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On periodic and antiperiodic spectra of non-self-adjoint Dirac operators

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Abstract. The necessary and sufficient conditions are given for a sequence of complex numbers to be the periodic (or antiperiodic) spectrum of non-self-adjoint Dirac operator.

1. Introduction

One of the important classes of inverse spectral problems is the problem of recovering a system of differential equations from spectral data. The solution of such problems are considered in many papers, see [12,18,29–35] and the references therein. The most studied cases are for the Dirac and the Dirac-type differential operators. In particular, such problems for the canonical Dirac system on a finite interval

$$B\mathbf{y}' + V\mathbf{y} = \lambda\mathbf{y}, \tag{1.1}$$

where $\mathbf{y} = \operatorname{col}(y_1(x), y_2(x))$,

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad V(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

in the self-adjoint case have been studied in detail. In the cases of the Dirichlet and the Neumann boundary conditions, the reconstruction of a continuous potential from two spectra was carried out in [6], from one spectrum and the norming constants in [5], and from the spectral function in [15]. The analogous results for the Dirac operator with summable potentials were established in [1]. The case of more general separated boundary conditions was considered in [3]. In the case of unseparated boundary conditions (including periodic, antiperiodic, and quasi-periodic conditions), the considered problem was solved in [17,19–22]. In the non-self-adjoint case, the problem of reconstructing the potential $V(x)$ from spectral data is much more complicated, since many methods successfully used to study self-adjoint operators are inapplicable. For

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example, the characterization of the spectra of the periodic (antiperiodic) problem for the operator (1.1) with real coefficients is given in [17] in terms of special conformal mappings, which do not exist for complex-valued potentials. The property that the eigenvalues of the corresponding Dirichlet and Neumann problems are interlaced, which is often used to prove the solvability of the basic equation, loses its meaning in the complex case. Non-self-adjoint inverse problems for the system (1.1) with regular boundary conditions and sufficiently smooth complex-valued coefficients were investigated in [24]. Analogous problems for non-self-adjoint Dirac systems with singularities in interior points were studied in [8]. In [2], for the integro-differential Dirac system with complex-valued coefficients, the authors obtained necessary and sufficient conditions of solvability of the inverse spectral problem. Various types of inverse spectral problems in the non-self-adjoint case were considered in the book [29]. A survey of papers on inverse problems of spectral analysis for non-self-adjoint systems of ordinary differential equations is given in [36].

Questions of uniqueness in inverse problems for operators of type (1.1) on a finite interval were studied in several papers (see, for instance, [28] and the references therein). In particular, the uniqueness of the inverse problem for general Dirac-type systems of order $2n$ was established in [13, 14]. Also, inverse theory was intensively developed for Dirac-type operators on the axis and semiaxis by many authors. A new inverse approach to such differential systems on the semiaxis based on the A-function concept was recently considered in [7].

The aim of this paper is to find necessary and sufficient conditions of solvability of the periodic (antiperiodic) inverse spectral problem for the system (1.1) with a nonsmooth complex-valued potential $V(x)$.

The paper is organized as follows. Section 2 contains some basic facts and definitions related to the considered problems. In Section 3, by using a modified version of the Gelfand–Levitan–Marchenko method, we prove the solvability of the basic equation and establish necessary and sufficient conditions for an entire function to be the characteristic determinant of the considered problem. The theory of sine-type entire functions is substantially used; in particular, deep results [10,11] on the zeros of these functions. Also, we repeatedly use the properties of the entire functions from the Paley–Winer class established in [26]. To prove sufficiency, we construct the Gelfand–Levitan type kernel $F(x, t)$ and obtain the unique solvability of the homogeneous Gelfand–Levitan type equation.

Further, we obtain necessary and sufficient conditions for a set of complex numbers to be the spectrum of the mentioned problem. Our reasoning is based on the properties of the entire functions of exponential type and the infinite products established, for example, in [9, 23].

2. Preliminaries

In the present paper, we consider the system (1.1), where complex-valued functions $p, q \in L_2(0,\pi)$ ($V \in L_2$), with periodic (antiperiodic) boundary conditions

$$U_1(\mathbf{y}) = y_1(0) - (-1)^\theta y_1(\pi) = 0, \quad U_2(\mathbf{y}) = y_2(0) - (-1)^\theta y_2(\pi) = 0, \quad (2.1)$$

where $\theta = 0, 1$. In what follows, we introduce the Euclidean norm

$$\|\mathbf{f}\| = (|f_1|^2 + |f_2|^2)^{1/2}$$

for vectors $\mathbf{f} = \operatorname{col}(f_1, f_2) \in \mathbb{C}^2$ and set

$$\langle \mathbf{f}, \mathbf{g} \rangle = f_1 \bar{g}_1 + f_2 \bar{g}_2.$$

If W is 2×2-matrix, then we set

$$\|W\| = \sup_{\|\mathbf{f}\|=1} \|W\mathbf{f}\|$$

and denote by $L_{2,2}(a,b)$ and $L_{2,2}^{2,2}(a,b)$, respectively, the spaces of 2-coordinate vector functions $\mathbf{f}(t) = \operatorname{col}(f_1(t), f_2(t))$ and 2×2-matrix functions $W(t)$ with finite norms

$$\|\mathbf{f}\|_{L_{2,2}(a,b)} = \left(\int_a^b \|\mathbf{f}(t)\|^2 dt \right)^{1/2}, \quad \|W\|_{L_{2,2}^{2,2}(a,b)} = \left(\int_a^b \|W(t)\|^2 dt \right)^{1/2}.$$

The operator $\mathbb{L}\mathbf{y} = B\mathbf{y}' + V\mathbf{y}$ is regarded as a linear operator in the space $L_{2,2}(0,\pi)$ with the domain

$$D(\mathbb{L}) = \{\mathbf{y} \in W_1^1[0,\pi] \times W_1^1[0,\pi] \colon \mathbb{L}\mathbf{y} \in L_{2,2}(0,\pi),\ U_j(\mathbf{y}) = 0\ \ (j = 1, 2)\}.$$

Denote by

$$E(x,\lambda) = \begin{pmatrix} c_1(x,\lambda) & -s_2(x,\lambda) \\ s_1(x,\lambda) & c_2(x,\lambda) \end{pmatrix} \quad (2.2)$$

the matrix of the fundamental solution system to equation (1.1) with boundary condition $E(0,\lambda) = I$, where I is the unit matrix.

It is well known that the entries of the matrix $E(x,\lambda)$ are related by the identity

$$c_1(x,\lambda)c_2(x,\lambda) + s_1(x,\lambda)s_2(x,\lambda) = 1, \quad (2.3)$$

which is valid for any x, λ. The eigenvalues of problem (1.1), (2.1) are the roots of the characteristic equation

$$\Delta(\lambda) = 0,$$

where

$$\Delta(\lambda) = \begin{vmatrix} U_1(E^{[1]}(\cdot,\lambda)) & U_1(E^{[2]}(\cdot,\lambda)) \\ U_2(E^{[1]}(\cdot,\lambda)) & U_2(E^{[2]}(\cdot,\lambda)) \end{vmatrix},$$

and $E^{[k]}(x,\lambda)$ is the k-th column of matrix (2.2).

The matrix $E(\pi,\lambda)$ is called the *monodromy matrix* of the operator $\mathbb{L}\mathbf{y}$. For its entries, we introduce the notation $c_j(\lambda) = c_j(\pi,\lambda)$, $s_j(\lambda) = s_j(\pi,\lambda)$, $j = 1,2$. We denote also the class of entire functions $f(z)$ of exponential type $\le \sigma$ such that $\|f\|_{L_2(R)} < \infty$ by PW_σ . It is known [27] that the functions $c_j(\lambda), s_j(\lambda)$ admit the representation

$$c_j(\lambda) = \cos\pi\lambda + g_j(\lambda), \quad s_j(\lambda) = \sin\pi\lambda + h_j(\lambda), \tag{2.4}$$

where $g_j, h_j \in \mathrm{PW}_\pi$, $j = 1,2$. For functions of type (2.4), the following statement is true:

Lemma 1 ([17]). *The functions $u(\lambda)$ and $v(\lambda)$ admit the representations*

$$u(\lambda) = \sin\pi\lambda + h(\lambda), \quad v(\lambda) = \cos\pi\lambda + g(\lambda),$$

where $h, g \in \mathrm{PW}_\pi$, if and only if

$$u(\lambda) = -\pi(\lambda_0 - \lambda)\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{\lambda_n - \lambda}{n},$$

where $\lambda_n = n + \epsilon_n, \{\epsilon_n\} \in \ell_2(\mathbb{Z})$,

$$v(\lambda) = \prod_{n=-\infty}^{\infty}\frac{\lambda_n - \lambda}{n - 1/2},$$

where $\lambda_n = n - 1/2 + \kappa_n, \{\kappa_n\} \in \ell_2(\mathbb{Z})$.

It is well known that the characteristic determinant of problem (1.1), (2.1) can be reduced to the form

$$\Delta(\lambda) = (-1)^{\theta+1} + \frac{c_1(\lambda) + c_2(\lambda)}{2}, \tag{2.5}$$

and the eigenvalues are specified by the asymptotic formulas

$$\lambda_{n,j} = 2n + \theta + \varepsilon_{n,j}, \tag{2.6}$$

where $\{\varepsilon_{n,j}\} \in \ell_2(\mathbb{Z}), n \in \mathbb{Z}, j = 1,2$.

Denote by $E_0(x, \lambda)$ the fundamental-solution system to the equation $B\mathbf{y}' = \lambda\mathbf{y}$ with boundary condition $E_0(0, \lambda) = I$. Obviously,

$$E_0(x, \lambda) = \begin{pmatrix} \cos \lambda x & -\sin \lambda x \\ \sin \lambda x & \cos \lambda x \end{pmatrix}.$$

Denote also the second column of the matrix $E_0(x, \lambda)$ by

$$Y_0(x, \lambda) = \begin{pmatrix} -\sin \lambda x \\ \cos \lambda x \end{pmatrix}.$$

Further, $\Gamma(z, r)$ denotes a disk of radius r centered at the point z.

3. Main results

3.1. Characteristic determinant

Theorem 3.1. *For a function $\Psi(\lambda)$ to be the characteristic determinant of problem* (1.1), (2.1), *it is necessary and sufficient that it can be represented in the form*

$$\Psi(\lambda) = (-1)^{\theta+1} + \cos \pi\lambda + f(\lambda),$$

where $f \in \mathrm{PW}_\pi$, and

$$\sum_{n=-\infty}^{\infty} |f(n)| < \infty. \tag{3.1}$$

Proof. Necessity. Evidently, relations (2.4) and (2.5) imply that $f \in \mathrm{PW}_\pi$. To check inequality 3.1, we consider the monodromy matrix of problem (1.1), (2.1). Let the corresponding function $s_2(\lambda)$ have the roots λ_n; hence, by [26, Lemma 2.2],

$$\lambda_n = n + \delta_n, \tag{3.2}$$

where $\{\delta_n\} \in \ell_2(\mathbb{Z})$, $n \in \mathbb{Z}$. Since

$$c_j(\lambda_n) = \cos \pi\lambda_n + g_j(\lambda_n), \tag{3.3}$$

it follows from (2.4) and [26, Lemma 2.1] that

$$\sum_{n=-\infty}^{\infty} |g_j(\lambda_n)|^2 < \infty. \tag{3.4}$$

Denote

$$\chi(\lambda) = \Psi(\lambda) - (-1)^{\theta+1} = \cos \pi\lambda + f(\lambda). \tag{3.5}$$

By virtue of (2.5),

$$c_1(\lambda_n) + c_2(\lambda_n) = 2\chi(\lambda_n).$$

It follows from (2.3) that $c_1(\lambda_n)c_2(\lambda_n) = 1$; consequently, the numbers $c_1(\lambda_n), c_2(\lambda_n)$ are the roots of the quadratic equation

$$w^2 - 2\chi(\lambda_n)w + 1 = 0. \tag{3.6}$$

Therefore, we have

$$\begin{aligned} &c_1(\lambda_n), c_2(\lambda_n) \\ &\quad = \chi(\lambda_n) \pm \sqrt{\chi^2(\lambda_n) - 1} \\ &\quad = \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{(\cos \pi\lambda_n + f(\lambda_n))^2 - 1} \\ &\quad = \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{\cos^2 \pi\lambda_n + 2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - 1} \\ &\quad = \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - \sin^2 \pi\lambda_n}. \end{aligned} \tag{3.7}$$

It follows from (3.3) and (3.7) that

$$(g_1(\lambda_n) - f(\lambda_n))^2 = 2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - \sin^2 \pi\lambda_n;$$

hence,

$$2\cos \pi\lambda_n f(\lambda_n) = g_1^2(\lambda_n) - 2g_1(\lambda_n)f(\lambda_n) + \sin^2 \pi\delta_n. \tag{3.8}$$

It follows from (3.2) that for all sufficiently large $|n|$ the inequality $|\cos \pi\lambda_n| > 1/2$ holds. This, together with (3.2), (3.4), and [26, Lemma 2.1] implies

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)| < \infty. \tag{3.9}$$

Since $f' \in \mathrm{PW}_\pi$, then

$$\begin{aligned} |f(n)| &\le |f(\lambda_n)| + |f(n) - f(\lambda_n)| \\ &\le |f(\lambda_n)| + |\delta_n||\tau_n| \\ &\le |f(\lambda_n)| + (|\delta_n|^2 + |\tau_n|^2)/2, \end{aligned}$$

where

$$\tau_n = \max_{\lambda \in \Gamma(n, |\delta_n|)} |f'(\lambda)|.$$

By [26, Lemma 2.1], $\{\tau_n\} \in \ell_2(\mathbb{Z})$. This and (3.9) imply (3.1).

Sufficiency. Let $f \in \mathrm{PW}_\pi$ satisfy condition (3.1). It follows from the Paley–Wiener theorem and [16, Lemma 1.3.1] that

$$\lim_{|\lambda| \to \infty} e^{-\pi|\mathrm{Im}\,\lambda|} f(\lambda) = 0, \tag{3.10}$$

hence there exists a positive integer N_0 large enough that $|f(\lambda)| < 1/100$ if $\operatorname{Im}\lambda = 0$, $|\operatorname{Re}\lambda| \geq N_0$. Let λ_n $(n \in \mathbb{Z})$ be a strictly monotone increasing sequence of real numbers such that for any $n \neq 0$ $\lambda_n = \lambda_{-n}$, $|\lambda_n - (N_0 + 1/2)| < 1/100$ if $0 \leq n \leq N_0$, and $\lambda_n = n$ if $n > N_0$. Denote

$$s(\lambda) = -\pi(\lambda_0 - \lambda) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - \lambda}{n}. \tag{3.11}$$

It follows from Lemma 1 that

$$s(\lambda) = \sin \pi\lambda + h(\lambda), \tag{3.12}$$

where $h \in \mathrm{PW}_\pi$; hence,

$$|s(\lambda)| \geq C_1 e^{\pi |\operatorname{Im}\lambda|} \tag{3.13}$$

if $|\operatorname{Im}\lambda| \geq M$, where M is sufficiently large. It follows from (3.11) that

$$\dot{s}(\lambda_0) = \pi \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\lambda_n - \lambda_0}{n} > 0.$$

One can readily see that the inequality $\dot{s}(\lambda_n)\dot{s}(\lambda_{n+1}) < 0$ holds for all $n \in \mathbb{Z}$. It follows from the two last inequalities that

$$(-1)^n \dot{s}(\lambda_n) > 0. \tag{3.14}$$

Relation (3.12) and [26, Lemma 2.1] imply that

$$\dot{s}(\lambda_n) = \pi(-1)^n + \tau_n, \tag{3.15}$$

where $\{\tau_n\} \in \ell_2(\mathbb{Z})$; hence,

$$\frac{1}{\dot{s}(\lambda_n)} = \frac{(-1)^n}{\pi} + \sigma_n, \tag{3.16}$$

where $\{\sigma_n\} \in \ell_2(\mathbb{Z})$.

Equation (3.6) has the roots

$$\begin{aligned} c_n^{\pm} &= \chi(\lambda_n) \pm \sqrt{\chi^2(\lambda_n) - 1} \\ &= \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{(\cos \pi\lambda_n + f(\lambda_n))^2 - 1} \\ &= \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{\cos^2 \pi\lambda_n + 2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - 1} \\ &= \cos \pi\lambda_n + f(\lambda_n) \pm \sqrt{2\cos \pi\lambda_n f(\lambda_n) + f^2(\lambda_n) - \sin^2 \pi\lambda_n}. \end{aligned} \tag{3.17}$$

It follows from (3.17) that if $0 < |n| \le N_0$, then the numbers c_n^+ are contained within the disk $\Gamma(i, 1/10)$ and the numbers c_n^- are contained within the disk $\Gamma(-i, 1/10)$; and if $|n| > N_0$, then the numbers $c_n^{\pm}$ are contained within the disk $\Gamma(1, 1/10)$ for even n and the numbers $c_n^{\pm}$ are contained within the disk $\Gamma(-1, 1/10)$ for odd n. Denote $c_n = c_n^+$ for even n and $c_n = c_n^-$ for odd n. Denote also

$$z_n = \frac{c_n}{\dot{s}(\lambda_n)}.$$

It follows from (3.14) that the numbers z_n lie strictly above the line $l\colon \operatorname{Im}\lambda = -\operatorname{Re}\lambda$.

Evidently,

$$\lambda_n = n + \rho_n, \tag{3.18}$$

where $\{\rho_n\} \in \ell_2(\mathbb{Z})$. It follows from (3.17), (3.18), and condition (3.1) that

$$c_n = (-1)^n + \vartheta_n, \tag{3.19}$$

where $\{\vartheta_n\} \in \ell_2(\mathbb{Z})$. Let $\beta_n = c_n - \cos\pi\lambda_n$; then $\{\beta_n\} \in \ell_2(\mathbb{Z})$. Let us consider the function

$$g(\lambda) = s(\lambda) \sum_{n=-\infty}^{\infty} \frac{\beta_n}{\dot{s}(\lambda_n)(\lambda - \lambda_n)}.$$

By [9, p. 120], $g \in PW_\pi$ and $g(\lambda_n) = \beta_n$. Denote $c(\lambda) = \cos\pi\lambda + g(\lambda)$. Then $c(\lambda_n) = c_n \ne 0$; hence, the functions $s(\lambda)$ and $c(\lambda)$ have disjoint zero sets.

Denote

$$F(x,t) = \sum_{n=-\infty}^{\infty} \Big(\frac{c_n}{\dot{s}(\lambda_n)} (Y_0(x,\lambda_n) Y_0^T(t,\lambda_n)) - \frac{1}{\pi} Y_0(x,n) Y_0^T(t,n) \Big).$$

It follows from [27] that

$$\|F(\cdot,x)\|_{L^{2,2}_{2,2}(0,\pi)} + \|F(x,\cdot)\|_{L^{2,2}_{2,2}(0,\pi)} < C_2,$$

where C_2 not depending on x.

Using the properties of the numbers z_n established above, we prove that for every $x \in [0,\pi]$ the homogeneous Gelfand–Levitan type equation

$$\mathbf{f}^T(t) + \int_0^x \mathbf{f}^T(s) F(s,t)\,ds = 0, \tag{3.20}$$

where $\mathbf{f}(t) = \operatorname{col}(f_1(t), f_2(t))$, $\mathbf{f} \in L_{2,2}(0,x)$, $\mathbf{f}(t) = 0$ if $x < t \le \pi$, has the trivial solution only.

Multiplying equation (3.20) by $\overline{\mathbf{f}^T(t)}$ and integrating the resulting equation over segment $[0, x]$, we obtain

$$\|\mathbf{f}\|^2_{L_{2,2}(0,x)} + \int_0^x \left\langle \int_0^x \mathbf{f}^T(s)F(s,t)ds, \mathbf{f}^T(t) \right\rangle dt = 0. \tag{3.21}$$

Simple computations show

$$\begin{aligned}
\mathbf{f}^T(s)F(s,t) = \sum_{n=-\infty}^{\infty} \{ & z_n[f_1(s)\sin\lambda_n s \sin\lambda_n t - f_2(s)\cos\lambda_n s \sin\lambda_n t, \\
& - f_1(s)\sin\lambda_n s\cos\lambda_n t + f_2(s)\cos\lambda_n s\cos\lambda_n t] \\
& - \frac{1}{\pi}[f_1(s)\sin ns \sin nt - f_2(s)\cos ns \sin nt, \\
& - f_1(s)\sin ns\cos nt + f_2(s)\cos ns\cos nt]\} \\
= \sum_{n=-\infty}^{\infty} \{ & z_n[f_1(s)\sin\lambda_n s \sin\lambda_n t - f_2(s)\cos\lambda_n s \sin\lambda_n t] \\
& \quad - \frac{1}{\pi}[f_1(s)\sin ns \sin nt - f_2(s)\cos ns \sin nt], \\
& \quad z_n[-f_1(s)\sin\lambda_n s\cos\lambda_n t + f_2(s)\cos\lambda_n s\cos\lambda_n t] \\
& \quad - \frac{1}{\pi}[-f_1(s)\sin ns\cos nt + f_2(s)\cos ns\cos nt]\},
\end{aligned} \tag{3.22}$$

therefore, substituting the right-hand side of (3.22) into the second term in the left-hand side of (3.21), transforming the iterated integrals into products of integrals, and using the reality of all numbers λ_n, we obtain

$$\begin{aligned}
&\int_0^x \left\langle \int_0^x \mathbf{f}^T(s)F(s,t)ds, \mathbf{f}^T(t) \right\rangle dt \\
&= \sum_{n=-\infty}^{\infty} \int_0^x \left(\int_0^x \{z_n[f_1(s)\sin\lambda_n s \sin\lambda_n t - f_2(s)\cos\lambda_n s \sin\lambda_n t] \right. \\
&\qquad \left. - \frac{1}{\pi}[f_1(s)\sin ns \sin nt - f_2(s)\cos ns \sin nt]\} ds \right) \overline{f_1(t)} dt \\
&\quad + \sum_{n=-\infty}^{\infty} \int_0^x \left(\int_0^x \{z_n[-f_1(s)\sin\lambda_n s\cos\lambda_n t + f_2(s)\cos\lambda_n s\cos\lambda_n t] \right. \\
&\qquad \left. - \frac{1}{\pi}[-f_1(s)\sin ns\cos nt + f_2(s)\cos ns\cos nt]\} ds \right) \overline{f_2(t)} dt
\end{aligned}$$

$$= \sum_{n=-\infty}^{\infty} \Bigg(z_n \int_0^x [f_1(s)\sin\lambda_n s - f_2(s)\cos\lambda_n s]ds \int_0^x \sin\lambda_n t\,\overline{f_1(t)}dt$$

$$- \frac{1}{\pi} \int_0^x [f_1(s)\sin ns - f_2(s)\cos ns]ds \int_0^x \sin nt\,\overline{f_1(t)}dt \Bigg)$$

$$+ \sum_{n=-\infty}^{\infty} \Bigg(z_n \int_0^x [-f_1(s)\sin\lambda_n s + f_2(s)\cos\lambda_n s]ds \int_0^x \cos\lambda_n t\,\overline{f_2(t)}dt$$

$$- \frac{1}{\pi} \int_0^x [-f_1(s)\sin ns + f_2(s)\cos ns]ds \int_0^x \cos nt\,\overline{f_2(t)}dt \Bigg)$$

$$= \sum_{n=-\infty}^{\infty} z_n \Bigg(\int_0^x [f_1(s)\sin\lambda_n s - f_2(s)\cos\lambda_n s]ds \int_0^x \sin\lambda_n t\,\overline{f_1(t)}dt$$

$$+ \int_0^x [-f_1(s)\sin\lambda_n s + f_2(s)\cos\lambda_n s]ds \int_0^x \cos\lambda_n t\,\overline{f_2(t)}dt \Bigg)$$

$$- \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \Bigg(\int_0^x [f_1(s)\sin ns - f_2(s)\cos ns]ds \int_0^x \sin nt\,\overline{f_1(t)}dt$$

$$+ \int_0^x [-f_1(s)\sin ns + f_2(s)\cos ns]ds \int_0^x \cos nt\,\overline{f_2(t)}dt \Bigg)$$

$$= \sum_{n=-\infty}^{\infty} z_n \Bigg(\int_0^x [f_1(t)\sin\lambda_n t - f_2(t)\cos\lambda_n t]dt \int_0^x \sin\lambda_n t\,\overline{f_1(t)}dt$$

$$+ \int_0^x [-f_1(t)\sin\lambda_n t + f_2(t)\cos\lambda_n t]dt \int_0^x \cos\lambda_n t\,\overline{f_2(t)}dt \Bigg)$$

$$- \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \Bigg(\int_0^x [f_1(t)\sin nt - f_2(t)\cos nt]dt \int_0^x \sin nt\,\overline{f_1(t)}dt$$

$$+ \int_0^x [-f_1(t)\sin nt + f_2(t)\cos nt]dt \int_0^x \cos nt\,\overline{f_2(t)}dt \Bigg)$$

$$= \sum_{n=-\infty}^{\infty} z_n \int_0^x [f_1(t)\sin\lambda_n t - f_2(t)\cos\lambda_n t]dt$$

$$\times \int_0^x [\overline{f_1(t)}\sin\lambda_n t - \overline{f_2(t)}\cos\lambda_n t]dt$$

$$-\sum_{n=-\infty}^{\infty}\frac{1}{\pi}\int_0^x [f_1(t)\sin nt - f_2(t)\cos nt]dt \int_0^x [\overline{f_1(t)}\sin nt - \overline{f_2(t)}\cos nt]dt$$
$$= \sum_{n=-\infty}^{\infty} z_n \left|\int_0^x \langle \mathbf{f}(t), Y_0(t,\lambda_n)\rangle dt\right|^2 - \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\left|\int_0^x \langle \mathbf{f}(t), Y_0(t,n)\rangle dt\right|^2. \tag{3.23}$$

It is well known that the function system $\{\frac{1}{\sqrt{\pi}}Y_0(t,n)\}$ $(n \in \mathbb{Z})$ is an orthonormal basis in $L_{2,2}(0,\pi)$; hence it follows from the Parseval equality that

$$\|\mathbf{f}\|^2_{L_{2,2}(0,x)} = \sum_{n=-\infty}^{\infty}\frac{1}{\pi}\left|\int_0^x \langle \mathbf{f}(t), Y_0(t,n)\rangle dt\right|^2. \tag{3.24}$$

It follows from (3.21), (3.23), and (3.24) that

$$\sum_{n=-\infty}^{\infty} z_n \left|\int_0^x \langle \mathbf{f}(t), Y_0(t,\lambda_n)\rangle dt\right|^2 = 0.$$

Since all the numbers z_n are located strictly in the same half-plane relative to a line which passes through the origin, we see that

$$\int_0^x \langle \mathbf{f}(t), Y_0(t,\lambda_n)\rangle dt = 0$$

for all $n \in \mathbb{Z}$. It follows from (3.12) that the function $s(\lambda)$ is a sin-type function [10], therefore [1, Lemma 5.3], the system $Y_0(t,\lambda_n)$ is a Riesz basis of $L_{2,2}(0,\pi)$; hence the system $Y_0(t,\lambda_n)$ is complete in $L_{2,2}(0,\pi)$. It follows now that $\mathbf{f}(t) \equiv 0$.

By [27, Theorem 5.1], the functions $c(\lambda)$ and $-s(\lambda)$ are the entries of the first line of the monodromy matrix

$$\widetilde{E}(\pi,\lambda) = \begin{pmatrix} \tilde{c}_1(\pi,\lambda) & -\tilde{s}_2(\pi,\lambda) \\ \tilde{s}_1(\pi,\lambda) & \tilde{c}_2(\pi,\lambda) \end{pmatrix}$$

for problem (1.1), (2.1) with a potential $\widetilde{V} \in L_2$, i.e.,

$$c(\lambda) = \tilde{c}_1(\pi,\lambda), s(\lambda) = \tilde{s}_2(\pi,\lambda). \tag{3.25}$$

The corresponding characteristic determinant is

$$\widetilde{\Delta}(\lambda) = (-1)^{\theta+1} + (\tilde{c}_1(\pi,\lambda) + \tilde{c}_2(\pi,\lambda))/2 = (-1)^{\theta+1} + \cos\pi\lambda + \tilde{f}(\lambda),$$

where $\tilde{f} \in \mathrm{PW}_\pi$. It follows from (2.3), (3.5), (3.6), and (3.25) that

$$\begin{aligned}\widetilde{\Delta}(\lambda_n) &= (-1)^{\theta+1} + (\tilde{c}_1(\pi,\lambda_n) + \tilde{c}_2(\pi,\lambda_n))/2 \\ &= (-1)^{\theta+1} + \Big(\tilde{c}_1(\pi,\lambda_n) + \frac{1}{\tilde{c}_1(\pi,\lambda_n)}\Big)/2 \\ &= (-1)^{\theta+1} + \Big(c(\lambda_n) + \frac{1}{c(\lambda_n)}\Big)/2 \\ &= (-1)^{\theta+1} + \chi(\lambda_n) = \Psi(\lambda_n).\end{aligned}$$

This implies that the function

$$\Phi(\lambda) = \frac{\Psi(\lambda) - \widetilde{\Delta}(\lambda)}{s(\lambda)} = \frac{f(\lambda) - \tilde{f}(\lambda)}{s(\lambda)}$$

is an entire function in the whole complex plane. Since, by the Paley–Wiener theorem,

$$|f(\lambda) - \tilde{f}(\lambda)| < C_3 e^{\pi|\mathrm{Im}\,\lambda|}, \tag{3.26}$$

then by (3.13) $|\Phi(\lambda)| \le C_4$ if $|\mathrm{Im}\,\lambda| \ge M$. We denote by Ω the set

$$\Gamma(N_0 + 1/2, 1/10) \cup \Gamma(-N_0 - 1/2, 1/10)) \cup \Gamma_{|n|>N_0}(n, 1/10).$$

Since the function $s(\lambda)$ is a sin-type function [11], then $|s(\lambda)| > C_5 > 0$ if $\lambda \notin \Omega$. From this inequality, (3.26), and the Maximum Principle, we obtain that $|\Phi(\lambda)| < C_6$ in the strip $|\mathrm{Im}\,\lambda| \le M$; hence the function $\Phi(\lambda)$ is bounded in the whole complex plane and, by virtue of Liouville theorem, it is a constant. Let $|\mathrm{Im}\,\lambda| = M$; then it follows from (3.10) that $\lim_{|\lambda|\to\infty}(f(\lambda) - \tilde{f}(\lambda)) = 0$. Consequently $\Phi(\lambda) \equiv 0$; therefore $\Psi(\lambda) \equiv \widetilde{\Delta}(\lambda)$. ∎

Remark 3.1. The necessity of condition (3.1) for the Dirac operators with skew-symmetric potentials by another method was established in [26].

Remark 3.2. An analysis of the function $f(\lambda) = \frac{\sin \pi\lambda}{\lambda}$ shows that condition (3.1) is not equivalent to condition $\|f\|_{L_1(R)} < \infty$.

Remark 3.3. We had to impose an additional condition (3.1) on the function $f(\lambda)$, in order to correctly construct the kernel $F(x,t)$, namely to obtain relation (3.19).

3.2. Spectrum

Theorem 3.2. *For a set Λ to be the spectrum of some Dirac operator (1.1), (2.1) with a complex-valued potential $V \in L_2(0,\pi)$, it is necessary and sufficient that it consists of two sequences of eigenvalues $\lambda_{n,j}$ satisfying condition (2.6) and the inequality*

$$\sum_{k=-\infty}^{\infty} \Big| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_{n,1} + \varepsilon_{n,2}}{2n - 2k - 1} \Big| < \infty. \tag{3.27}$$

Proof. The proof of the theorem is carried out in the same lines for the periodic and the antiperiodic cases, and here we present the reasoning only for the periodic one. The main idea of our reasoning is to prove that the difference between the characteristic determinant of problem (1.1), (2.1) and the characteristic determinant of corresponding nonperturbed problem is an entire function, satisfying all conditions of the Theorem 3.1.

Sufficiency. Let two sequences $\lambda_{n,j}$ satisfy conditions (2.6) and (3.27). Evidently, there exists a constant M such that

$$\sup_{n,j} |\varepsilon_{n,j}| < M, \qquad \sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\varepsilon_{n,j}|^2 < M. \tag{3.28}$$

It is well known that

$$\sin \pi\lambda = \pi\lambda \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{n-\lambda}{n} = \pi\lambda \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(1 - \frac{\lambda}{n}\Big);$$

hence,

$$\sin^2 \frac{\pi\lambda}{2} = \frac{\pi^2\lambda^2}{4} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(\frac{2n-\lambda}{2n}\Big)^2 = \frac{\pi^2\lambda^2}{4} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \Big(1 - \frac{\lambda}{2n}\Big)^2.$$

Therefore, the function $\Delta_0(\lambda) = -1 + \cos \pi\lambda$ has the representation

$$\Delta_0(\lambda) = -\frac{\pi^2\lambda^2}{2} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(2n-\lambda)(2n-\lambda)}{4n^2}. \tag{3.29}$$

Evidently,

$$|\Delta_0(\lambda)| < c_1 e^{\pi|\operatorname{Im}\lambda|}. \tag{3.30}$$

Denote

$$\Delta(\lambda) = -\frac{\pi^2}{2}(\lambda_{0,1} - \lambda)(\lambda_{0,2} - \lambda) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(\lambda_{n,1} - \lambda)(\lambda_{n,2} - \lambda)}{4n^2}.$$

Let $f(\lambda) = \Delta(\lambda) - \Delta_0(\lambda)$. Let us prove that $f \in \mathrm{PW}_\pi$ and satisfies condition (3.1). Our investigation of the properties of the function $f(\lambda)$ is based on the following propositions.

Proposition 1. *The function $f(\lambda)$ is an entire function of exponential type not exceeding π.*

Denote Γ the union of the disks $\Gamma(2n, 1/4)$ $(n \in Z)$. If $\lambda \notin \Gamma$, then

$$f(\lambda) = -\Delta_0(\lambda)\Big(1 - \frac{\Delta(\lambda)}{\Delta_0}\Big) = -\Delta_0(\lambda)(1 - \phi(\lambda)), \tag{3.31}$$

where

$$\begin{aligned}\phi(\lambda) &= \frac{(\lambda_{0,1} - \lambda)(\lambda_{0,2} - \lambda)}{\lambda^2} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(\lambda_{n,1} - \lambda)(\lambda_{n,2} - \lambda)}{(2n - \lambda)(2n - \lambda)} \\ &= \Big(1 - \frac{\lambda_{0,1}}{\lambda}\Big)\Big(1 - \frac{\lambda_{0,2}}{\lambda}\Big) \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \prod_{j=1}^{2} \Big(1 + \frac{\varepsilon_{n,j}}{2n - \lambda}\Big) \\ &= \prod_{n=-\infty}^{\infty} \prod_{j=1}^{2} (1 + \alpha_{n,j}(\lambda)),\end{aligned}$$

where $\alpha_{0,j}(\lambda) = \frac{-\lambda_{0,j}}{\lambda}$, $\alpha_{n,j}(\lambda) = \frac{\varepsilon_{n,j}}{2n-\lambda}$. Let us estimate the function $\phi(\lambda)$. It follows from (3.28) that

$$\sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\alpha_{n,j}(\lambda)| \leq c_2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{j=1}^{2} (|\varepsilon_{n,j}|^2 + |2n - \lambda|^{-2})/2 < c_3. \tag{3.32}$$

It is easy to see that, for all $|n| > n_0$, where n_0 is a sufficiently large number, we have

$$|\alpha_{n,j}(\lambda)| < 1/4 \tag{3.33}$$

for all $\lambda \notin \Gamma$. If $|n| \leq n_0$, then inequality (3.33) holds for all sufficiently large $|\lambda|$; hence inequality (3.33) is valid for all $|\lambda| \geq C_0$. It follows from (3.32), (3.33), and the elementary inequality

$$|\ln(1 + z)| \leq 2|z|, \tag{3.34}$$

which is valid if $|z| \leq 1/4$, that

$$\sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\ln(1 + \alpha_{n,j}(\lambda))| \leq c_4.$$

Here and throughout the following, we choose the branch of $\ln(1 + z)$ that is zero for $z = 0$. In view of [9, p. 433], we rewrite the last relation in the form

$$|\phi(\lambda)| \leq \prod_{n=-\infty}^{\infty} \prod_{j=1}^{2} |1 + \alpha_{n,j}(\lambda)| \leq e^{c_4}. \tag{3.35}$$

It follows from (3.30), (3.31), (3.35) that

$$|f(\lambda)| < c_5 e^{\pi |\operatorname{Im}\lambda|} \tag{3.36}$$

outside the domain $\Gamma' = \Gamma \cup \{|\lambda| < C_0\}$. In particular, inequality (3.36) is valid if λ belongs the lines $\operatorname{Im}\lambda = \pm C_0$ and the vertical segments with vertexes $(2k-1, -C_0)$, $(2k-1, C_0)$, $|2k-1| > C_0$, $k \in \mathbb{Z}$. By the Maximum Principle, inequality (3.36) holds in the whole complex plane. This completes the proof of Proposition 1.

Proposition 2. *The function f belongs to* PW_π.

Denote

$$W(\lambda) = \ln \phi(\lambda) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} \ln(1 + \alpha_{n,j}(\lambda)),$$

then

$$f(\lambda) = -\Delta_0(\lambda)(1 - e^{W(\lambda)}). \tag{3.37}$$

Let us estimate the function $W(\lambda)$ if $\lambda \notin \Gamma'$. It follows from (3.28), (3.31), and (3.34) that

$$\begin{aligned}
|W(\lambda)| &\le \sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\ln(1 + \alpha_{n,j}(\lambda))| \\
&\le 2 \sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\alpha_{n,j}(\lambda)| \\
&\le \frac{2M}{|\lambda|} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{j=1}^{2} \left(\frac{|\varepsilon_{n,j}|^2}{10M} + \frac{10M}{|2n-\lambda|^2} \right) \\
&\le \frac{2M}{|\lambda|} + 1/10 + 20M \sum_{n=0}^{\infty} \frac{1}{n^2 + |\operatorname{Im}\lambda|^2} \\
&\le \frac{2M}{|\lambda|} + 1/10 + 20M \left(\frac{2}{|\operatorname{Im}\lambda|^2} + \int_1^{\infty} \frac{dx}{x^2 + |\operatorname{Im}\lambda|^2} \right) \\
&\le \frac{2M}{|\operatorname{Im}\lambda|} + 1/10 + 20M \left(\frac{2}{|\operatorname{Im}\lambda|^2} + \frac{\pi}{2|\operatorname{Im}\lambda|} \right).
\end{aligned}$$

The last inequality implies that

$$|W(\lambda)| < 1/4 \tag{3.38}$$

if $|\operatorname{Im}\lambda| \geq M_1 = 10(\pi + 2 + 22M) + C_0$. Then, from the trivial inequality

$$\frac{|z|}{2} \leq |1 - e^z| \leq 2|z|, \tag{3.39}$$

which holds for $|z| \leq 1/4$, we obtain the inequality $|1 - e^{W(\lambda)}| \leq 2|W(\lambda)|$, which, together with (3.30) and (3.37) implies that

$$|f(\lambda)| \leq c_6|W(\lambda)| \tag{3.40}$$

for $\lambda \in l$, where l is the line $\operatorname{Im}\lambda = M_1$. Let us prove that

$$\int_l |W(\lambda)|^2 d\lambda < \infty. \tag{3.41}$$

From the elementary inequality $|\ln(1+z) - z| \leq |z|^2$ true for $|z| \leq 1/2$, we obtain

$$\ln(1+z) - z = r(z),$$

where $|r(z)| \leq |z|^2$; hence,

$$W(\lambda) = S_1(\lambda) + S_2(\lambda), \tag{3.42}$$

where

$$S_1(\lambda) = \sum_{n=-\infty}^{\infty}\sum_{j=1}^{2} \alpha_{n,j}(\lambda), \quad |S_2(\lambda)| \leq \sum_{n=-\infty}^{\infty}\sum_{j=1}^{2} |\alpha_{n,j}(\lambda)|^2.$$

Evidently,

$$|W(\lambda)| \leq |S_1(\lambda)| + |S_2(\lambda)|. \tag{3.43}$$

Set

$$I_m = \int_l |S_m(\lambda)|^2 d\lambda \quad (m = 1, 2).$$

First, consider the integral I_1. It follows from [25, p. 221] that

$$I_1 = \int_l \left| \sum_{n=-\infty}^{\infty} \frac{\varepsilon_{n,1} + \varepsilon_{n,2}}{2n - \lambda} \right|^2 d\lambda < \infty. \tag{3.44}$$

It is readily seen that

$$|S_2(\lambda)| \leq \sum_{n=-\infty}^{\infty} \frac{|\varepsilon_{n,1}|^2 + |\varepsilon_{n,2}|^2}{|2n - \lambda|^2} < c_7;$$

hence,

$$I_2 \le c_7 \int\limits_l \Big(\sum_{n=-\infty}^{\infty} \frac{|\varepsilon_{n,1}|^2 + |\varepsilon_{n,2}|^2}{|2n-\lambda|^2} \Big) d\lambda$$
$$= c_8 \sum_{n=-\infty}^{\infty} (|\varepsilon_{n,1}|^2 + |\varepsilon_{n,2}|^2) \int\limits_l \frac{d\lambda}{|2n-\lambda|^2}$$
$$< c_9 \sum_{n=-\infty}^{\infty} (|\varepsilon_{n,1}|^2 + |\varepsilon_{n,2}|^2) < c_{10}. \tag{3.45}$$

Relations (3.43)–(3.45) imply (3.41). It follows from (3.40), (3.41), and [23, p. 115] that

$$\int\limits_R |f(\lambda)|^2 d\lambda < \infty. \tag{3.46}$$

The validity of the Proposition 2 is proved.

Proposition 3. *The function* $f(\lambda)$ *satisfies condition* (3.1).

Obviously, $\Delta_0(2k) = 0$; hence $f(2k) = \Delta(2k)$. Since the function $\Delta(\lambda)$ is bounded in the strip $|\mathrm{Im}\,\lambda| \le 1$, and for all sufficiently large $|k|$ the inequality $|\varepsilon_{k,j}| < 1/2$ takes place, by the Maximum Principle, we have

$$|f(2k)| = |\Delta(2k)| \le |\varepsilon_{k,1}||\varepsilon_{k,2}| \max_{|2k-\lambda|=1} \left| \frac{\Delta(\lambda)}{(\lambda_{k,1}-\lambda)(\lambda_{k,2}-\lambda)} \right|$$
$$\le c_{11}(|\varepsilon_{k,1}|^2 + |\varepsilon_{k,2}|^2). \tag{3.47}$$

Let us estimate $|f(2k+1)|$. Obviously, $\Delta_0(2k+1) = -2$. Denote

$$\epsilon_n = \max(|\varepsilon_{n,1}|, |\varepsilon_{n,2}|).$$

There exists a number $n_0 > 0$ such that

$$\sum_{|n|>n_0} \epsilon_n^2 < 1/1000,$$

and for any $|n| > n_0$ the inequality $\epsilon_n^{2/3} < 1/1000$ holds. Let $\lambda \notin \Gamma'$. Supplementary suppose that

$$|\lambda| > M_2 = 1000(2n_0+1)n_0 M.$$

Then, using the well-known inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0,\ p, q > 1,\ 1/p + 1/q = 1),$$

we obtain

$$\sum_{n=-\infty}^{\infty}\sum_{j=1}^{2}|\alpha_{n,j}(\lambda)| \le 2\Big(\sum_{|n|\le n_0}\frac{\epsilon_n}{|2n-\lambda|}+\sum_{|n|>n_0}\frac{\epsilon_n}{|2n-\lambda|}\Big)$$
$$\le 2M\sum_{|n|\le n_0}\frac{1}{|2n-\lambda|}+2\sum_{|n|>n_0}\Big(\epsilon_n^2+\frac{\epsilon_n^{2/3}}{|2n-\lambda|^{4/3}}\Big)$$
$$\le 1/50+1/500\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}<1/10; \tag{3.48}$$

hence inequality (3.38) is valid for all λ belonging to the considered domain. Arguing as above, we see that

$$|f(\lambda)| \le c_{12}\Big(\Big|\sum_{n=-\infty}^{\infty}\sum_{j=1}^{2}\alpha_{n,j}(\lambda)\Big|+\sum_{n=-\infty}^{\infty}\sum_{j=1}^{2}|\alpha_{n,j}(\lambda)|^2\Big). \tag{3.49}$$

The last inequality implies that for all $|2k+1|>k_0$, where $k_0=\max(C_0,M_2)$,

$$|f(2k+1)| \le c_{13}\Big(\Big|\sum_{n=-\infty}^{\infty}\frac{\varepsilon_{n,1}+\varepsilon_{n,2}}{2n-2k-1}\Big|+\sum_{n=-\infty}^{\infty}\frac{|\varepsilon_{n,1}|^2+|\varepsilon_{n,2}|^2}{|2n-2k-1|^2}\Big). \tag{3.50}$$

Clearly,

$$\sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{|\varepsilon_{n,1}|^2+|\varepsilon_{n,2}|^2}{|2n-2k-1|^2}=\sum_{n=-\infty}^{\infty}(|\varepsilon_{n,1}|^2+|\varepsilon_{n,2}|^2)\sum_{k=-\infty}^{\infty}\frac{1}{|2n-2k-1|^2}$$
$$< c_{14}\sum_{n=-\infty}^{\infty}(|\varepsilon_{n,1}|^2+|\varepsilon_{n,2}|^2)<c_{15}. \tag{3.51}$$

It follows from (3.27), (3.47), (3.50), and (3.51) that (3.1) holds. This completes the proof of Proposition 3.

Thus, the function $f(\lambda)$ satisfies all conditions of Theorem 3.1, and the function $\Delta(\lambda)$ is the characteristic determinant of some problem (1.1), (2.1).

Remark 3.4. The proof of an inequality similar to (3.40), carried out in [25], is based on the boundedness of the Hilbert transformation in $\ell_2(\mathbb{Z})$.

Necessity. If a set $\{\Lambda\}$ is the spectrum of a Dirac operator (1.1), (2.1), then relation (2.6) takes place [4, Theorem 6.5]. Let us prove that condition (3.27) holds. Since $f(\lambda)=\Delta(\lambda)-\Delta_0(\lambda)$, then, by Theorem 3.1, relation (3.1) is valid.

Let $\lambda=2k+1$, $k\in\mathbb{Z}$, $|2k+1|>k_0$, hence inequality (3.48) holds. Since $\Delta_0(2k+1)=-2$, it follows from (3.37) and (3.39) that

$$|W(2k+1)| \le |f(2k+1)|.$$

This, together with (3.42), implies

$$|S_1(2k+1)| \le |f(2k+1)| + \sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\alpha_{n,j}(2k+1)|^2. \tag{3.52}$$

Using (3.51), we find that

$$\sum_{n=-\infty}^{\infty} \sum_{j=1}^{2} |\alpha_{n,j}(2k+1)|^2 < c_{16}. \tag{3.53}$$

It follows from (3.52), (3.53), and (3.1) that

$$\sum_{|2k+1|>k_0} |S_1(2k+1)| < c_{17}.$$

It is easy to see that

$$\sum_{|2k+1|\le k_0} |S_1(2k+1)| < k_0 c_{18}.$$

The last two inequalities imply (3.27). ■

Remark 3.5. One can see that, in addition to the well-known asymptotic formulas for the eigenvalues, the formulation of the theorem contains an additional condition (3.27). In fact, this condition means that the Hilbert transform of the sequence $\{\varepsilon_{n,1} + \varepsilon_{n,2}\}$ is bounded in $\ell_1(\mathbb{Z})$. Inequality (3.27) is essentially used to prove the satisfiability of condition (3.1), which in turn is required for the correct construction of the Gelfand–Levitan type kernel $F(x,t)$.

Example. We give an example when (2.6) holds, but (3.27) does not. Let $\varepsilon_{n,1} = \varepsilon_{n,2} = 1/m$ if $n = 2^m$ and $\varepsilon_{n,1} = \varepsilon_{n,2} = 0$ if $n \ne 2^m$, $m = 1, 2, \ldots, n \in \mathbb{Z}$. Denote

$$\gamma_k = \sum_{n=-\infty}^{\infty} \frac{\varepsilon_{n,1} + \varepsilon_{n,2}}{2n - 2k - 1}, \qquad k \in \mathbb{Z};$$

hence,

$$\gamma_k = 2 \sum_{m=1}^{\infty} \frac{1}{m(2^{m+1} - 2k - 1)}.$$

Let $k = 2^p$, $p \in \mathbb{N}$. Then

$$\gamma_{2^p} = 2 \sum_{m=1}^{\infty} \frac{1}{m(2^{m+1} - 2^{p+1} - 1)} = -\frac{2}{p} + \sigma_{p,1} + \sigma_{p,2},$$

where

$$\sigma_{p,1} = 2\sum_{m=1}^{p-1} \frac{1}{m(2^{m+1} - 2^{p+1} - 1)}, \quad \sigma_{p,2} = 2\sum_{m=p+1}^{\infty} \frac{1}{m(2^{m+1} - 2^{p+1} - 1)}.$$

A simple computation shows that

$$|\sigma_{p,1}| \leq \frac{1}{2^p} \sum_{m=1}^{p-1} \frac{1}{m(1 - 2^{m-p} - 1/2^{p+1})} \leq \frac{4}{2^p} \sum_{m=1}^{p-1} \frac{1}{m} \leq \frac{4(1 + \ln p)}{2^p} \tag{3.54}$$

and

$$\begin{aligned} |\sigma_{p,2}| &\leq 2\sum_{l=1}^{\infty} \frac{1}{(l + p)(2^{l+p+1} - 2^{p+1} - 1)} \\ &\leq \frac{1}{p2^p} \sum_{l=1}^{\infty} \frac{1}{2^l - 1 - 1/2^{p+1}} \leq \frac{4}{p2^p}. \end{aligned} \tag{3.55}$$

It follows from (3.54) and (3.55) that $|\sigma_{p,1} + \sigma_{p,2}| \leq 1/p$ if $p \geq 10$; hence $|\gamma_{2^p}| > 1/p$. Therefore, the series

$$\sum_{k \in \mathbb{Z}} |\gamma_k|$$

diverges.

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