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A growth estimate for the monodromy matrix of a canonical system

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Abstract. We investigate the spectrum of 2-dimensional canonical systems in the limit circle case. It is discrete and, by the Krein–de Branges formula, cannot be more dense than the integers. But in many cases it will be more sparse. The spectrum of a particular selfadjoint realisation coincides with the zeroes of one entry of the monodromy matrix of the system. Classical function theory thus establishes an immediate connection between the growth of the monodromy matrix and the distribution of the spectrum.

We prove a general and flexible upper estimate for the monodromy matrix, use it to prove a bound for the case of a continuous Hamiltonian, and construct examples which show that this bound is sharp. The first two results run along the lines of earlier work by R. Romanov, but significantly improve upon these results. This is seen even on the rough scale of exponential order.

1. Introduction

We investigate the spectral theory of 2-dimensional *canonical systems*

$$y'(t) = zJH(t)y(t), \quad t \in I, \tag{1.1}$$

where

- $I = [\alpha, \beta]$ is a finite interval with nonempty interior,
- $H: I \to \mathbb{R}^{2\times 2}$ is a (Lebesgue-) measurable function which is integrable and does not vanish on any set of positive measure,
- $H(t) \geq 0$ for almost all $t \in I$,
- J is the symplectic matrix $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,
- z is a complex parameter (the eigenvalue parameter).

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The function H is called the *Hamiltonian* of the system (1.1). Systems of this form are intensively investigated since they can be seen as a unifying framework which includes, e.g., Schrödinger operators, Jacobi matrices, Dirac operators, and many others. Some recent standard literature is [1, 20, 21].

With the system (1.1) one can associate an operator model. It consists of a Hilbert space $L^2(H)$, the maximal and minimal operators $T_{\max}(H)$ and $T_{\min}(H)$, and a boundary value map $\Gamma(H)\colon T_{\max}(H) \to \mathbb{C}^2 \times \mathbb{C}^2$ (here we understand $T_{\max}(H)$ as its graph). Selfadjoint realisations of (1.1) have compact resolvents, and are obtained by specifying boundary conditions on the right and left endpoints α and β. Each two of them are finite rank perturbations of each other, and the rank of the perturbation is at most 2. Pick one and denote its eigenvalues as (this sequence need not be two-sided infinite)

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

By the Krein–de Branges formula, we have (understanding the limit for a finite sequence as 0)

$$\lim_{n\to\infty} \frac{n}{\lambda_{|n|}} = \frac{1}{\pi} \int_\alpha^\beta \sqrt{\det H(t)}\, \mathrm{d}t, \tag{1.2}$$

cf. [7, Theorem VI.6.1, (6.5′)] (and [4, Lemma 1.5.1]). If $H(t)$ is invertible on some set of positive measure, this formula gives good information about the distribution of the eigenvalues. On the other hand, if $\det H(t) = 0$ almost everywhere, then it does not say anything other than that $\sigma(A)$ is sparse compared to the integers.

Denote $\xi_\alpha := \binom{\cos\alpha}{\sin\alpha}$. A Hamiltonian with zero determinant can always be written in the form

$$H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)} \xi_{\phi(t)}^T,$$

where $\phi\colon I \to \mathbb{R}$ is a measurable function (determined up to integer multiples of π). We shall refer to ϕ as the *rotation angle* of H.

The basic question is how the distribution (density, asymptotics, etc.) of eigenvalues of selfadjoint realisations of (1.1) relate to the rotation angle of H.

Let us view this question from another angle which allows us to invoke function theory. We denote by $W(t,z)\colon I \times \mathbb{C} \to \mathbb{C}^{2\times 2}$ the unique solution of the initial value problem

$$\begin{cases} \dfrac{\partial}{\partial t} W(t,z) J = z W(t,z) H(t), & t \in I \text{ a.e.}, \\ W(\alpha, z) = I, \end{cases}$$

and call $W(t,z)$ the *fundamental solution* of the system (for technical reasons we have passed to transposes, so that the rows of $W(t,z)$ give the solutions of (1.1)).

The matrix $W_H(z) := W(\beta, z)$ is called the *monodromy matrix* of the system. It is an entire function in the spectral parameter z.

Understanding spectral properties amounts to understanding the monodromy matrix as an entire function because of the following central connection: there exists a selfadjoint realisation, call it A_H, such that all eigenvalues of A_H are simple and

$$\sigma(A_H) = \{x \in \mathbb{R} \mid w_{22}(x) = 0\} \quad \text{where } w_{22}(z) := (0, 1) W_H(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, using classical theorems of complex analysis (see e.g. [4, 14]), we have the immediate connection

$$\textit{spectral distribution of selfadjoint realisations}$$
$$\longleftrightarrow$$
$$\textit{growth of the monodromy matrix}$$

and the correlation is that the slower the monodromy matrix grows, the less dense the spectrum will be.

In the present paper we prove results which provide bounds for the growth of $W_H(z)$. Our main results are the three theorems described below.

Theorem 4.1. In this theorem we provide a general method to obtain upper bounds for $\log \|W_H(z)\|$. It should be seen as an improvement of [22, Theorem 1]. Formulation and proof are fairly similar, still Theorem 4.1 turns out to be a significant improvement of Romanov's Theorem. This can be witnessed even on the rough scale of exponential order, cf. Remark 5.4. As in Romanov's Theorem there is a lot of freedom when applying the result, and using this freedom in a clever way is essential to obtain strong estimates.

Theorem 5.2. We give an upper bound for the growth of $\log \|W_H\|$ for a Hamiltonian with continuous rotation angle. This is a perfect example for a (not too complicated) application of Theorem 4.1.

Theorem 6.1. In our third theorem we prove that the bound given in Theorem 5.2 is nearly sharp: we construct examples where the bound coming from Theorem 5.2 is equal to the maximum modulus up to a logarithmic factor. The proof requires major effort; among other things it relies on an auxiliary operator theoretic result which is of interest on its own right, cf. Theorem 3.4.

The sharpness result Theorem 6.1 is related to the following – still open – problem: is it always possible to obtain the exact growth of W_H by an application of the bound obtained from Romanov's Theorem (naturally, in the form of the present improvement Theorem 4.1)? There are several hints which indicate that the answer may be affirmative: [22, Theorem 2] which deals with diagonal Hamiltonians, [18, Theorem 2.22] which deals with piecewise constant Hamiltonians, and the present Theorem 6.1 which deals with continuous Hamiltonians.

To close this introduction, let us briefly describe the organisation of the content. We start with two sections containing auxiliary results. Those are needed only in the proof of the sharpness theorem, and therefore the reader may skip Sections 2 and 3 until reaching Theorem 6.1. Then we proceed to the stated main results: in Section 4 we give the improvement of Romanov's Theorem, in Section 5 we apply it to obtain an upper bound for continuous rotation angles, and in Section 6 we prove sharpness for this case.

2. Revisiting a lower bound for a Hamburger Hamiltonian

In this section we discuss a lower bound for the growth of the monodromy matrix of Hamiltonians of a special form. This bound was used previously in [18, 19] and (in a different language and with a different proof) in [2]. A weaker variant appears already in [15].

Recall that a *Hamburger Hamiltonian* is a Hamiltonian of the form

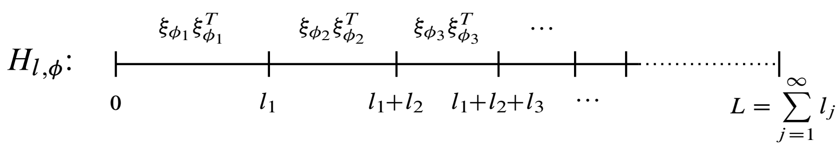
$$H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)} \xi_{\phi(t)}^T$$

whose rotation angle $\phi(t)$ is piecewise constant with constancy intervals accumulating only at the right endpoint. More precisely:

2.1 Definition. Let $(l_j)_{j=1}^\infty$ be a summable sequence of positive numbers and $(\phi_j)_{j=1}^\infty$ be a sequence of real numbers. Set $L := \sum_{j=1}^\infty l_j$, and define a Hamiltonian $H_{l,\phi}$ on the interval $[0, L]$ as

$$H_{l,\phi}(t) := \xi_{\phi_j} \xi_{\phi_j}^T \quad \text{for } j \in \mathbb{N} \text{ and } \sum_{i=1}^{j-1} l_i \le t < \sum_{i=1}^{j} l_i .$$

A Hamiltonian $H_{l,\phi}$ thus can be pictured as



We refer to the numbers l_j and ϕ_j defining a Hamburger Hamiltonian as its *lengths* and *angles*.[1]

[1] Angles are determined only up to integer multiples of π.

This terminology is motivated from the connection with the Hamburger moment problem, see e.g., [12].

The intuition concerning the growth of the monodromy matrix is that it grows slow if lengths decay fast, jumps of angles are small, and angles converge quickly. This reflects in the following result, which is the announced lower bound (it will also perfectly reflect in our later upper bounds).

2.2 Proposition. *Let H be a Hamburger Hamiltonian with lengths $(l_j)_{j=1}^{\infty}$ and angles $(\phi_j)_{j=1}^{\infty}$, and assume that $\phi_1 \not\equiv \frac{\pi}{2} \bmod \pi$. Set*

$$F(z) := \sum_{n=0}^{\infty} \Big[\prod_{j=1}^{n} l_{j+1} l_j \sin^2(\phi_{j+1} - \phi_j) \Big] z^n, \tag{2.1}$$

then

$$\log(\max_{|z|=r} \|W_H(z)\|) \geq \frac{1}{2} \log F(r^2) + \mathrm{O}(\log r).$$

The assumption on ϕ_1 is no loss of generality, since adding a certain offset to the sequences of angles does not change the function (2.1) and changes $\log \|W_H(z)\|$ only up to a summand which is a $\mathrm{O}(\log |z|)$.

The proof of Proposition 2.2 is obtained by repeating the “Alternative proof of Proposition 2.15” given in the extended preprint [17, p. 15].

Proof of Proposition 2.2 (cf. [17]). For $t \geq 0, \phi \in \mathbb{R}$ and $z \in \mathbb{C}$ set

$$W_\phi(t, z) = I - zt\xi_\phi \xi_\phi^T J,$$

and note that $W_\phi(t, z)\xi_\phi = \xi_\phi$. Set $t_n := \sum_{j=1}^{n} l_j$; then the fundamental solution of H is given as

$$W_H(t, z) = W_{\phi_1}(l_1, z) W_{\phi_2}(l_2, z) \dots W_{\phi_{n-1}}(l_{n-1}, z) W_{\phi_n}(t - t_{n-1}, z),$$

for $n \in \mathbb{N},\ t_{n-1} \leq t \leq t_n$. The function $(1, 0)W_H(t, z)\xi_{\phi_n}$ is constant on the interval $[t_{n-1}, t_n]$, and hence we can compute (writing $W_H(z) = (w_{ij}(z))_{i,j=1}^2$)

$$\begin{aligned}
\frac{w_{12}(z)\overline{w_{11}(z)} - w_{11}(z)\overline{w_{12}(z)}}{z - \overline{z}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \int_0^L W_H(t, z) H(t) W_H(t, z)^* \,\mathrm{d}t \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} (1, 0) W_H(t, z)\xi_{\phi_n} \cdot \xi_{\phi_n}^* W_H(t, z)^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}t \\
&= \sum_{n=1}^{\infty} \big| (1, 0) W_H(t_{n-1}, z)\xi_{\phi_n} \big|^2 \cdot l_n.
\end{aligned}$$

The function $p_n(z) := (1,0)W_H(t_{n-1},z)\xi_{\phi_n}$ is a polynomial of degree $n-1$ with real coefficients and has only real zeroes. Therefore, we have the estimate

$$|p_n(iy)| \geq y^{n-1}|c_{n-1}|, \quad y > 0,$$

where c_{n-1} denotes the leading coefficient of $p_n(z)$. This coefficient computes as

$$\begin{aligned} c_{n-1} &:= (-1)^{n-1} l_1 \cdot \dots \cdot l_{n-1} \cdot (1,0) \cdot \xi_{\phi_1}\xi_{\phi_1}^T J \cdot \dots \cdot \xi_{\phi_{n-1}}\xi_{\phi_{n-1}}^T J \cdot \xi_{\phi_n} \\ &= \Big(\prod_{j=1}^{n-1} l_j\Big) \cdot \cos\phi_1 \cdot \Big(\prod_{j=1}^{n-1} \sin(\phi_{j+1}-\phi_j)\Big). \end{aligned}$$

It follows that

$$\begin{aligned} &|w_{11}(iy)|^2 \\ &\quad = \Big(\frac{1}{y}\operatorname{Im}\frac{w_{12}(iy)}{w_{11}(iy)}\Big)^{-1} \cdot \frac{w_{12}(iy)\overline{w_{11}(iy)} - w_{11}(iy)\overline{w_{12}(iy)}}{2iy} \\ &\quad \geq \Big(\frac{1}{y}\operatorname{Im}\frac{w_{12}(iy)}{w_{11}(iy)}\Big)^{-1} \cdot \sum_{n=1}^{\infty} y^{2(n-1)}\Big[\cos\phi_1 \prod_{j=1}^{n-1} l_j \sin(\phi_{j+1}-\phi_j)\Big]^2 l_n \\ &\quad = \Big(\frac{1}{y}\operatorname{Im}\frac{w_{12}(iy)}{w_{11}(iy)}\Big)^{-1} \cdot l_1\cos^2\phi_1 \cdot \sum_{n=0}^{\infty} y^{2n}\Big[\prod_{j=1}^{n} l_{j+1}l_j \sin^2(\phi_{j+1}-\phi_j)\Big]. \end{aligned}$$

Each quotient of the entries of a line or a column of $W_H(z)$ is (up to a sign) a Herglotz function. We obtain

$$\begin{aligned} \log(\max_{|z|=r}\|W_H(z)\|) &\geq \log\|W_H(ir)\| \\ &= \log|w_{11}(ir)| + O(\log r) \\ &\geq \frac{1}{2}\big(\log F(r^2) + O(\log r)\big) + O(\log r). \end{aligned}$$ ■

In Section 5 we use this lower bound for Hamburger Hamiltonians whose lengths and angles are nicely behaving in the sense of regular variation (in Karamata's sense). For the theory of regular variation we refer to the monograph [3]; precise references will be given in course of the presentation. One can think of regularly varying functions as functions which behave roughly like a power. In this place, let us just recall the definition: a function $\ell : [r_0,\infty) \to (0,\infty)$ defined on some ray is called *regularly varying*, if there exists $\rho \in \mathbb{R}$ such that

$$\lim_{r\to\infty}\frac{\ell(\lambda r)}{\ell(r)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The number ρ is called the *index* of ℓ, and we shall write $\operatorname{Ind}\ell$ for it.

One example which illustrates that regularly varying functions behave like powers in many respects is that they satisfy a variant of Stirlings approximation formula. We do not know an explicit reference and hence provide a proof.[2]

2.3 Lemma. *Let $\mathfrak{f}$ be regularly varying with index $\rho \in \mathbb{R}$. Then*

$$\Big(\prod_{j=1}^{n} \mathfrak{f}(j)\Big)^{\frac{1}{n}} \sim \frac{\mathfrak{f}(n)}{e^{\rho}}.$$

Proof. Write $\mathfrak{f}(r) = r^{\rho} \cdot \ell(r)$ with ℓ slowly varying. By Stirlings formula we have

$$\Big(\prod_{j=1}^{n} j^{\rho}\Big)^{\frac{1}{n}} \sim \Big(\frac{n}{e}\Big)^{\rho},$$

hence we only have to deal with the slowly varying part.

By the representation theorem [3, Theorem 1.3.1] we can write ℓ as

$$\ell(r) = c(r) \exp\bigg(\int_{1}^{r} \frac{\varepsilon(u)}{u}\,\mathrm{d}u\bigg),$$

where c and ε are bounded measurable functions such that $\lim_{r\to\infty} c(r)$ exists in $(0, \infty)$ and $\lim_{r\to\infty} \varepsilon(r) = 0$. We obtain

$$\begin{aligned} &\frac{1}{\ell(n)}\Big(\prod_{j=1}^{n} \ell(j)\Big)^{\frac{1}{n}} \\ &\quad = \frac{1}{c(n)}\Big(\prod_{j=1}^{n} c(j)\Big)^{\frac{1}{n}} \cdot \exp\bigg(\frac{1}{n}\sum_{j=1}^{n}\int_{1}^{j} \frac{\varepsilon(u)}{u}\,\mathrm{d}u - \int_{1}^{n} \frac{\varepsilon(u)}{u}\,\mathrm{d}u\bigg). \end{aligned} \tag{2.2}$$

[2]Here, and throughout the paper, we shall use the following shorthand notations:

$$\begin{aligned} f \sim g \;&\Longleftrightarrow\; \frac{f}{g} \to 1, \\ f \ll g \;&\Longleftrightarrow\; \frac{f}{g} \to 0 \\ f \lesssim g \;&\Longleftrightarrow\; \text{there exists } C > 0 \text{ such that } f \le Cg, \\ f \asymp g \;&\Longleftrightarrow\; (f \lesssim g \wedge g \lesssim f). \end{aligned}$$

The first factor on the right side tends to 1 because $c(r)$ has a positive and finite limit. We estimate, for $j_0 \geq 1$ and $n > j_0$,

$$\begin{aligned}
&\Big|\frac{1}{n}\sum_{j=1}^{n}\int_1^j \frac{\varepsilon(u)}{u}\,\mathrm{d}u - \int_1^n \frac{\varepsilon(u)}{u}\,\mathrm{d}u\Big| \\
&= \Big|\frac{1}{n}\sum_{j=1}^{n}\int_j^n \frac{\varepsilon(u)}{u}\,\mathrm{d}u\Big| \\
&\leq \Big|\frac{1}{n}\sum_{j=1}^{j_0}\int_j^n \frac{\varepsilon(u)}{u}\,\mathrm{d}u\Big| + \Big|\frac{1}{n}\sum_{j=j_0+1}^{n}\int_j^n \frac{\varepsilon(u)}{u}\,\mathrm{d}u\Big| \\
&\leq \frac{j_0 \log n}{n}\cdot \sup_{r\in[1,\infty)}|\varepsilon(r)| + \frac{1}{n}\sum_{j=2}^{n}\int_j^n \frac{1}{u}\,\mathrm{d}u \cdot \sup_{r\in[j_0+1,\infty)}|\varepsilon(r)|,
\end{aligned}$$

and

$$\sum_{j=2}^{n}\int_j^n \frac{1}{u}\,\mathrm{d}u \leq \int_1^n\int_j^n \frac{1}{u}\,\mathrm{d}u\,\mathrm{d}j = \int_1^n\int_1^u \frac{1}{u}\,\mathrm{d}j\,\mathrm{d}u \leq n.$$

Hence, also the second factor on the right side of (2.2) tends to 1. ∎

Further, recall an elementary lim-inf variant of the classical formula [14, Theorem I.2′] for the type with respect to a proximate order.

2.4 Lemma. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, let $r_0, s_0 > 0$ and $g\colon [r_0,\infty) \to [s_0,\infty)$ be an increasing bijection. Then*

$$\liminf_{r\to\infty} \frac{1}{g(r)}(\log \max_{|z|=r}|A(z)|) \geq \liminf_{n\to\infty} \log(g^{-1}(n)|a_n|^{\frac{1}{n}}).$$

Proof. For all $r > 0$ and $n \in \mathbb{N}$ it holds that $\max_{|z|=r}|A(z)| \geq r^n|a_n|$, and in turn

$$\frac{1}{n}\log(\max_{|z|=r}|A(z)|) \geq \log(r|a_n|^{\frac{1}{n}}).$$

Using this for $r_n := g^{-1}(n)$ gives

$$\liminf_{n\to\infty} \frac{1}{n}\log\Big(\max_{|z|=r_n}|A(z)|\Big) \geq \liminf_{n\to\infty} \log(g^{-1}(n)|a_n|^{\frac{1}{n}}).$$

Let $r \geq r_1$ and take $n \in \mathbb{N}$ such that $r_n \leq r < r_{n+1}$. Then

$$\begin{aligned}
\frac{1}{g(r)}\log(\max_{|z|=r}|A(z)|) &\geq \frac{1}{n+1}\log(\max_{|z|=r_n}|A(z)|) \\
&= \frac{n}{n+1}\cdot\frac{1}{n}\log(\max_{|z|=r_n}|A(z)|),
\end{aligned}$$

and it follows that

$$\liminf_{r\to\infty}\frac{1}{\mathfrak{g}(r)}\log(\max_{|z|=r}|A(z)|)\geq\liminf_{n\to\infty}\frac{1}{n}\log(\max_{|z|=r_n}|A(z)|). \qquad\blacksquare$$

Combining the above results yields the following lower bound for the maximum modulus of the monodromy matrix when lengths and angles (in common) cannot have excessive downward drops.

2.5 Corollary. *Let H be a Hamburger Hamiltonian with lengths $(l_j)_{j=1}^\infty$ and angles $(\phi_j)_{j=1}^\infty$. Let $\mathfrak{f}$ be a regularly varying function, and choose $\mathfrak{g}$ regularly varying with $(\mathfrak{f}\circ\mathfrak{g})(x)\sim(\mathfrak{g}\circ\mathfrak{f})(x)\sim x$, see* [3, Theorem 1.5.12]. *If*

$$l_{j+1}l_j\sin^2(\phi_{j+1}-\phi_j)\gtrsim\frac{1}{\mathfrak{f}(j)},\quad j\in\mathbb{N}, \tag{2.3}$$

then

$$\log\max_{|z|=r}\|W_H(z)\|\gtrsim\mathfrak{g}(r^2). \tag{2.4}$$

Proof. Since the sequence $(l_j)_{j=1}^\infty$ is summable, we also have $\sum_{j=1}^\infty\big(\frac{1}{\mathfrak{f}(j)}\big)^{\frac{1}{2}}<\infty$. This implies that the index of $\mathfrak{f}$, call it ρ, is at least 2.

Passing from $\mathfrak{f}$ to another regularly varying function $\tilde{\mathfrak{f}}$ with $\mathfrak{f}\asymp\tilde{\mathfrak{f}}$ changes $\mathfrak{g}$ only up to "$\asymp$", and hence does not change the truth value of either (2.3) or (2.4). We may use this freedom to assume without loss of generality that

i. $\mathfrak{f}$ is an increasing bijection of $[1,\infty)$ onto itself,

ii. $\mathfrak{g}=\mathfrak{f}^{-1}$,

iii. the assumption (2.3) holds with "$\geq$" instead of "$\gtrsim$".

Using Lemma 2.3, we obtain

$$\Big(\prod_{j=1}^n l_{j+1}l_j\sin^2(\phi_{j+1}-\phi_j)\Big)^{\frac{1}{n}}\geq\Big(\prod_{j=1}^n\frac{1}{\mathfrak{f}(j)}\Big)^{\frac{1}{n}}\sim\frac{e^\rho}{\mathfrak{f}(n)},$$

and hence

$$\liminf_{n\to\infty}\log\Big[\mathfrak{f}(n)\cdot\Big(\prod_{j=1}^n l_{j+1}l_j\sin^2(\phi_{j+1}-\phi_j)\Big)^{\frac{1}{n}}\Big]\geq\rho\geq 2.$$

It follows from Lemma 2.4 that the function $F(z)$ from Proposition 2.2 satisfies

$$\liminf_{r\to\infty}\frac{1}{\mathfrak{g}(r)}\log F(r)\geq 2.$$

Note here that $F(z)$ has positive coefficients, and hence $\max_{|z|=r}|F(z)| = F(r)$. Now, Proposition 2.2 gives

$$\log(\max_{|z|=r}\|W_H(z)\|) \geq \frac{1}{2}\log F(r^2) + \mathrm{O}(\log r)$$
$$\gtrsim g(r^2) + \mathrm{O}(\log r) \gtrsim g(r^2). \qquad \blacksquare$$

3. An auxiliary theorem from operator theory

In this section we provide an auxiliary theorem about the operator model of a canonical system. It establishes a very intuitive fact, namely, that cutting out pieces of a Hamiltonian cannot increase the growth of the monodromy matrix.

3.1. The operator model of a canonical system

To start with we briefly recall the definition and some properties of the operator model of the equation (1.1). Our standard reference in this respect is [8] and [1, Chapter 7]. The operator theory behind (1.1) goes back to B. C. Orcutt [16] and I. S. Kac [10, 11] (see also [13]), and in a different language to L. de Branges [5]. Further recent references are [20, 21].

Intervals where H has constant nontrivial kernel require particular attention.

3.1 Definition. Let $\phi \in \mathbb{R}$. A nonempty interval $(a,b) \subseteq I$ is called *H-indivisible of type ϕ*, if

$$H(t) = \operatorname{tr} H(t) \cdot \xi_\phi \xi_\phi^T, \quad t \in (a,b) \text{ a.e.}$$

The type ϕ of an H-indivisible interval is unique up to integer multiples of π. We shall assume throughout this section that the whole interval I is not H-indivisible. This case is in some respects trivial: the monodromy matrix is a linear polynomial.

We denote by $L^2(H(t)\,\mathrm{d}t)$ the usual L^2-space of equivalence classes of 2-vector functions generated by the 2×2-matrix measure $H(t)\,\mathrm{d}t$, see e.g. [6, pp. 1337–1346]. To simplify notation, we shall always suppress explicit distinction between equivalence classes and their representants. However, one must keep in mind that sometimes it is important to make this distinction (for example when talking about boundary values further below).

Now, we can define the model space associated with a Hamiltonian H.

3.2 Definition. The *model space* $L^2(H)$ is the linear subspace of $L^2(H(t)\,\mathrm{d}t)$ which consists of all functions f having the following property:

if (a,b) is H-indivisible of type ϕ, then $\xi_\phi^T f(t)$ is constant a.e. on (a,b).

The space $L^2(H)$ is a closed subspace of $L^2(H(t)\,\mathrm{d}t)$, hence itself a Hilbert space, see, e.g., [8, Lemma 3.7].[3]

Next we define the minimal- and maximal-model operators.

3.3 Definition. Write $I = (\alpha, \beta)$. The *maximal* and the *minimal operators* $T_{\max}(H)$ and $T_{\min}(H)$ are defined in terms of their graphs as

$$T_{\max}(H) := \left\{ (f,g) \in L^2(H) \times L^2(H) \;\middle|\; \begin{array}{l} f \text{ has an absolutely continuous} \\ \text{representant with } f' = JHg \text{ a.e.} \end{array} \right\},$$

$$T_{\min}(H) := \left\{ (f,g) \in T_{\max}(H) \;\middle|\; \begin{array}{l} f \text{ has an absolutely continuous} \\ \text{representant with } f(\alpha) = f(\beta) = 0 \end{array} \right\}.$$

They have the following properties.

- For each $(f,g) \in T_{\max}(H)$, the first component has a unique absolutely continuous representant with $f' = JHg$. Thus, the boundary values $f(\alpha)$ and $f(\beta)$ are well-= defined.
- An abstract Green's identity holds:

$$(g_1, f_2)_{L^2(H)} - (f_1, g_2)_{L^2(H)} = f_2(\alpha)^* J f_1(\alpha) - f_2(\beta)^* J f_1(\beta)$$

  for all $(f_1, g_1), (f_2, g_2) \in T_{\max}(H)$.
- In some situations, $T_{\max}(H)$ may be a multivalued operator. Despite this technical difficulty, it always holds that $T_{\max}(H) = T_{\min}(H)^*$.
- $T_{\min}(H)$ is a closed symmetric operator, is completely nonselfadjoint (i.e., satisfies $\bigcap_{z \in \mathbb{C}\setminus\mathbb{R}} \operatorname{ran}(T_{\min}(H) - z) = \{0\}$), and has deficiency index $(2,2)$.

As a consequence of the above, selfadjoint extensions of $T_{\min}(H)$ can be described by boundary conditions at the left and right endpoints. We use the following two extensions:

$$\begin{aligned} B_H &:= \{(f,g) \in T_{\max}(H) \mid f(\alpha) = 0\}, \\ dR_H &:= B_H^{-1}, \\ A_H &:= \{(f,g) \in T_{\max}(H) \mid (1,0) f(\alpha) = (0,1) f(\beta) = 0\}. \end{aligned}$$

The operator R_H is the Volterra integral operator

$$(R_H f)(t) := \int_\alpha^t JH(t) f(t)\,\mathrm{d}t, \quad f \in L^2(H),$$

[3] Caution: the notation in [8] is different. The space $L^2(H(t)\,\mathrm{d}t)$ is what is there called $L^2(H, \mathbb{R}^+)$, and our space $L^2(H)$ there is $L^2_s(H, \mathbb{R}^+)$.

while A_H is selfadjoint. Note that B_H and A_H are invertible since $\ker T_{\max}(H) = \operatorname{span}\{\binom{1}{0}, \binom{0}{1}\}$, and that A_H^{-1} is a rank-one perturbation of R_H.

3.2. Cutting out pieces of a Hamiltonian

The theorem announced at the beginning of this section reads as follows.

3.4 Theorem. *Let H be a Hamiltonian on $I = [\alpha, \beta]$. Let $\Delta \subseteq [\alpha, \beta]$ be a (Lebesgue-) measurable set with positive measure, and assume that for every H-indivisible interval $(a, b) \subseteq [\alpha, \beta]$ either $(a, b) \cap \Delta$ or $(a, b) \setminus \Delta$ has measure zero. Set*

$$\begin{aligned} \lambda(t) &:= \int_\alpha^t \mathbb{1}_\Delta(u)\,\mathrm{d}u, \quad t \in [\alpha, \beta], \\ \widetilde{L} &:= \lambda(\beta), \\ \kappa(s) &:= \min\{t \in [\alpha, \beta] \mid \lambda(t) = s\}, \quad s \in [0, \widetilde{L}], \\ \widetilde{H} &:= H \circ \kappa. \end{aligned}$$

Then the following statements hold.

i. *$\widetilde{H}$ is a Hamiltonian on $[0, \widetilde{L}]$, and satisfies*

$$(\widetilde{H} \circ \lambda) \cdot \mathbb{1}_\Delta = H \cdot \mathbb{1}_\Delta \text{ a.e.} \tag{3.1}$$

ii. *The map V acting as*

$$V : f \mapsto (f \circ \lambda) \cdot \mathbb{1}_\Delta$$

induces an isometry of $L^2(\widetilde{H})$ into $L^2(H)$.

iii. *Denote by $M_{\mathbb{1}_\Delta}$ the multiplication operator with $\mathbb{1}_\Delta$. Then we have that $\operatorname{ran} M_{\mathbb{1}_\Delta} R_H V \subseteq \operatorname{ran} V$ and $R_{\widetilde{H}} = V^{-1} M_{\mathbb{1}_\Delta} R_H V$.*

$$\begin{array}{ccccc} L^2(H) & \xrightarrow{R_H} & L^2(H) & \xrightarrow{M_{\mathbb{1}_\Delta}} & \operatorname{ran}(M_{\mathbb{1}_\Delta} R_H V) \\ {\scriptstyle V}\uparrow & & & & \downarrow{\scriptstyle \subseteq} \\ L^2(\widetilde{H}) & \xrightarrow[R_{\widetilde{H}}]{} & L^2(\widetilde{H}) & \overset{V}{\underset{V^{-1}}{\rightleftarrows}}\ \cong & \operatorname{ran} V \end{array}$$

Proof. The proof of (i) relies on some measure theoretic considerations. Let us denote the maximal constancy intervals which contain more than one point (if any) as Δ_j. There exist at most countably many such intervals and $\Delta_j \cap \Delta$ is a zero set for all j. We now show that (the complement is understood in $[\alpha, \beta]$)

$$\lambda(\Delta^c) \text{ is a zero set.}$$

Since λ is absolutely continuous, the set $\lambda(\Delta^c)$ is (Lebesgue-) measurable. The change of variables formula gives

$$\int_0^{\widetilde{L}} \mathbb{1}_{\lambda(\Delta^c)}(s)\,\mathrm{d}s = \int_\alpha^\beta \big(\mathbb{1}_{\lambda(\Delta^c)} \circ \lambda\big)(t) \cdot \mathbb{1}_\Delta(t)\,\mathrm{d}t = \int_\alpha^\beta \mathbb{1}_{\lambda^{-1}(\lambda(\Delta^c))\cap\Delta}(t)\,\mathrm{d}t.$$

We have

$$\lambda^{-1}(\lambda(\Delta^c)) = \Delta^c \cup \bigcup\{\Delta_j \mid \Delta_j \cap \Delta \neq \emptyset\},$$

and hence the integral on the right vanishes.

In the second step we show that the function $\kappa\colon [0,\widetilde{L}] \to [\alpha,\beta]$, which is defined as

$$\kappa(s) := \min\{t \in [\alpha,\beta] \mid \lambda(t) = s\}, \quad s \in [0,\widetilde{L}],$$

is Lebesgue-to-Lebesgue measurable. Clearly, κ is nondecreasing and a right inverse of λ. Monotonicity implies that it is Borel-to-Borel measurable. Let E be a Lebesgue measurable subset of $[\alpha,\beta]$, and choose Borel sets $A, B \subseteq [\alpha,\beta]$ with $A \subseteq E \subseteq B$ and $B \setminus A$ being a zero set. Then $\kappa^{-1}(A) \subseteq \kappa^{-1}(E) \subseteq \kappa^{-1}(B)$ and

$$\kappa^{-1}(B) \setminus \kappa^{-1}(A) = \kappa^{-1}(B \setminus A) \subseteq \lambda(B \setminus A).$$

The set on the right is a zero set since λ is absolutely continuous, and we conclude that $\kappa^{-1}(E)$ is Lebesgue measurable.

Now, we define

$$\widetilde{H} := H \circ \kappa\colon [0,\widetilde{L}] \to \mathbb{R}^{2\times 2}.$$

Obviously, $\widetilde{H}$ takes nonnegative matrices as values and is (Lebesgue-) measurable. Moreover, we have

$$\{t \in [\alpha,\beta] \mid (\kappa \circ \lambda)(t) \neq t\} \subseteq \bigcup_j \Delta_j,$$

and hence

$$(\widetilde{H} \circ \lambda)(t)\mathbb{1}_\Delta(t) = H(\kappa \circ \lambda(t))\mathbb{1}_\Delta(t) = H(t)\mathbb{1}_\Delta(t) \quad \text{a.e.}$$

This is (3.1). We need to check that $\widetilde{H}$ is a Hamiltonian. Let $B \subseteq [0,\widetilde{L}]$ be measurable; then

$$\begin{aligned}\int_0^{\widetilde{L}} \operatorname{tr} \widetilde{H}(s)\mathbb{1}_B(s)\,\mathrm{d}s &= \int_\alpha^\beta \operatorname{tr}(\widetilde{H} \circ \lambda)(t)(\mathbb{1}_B \circ \lambda)(t)\mathbb{1}_\Delta(t)\,\mathrm{d}t \\ &= \int_\alpha^\beta \operatorname{tr} H(t)\mathbb{1}_{\lambda^{-1}(B)\cap\Delta}\,\mathrm{d}t.\end{aligned}$$

Choosing $B = [0, \widetilde{L}]$ already shows that $\widetilde{H}$ is integrable. Assume now that B is some set with positive measure. Since $\operatorname{tr} H(t) > 0$ a.e., measurability of the integrand in the last integral implies that the set $\lambda^{-1}(B) \cap \Delta$ is measurable. Moreover, $B \setminus \lambda(\lambda^{-1}(B) \cap \Delta)$ is a zero set since it is contained in $\lambda(\Delta^c)$. Therefore, $\lambda^{-1}(B) \cap \Delta$ must have positive measure, and the integral on the right is positive.

We come to the proof of (ii). The first step is to observe that V is isometric. This follows simply by making a change of variable. Let $f\colon [0, \widetilde{L}] \to \mathbb{C}^2$ be any measurable function, then we have

$$\int_0^{\widetilde{L}} f(s)^* \widetilde{H}(s) f(s)\,\mathrm{d}s = \int_\alpha^\beta (f \circ \lambda)(t)^* (\widetilde{H} \circ \lambda)(t)(f \circ \lambda)(t) \cdot \mathbb{1}_\Delta(t)\,\mathrm{d}t$$
$$= \int_\alpha^\beta (Vf)(t)^* H(t)(Vf)(t)\,\mathrm{d}t.$$

Note that isometry implies

$$\widetilde{H} f_1 = \widetilde{H} f_2 \text{ a.e.} \implies H(Vf_1) = H(Vf_2) \text{ a.e.} \tag{3.2}$$

We have to check the constancy condition from Definition 3.2 for indivisible intervals. Let $f\colon [0, \widetilde{L}] \to \mathbb{C}^2$ be a measurable function which satisfies the condition for $\widetilde{H}$. We have to show that Vf satisfies it for H.

Let $(a, b) \subseteq [\alpha, \beta]$ be an H-indivisible interval, and let ϕ be its type. By the assumption of the theorem, either $(a, b) \cap \Delta$ or $(a, b) \setminus \Delta$ is a zero set. In the first case, we have $(Vf)(t) = 0$ for $t \in (a, b)$ a.e., and are done. Consider the second case. Then $\mathbb{1}_\Delta(t) = 1$ for $t \in (a, b)$ a.e., and hence

$$\widetilde{H}(\lambda(t)) = H(t) = \operatorname{tr} H(t) \cdot \xi_\phi \xi_\phi^T, \quad t \in (a, b) \text{ a.e.} \tag{3.3}$$

Since λ is absolutely continuous and nondecreasing, we have $(\lambda(a), \lambda(b)) \subseteq \lambda((a, b))$ and the image of the exceptional set in (3.3) is a zero set. Hence,

$$\widetilde{H}(s) = \operatorname{tr} \widetilde{H}(s) \cdot \xi_\phi \xi_\phi^T, \quad s \in (\lambda(a), \lambda(b)) \text{ a.e.}$$

This means that $(\lambda(a), \lambda(b))$ is $\widetilde{H}$-indivisible of type ϕ, and hence that $\xi_\phi^T f(s)$ is constant on $(\lambda(a), \lambda(b))$ a.e. Say, we have $\xi_\phi^T f(s) = \gamma$ for a.a. $s \in (\lambda(a), \lambda(b))$. It follows that $\widetilde{H}(s) f(s) = \widetilde{H}(s)(\gamma \xi_\phi)$ for a.a. $s \in (\lambda(a), \lambda(b))$, in other words, the functions

$$f_1 := \mathbb{1}_{(\lambda(a), \lambda(b))} f, \quad f_2 := \mathbb{1}_{(\lambda(a), \lambda(b))}(\gamma \xi_\phi)$$

satisfy $\widetilde{H} f_1 = \widetilde{H} f_2$ for a.a. $s \in [0, \widetilde{L}]$. Applying (3.2) yields

$$H(t)[(\mathbb{1}_{(\lambda(a), \lambda(b))} \circ \lambda)(t)(f \circ \lambda)(t) \mathbb{1}_\Delta(t)] = H(t)[(\mathbb{1}_{(\lambda(a), \lambda(b))} \circ \lambda)(t) \gamma \xi_\phi \mathbb{1}_\Delta(t)]$$

for a.a. $t \in [\alpha,\beta]$. Since $(a,b)\setminus\Delta$ is a zero set, we have $\mathbb{1}_\Delta(t)=1$ for a.a. $t\in(a,b)$. The function λ is strictly increasing on (a,b), and hence $(\mathbb{1}_{(\lambda(a),\lambda(b))}\circ\lambda)(t)=1$ for all $t\in(a,b)$. It follows that

$$\operatorname{tr} H(t)\cdot\xi_\phi^T(Vf)(t)=\operatorname{tr} H(t)\cdot\xi_\phi^T(\gamma\xi_\phi),\quad t\in(a,b) \text{ a.e.},$$

and hence $\xi_\phi^T(Vf)(t)=\gamma$ again for $t\in(a,b)$ a.e.

Finally, we come to the proof of (iii). First note that, by our assumption on indivisible intervals, $M_{\mathbb{1}_\Delta}$ maps $L^2(H)$ into itself (in fact, is an orthogonal projection). Now, let $f\in L^2(\widetilde{H})$. Then

$$\begin{aligned}\big[(R_H\circ V)(f)\big](t)&=\int_\alpha^t JH(u)\cdot(f\circ\lambda)(u)\mathbb{1}_\Delta(u)\,\mathrm{d}u\\&=\int_\alpha^t J(\widetilde{H}\circ\lambda)(u)(f\circ\lambda)(u)\mathbb{1}_\Delta(u)\,\mathrm{d}u\\&=\int_0^{\lambda(t)} J\widetilde{H}(r)f(r)\,\mathrm{d}r=[(R_{\widetilde{H}}f)\circ\lambda](t).\end{aligned}$$

We see that $M_{\mathbb{1}_\Delta}R_HV=VR_{\widetilde{H}}$, and the assertion follows. ∎

Let us note that $\widetilde{H}$ defined above is the unique Hamiltonian with (3.1). To see this, assume we have $\widehat{H}$ with (3.1). Then

$$\widehat{H}\cdot(\mathbb{1}_\Delta\circ\kappa)=[(\widehat{H}\circ\lambda)\cdot\mathbb{1}_\Delta]\circ\kappa=(H\cdot\mathbb{1}_\Delta)\circ\kappa=\widetilde{H}\cdot(\mathbb{1}_\Delta\circ\kappa).$$

We have $\mathbb{1}_\Delta\circ\kappa=\mathbb{1}_{\kappa^{-1}(\Delta)}$, and since $\kappa^{-1}(\Delta^c)\subseteq\lambda(\Delta^c)$ this is equal to 1 a.e.

Passing to growth properties of W_H can easily be done using the usual function theoretic tools.

3.5 Corollary. *Consider the situation described in Theorem 3.4. Moreover, let $\mathfrak{f}$ be a regularly varying function with index $\rho\in(0,1)$. Then*

$$\limsup_{|z|\to\infty}\frac{\log\|W_{\widetilde{H}}(z)\|}{\mathfrak{f}(|z|)}\lesssim\limsup_{|z|\to\infty}\frac{\log\|W_H(z)\|}{\mathfrak{f}(|z|)}.$$

The constant implicit in this relation depends only on ρ.

Proof. For a compact operator T, we denote by $s_n(T)$ its n-th s-number and let $n_T(r)$ be the counting function

$$n_T(r):=\#\Big\{n\;\Big|\;s_n(T)\geq\frac{1}{r}\Big\},\quad r>0.$$

Due to Theorem 3.4, we have $s_n(R_{\tilde{H}}) \leq s_n(R_H)$ for all n, and hence $n_{R_{\tilde{H}}}(r) \leq n_{R_H}(r)$ for all $r > 0$.

The operator A_H^{-1} is a rank-one perturbation of R_H, and the same for $A_{\tilde{H}}^{-1}$ and $R_{\tilde{H}}$. Hence, we have

$$n_{A_{\tilde{H}}^{-1}}(r) \leq n_{R_{\tilde{H}}}(r) + 1 \leq n_{R_H}(r) + 1 \leq n_{A_H^{-1}}(r) + 2.$$

The spectrum of A_H coincides with the zero set of the entire function $w_{22}(z) := (0,1)W_H(z)\binom{0}{1}$, and the spectrum of $A_{\tilde{H}}$ with the zero set of $\tilde{w}_{22}(z) := (0,1)\widetilde{W}(z)\binom{0}{1}$. Thus, we have (now using the notation $n_f(r)$ for the counting function of the zeroes of an entire function f)

$$n_{\tilde{w}_{22}}(r) \leq n_{w_{22}}(r) + 2, \quad r > 0.$$

Due to [3, Proposition 7.4.1], we can assume without loss of generality that $\mathfrak{f}$ is a proximate order. Now, [14, Theorem I.17] is applicable, and yields

$$\limsup_{|z|\to\infty} \frac{\log|\tilde{w}_{22}(z)|}{\mathfrak{f}(|z|)} \asymp \limsup_{r\to\infty} \frac{n_{\tilde{w}_{22}}(r)}{\mathfrak{f}(r)} \leq \limsup_{r\to\infty} \frac{n_{w_{22}}(r)}{\mathfrak{f}(r)} \asymp \limsup_{|z|\to\infty} \frac{\log|\tilde{w}_{22}(z)|}{\mathfrak{f}(|z|)}.$$

By the proof of [14, Theorem I.17], the constants implicit in this relation depend only on the index of $\mathfrak{f}$. ■

4. A general estimate from above

In the below theorem we provide a method to estimate the monodromy matrix of a canonical system. This result is an improvement of a theorem due to R. Romanov in [22]. The proof follows the very same idea as [22, Theorem 1] and – despite the result being stronger – the argument is equally simple: it merely uses multiplicativity of the fundamental solution and Grönwall's lemma. Similar as for [22, Theorem 1], the power of Theorem 4.1 is its flexibility. Applying it in a clever way is at least as important as the theorem itself.

For practical reasons, we throughout use the spectral norm on $\mathbb{C}^{2\times 2}$. This norm has the advantage to be invariant under unitary transformations.

4.1 Theorem. *Let H be a Hamiltonian on a compact interval I with* $\det H = 0$ *a.e., and write $H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)}\xi_{\phi(t)}^T$ with a measurable function $\phi\colon I \to \mathbb{R}$. Assume we are given*

- *a partition $(y_0, \dots, y_N)$ of I, i.e.,*

$$N \in \mathbb{N}, \quad \min I = y_0 < y_1 < \dots < y_N = \max I,$$

- *rotation parameters* $\psi_1, \ldots, \psi_N \in \mathbb{R}$,
- *distortion parameters* $a_1, \ldots, a_N \in (0, 1]$,

and set

$$A_1 := \sum_{j=1}^{N} a_j^2 \int_{y_{j-1}}^{y_j} \cos^2(\phi(t) - \psi_j) \cdot \operatorname{tr} H(t)\, \mathrm{d}t,$$

$$A_2 := \sum_{j=1}^{N} \frac{1}{a_j^2} \int_{y_{j-1}}^{y_j} \sin^2(\phi(t) - \psi_j) \cdot \operatorname{tr} H(t)\, \mathrm{d}t,$$

$$A_3 := \sum_{j=1}^{N-1} \log\Big(\max\Big\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\Big\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\Big),$$

$$A_4 := -\log a_1 - \log a_N.$$

Then

$$\log \|W_H(z)\| \le |z| \cdot (A_1 + A_2) + A_3 + A_4 \quad \textit{for all } z \in \mathbb{C}, \tag{4.1}$$

where $\|\llcorner\!\lrcorner\|$ *denotes the spectral norm on* $\mathbb{C}^{2\times 2}$.

The following remark is essential for successful application of Theorem 4.1.

4.2 Remark. On first sight, the estimate (4.1) may seem quite useless. We know from the Krein–de Branges formula that $W_H(z)$ is of minimal exponential type, and hence of course an estimate $\log \|W_H(z)\| \lesssim |z|$ holds. The significance of Theorem 4.1 lies in a quantitative aspect. Namely, (4.1) holds for *all* choices of data y_j, ψ_j, a_j for *all* complex numbers z.

Now, reverse the viewpoint. Consider z as fixed, choose the data $y_j(z)$, $\psi_j(z)$, and $a_j(z)$ in dependence of z, and use (4.1) only for the given point z. If we manage to make the z-dependent choice of data in such a way that A_1 and A_2 decay when $|z|$ increases to ∞, and that A_3 and A_4 do not grow too fast, we may get a bound for $\log \|W_H(z)\|$ which is significantly smaller than $|z|$.

For the proof of Theorem 4.1, we start with an application of Grönwall's lemma.

4.3 Lemma. *Let H be a Hamiltonian on a compact interval I. Assume we are given a partition $(y_0, \ldots, y_N)$ of I and matrices* $\Omega_1, \ldots, \Omega_N \in \mathrm{GL}(2, \mathbb{R})$. *Then (for any submultiplicative norm),*

$$\|W_H(z)\| \le \exp\Big(|z| \sum_{j=1}^{N} \int_{y_{j-1}}^{y_j} \|\Omega_j H(t) J \Omega_j^{-1}\|\, \mathrm{d}t\Big) \|\Omega_1^{-1}\| \|\Omega_N\| \prod_{j=1}^{N-1} \|\Omega_j \Omega_{j+1}^{-1}\|. \tag{4.2}$$

Proof. For $j \in \{1, \dots, N\}$ let $W_j(t,z)$ be the fundamental solution of $H|_{[y_{j-1},y_j]}$, and let $W_j := W_j(y_j, z)$ be the corresponding monodromy matrices. Then

$$W_H(z) = W_1(z) \cdot W_2(z) \cdot \dots \cdot W_N(z).$$

We insert the matrices Ω_j and get

$$W_H(z) = \Omega_1^{-1} \cdot (\Omega_1 W_1(z) \Omega_1^{-1}) \cdot \Omega_1 \Omega_2^{-1} \cdot \dots \cdot (\Omega_N W_N(z) \Omega_N^{-1}) \cdot \Omega_N.$$

Applying Grönwall's lemma to the differential equation

$$\frac{\partial}{\partial x} \Omega_j W_j(x,z) \Omega_j^{-1} = -z \cdot \Omega_j W_j(x,z) \Omega_j^{-1} \cdot \Omega_j H(x) J \Omega_j^{-1}, \quad x \in [y_{j-1}, y_j],$$

yields that

$$\|\Omega_j W_j(z) \Omega_j^{-1}\| \leq \exp\Big(|z| \int_{y_{j-1}}^{y_j} \|\Omega_j H(t) J \Omega_j^{-1}\| \,\mathrm{d}t \Big).$$

The assertion of the lemma follows. ■

There happens no loss in precision when using only matrices Ω_j of a particular form.

4.4 Definition. For $a, b > 0$ set $D(a,b) := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and denote, for $a > 0$ and $\psi \in \mathbb{R}$,

$$\Omega(a,\psi) := D(a, a^{-1}) \exp(-\psi J) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}.$$

Geometrically, the matrix $\Omega(a,\psi)$ is a rotation followed by a distortion.

4.5 Remark. To see that we may restrict to matrices of the form $\Omega(a,\psi)$, observe that the right side of (4.2) remains unchanged when the matrices Ω_j are multiplied with real nonzero scalars α_j or multiplied from the left with matrices $C_j \in \mathrm{GL}(2,\mathbb{R})$ satisfying $\|C_j\| = \|C_j^{-1}\| = 1$. Using these two transformations, every matrix $\Omega \in \mathrm{GL}(2,\mathbb{R})$ can be brought to the form $\Omega(a,\psi)$.

In the next lemma we compute the relevant norms for matrices $\Omega(a,\psi)$.

4.6 Lemma. *Let $a, b > 0$ and $\psi, \phi \in \mathbb{R}$.*

i. $\|\Omega(a,\psi)\| = \|\Omega(a,\psi)^{-1}\| = \max\{a, a^{-1}\}$.

ii. $\|\Omega(a,\psi)\xi_\phi \xi_\phi^T J \Omega(a,\psi)^{-1}\| = a^2 \cos^2(\phi - \psi) + \frac{1}{a^2} \sin^2(\phi - \psi)$.

iii. *Set*

$$v_+ := \begin{pmatrix} \max\{\frac{a}{b}, \frac{b}{a}\}|\cos(\phi-\psi)| \\ \max\{ab, \frac{1}{ab}\}|\sin(\phi-\psi)| \end{pmatrix},$$

$$v_- := \begin{pmatrix} \min\{\frac{a}{b}, \frac{b}{a}\}|\cos(\phi-\psi)| \\ \min\{ab, \frac{1}{ab}\}|\sin(\phi-\psi)| \end{pmatrix},$$

and denote by $\|\llcorner\!\lrcorner\|_p$*,* $p \in \{1,2\}$*, the* p*-norm on* $\mathbb{R}^2$*. Then*

$$\begin{aligned} \|v_+\|_2^2 &\leq \|\Omega(a,\psi)\Omega(b,\phi)^{-1}\|^2 \\ &= 1 + \|v_+ - v_-\|_2 \cdot \frac{\|v_+ - v_-\|_2 + \|v_+ + v_-\|_2}{2} \\ &\leq \|v_+\|_1^2 \leq 2\|v_+\|_2^2. \end{aligned}$$

Proof. For the proof of (i), it is enough to note that $\exp(-\psi J)$ is unitary. This implies that

$$\|\Omega(a,\psi)\| = \|D(a,a^{-1})\exp(-\psi J)\| = \|D(a,a^{-1})\| = \max\{a, a^{-1}\},$$

and the analogous formula for $\Omega(a,\psi)^{-1}$.

We come to the proof of (ii). Note the relations $J\exp(\phi J) = \exp(\phi J)J$ and $\xi_\phi\xi_\phi^T = \exp(\phi J)\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\exp(-\phi J)$, which are easily verified. Moreover, set

$$\sigma := \phi - \psi.$$

Then

$$\begin{aligned} B &:= \Omega(a,\psi)\xi_\phi\xi_\phi^T J\Omega(a,\psi)^{-1} \\ &= D(a,a^{-1})\exp(-\psi J)\xi_\phi\xi_\phi^T J\exp(\psi J)D\big(a^{-1},a\big) \\ &= D(a,a^{-1})\exp(-\psi J)\exp(\phi J)\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\exp(-\phi J)\exp(\psi J)JD(a^{-1},a) \\ &= D(a,a^{-1})\xi_\sigma\xi_\sigma^T JD\big(a^{-1},a\big) = \begin{pmatrix} \cos(\sigma)\sin(\sigma) & -a^2\cos^2(\sigma) \\ \frac{1}{a^2}\sin^2(\sigma) & -\cos(\sigma)\sin(\sigma) \end{pmatrix}. \end{aligned}$$

A direct computation shows

$$B^T B = \begin{pmatrix} \frac{1}{a^4}\sin^4(\sigma) + \cos^2(\sigma)\sin^2(\sigma) & * \\ * & a^4\cos^4(\sigma) + \cos^2(\sigma)\sin^2(\sigma) \end{pmatrix},$$

from which we see that $\operatorname{tr}(B^T B) = (a^2\cos^2(\sigma) + \frac{1}{a^2}\sin^2(\sigma))^2$. Since $\det B = 0$, we have $\|B\| = \sqrt{\operatorname{tr}(B^T B)}$.

Finally, we turn to (iii). We compute

$$C := \Omega(a,\psi)\Omega(b,\phi)^{-1} = D(a,a^{-1})\exp((\phi-\psi)J)D(b^{-1},b)$$
$$= \begin{pmatrix} \frac{a}{b}\cos\sigma & -ab\sin\sigma \\ \frac{1}{ab}\sin\sigma & \frac{b}{a}\cos\sigma \end{pmatrix} = \cos\sigma\begin{pmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{pmatrix} + \sin\sigma\begin{pmatrix} 0 & -ab \\ \frac{1}{ab} & 0 \end{pmatrix}.$$

The asserted estimate from above follows:

$$\|C\| \leq |\cos\sigma|\cdot\left\|\begin{pmatrix} \frac{a}{b} & 0 \\ 0 & \frac{b}{a} \end{pmatrix}\right\| + |\sin\sigma|\cdot\left\|\begin{pmatrix} 0 & -ab \\ \frac{1}{ab} & 0 \end{pmatrix}\right\|$$
$$= |\cos\sigma|\cdot\max\Big\{\frac{a}{b},\frac{b}{a}\Big\} + |\sin\sigma|\cdot\max\Big\{ab,\frac{1}{ab}\Big\} = \|v_+\|_1.$$

A calculation shows

$$C^TC = \begin{pmatrix} \cos^2(\sigma)\big(\frac{a}{b}\big)^2 + \sin^2(\sigma)\big(\frac{1}{ab}\big)^2 & * \\ * & \cos^2(\sigma)\big(\frac{b}{a}\big)^2 + \sin^2(\sigma)(ab)^2 \end{pmatrix},$$

and we see that

$$\operatorname{tr}(C^TC) = \cos^2\sigma\cdot\Big[\Big(\frac{a}{b}\Big)^2 + \Big(\frac{b}{a}\Big)^2\Big] + \sin^2\sigma\cdot\Big[(ab)^2 + \Big(\frac{1}{ab}\Big)^2\Big].$$

We have $\det(C^TC) = 1$, and hence the eigenvalues of C^TC are the solutions of the equation

$$\lambda + \frac{1}{\lambda} = \operatorname{tr}(C^TC). \tag{4.3}$$

To shorten notation, set $\tau := \operatorname{tr}(C^TC)$. Computing the larger of the solutions of (4.3) gives

$$\|C^TC\| = \frac{1}{2}(\tau + \sqrt{\tau^2-4})$$
$$= 1 + \frac{1}{2}\Big((\tau-2) + \sqrt{(\tau-2)(\tau+2)}\Big)$$
$$= 1 + (\tau-2)^{\frac{1}{2}}\cdot\frac{(\tau-2)^{\frac{1}{2}} + (\tau+2)^{\frac{1}{2}}}{2}.$$

Now, note that

$$\tau - 2 = \cos^2\sigma\cdot\Big(\frac{a}{b}-\frac{b}{a}\Big)^2 + \sin^2\sigma\cdot\Big(ab-\frac{1}{ab}\Big)^2 = \|v_+ - v_-\|_2^2,$$
$$\tau + 2 = \cos^2\sigma\cdot\Big(\frac{a}{b}+\frac{b}{a}\Big)^2 + \sin^2\sigma\cdot\Big(ab+\frac{1}{ab}\Big)^2 = \|v_+ + v_-\|_2^2.$$

It remains to show the estimate from below. To this end, we use that the function

$$f\colon [1,\infty) \to [2,\infty), \quad x \mapsto x + \frac{1}{x},$$

is increasing, continuous, and convex. Its inverse function f^{-1} thus exists and is concave, and we obtain

$$\begin{aligned}\|C\|^2 &= \|C^T C\| \\ &= f^{-1}(\operatorname{tr}(C^T C)) \\ &= f^{-1}\Big(\cos^2\sigma \cdot f\Big(\max\Big\{\Big(\frac{a}{b}\Big)^2, \Big(\frac{b}{a}\Big)^2\Big\}\Big) + \sin^2\sigma \cdot f\Big(\max\Big\{(ab)^2, \Big(\frac{1}{ab}\Big)^2\Big\}\Big)\Big) \\ &\geq \cos^2\sigma \cdot \max\Big\{\Big(\frac{a}{b}\Big)^2, \Big(\frac{b}{a}\Big)^2\Big\} + \sin^2\sigma \cdot \max\Big\{(ab)^2, \Big(\frac{1}{ab}\Big)^2\Big\} \\ &= \|v_+\|_2^2. \qquad \blacksquare\end{aligned}$$

The proof of the theorem is now easily completed.

Proof of Theorem 4.1. Given data as in the theorem, we apply Lemma 4.3 with the matrices

$$\Omega_j := \Omega(a_j, \psi_j), \quad j = 1, \ldots, N,$$

and use Lemma 4.6. This yields

$$\begin{aligned}&\log \|W_H(z)\| \\ &\leq |z| \sum_{j=1}^{N} \int_{y_{j-1}}^{y_j} \|\Omega(a_j, \psi_j) H(t) J \Omega(a_j, \psi_j)^{-1}\| \,\mathrm{d}t \\ &\quad + \sum_{j=1}^{N-1} \log \|\Omega(a_j, \psi_j)\Omega(a_{j+1}, \psi_{j+1})^{-1}\| \\ &\quad + \log \|\Omega(a_1, \psi_1)^{-1}\| + \log \|\Omega(a_N, \psi_N)\| \\ &\leq |z| \sum_{j=1}^{N} \Big(\int_{y_{j-1}}^{y_j} \Big(a_j^2 \cos^2(\phi(t) - \psi_j) + \frac{1}{a_j^2} \sin^2(\phi(t) - \psi_j)\Big) \cdot \operatorname{tr} H(t) \,\mathrm{d}t \Big) \\ &\quad + \sum_{j=1}^{N-1} \log\Big(\max\Big\{\frac{a_j}{a_{j+1}}, \frac{a_{j+1}}{a_j}\Big\} \cdot |\cos(\psi_j - \psi_{j+1})| + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j a_{j+1}}\Big) \\ &\quad + \log \frac{1}{a_1} + \log \frac{1}{a_N} \\ &= |z|(A_1 + A_2) + A_3 + A_4. \qquad \blacksquare\end{aligned}$$

4.7 Remark. The estimate stated in the theorem could be slightly improved on the cost of writing a much more cumbersome expression A_3' instead of A_3. Namely, by using the exact value for the norm in Lemma 4.6 (iii) instead of the upper estimate given there. Doing this would turn the inequality on the fourth line of the above estimate into an equality.

The upper and lower bounds for the norm in Lemma 4.6 (iii) differ only at most by the universal multiplicative constant $\sqrt{2}$. Hence, the potential improvement is limited by

$$A_3 \le A_3' + (N-1) \cdot \frac{1}{2} \log 2.$$

Let us now show that [22, Theorem 1] can indeed be deduced from Theorem 4.1. Recall the statement (for convenience we formulate Romanov's theorem in a notation already fitting Theorem 4.1).

4.8 Theorem ([22]). *Let $H(t) = \xi_{\phi(t)}\xi_{\phi(t)}^T$ be a Hamiltonian on an interval $[0, L]$, and let $d \in (0, 1)$. Assume that we are given a constant $C > 0$, and for each sufficiently large R*

- *a partition $(y_0, \ldots, y_{N(R)})$ of $[0, L]$,*
- *rotation parameters $\psi_1(R), \ldots, \psi_{N(R)}(R) \in \mathbb{R}$,*
- *distortion parameters $a_1(R), \ldots, a_{N(R)}(R) \in (0, 1]$,*

such that

(1) $$\sum_{j=1}^{N(R)} \frac{1}{a_j(R)^2} \int_{y_{j-1}(R)}^{y_j(R)} \|H(t) - \xi_{\psi_j(R)}\xi_{\psi_j(R)}^T\| \, \mathrm{d}t \le CR^{d-1},$$

(2) $$\sum_{j=1}^{N(R)} a_j(R)^2\big(y_j(R) - y_{j-1}(R)\big) \le CR^{d-1},$$

(3) $$\sum_{j=1}^{N(R)-1} \log\Big(1 + \frac{|\sin(\psi_j(R) - \psi_{j-1}(R))|}{a_j(R)a_{j+1}(R)}\Big) \le CR^d,$$

(4) $$\log \frac{1}{a_1(R)} + \log \frac{1}{a_{N(R)}} + \sum_{j=1}^{N(R)-1} \Big|\log \frac{a_{j+1}(R)}{a_j(R)}\Big| \le CR^{d-1}.$$

Then there exists a constant $K > 0$ such that

$$\log \|W_H(z)\| \le K|z|^d \quad \textit{for all } z \in \mathbb{C}. \tag{4.4}$$

Deduction from Theorem 4.1. Let $R > 0$ and assume that we have data $y_j(R)$, $\psi_j(R)$, and $a_j(R)$ satisfying (i)–(iv). We are going to estimate the expressions $A_1, \ldots, A_4$ from Theorem 4.1.

First, it is clear that

$$A_1(R) = \sum_{j=1}^{N(R)} a_j(R)^2 \int_{y_{j-1}}^{y_j} \cos^2\big(\phi(t) - \psi_j(R)\big)\,\mathrm{d}t \le \sum_{j=1}^{N(R)} a_j(R)^2\big(y_j(R) - y_{j-1}(R)\big) \le CR^{d-1}.$$

Next, observe that for all $\phi, \psi \in \mathbb{R}$

$$\xi_\phi \xi_\phi^T - \xi_\psi \xi_\psi^T = \sin(\phi - \psi) \cdot \begin{pmatrix} -\sin(\phi+\psi) & \cos(\phi+\psi) \\ \cos(\phi+\psi) & \sin(\phi+\psi) \end{pmatrix}.$$

Since the matrix on the right side is unitary, it follows that

$$\|\xi_\phi \xi_\phi^T - \xi_\psi \xi_\psi^T\| = |\sin(\phi - \psi)|.$$

From this, we obtain

$$A_2(R) = \sum_{j=1}^{N(R)} \frac{1}{a_j(R)^2} \int_{y_{j-1}}^{y_j} \sin^2\big(\phi(t) - \psi_j(R)\big)\,\mathrm{d}t \le \sum_{j=1}^{N(R)} \frac{1}{a_j(R)^2} \int_{y_{j-1}}^{y_j} |\sin(\phi(t) - \psi_j(R))|\,\mathrm{d}t \le CR^{d-1}. \tag{4.5}$$

Finally, we have

$$\begin{aligned} A_3(R) + A_4(R) &\le \sum_{j=1}^{N(R)-1} \log\Big[\max\Big\{\frac{a_j(R)}{a_{j+1}(R)}, \frac{a_{j+1}(R)}{a_j(R)}\Big\}\Big(1 + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j(R)a_{j+1}(R)}\Big)\Big] \\ &\quad - \log a_1(R) - \log a_{N(R)}(R) \\ &= \sum_{j=1}^{N(R)-1} \Big|\log \frac{a_{j+1}(R)}{a_j(R)}\Big| + \sum_{j=1}^{N(R)-1} \Big(1 + \frac{|\sin(\psi_j - \psi_{j+1})|}{a_j(R)a_{j+1}(R)}\Big) \\ &\quad + \log\frac{1}{a_1(R)} + \log\frac{1}{a_{N(R)}(R)} \le 2CR^d. \end{aligned}$$

Now, (4.1) yields

$$\log\|W_H(z)\| \le |z| \cdot 2CR^{d-1} + 2CR^d \quad \text{for all } z \in \mathbb{C}.$$

We use this for $z \in \mathbb{C}$ with $|z| = R$ and obtain

$$\log\|W_H(z)\| \le 4CR^d \quad \text{for } |z| = R.$$

By the assumption of the theorem, the above argument can be made for all sufficiently large R. Hence, (4.4) follows. ■

4.9 Remark. The improvement of Theorem 4.1 compared to [22, Theorem 1] mainly happens in (4.5): clearly,

$$\sin^2(\phi(t) - \psi_j(R)) \ll \big|\sin(\phi(t) - \psi_j(R))\big|$$

when $\psi_j(R)$ is a good approximation of $\phi(t)$.

For this reason, we also refer to Theorem 4.1 as the *sine-square improvement* of Romanov's Theorem 1. We will see in Remark 5.4 below that it is indeed a significant improvement.

5. Hamiltonians with continuous rotation angle

In this section we consider Hamiltonians of the form

$$H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)}\xi_{\phi(t)}^T, \quad t \in I, \tag{5.1}$$

with a continuous rotation angle $\phi\colon I \to \mathbb{R}$, and prove an upper bound for $\log \|W_H(z)\|$. As a corollary, we obtain a bound for the exponential order of the monodromy matrix of a Hölder continuous Hamiltonian which improves [22, Corollary 4 (1)]. The proof of the upper estimate is an application of Theorem 4.1, and nicely illustrates how concrete growth estimates can be deduced from the general estimate.

We use the following notation which involves the modulus of continuity of a function ϕ. The case that ϕ is constant will be excluded, but this is no loss of generality: if ϕ in (5.1) is constant, then W is a linear polynomial and hence $\log \|W_H(z)\| = \mathrm{O}(\log |z|)$.

5.1 Definition. For $\alpha \in \mathbb{R} \setminus \{0\}$ denote by $p_\alpha\colon (0,\infty) \to (0,\infty)$ the power function $p_\alpha(t) := t^\alpha$.

i. Let $\omega\colon [0,\infty) \to [0,\infty)$ be a nondecreasing continuous function with $\omega(0) = 0$ and $\omega(\delta) > 0$ for all $\delta > 0$. Then we define an increasing bijection $\Gamma_\omega\colon (0,\infty) \to (0,\infty)$ as

$$\Gamma_\omega := p_{-1} \circ (p_1 \cdot \omega)^{-1} \circ p_{-1}, \tag{5.2}$$

where $(p_1 \cdot \omega)^{-1}$ denotes the inverse function of $p_1 \cdot \omega$ (note here that $p_1 \cdot \omega$ is an increasing bijection of $[0,\infty)$ onto itself).

ii. Let H be a Hamiltonian of the form (5.1) with continuous and non-constant rotation angle ϕ. Then we write ω_H for the modulus of uniform continuity of ϕ, i.e.

$$\omega_H(\delta) := \sup\{|\phi(t) - \phi(s)| \mid t,s \in I, |t-s| \le \delta\}, \quad \delta \ge 0,$$

and let $\Gamma_H := \Gamma_{\omega_H}$ be the function corresponding to ω_H by the construction in item (i).

The assignment $\omega \mapsto \Gamma_\omega$ defined by (5.2) is injective. In fact, ω can be recovered from Γ_ω by the formula

$$\omega = p_{-1} \cdot (p_{-1} \circ \Gamma_\omega^{-1} \circ p_{-1}).$$

Moreover, we have the following monotonicity property:

$$\omega_1 \leq \omega_2 \implies \Gamma_{\omega_1} \leq \Gamma_{\omega_2}.$$

Given a Hamiltonian H, the growth of the functions ω_H and Γ_H is limited: simply because ω_H is the modulus of uniform continuity of some continuous function on a compact interval, we have

$$\omega_H(\delta) = \mathrm{o}(1) \quad \text{and} \quad \delta = \mathrm{O}(\omega_H(\delta)) \quad \text{for } \delta \to 0.$$

From this, it follows that

$$\Gamma_H(r) = \mathrm{o}(r) \quad \text{and} \quad \sqrt{r} = \mathrm{O}(\Gamma_H(r)) \quad \text{for } r \to \infty. \tag{5.3}$$

Our bound for the monodromy matrix can now be formulated as follows.

5.2 Theorem. *Let $H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)}\xi_{\phi(t)}^T$ be a Hamiltonian on a compact interval $I = [\alpha, \beta]$ whose rotation angle ϕ is continuous and not constant. Set $l := \beta - \alpha$ and $L := \int_I \operatorname{tr} H(t)\,\mathrm{d}t$. Then*

$$\log \|W_H(z)\| \leq 3l \cdot \Gamma_H\Big(\frac{L}{l}|z|\Big) + \mathrm{O}(\log|z|). \tag{5.4}$$

Proof. We are going to apply Theorem 4.1. Let $\delta \in (0, l)$ and $a \in (0, 1]$; a specific choice will be made later in dependence of $|z|$. The data y_j, ψ_j, a_j in Theorem 4.1 are now specified as follows.

- Let N be the unique positive integer with $N - 1 < \frac{l}{\delta} \leq N$, and define a partition $(y_0, \ldots, y_N)$ of I as

$$y_j := \begin{cases} \alpha + j \cdot \delta & \text{if } j \in \{0, \ldots, N-1\}, \\ \beta & \text{if } j = N. \end{cases}$$

- Rotation parameters are

$$\psi_j := \phi(y_j), \quad j = 1, \ldots, N.$$

- Distortion parameters are $a_j := a$, $j = 1, \ldots, N$.

The choice of rotation parameters implies that

$$|\phi(t) - \psi_j| \le \omega_H(\delta), \quad t \in [y_{j-1}, y_j].$$

The constants A_1, A_2, A_3 from Theorem 4.1 can be estimated as follows:

$$\begin{aligned} A_1 &\le a^2 L =: B_1(a), \\ A_2 &\le \frac{1}{a^2}\omega_H(\delta)^2 L =: B_2(a,\delta), \\ A_3 &\le \sum_{j=1}^{N-1} \log\Big(1 + \frac{|\sin(\psi_j - \psi_{j+1})|}{a^2}\Big) \\ &\le (N-1)\frac{1}{a^2}\omega_H(\delta) < \frac{l}{\delta}\frac{1}{a^2}\omega_H(\delta) =: B_3(a,\delta). \end{aligned}$$

Given $z \in \mathbb{C}$, we specify the parameters δ and a as

$$\delta := \Big[\Gamma_H\Big(\frac{L}{l}|z|\Big)\Big]^{-1}, \quad a := \omega_H(\delta)^{\frac{1}{2}}. \tag{5.5}$$

These formulas are found by minimising the maximum of the expressions $B_1(a)$, $B_2(a,\delta)$, $B_3(a,\delta)$. Observe that, by the properties of Γ_H noted in (5.3), we have $\delta < l$ and $a \le 1$ for all sufficiently large $|z|$. We have

$$B_1(a) = B_2(a,\delta) = \omega_H(\delta)L, \quad B_3(a,\delta) = \frac{l}{\delta}, \quad \delta\omega_H(\delta) = \frac{l}{L|z|}, \tag{5.6}$$

and hence

$$|z|B_2(a,\delta) = |z|B_1(a) = |z|\omega_H(\delta)L = \frac{l}{\delta} = B_3(a,\delta).$$

Theorem 4.1 implies that

$$\begin{aligned} \log\|W_H(z)\| &\le |z|\big(B_1(a) + B_2(a,\delta)\big) + B_3(a,\delta) + \log\frac{1}{a^2} \\ &= 3 \cdot l\,\Gamma_H\Big(\frac{L}{l}|z|\Big) + \log\frac{1}{\omega_H(\delta)}. \end{aligned}$$

By the last relation in (5.6),

$$\log\frac{1}{\omega_H(\delta)} = \log\frac{L}{l} + \underbrace{\log\delta}_{<\log l} + \log|z| = \log|z| + \mathrm{O}(1) = \mathrm{O}(\log|z|),$$

and the bound (5.4) follows. ∎

Applying Theorem 5.2 to Hölder continuous functions leads to the following corollary. To fix notation, recall that a function $\phi\colon I \to \mathbb{R}$ is called *Hölder continuous with exponent* $\alpha \in [0,1]$ if there exists $c > 0$ such that

$$|\phi(t) - \phi(s)| \le c|t-s|^{\alpha} \quad \text{for all } t, s \in I. \tag{5.7}$$

The Hölder exponent $\alpha_1(\phi)$ of ϕ is

$$\alpha_1(\phi) := \sup\{\alpha \in [0,1] \mid \phi \text{ is Hölder continuous with exponent } \alpha\}.$$

5.3 Corollary. *Let* $\alpha \in (0,1]$ *and let* $H(t) = \operatorname{tr} H(t) \cdot \xi_{\phi(t)}\xi_{\phi(t)}^T$, $t \in I$, *be a Hamiltonian on a compact interval* I *whose rotation angle is Hölder continuous with exponent* α. *Then*

$$\log\bigl(\max_{|z|=r} \|W_H(z)\|\bigr) \lesssim r^{\frac{1}{1+\alpha}}.$$

Consequently, the exponential order of $W(z)$ *does not exceed* $\frac{1}{1+\alpha_1(\phi)}$.

Proof. Let c, α be as in (5.7) and set $\omega(\delta) := c\delta^{\alpha}$. Then $\Gamma_\omega(r) = c^{\frac{1}{1+\alpha}} r^{\frac{1}{1+\alpha}}$. We have $\omega_H \le \omega$, and thus also $\Gamma_H \le \Gamma_\omega$. Theorem 5.2 gives

$$\begin{aligned}\log\bigl(\max_{|z|=r} \|W(z)\|\bigr) &\lesssim \Gamma_H\Bigl(\frac{L}{l}r\Bigr) + \mathrm{O}(\log r)\\ &\le \Gamma_\omega\Bigl(\frac{L}{l}r\Bigr) + \mathrm{O}(\log r) \asymp r^{\frac{1}{1+\alpha}}.\end{aligned}$$ ■

This corollary shows that the present general estimate is an improvement of Romanov's Theorem even on the scale of exponential order.

5.4 Remark. In [22, Corollary 4 (1)] it is shown that for a Hölder continuous (trace normed) Hamiltonian with Hölder exponent $\alpha \in (0,1]$ the order of the entire function $W_H(z)$ does not exceed $1 - \frac{\alpha}{2}$. The above corollary improves this:

$$\frac{1}{1+\alpha} < 1 - \frac{\alpha}{2} \quad \text{for all } \alpha \in (0,1).$$

Theorem 5.2 is limited to orders in $[\frac{1}{2}, 1]$: due to (5.3) the bound (5.4) cannot go below $r^{\frac{1}{2}}$. To show that this really is a limitation, we should give an example of a Hamiltonian with continuous rotation angle and small order.

5.5 Example. We start from the example given in [22, Section 7.3]. Let $p \in (0,1)$, and let μ be a probability measure on $[0,1]$ which has no point masses, whose topological support has zero Lebesgue measure and is such that the connected components of $[0,1] \setminus \operatorname{supp}\mu$, call them $I_j = (\alpha_j, \beta_j)$, satisfy

$$\sum_j (\beta_j - \alpha_j)^p < \infty.$$

Set $\Delta := \bigcup_j \big(\mu([0,\alpha_j]) + I_j\big) \subseteq [0,2]$, and note that Δ has measure 1. Let us show that Δ is dense in $[0,2]$. By our assumption that μ has no point masses, the function $f(x) := \mu([0,x]) + x$ is continuous. Given $t \in [0,2]$, we thus find $x \in [0,1]$ with $f(x) = t$. The support of μ has empty interior, hence we can choose $x_n \in I_{j_n}$ such that $\lim_{n\to\infty} x_n = x$. It follows that $t = f(x) = \lim_{n\to\infty} f(x_n)$, and since the distribution function of μ is constant on intervals I_j

$$f(x_n) = \mu([0,x_n]) + x_n = \mu([0,\alpha_{j_n}]) + x_n \in \mu([0,\alpha_{j_n}]) + I_{j_n} \subseteq \Delta.$$

Let $H\colon [0,2] \to \mathbb{R}^{2\times 2}$ be the Hamiltonian defined as

$$H(t) := \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & \text{if } t \in \Delta, \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } t \in [0,2] \setminus \Delta. \end{cases}$$

We set $h_1 = \mathbb{1}_\Delta$ and $h_2 = \mathbb{1}_{\Delta^c}$, so that $H = \left(\begin{smallmatrix} h_1 & 0 \\ 0 & h_2 \end{smallmatrix}\right)$. By [22, Section 7.3] (we write $\rho(\cdot)$ for the order of an entire function),

$$\rho(W_H) \leq \frac{2p}{p+1}.$$

Now, we apply the general procedure [9, Section 4] to construct a non-diagonal Hamiltonian. Set

$$m_j(t) := \int_0^t h_j(s)\,\mathrm{d}s, \quad j = 1,2.$$

Then $m_j\colon [0,2] \to [0,1]$ are continuous, nondecreasing, and surjective. Since Δ is open and dense, its intersection with any nonempty open interval has positive measure. Hence, m_1 is even an increasing bijection. This allows us to define a continuous Hamiltonian $\tilde{H}\colon [0,1] \to \mathbb{R}^{2\times 2}$ by

$$m := m_2 \circ m_1^{-1}, \quad \tilde{H} := \begin{pmatrix} 1 & -m \\ -m & m^2 \end{pmatrix}.$$

By [9, Lemma 4.1], we have

$$(0,1) W_H(z) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (0,1) W_{\tilde{H}}(z^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and it follows that

$$\rho(W_{\tilde{H}}) \leq \frac{p}{p+1}.$$

Making an approriate choice of p, this becomes arbitrarily small.

Note that we can write $\tilde{H}$ in the form (5.1) with the continuous rotation angle

$$\tilde{\phi}(x) := -\arctan m(x), \quad x \in [0,1].$$

One interesting observation about the statement in Theorem 5.2 is that passing from H to a reparameterisation does not change the monodromy matrix, but may drastically change the modulus of continuity of the rotation angle and with it the bound on the right side of (5.4). This fact can be used to improve the bound.

Methodologically, this is not a surprise; it reflects that in the proof of Theorem 5.2 we applied the general estimate only with equidistant partitions and the modulus of uniform continuity. Making a change of scale, we can try to flatten out the rotation angle on sections where it heavily oscillates, and by this make the quality of its continuity more even across the whole interval. The other way to achieve this effect would be to use arbitrary partitions. Maybe this would be more effective, but certainly it is computationally harder to handle.

At this point let us just illustrate by an example that working with reparameterisations indeed can leed to a significant improvement.

5.6 Example. For $\gamma, \beta > 0$, let $\phi_{\gamma,\beta}\colon [0,1] \to \mathbb{R}$ be the chirp function

$$\phi_{\gamma,\beta}(t) := \begin{cases} t^\gamma \sin(\frac{1}{t^\beta}) & \text{if } t \in (0,1], \\ 0 & \text{if } t = 0, \end{cases}$$

and consider the Hamiltonian

$$H_{\gamma,\beta}(t) := \xi_{\phi_{\gamma,\beta}} \xi_{\phi_{\gamma,\beta}}^T, \quad t \in [0,1].$$

We require in the following that $\gamma \leq \beta$, so that $\phi_{\gamma,\beta}$ is not of bounded variation. This is done to rule out an application of [22, Corollary 4 (2)] which would imply at once that the order of the monodromy matrix is at most $\frac{1}{2}$ (and we could not go below order $\frac{1}{2}$ anyway). Our aim is to show that the order of the monodromy matrix $W_{H_{\gamma,\beta}}(z)$ is bounded by

$$\rho(W_{H_{\gamma,\beta}}) \leq \frac{\beta}{\beta+\gamma}. \tag{5.8}$$

The Hölder exponent of $\phi_{\gamma,\beta}$ is $\frac{\gamma}{\beta+1}$. Hence, Corollary 5.3 gives

$$\rho(W_{H_{\gamma,\beta}}) \leq \frac{1}{1+\frac{\gamma}{\beta+1}} = \frac{\beta+1}{\beta+\gamma+1}.$$

For $\kappa > 1$, set $\psi_\kappa(t) := t^\kappa$. Then ψ_κ is an absolutely continuous increasing bijection of $[0,1]$ onto itself whose derivative is positive almost everywhere. It thus qualifies for being used as a reparameterisation. Denote

$$H_{\gamma,\beta}^{[\kappa]}(t) := (H_{\gamma,\beta} \circ \psi_\kappa)(t) \cdot \psi_\kappa'(t), \quad t \in [0,1].$$

Apparently, $H_{\gamma,\beta}^{[\kappa]} = \kappa t^{\kappa-1} \cdot H_{\gamma\kappa,\beta\kappa}$, and hence

$$\rho(W_{H_{\gamma,\beta}}) = \rho(W_{H_{\gamma,\beta}^{[\kappa]}}) \leq \frac{\beta\kappa + 1}{\beta\kappa + \gamma\kappa + 1} = \frac{\beta + \frac{1}{\kappa}}{\beta + \gamma + \frac{1}{\kappa}}.$$

Sending κ to infinity, (5.8) follows.

6. Sharpness in Theorem 5.2

Remember Example 5.5 where the bound from Theorem 5.2 cannot possibly give the correct growth of the monodromy matrix. Our aim in this section is to construct examples where (5.4) gives the correct growth, at least up to an error of logarithmic size. In particular, in these examples, (5.4) will give the correct order. We formulate this fact in a fairly general way.

6.1 Theorem. *Let $\mathfrak{g}$ and $\mathfrak{m}$ be regularly varying function with*

$$\frac{1}{2} < \operatorname{Ind} \mathfrak{g} < 1 \quad \textit{and} \quad \int_1^\infty \frac{1}{\mathfrak{m}(t)}\,\mathrm{d}t < \infty,$$

and let $\mathfrak{n}$ be regularly varying with $(\mathfrak{n} \circ \mathfrak{m})(x) \sim (\mathfrak{m} \circ \mathfrak{n})(x) \sim x$.

Then there exists a Hamiltonian $H(t) = \xi_{\phi(t)}\xi_{\phi(t)}^T$ whose rotation angle $\phi(t)$ is continuous, such that

$$(\mathfrak{n} \circ \mathfrak{g})(r) \lesssim \log(\max_{|z|=r} \|W_H(z)\|) \lesssim \mathfrak{g}(r). \tag{6.1}$$

Note that the gap left by (6.1) is indeed rather small: we could choose for examples $\mathfrak{m}(r) := r(\log r)(\log\log r)^2$. Then $\mathfrak{n}(r) \sim \frac{r}{(\log r)(\log\log r)^2}$, and hence the lower bound (6.1) satisfies

$$(\mathfrak{n} \circ \mathfrak{g})(r) \asymp \frac{\mathfrak{g}(r)}{(\log r)(\log\log r)^2}.$$

In particular, we see that in the Hölder continuous situation the bound for order given in Corollary 5.3 is sharp.

For the proof of Theorem 6.1, we have to construct a function $\phi(t)$ whose modulus of continuity is prescribed and such that the growth of the corresponding monodromy matrix can be estimated from below.

Finding just some function with given modulus of continuity is of course easy. Every continuous increasing and subadditive function $\omega\colon [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ is the modulus of continuity of itself. However, using such functions for the rotation angle $\phi(t)$ of a Hamiltonian will not lead to a required example: the order of the monodromy matrix cannot exceed $\frac{1}{2}$ by [22, Corollary 4 (2)].

It turns out that the following example of an oscillating function with prescribed modulus of continuity does the job. We want to point out that placing constancy intervals is crucial, at least for our argument.

6.2 Example. Assume we are given

i. a sequence $(l_j)_{j=1}^{\infty}$ of positive numbers with $\sum_{j=1}^{\infty} l_j < \infty$,

ii. a nonincreasing sequence $(m_j)_{j=1}^{\infty}$ of positive numbers with $m_j \leq l_j$ for all $j \in \mathbb{N}$,

iii. a continuous function $\pi\colon (0,\infty) \to (0,\infty)$, such that π is nondecreasing on $(0, m_1)$, the function $p_{-1} \cdot \pi$ (again $p_\alpha(x) := x^\alpha$) is nonincreasing on $(0, m_1)$, and

$$\lim_{x\to 0} \pi(x) = 0, \quad \sup_{j\in\mathbb{N}} \frac{\pi(m_j)}{\pi(m_{j+1})} < \infty, \quad \pi(m_1) < \frac{\pi}{2}.$$

Set

$$r_n := \sum_{j=1}^{n-1} (l_j + m_j),$$

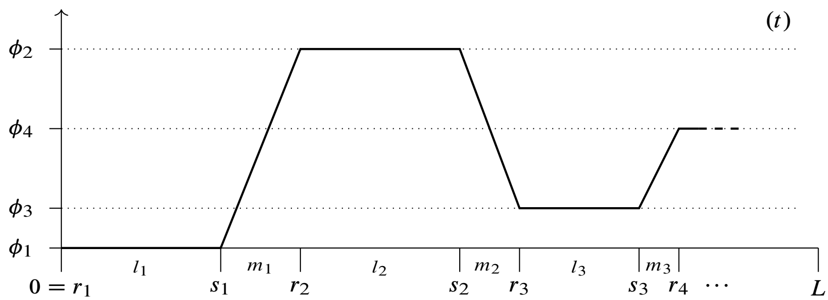
$$s_n := r_n + l_n,$$

$$L := \sum_{j=1}^{\infty} (l_j + m_j),$$

$$\phi_n := \sum_{j=1}^{n-1} (-1)^{j+1} \pi(m_j),$$

and let $\phi\colon [0, L) \to [0, \frac{\pi}{2})$ be the piecewise linear path connecting the points

$$(r_1, \quad \phi_1), \quad (s_1, \phi_1), \quad (r_2, \phi_2), \quad (s_2, \phi_2), \quad (r_3, \phi_3), \quad \ldots.$$



We assert that the modulus of uniform continuity ω_ϕ of the function ϕ satisfies

$$\omega_\phi(\delta) \asymp \pi(\delta) \quad \text{for } \delta \leq m_1. \tag{6.2}$$

In order to prove this, we first show that

$$\omega_\phi(m_n) = \pi(m_n). \tag{6.3}$$

The inequality "$\geq$" follows since we have

$$r_{n+1} - s_n = m_n \text{ and } |\phi(r_{n+1}) - \phi(s_n)| = |\phi_{n+1} - \phi_n| = \pi(m_n).$$

Consider two points t, s with $t < r_n$ and $t \leq s \leq t + m_n$. Then $s \leq s_n$. The function $\phi|_{[0,s_n]}$ is a polygonal path with maximal slope $\frac{\pi(m_{n-1})}{m_{n-1}}$, and we obtain

$$|\phi(t) - \phi(s)| \leq \frac{\pi(m_{n-1})}{m_{n-1}} \cdot m_n \leq \pi(m_n).$$

For each two points t, s with $r_n \leq t \leq s \leq t + m_n$ we have

$$|\phi(t) - \phi(s)| \leq |\phi_{n+1} - \phi_n| = \pi(m_n),$$

and "$\leq$" in (6.3) follows.

For the proof of (6.2), let $\delta \leq m_1$ be given. Let $n \in \mathbb{N}$ be such that $m_{n+1} < \delta \leq m_n$, then

$$\frac{\pi(m_{n+1})}{\pi(m_n)} = \frac{\pi(m_{n+1})}{\omega_\phi(m_n)} \leq \frac{\pi(\delta)}{\omega_\phi(\delta)} \leq \frac{\pi(m_n)}{\omega_\phi(m_{n+1})} = \frac{\pi(m_n)}{\pi(m_{n+1})}.$$

Proof of Theorem 6.1. Based on [3, Theorems 1.8.2 and 1.8.5] we may assume without loss of generality that g and m are increasing bijections of $(0, \infty)$ onto itself. Moreover, we may say that $m(1)$ is as large as it pleases us (and a concrete request will be put later).

We use Example 6.2 with the data

$$m_j = l_j := \frac{1}{m(j)}, \quad \pi := p_{-1} \cdot \big(p_{-1} \circ g^{-1} \circ p_{-1}\big).$$

We have to check that the conditions required in Example 6.2 (i)–(iii) are fulfilled. First,

$$\sum_{j=1}^{\infty} l_j \leq l_1 + \int_1^\infty \frac{1}{m(t)}\,\mathrm{d}t < \infty,$$

and m_j $(= l_j)$ is decreasing. Second, π is continuous and regularly varying (at 0) with

$$\operatorname{Ind}\pi = \frac{1}{\operatorname{Ind} g} - 1 \in (0, 1).$$

Hence, sufficiently close to 0, π is increasing and $p_{-1} \circ \pi$ is decreasing. Now, we assume (without loss of generality) that $m(1)$ is so large that $\frac{1}{m(1)}$ is already sufficiently close to 0 in the above sense and $< \frac{\pi}{2}$. Also, the function $\pi \circ p_{-1} \circ m$ is regularly varying, and hence

$$\lim_{j \to \infty} \frac{\pi(m_j)}{\pi(m_{j+1})} = \lim_{j \to \infty} \frac{(\pi \circ p_{-1} \circ m)(j)}{(\pi \circ p_{-1} \circ m)(j+1)} = 1.$$

In particular, the quotient is bounded.

Let ϕ be the function constructed in Example 6.2. Then $\omega_\phi \asymp \pi$, and since π is regularly varying it follows that $\Gamma_{\omega_\phi} \asymp \Gamma_\pi$. The latter function computes as

$$\Gamma_\pi = p_{-1} \circ (p_1 \cdot \pi)^{-1} \circ p_{-1} = \mathfrak{g}.$$

Let H be the Hamiltonian $H(t) := \xi_{\phi(t)} \xi_{\phi(t)}^T$. The upper bound in (6.1) is just (5.4). In order to show the lower bound, we aim at an application of Corollaries 2.5 and 3.5. We use the set

$$\Delta := \bigcup_{j=1}^{\infty} (s_j, r_j)$$

in Corollary 3.5. The Hamiltonian $\widetilde{H}$ constructed there is in our situation the Hamburger Hamiltonian with lengths $(l_j)_{j=1}^\infty$ and angles $(\phi_j)_{j=1}^\infty$. Now, Corollary 2.5 comes into play: we have

$$l_{j+1} l_j \sin^2(\phi_{j+1} - \phi_j) = \frac{1}{m(j+1)m(j)} \sin^2(\pi(m_j)) \sim \frac{1}{m(j)^2} (\pi \circ p_{-1} \circ m)(j)^2,$$

and hence we can use

$$\mathfrak{f} := p_{-2} \circ (p_1 \cdot \pi) \circ p_{-1} \circ m$$

in (2.3). The right side of (2.4) then is

$$\mathfrak{f}^{-1} \circ p_2 = m^{-1} \circ p_{-1} \circ (p_1 \cdot \pi)^{-1} \circ p_{-\frac{1}{2}} \circ p_2 = m^{-1} \circ \mathfrak{g}.$$

Thus, by Corollary 2.5,

$$\log \max_{|z|=r} \|W_{\widetilde{H}}(z)\| \gtrsim (m^{-1} \circ \mathfrak{g})(r).$$

Corollary 3.5 implies that also

$$\log \max_{|z|=r} \|W_H(z)\| \gtrsim (m^{-1} \circ \mathfrak{g})(r).$$

Note here that $\operatorname{Ind} m \geq 1$, and hence $\operatorname{Ind}(m^{-1} \circ \mathfrak{g}) \in (0, 1)$. ■

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