = (f_\infty, 0, \dots, \mathbb{1}_{[2,\infty)} f_n, 0, \dots) \to P_{\infty,\min} f = (f_\infty, 0, \dots)$$

for all $f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H}_{\min}$. It does not hold in the natural parent space $\mathscr{H} = \mathsf{L}_2(X)$, as we have seen in Example 4.1.

The identification operator J_n is an isometry. If $X_n \subset X_\infty$ holds for all $n \in \mathbb{N}$, then we can choose

$$\mathscr{H}_{\min} = \mathscr{H}_\infty = \mathsf{L}_2(X_\infty)$$

as minimal parent space and $\iota_n \colon \mathscr{H}_n \to \mathscr{H}_\infty$ as $\iota_n f_n = f_n \oplus 0_{X_\infty \setminus X_n}$ for $n \in \mathbb{N}$ and $\iota_\infty = \mathrm{id}_{\mathscr{H}_\infty}$. Moreover, Weidmann's generalised resolvent convergence means that

$$J_n R_n J_n^* - R_\infty = \iota_n R_n \iota_n^* - R_\infty = D_n = R_n \oplus 0_{\mathsf{L}_2(X_\infty \setminus X_n)} - R_\infty \tag{4.4}$$

converges in operator norm in $\mathsf{L}_2(X_\infty)$ to 0. Here, $0_{\mathsf{L}_2(X_\infty \setminus X_n)}$ is the 0-operator on $\mathsf{L}_2(X_\infty \setminus X_n)$.

In the QUE-setting, $J_n = \iota_n$ is an isometry, i.e., $J_n^* J_n = \mathrm{id}_{\mathscr{H}_n}$ and for the QUE-convergence we need only the second estimate in (1.7b). It is equivalent with

$$\begin{aligned}\|(\mathrm{id}_{\mathscr{H}_\infty} - J_n J_n^*) g\|^2_{\mathsf{L}_2(X_\infty)} &= \int_{X_\infty \setminus X_n} |g|^2 \,\mathrm{d}\mu = \|g\|^2_{\mathsf{L}_2(X_\infty \setminus X_n)} \\ &\le \delta_n^2 \|(A_\infty + 1) g\|^2_{\mathsf{L}_2(X_\infty)} \end{aligned} \tag{4.5}$$

for all $g \in \operatorname{dom} A_\infty$ (substitute $g = R_\infty f$). Moreover, the last estimate (1.7c) is equivalent with the operator norm in (4.4) being not greater than δ_n.

Example 4.5 (Laplacians and fading obstacles). We present here briefly results of [1, Sections 4 and 5]. Assume that X is a Riemannian manifold of bounded geometry, see [1, Section 3.2] for details. For simplicity, assume that X has dimension $d = 3$. Assume furthermore that $B_n \subset X$ is a closed subset of X, for example the disjoint union of balls of radius $1/n$, at distance at least $2/n^\alpha$ for some $\alpha \in (0, 1)$. It is shown in [1, Theorem 4.7] that the Neumann Laplacian on X_n QUE-converges to the Laplacian on X with convergence speed $\delta_n \in \mathrm{O}(1/n^{1-\alpha})$.

The Dirichlet Laplacian on X_n QUE-converges to the Laplacian on X only if $\alpha \in (0, 1/3)$ with convergence speed $\delta_n \in \mathrm{O}(1/n^{(1/3-\alpha)/2})$; for values $\alpha \in [1/3, 1)$, the behaviour is different, see Example 4.6. In both cases, the limit operator A_∞ does not see the obstacles any more (hence we called them *fading*). The natural identification operator J_n here is an isometry, as $X_n \subset X$.

From Theorems 2.4 and 2.12, we conclude that in both cases, the established QUE-convergence is equivalent with the Weidmann-convergence (i.e., the resolvents R_n of A_n fulfil (4.4)) with the same convergence speed.

The identification operator J_n is a co-isometry. Assume now that X_∞ is eventually contained in X_n. For simplicity, we assume that $X_\infty \subset X_n$ for all $n \in \mathbb{N}$. In this case, obviously J_n^* is an isometry, i.e., J_n is a co-isometry, and we have

$$J_n f_n = f_n \restriction_{X_\infty} \quad \text{and} \quad J_n^* f_\infty = f_\infty \oplus 0_{X_n \setminus X_\infty}$$

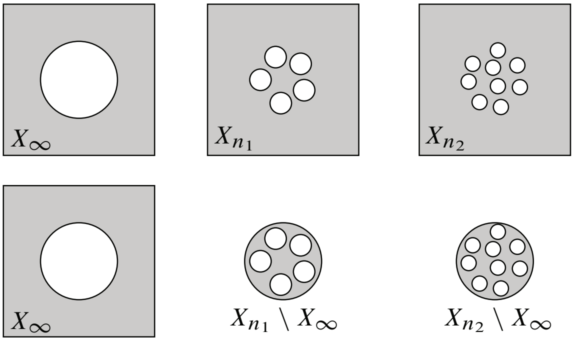

Figure 2. The Dirichlet solidifying case. *Top row.* The parent space X is the entire square, $X_\infty = X \setminus S$ is the limit space, where S is the centred ball. Moreover, $X_{n_j} = X \setminus B_{n_j}$ are the approximating spaces for $n_1 < n_2$, where the obstacles B_{n_j} (balls of radius $1/n_j$) are taken out. *Bottom row.* Three components of the abstract minimal parent space $\mathscr{H}_{\min}$, see Remark 4.3.

A parent space is either X, or the (possibly smaller) set $\bigcup_{n\in\mathbb{N}} X_n \subset X$, or, using the abstract construction, it is as in (4.2).

Example 4.6 (Dirichlet Laplacian on solidifying obstacles). We consider again the situation of Example 4.5. Assume that the obstacle set $B_n = \bigcup_{p\in I_n} \overline{B}(p, 1/n)$ (points in $I_n \subset X$ have distance at least $2/n^\alpha$) fulfils $B_n \subset S$ for some compact subset $S \subset X$ with smooth boundary (see Figure 2). Assume furthermore that the balls in B_n are "dense enough" in the sense that S is covered by the $1/n^\alpha$-neighbourhood of I_n and that $\alpha \in (1/2, 1)$. The limit operator A_∞ is (minus) the Dirichlet Laplacian on $X \setminus S$.

Then $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with convergence speed $\delta_n \in \mathrm{O}(1/n^{(1/3-\alpha)/4})$ (for details we refer to [1, Corollary 6.18]). We can understand the result in the sense that the obstacles B_n "solidify" to the set S, leading to the Dirichlet condition "on S." Unfortunately, the case $\alpha \in (1/3, 1/2]$ closer to the critical case $\alpha = 1/3$ cannot be treated by the results of [1].

Again, we conclude from Theorems 2.4 and 2.12 that the established QUE-convergence is equivalent Weidmann's convergence with the same convergence speed. Note that the parent space has to be larger than the limit space $\mathsf{L}_2(X_\infty)$ here as $X_\infty \subset X_n$. Nevertheless, there is a natural candidate for the parent space, namely $\mathscr{H} = \mathsf{L}_2(X)$.

4.2. No common natural parent space I: Graph-like spaces converging to metric graphs, a simplified model

The remaining examples we present here are cases, where the approximating and limit space are of different nature; namely in the limit there is a change in dimension.

We begin with the QUE-setting and then construct the abstract minimal parent space, which is not naturally given.

The next example was actually the first example for which the concept of QUE-convergence was established (cf. [15]).

A metric graph as the limit space. Let X_∞ be a (for simplicity) compact metric graph, i.e., a topological graph where each edge $e \in E$ is assigned a length $\ell_e \in (0, \infty)$. Moreover, X_∞ carries a natural measure: the Lebesgue measure on each line segment. We decompose the metric graph into its line segments $X_{\infty,e} \cong [0, \ell_e]$, i.e., we have

$$X_\infty = \bigcup_{e \in E} X_{\infty,e}$$

(see Figure 3 left). We use $x \in [0, \ell_e]$ also as coordinate of $X_{\infty,e}$, suppressing the formally necessary isometry $\varphi_{\infty,e} \colon X_{\infty,e} \to [0, \ell_e]$ in the notation. Moreover, we refer to the coordinate x as the *longitudinal* direction.

A graph-like space converging to a metric graph. For simplicity, we assume that X_∞ is embedded in $\mathbb{R}^d$ and that the edges are straight line segments in X_∞. Let now X_n be the $1/n$-neighbourhood of X_∞ in $\mathbb{R}^d$ together with its natural Lebesgue measure (see Figure 3). In particular, there is a number $a \in (0, \infty)$ such that we can decompose X_n into so-called *shortened edge neighbourhoods* $X_{n,e}$ and *vertex neighbourhoods* $X_{n,v}$. Each shortened edge neighbourhood $X_{n,e}$ is isometric with $[a/n, \ell_e - a/n] \times B_n$, where $B_n = \{y \in \mathbb{R}^{d-1} \mid |y| \le 1/n\}$ is called the *transversal* direction. Moreover, the "area" $|B_n|$ of B_n is $|B_n| = |B_1| n^{-(d-1)}$. Each vertex neighbourhood $X_{n,v}$ is $(1/n)$-homothetic with a building block $X_{1,v}$.

The "bumpy" graph-like space as simplification. In order to simplify the arguments, we consider the *(full) edge neighbourhood* $\check{X}_{n,e}$ isometric with $[0, \ell_e] \times B_n$ instead of the shortened edge neighbourhood $X_{n,e}$. We consider the resulting space $\check{X}_n$ as being glued together from the so-called *vertex neighbourhoods* $X_{n,v}$ $(v \in V)$ and the full edge neighbourhoods $\check{X}_{n,e}$ $(e \in E)$ in such a way that

$$\check{X}_n = \bigcup_{v \in V} X_{n,v} \cup \bigcup_{e \in E} \check{X}_{n,e}$$

(see Figure 4). Moreover, we assume that $X_{n,v} \cap \check{X}_{n,e} = \emptyset$ if e and v are not incident and that $X_{n,v} \cap \check{X}_{n,e}$ is $(d-1)$-dimensional and isometric with B_n if e and v are incident.

If the original graph X_∞ is embedded in $\mathbb{R}^2$, one might think of $\check{X}_n$ as a paper model, where the edge neighbourhoods have a "bump" in direction of a third dimension, in order to have space for the extra longitudinal length (see Figure 4). Therefore, we call $\check{X}_n$ the *bumpy graph-like space*. We will treat the original space X_n with the

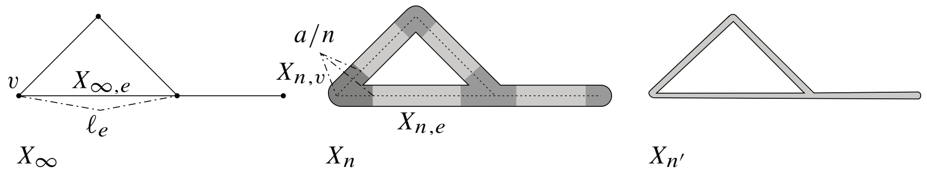

Figure 3. *Left.* A metric graph X_∞ with four edges and four vertices. The edge e is identified with the interval $X_{\infty,e} = [0, \ell_e]$ (fat). *Middle:* The embedded graph-like space as subset of $\mathbb{R}^2$ ($d = 2$), the edge neighbourhood $X_{n,e}$ (light grey) is isometric with $[0, \ell_e - 2a/n] \times [-1/n, 1/n]$, and the vertex neighbourhood $X_{n,v}$ (dark grey) is $(1/n)$-homothetic with a fixed set $X_{1,v}$. *Right.* An embedded graph-like space $X_{n'}$ for $n' > n$.

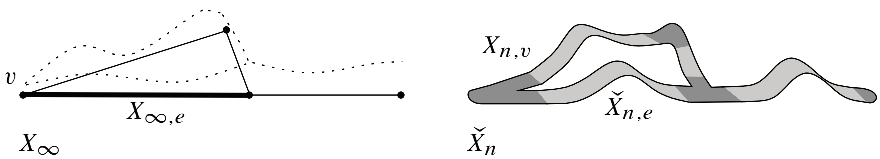

Figure 4. *Left.* A metric graph X_∞, the dotted lines represent a function f_∞ on X_∞. *Right.* A bumpy graph-like space. Here, the edge neighbourhood $\check{X}_{n,e}$ is isometric with $[0, \ell_e] \times [-1/n, 1/n]$ (for $d = 2$), and the vertex neighbourhood $X_{n,v}$ is the same as in the embedded case.

slightly shortened edge neighbourhoods $X_{n,e}$ as a perturbation of the bumpy graph-like space $\check{X}_n$ in Section 4.3, see also [17, Proposition 5.3.7].

As the components of the decomposition are disjoint up to sets of (d-dimensional) measure 0, we have

$$\check{\mathscr{H}}_n := \mathsf{L}_2(\check{X}_n) = \bigoplus_{v \in V} \mathsf{L}_2(X_{n,v}) \oplus \bigoplus_{e \in E} \mathsf{L}_2(\check{X}_{n,e})$$

and

$$\mathscr{H}_\infty := \mathsf{L}_2(X_\infty) = \bigoplus_{e \in E} \mathsf{L}_2(X_{\infty,e}).$$

According to this decomposition, we write

$$f_{n,v} := f_n \restriction_{X_{n,v}}, \quad f_{n,e} := f_n \restriction_{\check{X}_{n,e}}, \quad f_{\infty,e} := f_\infty \restriction_{X_{\infty,e}}.$$

for $f_n \in \check{\mathscr{H}}_n$ and $f_\infty \in \mathscr{H}_\infty$ and use the notation $(\cdot)_{n,v}$, $(\cdot)_{n,e}$ and $(\cdot)_{\infty,e}$ as the corresponding restriction operators, respectively.

The QUE-setting and the identification operators. We now define the identification operators $\check{J}_n$ as follows:

$$\check{J}_n \colon \check{\mathscr{H}}_n = \mathsf{L}_2(\check{X}_n) \to \mathscr{H}_\infty = \mathsf{L}_2(X_\infty),$$
$$(\check{J}_n f_n)_{\infty,e}(x) = |B_n|^{-1/2} \int_{B_n} f_{n,e}(x,y)\,\mathrm{d}y.$$

Note that $(\check{J}_n f_n)_{\infty,e}$ is (up to a scaling) the transversal average of $f_{n,e}$. A straightforward calculation gives the adjoint $\check{J}_n^* g_\infty$ on the components as

$$(\check{J}_n^* g_\infty)_{n,v} = 0 \quad \text{and} \quad (\check{J}_n^* g_\infty)_{n,e}(x,y) = |B_n|^{-1/2} g_\infty(x).$$

Moreover, it is easily seen that $\check{J}_n^*$ is an isometry.

It was shown in [15] (see also [17]) that (minus) the Neumann Laplacian on $\check{X}_n$ denoted here by $\check{A}_n$ QUE-converges to (minus) the standard Laplacian A_∞ on X_∞, i.e., $\check{A}_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with convergence speed $\delta_n \in \mathrm{O}(n^{-1/2})$. The standard Laplacian in X_∞ is the operator acting as $(Af_\infty)_e = -f_e''$ with functions continuous at a vertex and $\sum_{e \sim v} f_e'(v) = 0$, where $f_e'(v)$ denotes the derivative along $X_{\infty,e}$ towards the point in X_∞ corresponding to v.

Weidman's setting: the corresponding defect operators, the minimal parent space and its isometries. As $\check{J}_n^*$ is an isometry, its defect operator fulfils

$$\check{W}_{\infty,n} = (\mathrm{id}_{\mathscr{H}_\infty} - \check{J}_n \check{J}_n^*)^{1/2} = 0.$$

It follows that $\check{W}_n = (\mathrm{id}_{\check{\mathscr{H}}_n} - \check{J}_n^* \check{J}_n)^{1/2}$ is an orthogonal projection onto $\ker \check{J}_n$ (see Theorem 3.9 (2)); hence we can leave out the square root. In particular,

$$\check{W}_n = \mathrm{id}_{\check{\mathscr{H}}_n} - \check{J}_n^* \check{J}_n = \bigoplus_{v \in V} \mathrm{id}_{\mathsf{L}_2(X_{v,n})} \oplus \bigoplus_{e \in E} \check{P}_{n,e}^{\perp}, \tag{4.6}$$

where

$$(\check{P}_{n,e}^{\perp} f_{n,e}(x,y) = f_{n,e}(x,y) - \frac{1}{|B_n|} \int_{B_n} f_{n,e}(x,y')\,\mathrm{d}y'$$

is the projection onto the orthogonal complement of $\mathsf{L}_2([0,\ell_e]) \otimes \mathbb{C}\mathbb{1}_{B_n}$ in $\mathsf{L}_2(\check{X}_{n,e}) \cong \mathsf{L}_2([0,\ell_e]) \otimes \mathsf{L}_2(B_n)$; i.e., the projection onto the space of functions with transversal average 0. Moreover,

$$\check{W}_n(\check{\mathscr{H}}_n) = \ker \check{J}_n \cong \bigoplus_{v \in V} \mathsf{L}_2(X_{n,v}) \oplus \bigoplus_{e \in E} \mathsf{L}_2([0,\ell_e]) \otimes (\mathsf{L}_2(B_n) \ominus \mathbb{C}\mathbb{1}_{B_n})$$

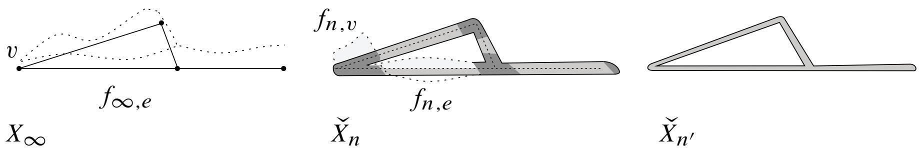

Figure 5. Three components of the minimal parent space (for simplicity not drawn as bumpy space): left the space $\mathsf{L}_2(X_\infty)$, in the middle the n-th component (only the functions on the vertex neighbourhoods $X_{n,v}$ are needed and the functions on $\check{X}_{n,e}$ with transversal average 0 on $\check{X}_{n,e}$). On the right, the underlying space of the n'-th component for some $n' > n$.

is a closed subspace of $\check{\mathscr{H}}_n$. In particular, we have

$$\check{\mathscr{H}}_{\min} = \mathsf{L}_2(X_\infty) \oplus \bigoplus_{n\in\mathbb{N}} \Big(\bigoplus_{v\in V} \mathsf{L}_2(X_{n,v}) \oplus \bigoplus_{e\in E} \mathsf{L}_2([0,\ell_e]) \otimes (\mathsf{L}_2(B_n) \ominus \mathbb{C}\mathbb{1}_{B_n}) \Big)$$

as abstract minimal parent space (see Lemma 3.15) with corresponding isometries $\check{\iota}_\infty \colon \mathsf{L}_2(X_\infty) \to \check{\mathscr{H}}_{\min}$ given by $\check{\iota}_\infty f_\infty = (f_\infty, 0, \dots)$ and $\check{\iota}_n \colon \mathsf{L}_2(\check{X}_n) \to \check{\mathscr{H}}_{\min}$ acting as

$$\check{\iota}_n f_n = \big(\check{J}_n f_n, 0, \dots, 0, (f_{n,v})_{v\in V} \oplus (\check{P}^\perp_{n,e} f_{n,e})_{e\in E}, 0, \dots\big)$$

for $f_n \in \check{\mathscr{H}}_n$ (see Figure 5). Note that $\check{\iota}_n f_n$ here is *not* positivity-preserving, as for any function $f_{n,e}(x,y) = g_e(x) h_n(y)$ we have

$$P^\perp_{n,e} f_{n,e}(x,y) = g_e(x) \Big(h_n(y) - \frac{1}{|B_n|} \int_{B_n} h_n(y')\, \mathrm{d}y' \Big).$$

If $g_e \geq 0$ and if $h_n \geq 0$ is not constant, then $f_{n,e} \geq 0$, but $\check{P}^\perp_{n,e} f_{n,e}$ must change sign for different y (and x with $g_e(x) > 0$). Nevertheless, note that $\check{J}_n = \check{\iota}^*_\infty \check{\iota}_n$ is positivity preserving.

The projections $\check{P}_\infty = \check{\iota}_\infty \check{\iota}^*_\infty$ and $\check{P}_n = \check{\iota}_n \check{\iota}^*_n$ are given by

$$\check{P}_\infty f = (f_\infty, 0, \dots)$$

and

$$\check{P}_n f = \big(f_\infty, 0, \dots, 0, (f_{n,v})_{v\in V} \oplus (\check{P}^\perp_{n,e} f_{n,e})_{e\in E}), 0, \dots\big)$$

for $f = (f_\infty, f_1, f_2, \dots) \in \mathscr{H}$.

As the projections commute (J^*_n is an isometry), the convergence speed is preserved: denote by δ_n the convergence speed of the QUE-convergence $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ shown e.g. in [15, 17]. We then conclude from Corollary 2.17:

Corollary 4.7 (Weidmann's convergence for graph-like spaces). *We have*

$$\check{A}_n \xrightarrow{\text{W-gnrc}} A_\infty$$

with convergence speed $2\delta_n \in \mathrm{O}(n^{-1/2})$.

4.3. No common natural parent space II: Graph-like spaces converging to metric graphs, the original model

Let us now show an example where we need to rescale the identification operators. We keep the notation from the previous subsection.

The original (embedded) graph-like space. Let us now consider the original graph-like space X_n with the slightly shortened edge neighbourhoods $X_{n,e}$ isometric with $[a/n, \ell_e - a/n] \times B_n$ (see Figure 3). We only present the differences with the abstract model in Section 4.2: we set $\mathscr{H}_n := \mathsf{L}_2(X_n)$ and

$$J_n \colon \mathscr{H}_n = \mathsf{L}_2(X_n) \to \mathscr{H}_\infty = \mathsf{L}_2(X_\infty),$$
$$(J_n f_n)_{\infty,e}(x) = |B_n|^{-1/2} \int\limits_{B_n} f_{n,e}(\Phi_{n,e}(x), y)\,\mathrm{d}y,$$

where

$$\Phi_{n,e} \colon [0, \ell_e] \to \Big[\frac{a}{n}, \ell_e - \frac{a}{n}\Big], \quad \Phi_{n,e}(x) = \frac{a}{n} + \Big(1 - \frac{2a}{n\ell_e}\Big)x.$$

Again, an easy computation shows that

$$(J_n^* g_\infty)_{n,v} = 0 \quad \text{and} \quad (J_n^* g_\infty)_{n,e}(\hat{x}, y) = |B_n|^{-1/2}\Big(1 - \frac{2a}{n\ell_e}\Big)^{-1} g_\infty(\Phi_{n,e}^{-1}(\hat{x})).$$

Moreover, we have here

$$\|J_n^* g_\infty\|^2_{\mathscr{H}_n} = \sum_{e \in E}\Big(1 - \frac{2a}{n\ell_e}\Big)^{-1} \int\limits_0^{\ell_e} |g_{\infty,e}(x)|^2\,\mathrm{d}x = \|K_{\infty,n} g_\infty\|^2_{\mathscr{H}_\infty},$$

where

$$K_{n,e} := \Big(1 - \frac{2a}{n\ell_e}\Big)^{-1/2} > 1$$

and $K_{\infty,n}$ is a multiplication operator. In particular, we have the norm equality $\|J_n\| = \|J_n^*\| = \|K_{\infty,n}\| = (1 - 2a/(n\ell_0))^{-1/2} > 1$, where ℓ_0 is the maximum of all edge lengths ℓ_e.

QUE-convergence in the embedded case: a change of identification operators. It was again shown in [15] (see also [17]) that A_n ((minus) the Neumann Laplacian on X_n) QUE-converges to (minus) the standard Laplacian A_∞ on the metric graph X_∞, i.e., that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with speed $\delta_n \in \mathrm{O}(n^{-1/2})$.

As here $\|J_n\| > 1$, we have to change the identification operator as in Section 3.4 to

$$\hat{J}_n := \frac{1}{\|J_n\|} J_n = \Bigl(1 - \frac{2a}{n\ell_0}\Bigr)^{1/2} J_n.$$

From Lemma 3.17 we conclude that $A_n \xrightarrow{\text{Q-gnrc}} A_\infty$ with identification operators $(\hat{J}_n)_n$ and convergence speed still of order in $\mathrm{O}(n^{-1/2})$.

Weidmann's convergence in the embedded case. Let us finally comment on Weidmann's convergence: we have

$$(\hat{J}_n \hat{J}_n^* f_\infty)_e(x) = \Bigl(1 - \frac{2a}{n\ell_0}\Bigr)\Bigl(1 - \frac{2a}{n\ell_e}\Bigr)^{-1} f_{\infty,e}(x),$$

hence J_n^* is only an isometry if $\ell_e = \ell_0$ for all $e \in E$ (i.e., if X_∞ is an *equilateral* metric graph). In particular, the defect operator associated with $\hat{J}_n^*$ is given by

$$(\widehat{W}_{\infty,n} f_\infty)_e = \widehat{w}_{n,e} \cdot f_{\infty,e},$$

where

$$\widehat{w}_{n,e} := \Bigl(1 - \frac{1 - 2a/(n\ell_0)}{1 - 2a/(n\ell_e)}\Bigr)^{1/2} \in \mathrm{O}\Bigl(\frac{1}{n^{1/2}}\Bigr),$$

and we have

$$\|\widehat{W}_{\infty,n}\|_{\mathfrak{L}(\mathscr{H}_\infty)} \to 0 \quad \text{as } n \to \infty.$$

For the defect operator associated with $\hat{J}_n$, we have

$$\widehat{W}_n = (\mathrm{id}_{\mathscr{H}_n} - \hat{J}_n^* \hat{J}_n)^{1/2} = \bigoplus_{v \in V} \mathrm{id}_{\mathsf{L}_2(X_{n,v})} \oplus \bigoplus_{e \in E} (\widehat{w}_{n,e} P_{n,e} \oplus P_{n,e}^\perp)\Bigr),$$

where $P_{n,e}$ is the orthogonal projection onto $\mathsf{L}_2([a/n, \ell_e - a/n]) \otimes \mathbb{C}\mathbb{1}_{B_n} \subset \mathsf{L}_2(X_{n,e})$ and $P_{n,e}^\perp$ its complement. The abstract minimal parent space as constructed in Section 3.1 is given by

$$\widehat{\mathscr{H}}_{\min} = \mathsf{L}_2(X_\infty) \oplus \bigoplus_{n \in \mathbb{N}} \Bigl(\bigoplus_{v \in V} \mathsf{L}_2(X_{n,v}) \oplus \bigoplus_{e \in E} \mathsf{L}_2(X_{n,e})\Bigr)$$

with isometries $\hat{\iota}_\infty \colon \mathsf{L}_2(X_\infty) \to \widehat{\mathscr{H}}_{\min}$ given by

$$\hat{\iota}_\infty f_\infty = (f_\infty, 0, \dots) \quad \text{and} \quad \hat{\iota}_n \colon \mathsf{L}_2(\hat{X}_n) \to \widehat{\mathscr{H}}_{\min}$$

acting as

$$\hat\iota_n f_n = \big(\hat J_n f_n, 0, \dots, 0, (f_{n,v})_{v\in V} \oplus \big((\hat w_{n,e} P_{n,e} \oplus \widehat P^{\perp}_{n,e}) f_{n,e}\big)_{e\in E}, 0, \dots\big)$$

for $f_n \in \widehat{\mathscr{H}}_n$. Again, $\hat\iota_n f_n$ is not positivity-preserving (but $\hat J_n$ is). The corresponding orthogonal projections $\widehat P_n$ and $\widehat P_\infty$ of the abstract parent space fulfil

$$(\widehat P_n \widehat P_\infty - \widehat P_\infty \widehat P_n) f = (-\widehat W_{\infty,n} \hat J_n f_n, 0, \dots, 0, \hat J_n^* \widehat W_{\infty,n} f_\infty, 0, \dots),$$

i.e., they do not commute. In particular, we conclude from Corollary 3.2 that the convergence speed in Weidmann's convergence will be slower:

Corollary 4.8 (Weidmann's convergence for embedded graph-like spaces). *We have* $\breve A_n \xrightarrow{\text{W-gnrc}} A_\infty$ *with convergence speed of order* $\mathrm{O}(n^{-1/4})$.

Remarks 4.9. 1. One could turn $\hat J_n^*$ into an isometry also for non-equilateral graphs by using the identification operator $\widetilde J_n := K_{\infty,n}^{-1/2} J_n = J_n K_{\infty,n}^{-1/2}$ and hence get the same convergence speed, but we want to illustrate the effect of non-commuting projections here.

2. Note that $\|\widehat P_n \widehat P_\infty - \widehat P_\infty \widehat P_n\|_{\mathfrak{L}(\mathscr{H})} \to 0$ as $\|\widehat W_{\infty,n}\| \to 0$ in operator norm and as $\|\hat J_n\| = 1$, cf. also Corollary 3.12. In contrast, the operator norm $\|\widehat P_n - \widehat P_\infty\|_{\mathfrak{L}(\mathscr{H})}$ of the projection difference does not converge to 0.

3. As already mentioned in Remark 2.11, it seems to be better to use (2.13) instead of (1.7b) (with J_n replaced by $\hat J_n$). In this concrete example, one can show that (2.13) holds with $\delta_n \in \mathrm{O}(n^{-1/2})$, hence following the argument in Remark 2.11 we would still end up with $A_n \xrightarrow{\text{W-gnrc}} A_\infty$, but now with convergence speed of order $\mathrm{O}(n^{-1/2})$.

4. Actually, when proving the QUE-convergence in [15, 17], we have shown that

$$\|(\mathrm{id}_{\mathscr{H}_n} - \hat J_n^* \hat J_n) R_n^{1/2}\| = \mathrm{O}(1/n^{1/2}).$$

It is not clear to us if $\|(\mathrm{id}_{\mathscr{H}_n} - \hat J_n^* \hat J_n) R_n\|$ would give a better estimate here. It could be helpful to analyse the resolvent difference $\widehat D_n = \hat\iota_n \widehat R_n \hat\iota_n^* - \hat\iota_\infty \widehat R_\infty \hat\iota_\infty^*$ (see (3.8) for a formula) directly in this example.

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