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Discrete approximations to Dirac operators and norm resolvent convergence

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Abstract. We consider continuous Dirac operators defined on $\mathbf{R}^d$, $d \in \{1,2,3\}$, together with various discrete versions of them. Both forward-backward and symmetric finite differences are used as approximations to partial derivatives. We also allow a bounded, Hölder continuous, and self-adjoint matrix-valued potential, which in the discrete setting is evaluated on the mesh. Our main goal is to investigate whether the proposed discrete models converge in norm resolvent sense to their continuous counterparts, as the mesh size tends to zero and up to a natural embedding of the discrete space into the continuous one. In dimension one we show that forward-backward differences lead to norm resolvent convergence, while in dimension two and three they do *not*. The same negative result holds in all dimensions when symmetric differences are used. On the other hand, strong resolvent convergence holds in all these cases. Nevertheless, and quite remarkably, a rather simple but non-standard modification to the discrete models, involving the mass term, ensures norm resolvent convergence in general.

1. Introduction

We study in detail in what sense continuous Dirac operators [8] can be approximated by a family of discrete operators indexed by the mesh size. To investigate spectral properties based on the discrete models, it is essential to know whether we can obtain norm resolvent convergence or only strong resolvent convergence of the discrete models (suitably embedded into the continuum) to the continuous Dirac operators.

In this paper we present a remarkable new phenomenon. In dimensions two and three we cannot obtain norm resolvent convergence of the discrete operators (embedded into the continuum) as the mesh size tends to zero, if we use the natural discretizations based on either symmetric first order differences or a pair of forward-backward first order differences. The models require a simple modification to obtain norm resolvent convergence. In dimension one the discretization using a pair of forward-backward first order differences does lead to norm resolvent convergence, whereas the model based on symmetric first order differences does not.

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These results are now described in some detail. To unify the notation, we define $\nu(1) = \nu(2) = 2$ and $\nu(3) = 4$. The Hilbert spaces used for the continuous Dirac operators are

$$\mathcal{H}^d = L^2(\mathbf{R}^d) \otimes \mathbf{C}^{\nu(d)}, \quad d = 1, 2, 3.$$

For mesh size $h > 0$, the corresponding discrete spaces are denoted by

$$\mathcal{H}_h^d = \ell^2(h\mathbf{Z}^d) \otimes \mathbf{C}^{\nu(d)}, \quad d = 1, 2, 3.$$

The norm on $\mathcal{H}_h^d$ is given by

$$\|u_h\|^2_{\mathcal{H}_h^d} = h^d \sum_{k \in \mathbf{Z}^d} |u_h(k)|^2, \quad u_h \in \mathcal{H}_h^d.$$

Here $|\cdot|$ denotes the Euclidean norm on $\mathbf{C}^{\nu(d)}$. We index u_h by $k \in \mathbf{Z}^d$; the h dependence is in the subscript of u_h.

To relate the spaces $\mathcal{H}_h^d$ and $\mathcal{H}^d$, we introduce embedding operators $J_h \colon \mathcal{H}_h^d \to \mathcal{H}^d$ and discretization operators $K_h \colon \mathcal{H}^d \to \mathcal{H}_h^d$, constructed from a pair of biorthogonal Riesz sequences, as in [2, Section 2]. We describe the construction briefly, with further details and assumptions given in Section 2. Let $\varphi_0, \psi_0 \in L^2(\mathbf{R}^d)$ and assume that $\{\varphi_0(\cdot - k)\}_{k \in \mathbf{Z}^d}$ and $\{\psi_0(\cdot - k)\}_{k \in \mathbf{Z}^d}$ are a pair of biorthogonal Riesz sequences in $L^2(\mathbf{R}^d)$. Define $\varphi_{h,k}(x) = \varphi_0((x - hk)/h)$, and $\psi_{h,k}(x) = \psi_0((x - hk)/h)$, $x \in \mathbf{R}^d$, $k \in \mathbf{Z}^d$, $h > 0$. The embedding operator J_h is then defined as

$$(J_h u_h)(x) = \sum_{k \in \mathbf{Z}^d} \varphi_{h,k}(x) u_h(k).$$

Note that here $\varphi_{h,k}(x)$ is a scalar multiplying a vector $u_h(k) \in \mathbf{C}^{\nu(d)}$. To construct the discretization operator, let $\widetilde{J}_h$ be defined as J_h with φ_0 replaced by ψ_0. The discretization operator is then defined as $K_h = (\widetilde{J}_h)^*$. For $d = 1, 2$, it can be written explicitly as

$$(K_h f)(k) = \frac{1}{h^d} \begin{bmatrix} \langle \psi_{h,k}, f^1 \rangle \\ \langle \psi_{h,k}, f^2 \rangle \end{bmatrix}, \quad f = \begin{bmatrix} f^1 \\ f^2 \end{bmatrix} \in \mathcal{H}^d.$$

A similar formula holds for $d = 3$. We have $K_h J_h = I_h$, where I_h is the identity in $\mathcal{H}_h^d$, and $J_h K_h$ is a projection in $\mathcal{H}^d$ onto $J_h \mathcal{H}_h^d$.

Let H_0 be the free Dirac operator in $\mathcal{H}^d$, $d = 1, 2, 3$, and let $H_{0,h}$ be an approximation defined on $\mathcal{H}_h^d$. We compare the operators

$$J_h (H_{0,h} - zI_h)^{-1} K_h \quad \text{and} \quad (H_0 - zI)^{-1}$$

acting on $\mathcal{H}^d$. The question of interest is in what sense will $J_h(H_{0,h} - zI_h)^{-1} K_h$ converge to $(H_0 - zI)^{-1}$ as $h \to 0$. We now summarize the results obtained. First we briefly define the operators considered.

Let σ_j, $j = 1, 2, 3$, denote the Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.1}$$

Let $m \geq 0$ denote the mass. To simplify, we do not indicate dependence on the mass in the notation for operators. In dimension $d = 1$, the free Dirac operator is given by the operator matrix

$$H_0 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1 + m\sigma_3$$

on $\mathscr{H}^1$. We consider two discrete approximations based on replacing $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ by finite difference operators. Let I_h denote the identity operator on $\ell^2(h\mathbf{Z})$. We define

$$H_{0,h}^{\mathrm{fb}} = \begin{bmatrix} m\mathsf{I}_h & D_h^- \\ D_h^+ & -m\mathsf{I}_h \end{bmatrix} \quad \text{and} \quad H_{0,h}^{\mathrm{s}} = \begin{bmatrix} m\mathsf{I}_h & D_h^{\mathrm{s}} \\ D_h^{\mathrm{s}} & -m\mathsf{I}_h \end{bmatrix}.$$

Here the forward and backward finite difference operators are defined as

$$(D_h^+ u_h)(k) = \frac{1}{\mathrm{i}h}(u_h(k+1) - u_h(k)), \tag{1.2a}$$

$$(D_h^- u_h)(k) = \frac{1}{\mathrm{i}h}(u_h(k) - u_h(k-1)), \tag{1.2b}$$

and satisfies $(D_h^+)^* = D_h^-$. The symmetric difference operator is the self-adjoint operator $D_h^{\mathrm{s}} = \frac{1}{2}(D_h^+ + D_h^-)$, i.e.,

$$(D_h^{\mathrm{s}} u_h)(k) = \frac{1}{2\mathrm{i}h}(u_h(k+1) - u_h(k-1)). \tag{1.3}$$

In dimension $d = 2$, the free Dirac operator is defined as

$$H_0 = -\mathrm{i}\frac{\partial}{\partial x_1}\sigma_1 - \mathrm{i}\frac{\partial}{\partial x_2}\sigma_2 + m\sigma_3$$

on $\mathscr{H}^2$. As in the $d = 1$ case, there are two natural discrete models given by

$$H_{0,h}^{\mathrm{fb}} = \begin{bmatrix} m\mathsf{I}_h & D_{h;1}^- - \mathrm{i}D_{h;2}^- \\ D_{h;1}^+ + \mathrm{i}D_{h;2}^+ & -m\mathsf{I}_h \end{bmatrix}$$

and

$$H_{0,h}^{\mathrm{s}} = \begin{bmatrix} m\mathsf{I}_h & D_{h;1}^{\mathrm{s}} - \mathrm{i}D_{h;2}^{\mathrm{s}} \\ D_{h;1}^{\mathrm{s}} + \mathrm{i}D_{h;2}^{\mathrm{s}} & -m\mathsf{I}_h \end{bmatrix}.$$

Here $D_{h;j}^{\pm}$ and $D_{h;j}^{\mathrm{s}}$ are the corresponding finite differences in the j'th coordinate. It turns out that these two discrete models *do not* lead to norm resolvent convergence, so we also define two modified versions. Let $-\Delta_h$ denote the discrete Laplacian; see (2.4). Then the modified operators are given by

$$\widetilde{H}^{\mathrm{fb}}_{0,h} = H^{\mathrm{fb}}_{0,h} - h\Delta_h\sigma_3 \quad \text{and} \quad \widetilde{H}^{\mathrm{s}}_{0,h} = H^{\mathrm{s}}_{0,h} - h\Delta_h\sigma_3.$$

Here $-h\Delta_h\sigma_3$ is understood to be the operator matrix

$$\begin{bmatrix} -h\Delta_h & 0 \\ 0 & h\Delta_h \end{bmatrix}.$$

The details on the discretizations in dimension $d = 3$ can be found in Section 5.

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two Hilbert spaces. The space of bounded operators from $\mathcal{K}_1$ to $\mathcal{K}_2$ is denoted by $\mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K}$, we write $\mathcal{B}(\mathcal{K}) = \mathcal{B}(\mathcal{K}, \mathcal{K})$. In the following theorem, we collect the positive results obtained on norm resolvent convergence in $\mathcal{B}(\mathcal{H}^d)$. We use the convention $(-0, 0) = \emptyset$ in the statements of results.

Theorem 1.1. *Let $H_{0,h}$ be equal to $H^{\mathrm{fb}}_{0,h}$, $d = 1$, or equal to $\widetilde{H}^{\mathrm{fb}}_{0,h}$, $d = 2, 3$, or equal to $\widetilde{H}^{\mathrm{s}}_{0,h}$, $d = 1, 2, 3$. Let H_0 denote the free Dirac operator in the corresponding dimension. Then the following result holds.*

Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that

$$\|J_h(H_{0,h} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^d)} \leq Ch \tag{1.4}$$

for all $z \in K$ and $h \in (0, 1]$.

Theorem 1.1 can be generalized to also include a potential, by following the approach in [2]. Let $V\colon \mathbf{R}^d \to \mathcal{B}(\mathbf{C}^{\nu(d)})$ be bounded and Hölder continuous. Assume $V(x)$ is self-adjoint for each $x \in \mathbf{R}^d$. Define the discretization as $V_h(k) = V(hk)$ for $k \in \mathbf{Z}^d$. Then we can define self-adjoint operators $H = H_0 + V$ on $\mathcal{H}^d$ and $H_h = H_{0,h} + V_h$ on $\mathcal{H}^d_h$ for all the discrete models. The results in Theorem 1.1 then generalize to H and H_h, with an estimate $Ch^{\theta'}$, where $0 < \theta' < 1$ depends on the Hölder exponent for V; see Section 7.

In the next theorem we summarize some negative results with non-convergence in the $\mathcal{B}(\mathcal{H}^d)$-operator norm in part (i), and in part (ii) a result using the Sobolev spaces $H^1(\mathbf{R}^d) \otimes \mathbf{C}^{\nu(d)}$ is given. In particular, the estimate (1.5) implies strong resolvent convergence in $\mathcal{B}(\mathcal{H}^d)$.

Theorem 1.2. *Let $H_{0,h}$ be equal to $H^{\mathrm{fb}}_{0,h}$, $d = 2, 3$, or equal to $H^{\mathrm{s}}_{0,h}$, $d = 1, 2, 3$. Let H_0 denote the free Dirac operator in the corresponding dimension. Then the following results hold.*

i. *Let $z \in (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$. Then $J_h(H_{0,h} - zI_h)^{-1}K_h$* does not converge *to $(H_0 - zI)^{-1}$ in the operator norm on $\mathcal{B}(\mathcal{H}^d)$ as $h \to 0$.*

ii. *Let* $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ *be compact. Then there exists* $C > 0$ *such that*

$$\|J_h(H_{0,h} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(H^1(\mathbf{R}^d)\otimes\mathbf{C}^{\nu(d)},\mathcal{H}^d)} \leq Ch \quad (1.5)$$

for all $z \in K$ *and* $h \in (0, 1]$.

The estimate (1.4) implies results on the spectra of the operators $H_{0,h}$ and H_0 and their relation, see [2, Section 5]. Such results are not obtainable from the strong convergence implied by the estimate (1.5). Thus, we are in the remarkable situation that in dimensions $d = 2, 3$ we need to modify the natural discretizations in order to obtain spectral information. Furthermore, in dimension $d = 1$ to obtain spectral information we must use either the forward-backward discretizations or the modified symmetric discretizations. Moreover, this is relevant for resolving the unwanted *fermion doubling* phenomenon that is present in some discretizations of Dirac operators [1].

Results of the type (1.4) were first obtained by Nakamura and Tadano [5] for $H = -\Delta + V$ on $L^2(\mathbf{R}^d)$ and $H_h = -\Delta_h + V_h$ on $\ell^2(h\mathbf{Z}^d)$ for a large class of real potentials V, including unbounded V. They used special cases of the J_h and K_h as defined here, i.e., the pair of biorthogonal Riesz sequences is replaced by a single orthonormal sequence. Recently, their results have been applied to quantum graph Hamiltonians [3]. In [4] the continuum limit is studied for a number of different problems. Here strong resolvent convergence is proved up to the spectrum and scattering results are derived.

In [2] the authors proved results of the type (1.4) for a class of Fourier multipliers H_0 and their discretizations $H_{0,h}$, and obtained results of the type (1.4) for perturbations $H = H_0 + V$ and $H_h = H_{0,h} + V_h$ with a bounded, real-valued, and Hölder continuous potential. Note that the results in [2] do not directly apply to Dirac operators, since the free Dirac operators do not satisfy an essential symmetry condition [2, Assumption 3.1(4)]. In [7] Schmidt and Umeda proved strong resolvent convergence for Dirac operators in dimension $d = 2$ using the discretization $H^{\mathrm{fb}}_{0,h}$. They allow a class of bounded non-self-adjoint potentials and also state corresponding results for dimensions $d = 1, 3$.

The remainder of this paper is organized as follows. Section 2 introduces additional notation and operators used in the paper. Sections 3, 4, and 5 prove Theorem 1.1 and Theorem 1.2(i) in the one-, two-, and three-dimensional cases, respectively. Since some of the arguments are very similar in the different dimensions, we will give the full details in dimension two, and omit parts of the proofs in dimensions one and three that are essentially the same verbatim. Theorem 1.2 (ii) is proved in Section 6. Finally, we show how a potential V can be added to our results in Section 7.

2. Preliminaries

In this section we collect a number of definitions and results used in the sequel.

2.1. Notation for identity operators

We use the following notation for identity operators on various spaces: I on $\mathcal{H}^d$, I_h on $\mathcal{H}_h^d$, I on $L^2(\mathbf{R}^d)$, I_h on $\ell^2(h\mathbf{Z}^d)$, $\mathbf{1}$ on $\mathbf{C}^2$, and $\mathtt{1}$ on $\mathbf{C}^4$. In Section 5, in the definitions of the operator matrices for the free Dirac operator and its discretizations, $\mathbf{1}$ denotes the identity on $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$ and $\mathbf{1}_h$ denotes the identity on $\ell^2(h\mathbf{Z}^3) \otimes \mathbf{C}^2$.

2.2. Finite differences

The forward, backward, and symmetric difference operators on $\mathcal{H}_h^1$ are defined in (1.2) and (1.3). Let $\{e_1, e_2, e_3\}$ be the canonical basis in $\mathbf{Z}^3$. The forward partial difference operators for mesh size h are defined by

$$(D_{h;j}^{+} u_h)(k) = \frac{1}{\mathrm{i}h}(u_h(k+e_j) - u_h(k)), \quad j = 1,2,3, \tag{2.1}$$

and backward partial difference operators by

$$(D_{h;j}^{-} u_h)(k) = \frac{1}{\mathrm{i}h}(u_h(k) - u_h(k-e_j)), \quad j = 1,2,3. \tag{2.2}$$

The symmetric difference operators are given by

$$(D_{h;j}^{\mathrm{s}} u_h)(k) = \frac{1}{2\mathrm{i}h}(u_h(k+e_j) - u_h(k-e_j)), \quad j = 1,2,3. \tag{2.3}$$

Note that $(D_{h;j}^{+})^* = D_{h;j}^{-}$ and $(D_{h;j}^{\mathrm{s}})^* = D_{h;j}^{\mathrm{s}}$.

The discrete Laplacian acting on $\ell^2(h\mathbf{Z}^d)$ is given by

$$(-\Delta_h v_h)(k) = \frac{1}{h^2}\sum_{j=1}^{d}(2v_h(k) - v_h(k+e_j) - v_h(k-e_j)). \tag{2.4}$$

2.3. Fourier transforms

We use Fourier transforms extensively. They are normalized to be unitary. Write $\widehat{\mathcal{H}}^d = L^2(\mathbf{R}^d) \otimes \mathbf{C}^{\nu(d)}$ and let $\mathcal{F}\colon \mathcal{H}^d \to \widehat{\mathcal{H}}^d$ be the Fourier transform given by

$$(\mathcal{F} f)(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x, \quad \xi \in \mathbf{R}^d,$$

with adjoint $\mathcal{F}^*\colon \widehat{\mathcal{H}}^d \to \mathcal{H}^d$. We suppress their dependence on d in the notation, as it will be obvious in which dimension they are used.

Let $\mathbf{T}_h^d = [-\frac{\pi}{h}, \frac{\pi}{h}]^d$, $d = 1,2,3$, and $\widehat{\mathcal{H}}_h^d = L^2(\mathbf{T}_h^d) \otimes \mathbf{C}^{\nu(d)}$. The discrete Fourier transform $\mathsf{F}_h\colon \mathcal{H}_h^d \to \widehat{\mathcal{H}}_h^d$ and its adjoint $\mathsf{F}_h^*\colon \widehat{\mathcal{H}}_h^d \to \mathcal{H}_h^d$ are given by

$$(\mathsf{F}_h u_h)(\xi) = \frac{h^d}{(2\pi)^{d/2}} \sum_{k\in\mathbf{Z}^d} u_h(k) \mathrm{e}^{-\mathrm{i}hk\cdot\xi}, \quad \xi \in \mathbf{T}_h^d,$$

$$(\mathsf{F}_h^* g)(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{T}_h^d} \mathrm{e}^{\mathrm{i}hk\cdot\xi} g(\xi)\, \mathrm{d}\xi, \quad k \in \mathbf{Z}^d,$$

for $d = 1,2,3$.

2.4. Embedding and discretization operators

We describe in some detail how the the embedding and discretization operators in [2, Section 2] are adapted to the Dirac case.

Let $\mathcal{K}$ be a Hilbert space. Let $\{u_k\}_{k\in\mathbf{Z}^d}$ and $\{v_k\}_{k\in\mathbf{Z}^d}$ be two sequences in $\mathcal{K}$. They are said to be *biorthogonal* if

$$\langle u_k, v_n\rangle = \delta_{k,n}, \quad k,n \in \mathbf{Z}^d,$$

where $\delta_{k,n}$ is Kronecker's delta.

A sequence $\{u_k\}_{k\in\mathbf{Z}^d}$ is called a *Riesz sequence* if there exist $A > 0$ and $B > 0$ such that

$$A \sum_{k\in\mathbf{Z}^d} |c_k|^2 \le \Big\| \sum_{k\in\mathbf{Z}^d} c_k u_k \Big\|^2 \le B \sum_{k\in\mathbf{Z}^d} |c_k|^2$$

for all $\{c_k\}_{k\in\mathbf{Z}^d} \in \ell^2(\mathbf{Z}^d)$.

Assumption 2.1. Let $d = 1, 2$, or 3. Let $\varphi_0, \psi_0 \in L^2(\mathbf{R}^d)$. Define

$$\varphi_{h,k}(x) = \varphi_0((x - hk)/h), \quad \psi_{h,k}(x) = \psi_0((x - hk)/h), \quad h > 0,\ k \in \mathbf{Z}^d.$$

Assume further that $\{\varphi_{1,k}\}_{k\in\mathbf{Z}^d}$ and $\{\psi_{1,k}\}_{k\in\mathbf{Z}^d}$ are biorthogonal Riesz sequences in $L^2(\mathbf{R}^d)$.

To simplify, we omit the dependence on d in the notation for embedding and discretization operators. The embedding operators $J_h\colon \mathcal{H}_h^d \to \mathcal{H}^d$ are defined by

$$J_h u_h = \sum_{k\in\mathbf{Z}^d} \varphi_{h,k} u_h(k), \quad u_h \in \mathcal{H}_h^d. \tag{2.5}$$

For $d = 1, 2$, and

$$u_h(k) = \begin{bmatrix} u_h^1(k) \\ u_h^2(k) \end{bmatrix},$$

the notation above means

$$\varphi_{h,k} u_h(k) = \begin{bmatrix} u_h^1(k)\varphi_{h,k} \\ u_h^2(k)\varphi_{h,k} \end{bmatrix},$$

with an obvious modification in case $d = 3$. As a consequence of the Riesz sequence assumption we get a uniform bound

$$\sup_{h>0} \|J_h\|_{\mathcal{B}(\mathcal{H}_h^d, \mathcal{H}^d)} < \infty.$$

The operators $\widetilde{J}_h$ are defined as above by replacing $\varphi_{h,k}$ by $\psi_{h,k}$ in (2.5). Then the discretization operators are defined as $K_h = (\widetilde{J}_h)^*$. Explicitly, for $d = 1, 2$,

$$(K_h f)(k) = \frac{1}{h^d} \begin{bmatrix} \langle \psi_{h,k}, f^1 \rangle \\ \langle \psi_{h,k}, f^2 \rangle \end{bmatrix}, \quad k \in \mathbf{Z}^d,$$

with an obvious modification for $d = 3$. We have the uniform bound

$$\sup_{h>0} \|K_h\|_{\mathcal{B}(\mathcal{H}^d, \mathcal{H}_h^d)} < \infty.$$

Biorthogonality implies that

$$K_h J_h = I_h$$

and that $J_h K_h$ is a projection onto $J_h \mathcal{H}_h^d$ in $\mathcal{H}^d$. A further assumption on the functions φ_0 and ψ_0 is needed.

Assumption 2.2 ([2, Assumption 2.8]). Let $\hat{\varphi}_0, \hat{\psi}_0 \in L^2(\mathbf{R}^d)$ be essentially bounded and satisfy Assumption 2.1. Assume further that there exists $c_0 > 0$ such that

$$\operatorname{supp}(\hat{\varphi}_0) \subseteq \Big[-\frac{3\pi}{2}, \frac{3\pi}{2}\Big]^d \quad \text{and} \quad |\hat{\varphi}_0(\xi)| \geq c_0, \quad \xi \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]^d,$$

and

$$\operatorname{supp}(\hat{\psi}_0) \subseteq \Big[-\frac{3\pi}{2}, \frac{3\pi}{2}\Big]^d \quad \text{and} \quad |\hat{\psi}_0(\xi)| \geq c_0, \quad \xi \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]^d.$$

For examples of φ_0 and ψ_0 satisfying Assumption 2.2, see [2, Section 2.1].

2.5. Two lemmas

We often use the following elementary result, where the identity matrix is denoted by I.

Lemma 2.3. *Let $G \in \mathcal{B}(\mathbf{C}^n)$ be a self-adjoint $n \times n$ matrix. Then*

$$\|G - \mathrm{i}I\|_{\mathcal{B}(\mathbf{C}^n)} = \|G^2 + I\|^{1/2}_{\mathcal{B}(\mathbf{C}^n)}, \tag{2.6}$$

$$\|(G - \mathrm{i}I)^{-1}\|_{\mathcal{B}(\mathbf{C}^n)} = \|(G^2 + I)^{-1}\|^{1/2}_{\mathcal{B}(\mathbf{C}^n)}. \tag{2.7}$$

Proof. It suffices to prove (2.6). We use the C^*-identity in $\mathcal{B}(\mathbf{C}^n)$ to get

$$\|G - \mathrm{i}I\|^2_{\mathcal{B}(\mathbf{C}^n)} = \|(G + \mathrm{i}I)(G - \mathrm{i}I)\|_{\mathcal{B}(\mathbf{C}^n)} = \|G^2 + I\|_{\mathcal{B}(\mathbf{C}^n)}. \qquad \blacksquare$$

The following lemma will be used in the proofs related to the non-convergence results; see, e.g., [6, Theorem XIII.83].

Lemma 2.4. *Let $d = 1$, 2, or 3. Assume that $M_h\colon \mathbf{T}^d_h \to \mathcal{B}(\mathbf{C}^{\nu(d)})$ is a continuous matrix-valued function. Let T_{M_h} denote the operator of multiplication by M_h,*

$$T_{M_h} = \int_{\mathbf{T}^d_h}^{\oplus} M_h(\xi)\,\mathrm{d}\xi,$$

on $\widehat{\mathcal{H}}^d_h \simeq L^2(\mathbf{T}^d_h; \mathbf{C}^{\nu(d)})$. Then

$$\|T_{M_h}\|_{\mathcal{B}(\widehat{\mathcal{H}}^d_h)} = \max_{\xi\in\mathbf{T}^d_h} \|M_h(\xi)\|_{\mathcal{B}(\mathbf{C}^{\nu(d)})}. \tag{2.8}$$

3. The 1D free Dirac operator

We state and prove results for the 1D Dirac operator. On $\mathcal{H}^1$ the one-dimensional free Dirac operator with mass $m \geq 0$ is given by the operator matrix

$$H_0 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\sigma_1 + m\sigma_3 = \begin{bmatrix} m\mathsf{I} & -\mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}x} \\ -\mathrm{i}\dfrac{\mathrm{d}}{\mathrm{d}x} & -m\mathsf{I} \end{bmatrix},$$

where I denotes the identity operator on $L^2(\mathbf{R})$.

3.1. The 1D forward-backward difference model

Using (1.2), the forward-backward difference model of H_0 is defined as

$$H_{0,h}^{\mathrm{fb}} = \begin{bmatrix} m\mathsf{I}_h & D_h^- \\ D_h^+ & -m\mathsf{I}_h \end{bmatrix},$$

where I_h denotes the identity operator on $\ell^2(h\mathbf{Z})$. The operators H_0 and $H_{0,h}^{\mathrm{fb}}$ are given as multipliers in Fourier space by the functions G_0 and $G_{0,h}^{\mathrm{fb}}$, respectively, where

$$G_0(\xi) = \begin{bmatrix} m & \xi \\ \xi & -m \end{bmatrix} \tag{3.1}$$

and

$$G_{0,h}^{\mathrm{fb}}(\xi) = \begin{bmatrix} m & -\dfrac{1}{\mathrm{i}h}(\mathrm{e}^{-\mathrm{i}h\xi} - 1) \\ \dfrac{1}{\mathrm{i}h}(\mathrm{e}^{\mathrm{i}h\xi} - 1) & -m \end{bmatrix}. \tag{3.2}$$

We define

$$g_0(\xi) = m^2 + \xi^2, \tag{3.3}$$

and

$$g_{0,h}^{\mathrm{fb}}(\xi) = m^2 + \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi\Big). \tag{3.4}$$

Then

$$G_0(\xi)^2 = g_0(\xi)\mathbf{1} \quad \text{and} \quad G_{0,h}^{\mathrm{fb}}(\xi)^2 = g_{0,h}^{\mathrm{fb}}(\xi)\mathbf{1}. \tag{3.5}$$

Lemma 3.1. *Assume $\xi \neq 0$. Then we have*

$$\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq \frac{1}{|\xi|}. \tag{3.6}$$

There exists $C > 0$ such that for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$ we have

$$\|(G_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq \frac{C}{|\xi|}. \tag{3.7}$$

Proof. Using Lemma 2.3 together with (3.3) and (3.5), we get

$$\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} = \|(G_0(\xi)^2 + \mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)}^{1/2} = \frac{1}{(1 + m^2 + \xi^2)^{1/2}} \leq \frac{1}{|\xi|},$$

proving (3.6).

To prove (3.7), we use Lemma 2.3, (3.4), and (3.5) to get

$$\begin{aligned}\|(G_0^{\mathrm{fb}}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} &= \|(G_0^{\mathrm{fb}}(\xi)^2 + \mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)}^{1/2} \\ &= \Big(1 + m^2 + \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi\Big)\Big)^{-1/2} \\ &\leq \Big(\frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi\Big)\Big)^{-1/2}.\end{aligned}$$

There exists $c > 0$ such that for $|\theta| \leq \frac{3\pi}{4}$ we have $|\sin(\theta)| \geq c|\theta|$. For $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$ then (3.7) follows. ∎

Lemma 3.2. *There exists $C > 0$ such that*

$$\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq Ch$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$.

Proof. We have

$$\begin{aligned}&(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}\mathbf{1})^{-1} \\ &\quad = (G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}(G_{0,h}^{\mathrm{fb}}(\xi) - G_0(\xi))(G_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}\mathbf{1})^{-1}.\end{aligned}$$

To estimate the 12 and 21 entries in $G_{0,h}^{\mathrm{fb}}(\xi) - G_0(\xi)$ we use Taylor's formula:

$$\mathrm{e}^{\mathrm{i}h\xi} = 1 + \mathrm{i}h\xi + (\mathrm{i}h\xi)^2 \int_0^1 \mathrm{e}^{\mathrm{i}ht\xi}(1-t)\,\mathrm{d}t.$$

It follows that the 12 and 21 entries are estimated by $Ch|\xi|^2$. Using Lemma 3.1 the result follows. ∎

Using Lemmas 3.1 and 3.2, we can adapt the arguments in [2] to obtain the following result. We omit the details here, and refer the reader to the proof of Theorem 4.4 where details of the adaptation are given.

Theorem 3.3. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(H_{0,h}^{\mathrm{fb}} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^1)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

3.2. The 1D symmetric difference model

The discrete model based on the symmetric difference operator (1.3) is

$$H^{\mathrm{s}}_{0,h} = \begin{bmatrix} m\mathsf{I}_h & D^{\mathrm{s}}_h \\ D^{\mathrm{s}}_h & -m\mathsf{I}_h \end{bmatrix}. \tag{3.8}$$

In Fourier space, it is a multiplier with symbol

$$G^{\mathrm{s}}_{0,h}(\xi) = \begin{bmatrix} m & \frac{1}{h}\sin(h\xi) \\ \frac{1}{h}\sin(h\xi) & -m \end{bmatrix}. \tag{3.9}$$

We have

$$G^{\mathrm{s}}_{0,h}(\xi)^2 = g^{\mathrm{s}}_{0,h}(\xi)\mathbf{1} \quad \text{where} \quad g^{\mathrm{s}}_{0,h}(\xi) = m^2 + \frac{1}{h^2}\sin^2(h\xi). \tag{3.10}$$

Lemma 3.4. *There exists $c > 0$ such that*

$$\max_{\xi\in\mathbf{T}^1_h} \|(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \geq c \tag{3.11}$$

for all $h \in (0, 1]$.

Proof. We have

$$\begin{aligned}(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})&[(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}](G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1}) \\ &= G^{\mathrm{fb}}_{0,h}(\xi) - G^{\mathrm{s}}_{0,h}(\xi) = \frac{1}{h}(1-\cos(h\xi))\sigma_2. \end{aligned} \tag{3.12}$$

From (3.5) and (3.10),

$$(1 + g^{\mathrm{fb}}_{0,h}(\xi))^{-1/2}(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1}) \quad \text{and} \quad (1 + g^{\mathrm{s}}_{0,h}(\xi))^{-1/2}(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})$$

are unitary matrices for all $\xi \in \mathbf{T}^1_h$. Since σ_2 is also unitary, (3.12) gives the norm equality

$$\|(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} = \frac{1-\cos(h\xi)}{h(1+g^{\mathrm{fb}}_{0,h}(\xi))^{1/2}(1+g^{\mathrm{s}}_{0,h}(\xi))^{1/2}}.$$

If we take $h\xi = \pi$, the right-hand side becomes

$$\frac{2}{\sqrt{(1+m^2)h^2+4}\sqrt{1+m^2}}.$$

Thus, for $0 < h \leq 1$ one can take $c = 2((1+m^2)^2 + 4(1+m^2))^{-1/2}$ in (3.11). This concludes the proof. ■

Using Lemmas 2.4 and 3.4 together with Theorem 3.3 and properties of J_h and K_h, we get the following result.

Theorem 3.5. *Let $z \in (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$. Then $J_h(H^{\mathrm{s}}_{0,h} - zI_h)^{-1}K_h$ does not converge to $(H_0 - zI)^{-1}$ in the operator norm on $\mathcal{B}(\mathcal{H}^1)$ as $h \to 0$.*

We can introduce a modified operator $\widetilde{H}^{\mathrm{s}}_{0,h}$ given by

$$\widetilde{H}^{\mathrm{s}}_{0,h} = H^{\mathrm{s}}_{0,h} + \begin{bmatrix} -h\Delta_h & 0 \\ 0 & h\Delta_h \end{bmatrix},$$

where $-\Delta_h$ is the 1D discrete Laplacian; see (2.4). We obtain norm resolvent convergence for the modified symmetric difference model, similar to the results in dimensions two and three; see Theorems 4.4 and 5.1. The proof is omitted as it is nearly identical to the proof of Theorem 4.4.

Theorem 3.6. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(\widetilde{H}^{\mathrm{s}}_{0,h} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^1)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

4. The 2D free Dirac operator

In two dimensions, the free Dirac operator on $\mathcal{H}^2$ with mass $m \geq 0$ is given by

$$H_0 = -\mathrm{i}\frac{\partial}{\partial x_1}\sigma_1 - \mathrm{i}\frac{\partial}{\partial x_2}\sigma_2 + m\sigma_3 = \begin{bmatrix} m\mathsf{I} & -\mathrm{i}\dfrac{\partial}{\partial x_1} - \dfrac{\partial}{\partial x_2} \\ -\mathrm{i}\dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2} & -m\mathsf{I} \end{bmatrix}, \tag{4.1}$$

where the Pauli matrices are given in (1.1). In $\widehat{\mathcal{H}}^2$ it is a Fourier multiplier with symbol

$$G_0(\xi) = \begin{bmatrix} m & \xi_1 - \mathrm{i}\xi_2 \\ \xi_1 + \mathrm{i}\xi_2 & -m \end{bmatrix}. \tag{4.2}$$

The corresponding discrete Dirac operator can be obtained by replacing the derivatives in (4.1) by finite differences.

4.1. The 2D symmetric difference model

We first consider the model obtained by using the symmetric difference operators; see (2.3) for the definition.

$$H^{\mathrm{s}}_{0,h} = \begin{bmatrix} m\mathsf{I}_h & D^{\mathrm{s}}_{h;1} - \mathrm{i}D^{\mathrm{s}}_{h;2} \\ D^{\mathrm{s}}_{h;1} + \mathrm{i}D^{\mathrm{s}}_{h;2} & -m\mathsf{I}_h \end{bmatrix}. \tag{4.3}$$

In, $\widehat{\mathscr{H}}_h^2$ it acts as a Fourier multiplier with symbol

$$G_{0,h}^{\mathrm{s}}(\xi) = \begin{bmatrix} m & \frac{1}{h}\sin(h\xi_1) - \frac{\mathrm{i}}{h}\sin(h\xi_2) \\ \frac{1}{h}\sin(h\xi_1) + \frac{\mathrm{i}}{h}\sin(h\xi_2) & -m \end{bmatrix}. \tag{4.4}$$

The 2D discrete Laplacian is defined in (2.4). We introduce the modified symmetric difference model as

$$\widetilde{H}_{0,h}^{\mathrm{s}} = H_{0,h}^{\mathrm{s}} + \begin{bmatrix} -h\Delta_h & 0 \\ 0 & h\Delta_h \end{bmatrix}.$$

We will show that $J_h(\widetilde{H}_{0,h}^{\mathrm{s}} - zI_h)^{-1}K_h$ converges in norm to $(H_0 - zI)^{-1}$.

In $\widehat{\mathscr{H}}_h^2$, the operator $\widetilde{H}_{0,h}^{\mathrm{s}}$ acts as a Fourier multiplier with symbol

$$\widetilde{G}_{0,h}^{\mathrm{s}}(\xi) = G_{0,h}^{\mathrm{s}}(\xi) + f_h(\xi)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.5}$$

where

$$f_h(\xi) = \frac{4}{h}\sin^2\Bigl(\frac{h}{2}\xi_1\Bigr) + \frac{4}{h}\sin^2\Bigl(\frac{h}{2}\xi_2\Bigr). \tag{4.6}$$

Related to the symbols G_0, $G_{0,h}^{\mathrm{s}}$, and $\widetilde{G}_{0,h}^{\mathrm{s}}$, we define

$$g_0(\xi) = m^2 + \xi_1^2 + \xi_2^2, \tag{4.7}$$

$$g_{0,h}^{\mathrm{s}}(\xi) = m^2 + \frac{1}{h^2}\sin^2(h\xi_1) + \frac{1}{h^2}\sin^2(h\xi_2), \tag{4.8}$$

and

$$\tilde{g}_{0,h}^{\mathrm{s}}(\xi) = (m + f_h(\xi))^2 + \frac{1}{h^2}\sin^2(h\xi_1) + \frac{1}{h^2}\sin^2(h\xi_2). \tag{4.9}$$

We have

$$G_0(\xi)^2 = g_0(\xi)\mathbf{1}, \quad G_{0,h}^{\mathrm{s}}(\xi)^2 = g_{0,h}^{\mathrm{s}}(\xi)\mathbf{1}, \quad \text{and} \quad \widetilde{G}_{0,h}^{\mathrm{s}}(\xi)^2 = \tilde{g}_{0,h}^{\mathrm{s}}(\xi)\mathbf{1}. \tag{4.10}$$

Lemma 4.1. *For $\xi \neq 0$, we have*

$$\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq \frac{1}{|\xi|}. \tag{4.11}$$

There exists $C > 0$ such that for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$ we have

$$\|(\widetilde{G}_{0,h}^{\mathrm{s}}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq \frac{C}{|\xi|}. \tag{4.12}$$

Proof. Lemma 2.3 and (4.10) imply

$$\|(G_0(\xi)-\mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} = \frac{1}{(1+g_0(\xi))^{1/2}} \le \frac{1}{|\xi|},$$

such that (4.11) holds.

To prove (4.12), we first use Lemma 2.3 and (4.10) to get

$$\|(\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-\mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} = \frac{1}{(1+\tilde{g}^{\mathrm{s}}_{0,h}(\xi))^{1/2}}.$$

Then note that there exists $c>0$ such that $|\sin(\theta)| \ge c|\theta|$ for $\theta \in [-\frac{3\pi}{4}, \frac{3\pi}{4}]$. Thus, for $|h\xi_j| \le \frac{3\pi}{4}$, $j=1,2$, we have

$$\frac{1}{h^2}\sin^2(h\xi_j) \ge c_1|\xi_j|^2, \quad j=1,2.$$

For $\frac{3\pi}{4} \le |h\xi_j| \le \frac{3\pi}{2}$ we have

$$\frac{1}{h}\sin^2\Big(\frac{h}{2}\xi_j\Big) \ge c_1 h|\xi_j|^2 \ge c_2|\xi_j|, \quad j=1,2.$$

Combining these estimates, we get

$$\tilde{g}^{\mathrm{s}}_{0,h}(\xi) \ge c|\xi|^2, \quad h\xi \in \Big[-\frac{3\pi}{2}, \frac{3\pi}{2}\Big]^2.$$

The estimate (4.12) follows. ■

Lemma 4.2. *There exists $C>0$ such that*

$$\|(G_0(\xi)-\mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-\mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \le Ch$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$.

Proof. We have

$$\begin{aligned}(G_0(\xi)-\mathrm{i}\mathbf{1})^{-1} &- (\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-\mathrm{i}\mathbf{1})^{-1}\\ &= (G_0(\xi)-\mathrm{i}\mathbf{1})^{-1}(\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-G_0(\xi))(\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-\mathrm{i}\mathbf{1})^{-1}.\end{aligned}$$

The 11 entry in $\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-G_0(\xi)$ is estimated using $|\sin(\theta)| \le |\theta|$. We get

$$|[\widetilde{G}^{\mathrm{s}}_{0,h}(\xi)-G_0(\xi)]_{11}| \le |f_h(\xi)| \le Ch|\xi|^2. \tag{4.13}$$

The same estimate holds for the 22 entry.

Taylor's formula yields

$$\sin(\theta) = \theta - \frac{1}{2}\theta^3 \int_0^1 \cos(t\theta)(1-t)^2 \, dt. \tag{4.14}$$

This result implies the estimates

$$\left| \frac{1}{h} \sin(h\xi_j) - \xi_j \right| \leq C h^2 |\xi_j|^3, \quad j = 1, 2,$$

that are used to estimate the 12 and 21 entries in $\widetilde{G}^{s}_{0,h}(\xi) - G_0(\xi)$.

Combining these results with the estimates from Lemma 4.1, we get

$$\begin{aligned} \|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{s}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} &\leq C \frac{h|\xi|^2 + h^2|\xi|^3}{|\xi|^2} \\ &= C(h + h^2|\xi|) \leq Ch, \end{aligned}$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$. ■

We state [2, Lemma 3.3] in a form adapted to the Dirac operators and outline its proof.

Lemma 4.3. *Let $d = 1, 2,$ or 3. Let H_0 be the free Dirac operator in $\mathcal{H}^d$. Let φ_0 and ψ_0 satisfy Assumption 2.2. Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|(J_h K_h - I)(H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^d)} \leq Ch,$$

for all $z \in K$ and $h \in (0, 1]$.

Proof. We assume $d = 2$. It suffices to consider $K = \{\mathrm{i}\}$, since $(H_0 - \mathrm{i}I)(H_0 - zI)^{-1}$ is bounded uniformly in norm for $z \in K$. Let $u \in \mathcal{S}(\mathbf{R}^2) \otimes \mathbf{C}^2$, the Schwartz space. Going through the computations in [2, Section 2] using that φ_0 and ψ_0 are scalar functions, we get the result

$$\begin{aligned} &(\mathcal{F}(J_h K_h - I)(H_0 - \mathrm{i}I)^{-1}\mathcal{F}^* u)(\xi) \\ &\quad = (2\pi)^d \hat{\varphi}_0(h\xi) \sum_{j \in \mathbf{Z}^2} \overline{\hat{\psi}_0(h\xi + 2\pi j)} \Big(G_0\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1} u\Big(\xi + \frac{2\pi}{h} j\Big) \\ &\qquad - (G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} u(\xi), \quad \xi \in \mathbf{R}^2. \end{aligned}$$

Here G_0 is given by (4.2). If $h\xi \in [-\frac{\pi}{2}, \frac{\pi}{2}]^2$, then the $j = 0$ term is the only non-zero term in the sum. Using [2, Lemma 2.7], we conclude that this term and the last term cancel. For $h\xi \notin [-\frac{\pi}{2}, \frac{\pi}{2}]^2$, we use Lemma 4.1 to get $\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq Ch$,

$0 < h \leq 1$. Since $\hat{\varphi}_0$ and $\hat{\psi}_0$ are assumed essentially bounded, we conclude that the $j = 0$ term in the sum and the last term are bounded by $Ch\|u\|_{\hat{\mathcal{H}}^2}$.

Due to the support assumptions on $\hat{\varphi}_0$ and $\hat{\psi}_0$, only the terms in the sum with $|j| \leq 1$ contribute. Assume $|j| = 1$ and $h\xi \in \operatorname{supp}(\hat{\varphi}_0) \cap \operatorname{supp}(\hat{\psi}_0(\cdot + 2\pi j))$. Then for some $c_0 > 0$ we have $|\xi + \frac{2\pi}{h} j| \geq \frac{c_0}{h}$, which by Lemma 4.1 implies

$$\Big\|\Big(G_0\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1}\Big\|_{\mathcal{B}(\mathbf{C}^2)} \leq Ch.$$

Again, using the boundedness of $\hat{\varphi}_0$ and $\hat{\psi}_0$ we conclude that

$$\begin{aligned}
&\Big|(2\pi)^d \hat{\varphi}_0(h\xi)\overline{\hat{\psi}_0(h\xi + 2\pi j)}\Big(G_0\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1} u\Big(\xi + \frac{2\pi}{h} j\Big)\Big| \\
&\quad \leq Ch\Big|u\Big(\xi + \frac{2\pi}{h} j\Big)\Big|,
\end{aligned}$$

$0 < h \leq 1$, $\xi \in \mathbf{R}^2$. Squaring and integrating the result gives an estimate of the form $Ch\|u\|_{\hat{\mathcal{H}}^2}$. By density, adding up the finite number of terms corresponding to $|j| \leq 1$ gives the final result. ■

We have now established the estimates necessary to repeat the arguments from [2]. Using the embedding operators J_h and discretization operators K_h defined in Section 2, we state the result and then show in some detail how the arguments in [2] are adapted to the Dirac case.

Theorem 4.4. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(\widetilde{H}^{\mathrm{s}}_{0,h} - zI_h)^{-1} K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^2)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

Proof. We start by proving the result for $K = \{\mathrm{i}\}$. We have

$$\begin{aligned}
&J_h(\widetilde{H}^{\mathrm{s}}_{0,h} - \mathrm{i}I_h)^{-1} K_h - (H_0 - \mathrm{i}I)^{-1} \\
&\quad = J_h(\widetilde{H}^{\mathrm{s}}_{0,h} - \mathrm{i}I_h)^{-1} K_h - J_h K_h (H_0 - \mathrm{i}I)^{-1} + (J_h K_h - I)(H_0 - \mathrm{i}I)^{-1}.
\end{aligned}$$

The last term is estimated using Lemma 4.3.

To estimate the remaining terms we go to Fourier space. We have

$$\begin{aligned}
&\mathcal{F}(J_h(\widetilde{H}^{\mathrm{s}}_{0,h} - \mathrm{i}I_h)^{-1} K_h - J_h K_h (H_0 - \mathrm{i}I)^{-1})\mathcal{F}^* \\
&\quad = \mathcal{F} J_h \mathsf{F}_h^* \mathsf{F}_h (\widetilde{H}^{\mathrm{s}}_{0,h} - \mathrm{i}I_h)^{-1} \mathsf{F}_h^* \mathsf{F}_h K_h \mathcal{F}^* - \mathcal{F} J_h K_h \mathcal{F}^* \mathcal{F}(H_0 - \mathrm{i}I)^{-1}\mathcal{F}^*.
\end{aligned} \tag{4.15}$$

Let $u \in \mathcal{S}(\mathbf{R}^2) \otimes \mathbf{C}^2$. We now use a modified version of the computation leading to [2, equation (2.11)]. For the first term we get

$$[\mathcal{F} J_h \mathsf{F}_h^* \mathsf{F}_h (\widetilde{H}_{0,h}^{\mathrm{s}} - \mathrm{i} I_h)^{-1} \mathsf{F}_h^* \mathsf{F}_h K_h \mathcal{F}^* u](\xi) \\ = (2\pi)^d \hat{\varphi}_0(h\xi)(\widetilde{G}_{0,h}^{\mathrm{s}}(\xi) - \mathrm{i}\mathbf{1})^{-1} \sum_{j\in\mathbf{Z}^2} \overline{\hat{\psi}_0(h\xi + 2\pi j)} u\Big(\xi + \frac{2\pi}{h} j\Big). \tag{4.16}$$

For the second term we get

$$[\mathcal{F} J_h K_h \mathcal{F}^* \mathcal{F}(H_0 - \mathrm{i} I)^{-1} \mathcal{F}^* u](\xi) \\ = (2\pi)^d \hat{\varphi}_0(h\xi) \sum_{j\in\mathbf{Z}^2} \overline{\hat{\psi}_0(h\xi + 2\pi j)} \Big(G_0\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1} u\Big(\xi + \frac{2\pi}{h} j\Big). \tag{4.17}$$

We need to rewrite (4.16). First, we note that

$$\widetilde{G}_{0,h}^{\mathrm{s}}(\xi) = \widetilde{G}_{0,h}^{\mathrm{s}}\Big(\xi + \frac{2\pi}{h} j\Big), \quad j \in \mathbf{Z}^2.$$

Next, we can rewrite part of (4.16) as follows, since $\hat{\psi}_0$ is a scalar-valued function:

$$(\widetilde{G}_{0,h}^{\mathrm{s}}(\xi) - \mathrm{i}\mathbf{1})^{-1} \sum_{j\in\mathbf{Z}^2} \overline{\hat{\psi}_0(h\xi + 2\pi j)} u\Big(\xi + \frac{2\pi}{h} j\Big) \\ = \sum_{j\in\mathbf{Z}^2} \overline{\hat{\psi}_0(h\xi + 2\pi j)} \Big(\widetilde{G}_{0,h}^{\mathrm{s}}\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1} u\Big(\xi + \frac{2\pi}{h} j\Big).$$

We now insert (4.17) and the rewritten (4.16) into (4.15) to get

$$\mathcal{F}(J_h(\widetilde{H}_{0,h}^{\mathrm{s}} - \mathrm{i} I_h)^{-1} K_h - J_h K_h (H_0 - \mathrm{i} I)^{-1}) \mathcal{F}^* u = \sum_{j\in\mathbf{Z}^2} q_{j,h},$$

where

$$q_{j,h}(\xi) = (2\pi)^d \hat{\varphi}_0(h\xi) \overline{\hat{\psi}_0(h\xi + 2\pi j)} \\ \times \Big(\Big(\widetilde{G}_{0,h}^{\mathrm{s}}\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1} - \Big(G_0\Big(\xi + \frac{2\pi}{h} j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1}\Big) \\ \times u\Big(\xi + \frac{2\pi}{h} j\Big).$$

Due to the support conditions on $\hat{\varphi}_0$ and $\hat{\psi}_0$ in Assumption 2.2, only terms with $|j| \le 1$ contribute. First, consider $j = 0$. We have assumed $\operatorname{supp}(\hat{\varphi}_0), \operatorname{supp}(\hat{\psi}_0) \subseteq [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$. Using Lemma 4.2 and Assumption 2.2, we get

$$\|q_{0,h}\|_{\widehat{\mathcal{H}}^2} \le C h \|u\|_{\widehat{\mathcal{H}}^2}.$$

Fix $j \in \mathbf{Z}^2$ with $|j| = 1$. Define

$$M = \operatorname{supp}(\hat{\varphi}_0) \cap \operatorname{supp}(\hat{\psi}_0(\,\cdot + 2\pi j)).$$

From the supports of $\hat{\varphi}_0$ and $\hat{\psi}_0$ we have

$$M \subseteq \Big\{\zeta \in \Big[-\frac{3\pi}{2}, \frac{3\pi}{2}\Big]^2 : |\zeta + 2\pi j| \geq \frac{\pi}{2}\Big\}.$$

Assume $h\xi \in M$, then Lemma 4.1 implies

$$\Big\|\Big(G_0\Big(\xi + \frac{2\pi}{h}j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1}\Big\|_{\mathcal{B}(\mathbf{C}^2)} \leq Ch,$$

and

$$\Big\|\Big(\widetilde{G}^{\mathrm{s}}_{0,h}\Big(\xi + \frac{2\pi}{h}j\Big) - \mathrm{i}\mathbf{1}\Big)^{-1}\Big\|_{\mathcal{B}(\mathbf{C}^2)} \leq Ch.$$

These estimates imply

$$\|q_{j,h}\|_{\widehat{\mathcal{H}}^2} \leq Ch\|u\|_{\widehat{\mathcal{H}}^2}, \quad |j| = 1.$$

Since we have a finite number of j with $|j| \leq 1$ and since u is in a dense set, the estimate in Theorem 4.4 follows in the $K = \{\mathrm{i}\}$ case. For the general case, we use the estimates

$$\|(G_0(\xi) - z\mathbf{1})(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq C$$

and

$$\|(\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - z\mathbf{1})(\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq C$$

for $z \in K$ where $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ is compact. Combining these estimates with Lemma 4.2, we get for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$

$$\|(\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - z\mathbf{1})^{-1} - (G_0(\xi) - z\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \leq Ch, \quad z \in K.$$

This is the crucial estimate used above. Further details are omitted. ∎

Next, we show that, without modification to the symmetric difference model, the norm convergence stated in the theorem fails.

Lemma 4.5. *There exists $c > 0$ such that*

$$\max_{\xi \in \mathbf{T}^2_h} \|(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \geq c \tag{4.18}$$

for all $h \in (0, 1]$.

Proof. Using the notation from (4.5) we have

$$\begin{aligned}(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})[(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}](\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})\\ = \widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - G^{\mathrm{s}}_{0,h}(\xi) = f_h(\xi)\sigma_3.\end{aligned}$$

From the same reasoning as in the proof of Lemma 3.4, we obtain

$$\begin{aligned}&\|(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)}\\ &= \frac{f_h(\xi)}{(1 + g^{\mathrm{s}}_{0,h}(\xi))^{1/2}(1 + \tilde{g}^{\mathrm{s}}_{0,h}(\xi))^{1/2}}.\end{aligned} \tag{4.19}$$

Here $g^{\mathrm{s}}_{0,h}(\xi)$ is given by (4.8), $\tilde{g}^{\mathrm{s}}_{0,h}(\xi)$ by (4.9), and $f_h(\xi)$ by (4.6).

Take $h\xi_1 = \pi$ and $h\xi_2 = \pi$, and insert them in the last term in (4.19). We get

$$\begin{aligned}&\max_{\xi\in\mathbf{T}^2_h}\|(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)}\\ &\geq \frac{8}{\sqrt{1+m^2}\sqrt{h^2 + (8+hm)^2}}.\end{aligned}$$

The result (4.18) then holds for $0 < h \leq 1$ with $c = 8[(1+m^2)(1+(8+m)^2)]^{-1/2}$. ■

Combining Theorem 4.4 with Lemma 2.4 and Lemma 4.5, we obtain the following result using the operators J_h and K_h introduced in Section 2.

Theorem 4.6. *Let $z \in (\mathbf{C}\setminus\mathbf{R}) \cup (-m, m)$. Then $J_h(H^{\mathrm{s}}_{0,h} - zI_h)^{-1}K_h$ does not converge to $(H_0 - zI)^{-1}$ in the operator norm on $\mathcal{B}(\mathcal{H}^2)$ as $h \to 0$.*

4.2. The 2D forward-backward difference model

We now consider the model for the discrete Dirac operator obtained by using the forward and backward difference operators; see (2.1) and (2.2) for definitions. The discretized operator is given by

$$H^{\mathrm{fb}}_{0,h} = \begin{bmatrix} m\mathbb{1}_h & D^-_{h;1} - \mathrm{i}D^-_{h;2}\\ D^+_{h;1} + \mathrm{i}D^+_{h;2} & -m\mathbb{1}_h\end{bmatrix}. \tag{4.20}$$

In $\mathcal{H}^2_h$ it is a Fourier multiplier with the symbol

$$G^{\mathrm{fb}}_{0,h}(\xi) = \begin{bmatrix} m & -\frac{1}{\mathrm{i}h}(\mathrm{e}^{-\mathrm{i}h\xi_1} - 1) + \frac{1}{h}(\mathrm{e}^{-\mathrm{i}h\xi_2} - 1)\\ \frac{1}{\mathrm{i}h}(\mathrm{e}^{\mathrm{i}h\xi_1} - 1) + \frac{1}{h}(\mathrm{e}^{\mathrm{i}h\xi_2} - 1) & -m\end{bmatrix}.$$

We also consider the modified model, where the modification is the same as in the symmetric case, i.e.,

$$\widetilde{H}^{\mathrm{fb}}_{0,h} = H^{\mathrm{fb}}_{0,h} + \begin{bmatrix} -h\Delta_h & 0 \\ 0 & h\Delta_h \end{bmatrix}.$$

The corresponding Fourier multiplier is

$$\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) = G^{\mathrm{fb}}_{0,h}(\xi) + f_h(\xi)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $f_h(\xi)$ is given by (4.6). We recall the expression

$$f_h(\xi) = \frac{4}{h}\sin^2\Big(\frac{h}{2}\xi_1\Big) + \frac{4}{h}\sin^2\Big(\frac{h}{2}\xi_2\Big).$$

Define

$$\begin{aligned} g^{\mathrm{fb}}_{0,h}(\xi) = m^2 &+ \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi_1\Big) + \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi_2\Big) \\ &+ \frac{2}{h^2}\sin(h(\xi_1-\xi_2)) - \frac{2}{h^2}\sin(h\xi_1) + \frac{2}{h^2}\sin(h\xi_2) \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \tilde{g}^{\mathrm{fb}}_{0,h}(\xi) = (m + f_h(\xi))^2 &+ \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi_1\Big) + \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi_2\Big) \\ &+ \frac{2}{h^2}\sin(h(\xi_1-\xi_2)) - \frac{2}{h^2}\sin(h\xi_1) + \frac{2}{h^2}\sin(h\xi_2). \end{aligned} \tag{4.22}$$

Straightforward computations show that

$$G^{\mathrm{fb}}_{0,h}(\xi)^2 = g^{\mathrm{fb}}_{0,h}(\xi)\mathbf{1} \quad \text{and} \quad \widetilde{G}^{\mathrm{fb}}_{0,h}(\xi)^2 = \tilde{g}^{\mathrm{fb}}_{0,h}(\xi)\mathbf{1}.$$

We now prove the analogue of (4.12) for $\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi)$.

Lemma 4.7. *There exists $C > 0$ such that for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$ we have*

$$\|(\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \le \frac{C}{|\xi|}.$$

Proof. We will show that we have a lower bound

$$\tilde{g}^{\mathrm{fb}}_{0,h}(\xi) \ge c|\xi|^2, \quad h\xi \in \Big[\frac{3\pi}{2}, \frac{3\pi}{2}\Big]^2. \tag{4.23}$$

The result then follows from Lemma 2.3. We start with the estimate

$$\begin{aligned} \tilde{g}^{\mathrm{fb}}_{0,h}(\xi) \ge &\frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi_1\Big) + \frac{4}{h^2}\sin^2\Big(\frac{h}{2}\xi_2\Big) \\ &+ \frac{2}{h^2}\sin(h(\xi_1-\xi_2)) - \frac{2}{h^2}\sin(h\xi_1) + \frac{2}{h^2}\sin(h\xi_2). \end{aligned} \tag{4.24}$$

We have

$$\frac{2}{h^2}\sin(h(\xi_1-\xi_2)) - \frac{2}{h^2}\sin(h\xi_1) + \frac{2}{h^2}\sin(h\xi_2)$$
$$= \frac{2}{h^2}(\sin(h\xi_1)(\cos(h\xi_2)-1) + \sin(h\xi_2)(1-\cos(h\xi_1))).$$

We recall the elementary estimates

$$|\sin(\theta)| \le |\theta| \quad \text{and} \quad 1-\cos(\theta) \le \frac{1}{2}\theta^2$$

for all $\theta \in \mathbf{R}$. Using these estimates we get

$$\frac{2}{h^2}|\sin(h\xi_1)(\cos(h\xi_2)-1) + \sin(h\xi_2)(1-\cos(h\xi_1))| \le h|\xi_1||\xi_2|^2 + h|\xi_2||\xi_1|^2.$$

Recall that there exists $c_0 > 0$ such that $\frac{4}{h^2}\sin^2(\frac{h}{2}\xi_j) \ge c_0|\xi_j|^2$ for all $h|\xi_j| \le \frac{3\pi}{2}$, $j = 1, 2$. Thus, using these estimates and (4.24), we find

$$\tilde{g}^{\mathrm{fb}}_{0,h}(\xi) \ge (c_0 - h|\xi_2|)|\xi_1|^2 + (c_0 - h|\xi_1|)|\xi_2|^2 \ge \frac{1}{2}c_0|\xi|^2$$

for $h|\xi_j| \le \frac{1}{2}c_0$, $j = 1, 2$.

For $\frac{1}{2}c_0 \le h|\xi_j| \le \frac{3\pi}{2}$, $j = 1, 2$, the estimate (4.23) is obtained as in the proof of Lemma 4.1. We omit the details. ■

Lemma 4.8. *There exists $C > 0$ such that*

$$\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \le Ch$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^2$.

Proof. We have

$$(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}$$
$$= (G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}(\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - G_0(\xi))(\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}.$$

The 11 and 22 entries in $\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - G_0(\xi)$ are estimated by $Ch|\xi|^2$; see (4.13). To estimate the 12 and 21 entries we use Taylor's formula:

$$\mathrm{e}^{\mathrm{i}h\xi_j} = 1 + \mathrm{i}h\xi_j + (\mathrm{i}h\xi_j)^2 \int_0^1 \mathrm{e}^{\mathrm{i}ht\xi_j}(1-t)\,\mathrm{d}t. \tag{4.25}$$

It follows that the 12 and 21 entries also are estimated by $Ch|\xi|^2$. Using Lemmas 4.1 and 4.7 the result follows. ■

We can now state the analogue of Theorem 4.4. The proof is omitted, since it is almost identical to the proof of Theorem 4.4; indeed the key ingredients are the estimates in Lemmas 4.7 and 4.8, that correspond to the results from Lemmas 4.1 and 4.2 with the modified symmetric difference model.

Theorem 4.9. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(\widetilde{H}^{\mathrm{fb}}_{0,h} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^2)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

The negative result in Theorem 4.6 for the symmetric model holds also in the forward-backward case.

Lemma 4.10. *There exists $c > 0$ such that*

$$\max_{\xi \in \mathbf{T}^2_h} \|(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \geq c$$

for all $h \in (0, 1]$.

Proof. As in the proof of Lemma 4.5 we get

$$\|(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} = \frac{f_h(\xi)}{(1 + g^{\mathrm{fb}}_{0,h}(\xi))^{1/2}(1 + \tilde{g}^{\mathrm{fb}}_{0,h}(\xi))^{1/2}}.$$

Using (4.6), (4.21), and (4.22) we get

$$f_h\Big(\frac{\pi}{2h}, -\frac{\pi}{2h}\Big) = \frac{4}{h}, \quad g^{\mathrm{fb}}_{0,h}\Big(\frac{\pi}{2h}, -\frac{\pi}{2h}\Big) = m^2, \quad \text{and} \quad \tilde{g}^{\mathrm{fb}}_{0,h}\Big(\frac{\pi}{2h}, -\frac{\pi}{2h}\Big) = \Big(m + \frac{4}{h}\Big)^2.$$

It follows that we have a lower bound

$$\begin{aligned}\max_{\xi \in \mathbf{T}^2_h} &\|(G^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \\ &\geq \frac{4}{(1 + m^2)^{1/2}(h^2 + (4 + hm)^2)^{1/2}} \\ &\geq \frac{4}{(1 + m^2)^{1/2}(1 + (4 + m)^2)^{1/2}}\end{aligned}$$

for $0 < h \leq 1$. ■

Theorem 4.9 combined with Lemma 2.4 and Lemma 4.10 gives the following result.

Theorem 4.11. *Let $z \in (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$. Then $J_h(H^{\mathrm{fb}}_{0,h} - zI_h)^{-1}K_h$* does not converge *to $(H_0 - zI)^{-1}$ in the operator norm on $\mathcal{B}(\mathcal{H}^2)$ as $h \to 0$.*

This result implies that the strong convergence result in [7] cannot be improved to a norm convergence result, without modifying the discretization.

5. The 3D free Dirac operator

Write $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Let $\mathbf{0}$ and $\mathbf{1}$ denote the 2×2 zero and identity matrices, and let $\mathsf{0}$ and $\mathsf{1}$ denote the corresponding 4×4 matrices.

For $U, W \in \mathbf{C}^3$, there is the following identity related to the Pauli matrices, where the "dot" does not involve complex conjugation:

$$(U \cdot \sigma)(W \cdot \sigma) = (U \cdot W)\mathbf{1} + \mathrm{i}(U \times W) \cdot \sigma. \tag{5.1}$$

The Dirac matrices $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β satisfy

$$\begin{aligned} \alpha_j \alpha_k + \alpha_k \alpha_j &= 2\delta_{j,k}\mathsf{1}, \\ \alpha_j \beta + \beta \alpha_j &= \mathsf{0}, \\ \beta^2 &= \mathsf{1}. \end{aligned}$$

We can choose

$$\beta = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}, \quad \alpha_j = \begin{bmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{bmatrix}, \quad j = 1, 2, 3.$$

The free Dirac operator with mass $m \geq 0$ in $\mathcal{H}^3$ is given by

$$H_0 = -\mathrm{i}\alpha \cdot \nabla + m\beta = \begin{bmatrix} m\mathbf{1} & -\mathrm{i}\sigma \cdot \nabla \\ -\mathrm{i}\sigma \cdot \nabla & -m\mathbf{1} \end{bmatrix}, \tag{5.2}$$

see, for instance, [8], where $\mathbf{1}$ in the context of (5.2) denotes the identity operator on $L^2(\mathbf{R}^3) \otimes \mathbf{C}^2$. In Fourier space $\widehat{\mathcal{H}}^3$ it is a multiplier with symbol

$$G_0(\xi) = \begin{bmatrix} m\mathbf{1} & \xi \cdot \sigma \\ \xi \cdot \sigma & -m\mathbf{1} \end{bmatrix}, \quad \xi \in \mathbf{R}^3. \tag{5.3}$$

Define

$$g_0(\xi) = m^2 + \xi_1^2 + \xi_2^2 + \xi_3^2, \tag{5.4}$$

then

$$G_0(\xi)^2 = g_0(\xi)\mathsf{1}. \tag{5.5}$$

As in dimension two there are two natural discretizations of (5.2), using either the pair of forward-backward partial difference operators or the symmetric partial difference operators.

5.1. The 3D symmetric difference model

The symmetric partial difference operators are defined in (2.3). We use the notation

$$\mathsf{D}_h^{\mathrm{s}} = (D_{h;1}^{\mathrm{s}}, D_{h;2}^{\mathrm{s}}, D_{h;3}^{\mathrm{s}})$$

for the discrete symmetric gradient. The symmetric discretization of the 3D Dirac operator is defined as

$$H_{0,h}^{\mathrm{s}} = \begin{bmatrix} m\mathbf{1}_h & \mathsf{D}_h^{\mathrm{s}}\cdot\sigma \\ \mathsf{D}_h^{\mathrm{s}}\cdot\sigma & -m\mathbf{1}_h \end{bmatrix}, \tag{5.6}$$

where $\mathbf{1}_h$ is the identity operator on $\ell^2(h\mathbf{Z}^3)\otimes\mathbf{C}^2$. In Fourier space this operator is a multiplier with symbol

$$G_{0,h}^{\mathrm{s}}(\xi) = \begin{bmatrix} m\mathbf{1} & \mathsf{S}_h^{\mathrm{s}}(\xi)\cdot\sigma \\ \mathsf{S}_h^{\mathrm{s}}(\xi)\cdot\sigma & -m\mathbf{1} \end{bmatrix}, \tag{5.7}$$

where

$$\mathsf{S}_h^{\mathrm{s}}(\xi) = \Big(\frac{1}{h}\sin(h\xi_1), \frac{1}{h}\sin(h\xi_2), \frac{1}{h}\sin(h\xi_3)\Big).$$

We have

$$G_{0,h}^{\mathrm{s}}(\xi)^2 = g_{0,h}^{\mathrm{s}}(\xi)1,$$

where

$$g_{0,h}^{\mathrm{s}}(\xi) = m^2 + \frac{1}{h^2}\sin^2(h\xi_1) + \frac{1}{h^2}\sin^2(h\xi_2) + \frac{1}{h^2}\sin^2(h\xi_3).$$

As in the two-dimensional case we also define a modified discretization. Let $-\Delta_h$ denote the 3D discrete Laplacian; see (2.4). Let $-\Delta_h\mathbf{1}$ denote the 2×2 diagonal operator matrix with the discrete Laplacian on the diagonal elements. Then define

$$\widetilde{H}_{0,h}^{\mathrm{s}} = H_{0,h}^{\mathrm{s}} + \begin{bmatrix} -h\Delta_h\mathbf{1} & \mathbf{0} \\ \mathbf{0} & h\Delta_h\mathbf{1} \end{bmatrix}.$$

Its symbol is

$$\widetilde{G}_{0,h}^{\mathrm{s}}(\xi) = G_{0,h}^{\mathrm{s}}(\xi) + \mathsf{f}_h(\xi)\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix},$$

where

$$\mathsf{f}_h(\xi) = \frac{4}{h}\sin^2\Big(\frac{h}{2}\xi_1\Big) + \frac{4}{h}\sin^2\Big(\frac{h}{2}\xi_2\Big) + \frac{4}{h}\sin^2\Big(\frac{h}{2}\xi_3\Big). \tag{5.8}$$

We have

$$\widetilde{G}_{0,h}^{\mathrm{s}}(\xi)^2 = \tilde{g}_{0,h}^{\mathrm{s}}(\xi)1,$$

where

$$\tilde{g}_{0,h}^{\mathrm{s}}(\xi) = (m + \mathsf{f}_h(\xi))^2 + \frac{1}{h^2}\sin^2(h\xi_1) + \frac{1}{h^2}\sin^2(h\xi_2) + \frac{1}{h^2}\sin^2(h\xi_3).$$

Since g_0, $g^{\mathrm{s}}_{0,h}$, and $\tilde{g}^{\mathrm{s}}_{0,h}$ have similar expressions in dimensions $d = 2, 3$, one can directly repeat the computations leading to Theorems 4.4 and 4.6, which yield the following results.

Theorem 5.1. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(\widetilde{H}^{\mathrm{s}}_{0,h} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^3)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

Theorem 5.2. *Let $z \in (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$. Then $J_h(H^{\mathrm{s}}_{0,h} - zI_h)^{-1}K_h$* does not converge *to $(H_0 - zI)^{-1}$ in the operator norm on $\mathcal{B}(\mathcal{H}^3)$ as $h \to 0$.*

5.2. The 3D forward-backward difference model

Using the definitions (2.1) and (2.2), we introduce the discrete forward and backward gradients as

$$\mathsf{D}^{\pm}_h = (D^{\pm}_{h;1}, D^{\pm}_{h;2}, D^{\pm}_{h;3}).$$

The forward-backward difference model is then given by

$$H^{\mathrm{fb}}_{0,h} = \begin{bmatrix} m\mathbf{1}_h & \mathsf{D}^{-}_h \cdot \sigma \\ \mathsf{D}^{+}_h \cdot \sigma & -m\mathbf{1}_h \end{bmatrix}. \tag{5.9}$$

The symbols of $D^{\pm}_{h;j}$ in Fourier space are

$$\pm\frac{1}{\mathrm{i}h}(\mathrm{e}^{\pm\mathrm{i}h\xi_j} - 1), \quad j = 1, 2, 3.$$

The symbols of the discrete gradients are then

$$\mathsf{S}^{\pm}_h(\xi) = \Big(\pm\frac{1}{\mathrm{i}h}(\mathrm{e}^{\pm\mathrm{i}h\xi_1} - 1), \pm\frac{1}{\mathrm{i}h}(\mathrm{e}^{\pm\mathrm{i}h\xi_2} - 1), \pm\frac{1}{\mathrm{i}h}(\mathrm{e}^{\pm\mathrm{i}h\xi_3} - 1)\Big), \tag{5.10}$$

such that the symbol of $H^{\mathrm{fb}}_{0,h}$ is

$$G^{\mathrm{fb}}_{0,h}(\xi) = \begin{bmatrix} m\mathbf{1} & \mathsf{S}^{-}_h(\xi) \cdot \sigma \\ \mathsf{S}^{+}_h(\xi) \cdot \sigma & -m\mathbf{1} \end{bmatrix}.$$

We also define the modified discretization as

$$\widetilde{H}^{\mathrm{fb}}_{0,h} = H^{\mathrm{fb}}_{0,h} + \begin{bmatrix} -h\Delta_h\mathbf{1} & \mathbf{0} \\ \mathbf{0} & h\Delta_h\mathbf{1} \end{bmatrix}$$

which has the symbol

$$\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) = G^{\mathrm{fb}}_{0,h}(\xi) + \mathsf{f}_h(\xi)\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix},$$

with f_h given in (5.8).

The arguments for norm resolvent convergence of the 3D modified forward-backward difference model do not follow as straightforwardly as in the symmetric difference case, since in particular $\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi)^2$ is not a diagonal matrix. A computation reveals that

$$\begin{bmatrix} \mathbf{0} & \mathsf{S}_h^-(\xi)\cdot\sigma \\ \mathsf{S}_h^+(\xi)\cdot\sigma & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix} = \begin{bmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathsf{S}_h^-(\xi)\cdot\sigma \\ \mathsf{S}_h^+(\xi)\cdot\sigma & \mathbf{0} \end{bmatrix}$$

which implies

$$\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi)^2 = (m+\mathsf{f}_h(\xi))^2 1 + \begin{bmatrix} (\mathsf{S}_h^-(\xi)\cdot\sigma)(\mathsf{S}_h^+(\xi)\cdot\sigma) & \mathbf{0} \\ \mathbf{0} & (\mathsf{S}_h^+(\xi)\cdot\sigma)(\mathsf{S}_h^-(\xi)\cdot\sigma) \end{bmatrix}. \tag{5.11}$$

We proceed to show the required estimates related to $\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi)$ in detail.

Lemma 5.3. *Assume $\xi \neq 0$. Then we have*

$$\|(G_0(\xi) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} \le \frac{1}{|\xi|}. \tag{5.12}$$

There exists $C > 0$ such that for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^3$ we have

$$\|(\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} \le \frac{C}{|\xi|}. \tag{5.13}$$

Proof. The estimate (5.12) follows from (5.4) and (5.5), together with Lemma 2.3.

To prove the estimate (5.13) use Lemma 2.3 and note that

$$\|(\widetilde{G}^{\mathrm{fb}}_{0,h}(\xi) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} = \frac{1}{(\tilde{\lambda}_{\min}(\xi))^{1/2}}$$

where $\tilde{\lambda}_{\min}(\xi)$ is the smallest eigenvalue of $1 + \widetilde{G}^{\mathrm{fb}}_{0,h}(\xi)^2$. Estimating $\tilde{\lambda}_{\min}(\xi) \ge c|\xi|^2$ for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^3$ will conclude the proof.

The matrices $(\mathsf{S}_h^-(\xi)\cdot\sigma)(\mathsf{S}_h^+(\xi)\cdot\sigma)$ and $(\mathsf{S}_h^+(\xi)\cdot\sigma)(\mathsf{S}_h^-(\xi)\cdot\sigma)$ have the same spectrum, so by (5.11) it is enough to focus on one of these blocks. Applying (5.1) to the top left block of (5.11), and noticing that $\mathsf{S}_h^+(\xi) = \overline{\mathsf{S}_h^-(\xi)}$, we thereby need to investigate the smallest eigenvalue of

$$(1 + (m+\mathsf{f}_h(\xi))^2 + |\mathsf{S}_h^-(\xi)|^2)\mathbf{1} + \mathrm{i}(\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)})\cdot\sigma.$$

The smallest eigenvalue of the last term is $-|\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)}|$, so we have

$$\tilde{\lambda}_{\min}(\xi) = 1 + (m+\mathsf{f}_h(\xi))^2 + |\mathsf{S}_h^-(\xi)|^2 - |\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)}|. \tag{5.14}$$

If $0 < \delta \le h|\xi|$ and $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^3$ for some $\delta > 0$, and as $|\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)}| \le |\mathsf{S}_h^-(\xi)|^2$, then (5.14) and (5.8) imply the lower bound

$$\tilde{\lambda}_{\min}(\xi) \ge \mathsf{f}_h(\xi)^2 \ge c|\xi|^2.$$

What remains is an estimate when $h|\xi| \le \delta$ for some small $\delta > 0$. Here we first need a bound on $|\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)}|$, where

$$\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)} = \frac{2\mathrm{i}}{h^2}\begin{bmatrix} \sin(h(\xi_3-\xi_2)) - \sin(h\xi_3) + \sin(h\xi_2) \\ \sin(h(\xi_1-\xi_3)) - \sin(h\xi_1) + \sin(h\xi_3) \\ \sin(h(\xi_2-\xi_1)) - \sin(h\xi_2) + \sin(h\xi_1) \end{bmatrix}. \tag{5.15}$$

Using the same estimate as in the proof of Lemma 4.7, we have

$$\frac{2}{h^2}|\sin(h(x-y)) - \sin(hx) + \sin(hy)| \le h|x||y|^2 + h|y||x|^2 \le 2\delta|\xi|^2,$$

for $h|(x,y)| \le h|\xi| \le \delta$. We have $|\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)}| \le 2\sqrt{3}\delta|\xi|^2$.

For $h|\xi| \le \delta_0 < 1$ there exists $c_0 > 0$ such that $|\mathsf{S}_h^-(\xi)|^2 \ge c_0|\xi|^2$. A fixed δ with $0 < \delta < \min\{\frac{c_0}{2\sqrt{3}}, \delta_0\}$ gives the estimate

$$\tilde{\lambda}_{\min}(\xi) \ge |\mathsf{S}_h^-(\xi)|^2 - |\mathsf{S}_h^-(\xi) \times \overline{\mathsf{S}_h^-(\xi)}| \ge c|\xi|^2$$

for $h|\xi| \le \delta$. ■

Lemma 5.4. *There exists $C > 0$ such that*

$$\|(G_0(\xi) - \mathrm{i}1)^{-1} - (\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} \le Ch$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]^3$.

Proof. We have

$$\begin{aligned}(G_0(\xi) - \mathrm{i}1)^{-1} &- (\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}1)^{-1} \\ &= (G_0(\xi) - \mathrm{i}1)^{-1}(\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi) - G_0(\xi))(\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}1)^{-1}.\end{aligned}$$

We estimate the entries in $\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi) - G_0(\xi)$ as in the proof of Lemma 4.8. Thus, the entries are estimated by $Ch|\xi|^2$. Using Lemma 5.3 the result follows. ■

Using Lemmas 5.3 and 5.4, we can adapt the arguments in [2] to obtain the following result. We omit the details here, and refer the reader to the proof of Theorem 4.4 where details of the adaptation are given.

Theorem 5.5. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(\widetilde{H}_{0,h}^{\mathrm{fb}} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^3)} \le Ch$$

for all $z \in K$ and $h \in (0, 1]$.

As in dimension two, the unmodified forward-backward difference model does not lead to norm resolvent convergence.

Lemma 5.6. *There exists $c > 0$ such that*

$$\max_{\xi \in \mathbf{T}_h^3} \|(G_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}1)^{-1} - (\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} \geq c$$

for all $h \in (0, 1]$.

Proof. Consider $\xi_h = (\frac{\pi}{2h}, -\frac{\pi}{2h}, 0)$ and notice that $\xi_h \in \mathbf{T}_h^3$ for $h > 0$. From (5.8), (5.10), and (5.15) then

$$\mathsf{f}_h(\xi_h) = \frac{4}{h} \quad \text{and} \quad |\mathsf{S}_h^-(\xi_h)|^2 = |\mathsf{S}_h^-(\xi_h) \times \overline{\mathsf{S}_h^-(\xi_h)}| = \frac{4}{h^2}.$$

From (5.14), the smallest eigenvalues $\widetilde{\lambda}_{\min}(\xi_h)$ of $1 + \widetilde{G}_{0,h}^{\mathrm{fb}}(\xi_h)^2$ and $\lambda_{\min}(\xi_h)$ of $1 + G_{0,h}^{\mathrm{fb}}(\xi_h)^2$ are

$$\widetilde{\lambda}_{\min}(\xi_h) = 1 + (m + \mathsf{f}_h(\xi_h))^2 + |\mathsf{S}_h^-(\xi_h)|^2 - |\mathsf{S}_h^-(\xi_h) \times \overline{\mathsf{S}_h^-(\xi_h)}| = 1 + \Big(m + \frac{4}{h}\Big)^2,$$
$$\lambda_{\min}(\xi_h) = 1 + m^2 + |\mathsf{S}_h^-(\xi_h)|^2 - |\mathsf{S}_h^-(\xi_h) \times \overline{\mathsf{S}_h^-(\xi_h)}| = 1 + m^2.$$

Thus, by Lemma 2.3 we have

$$\|(\widetilde{G}_{0,h}^{\mathrm{fb}}(\xi_h) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} = \frac{1}{(\widetilde{\lambda}_{\min}(\xi_h))^{1/2}} = \frac{h}{(h^2 + (mh + 4)^2)^{1/2}}$$

and

$$\|(G_{0,h}^{\mathrm{fb}}(\xi_h) - \mathrm{i}1)^{-1}\|_{\mathcal{B}(\mathbf{C}^4)} = \frac{1}{(\lambda_{\min}(\xi_h))^{1/2}} = \frac{1}{(1 + m^2)^{1/2}},$$

which conclude the proof with

$$c = \frac{1}{(1 + m^2)^{1/2}} - \frac{1}{(1 + (m + 4)^2)^{1/2}} > 0. \qquad \blacksquare$$

Combining Theorem 5.5 with Lemma 2.4 and Lemma 5.6 gives the following non-convergence result.

Theorem 5.7. *Let $z \in (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$. Then $J_h(H_{0,h}^{\mathrm{fb}} - zI_h)^{-1}K_h$* does not converge *to $(H_0 - zI)^{-1}$ in the operator norm on $\mathcal{B}(\mathcal{H}^3)$ as $h \to 0$.*

6. Sobolev space estimates and strong convergence

In Sections 3–5 we have shown that $J_h(H_{0,h} - zI_h)^{-1}K_h$ converges in the $\mathcal{B}(\mathcal{H}^d)$-operator norm to $(H_0 - zI)^{-1}$ for several choices of discrete model $H_{0,h}$, and we have also shown that in other cases this norm convergence does not hold. This section is dedicated to the cases where $J_h(H_{0,h} - zI_h)^{-1}K_h$ does not converge to $(H_0 - zI)^{-1}$ in the $\mathcal{B}(\mathcal{H}^d)$-operator norm, and instead we will prove that convergence holds in the $\mathcal{B}(H^1(\mathbf{R}^d) \otimes \mathbf{C}^{\nu(d)}, \mathcal{H}^d)$-operator norm. These latter results obviously imply strong convergence. In particular, we recover the result in [7] for $d = 2$ with the discretization $H^{\mathrm{fb}}_{0,h}$.

6.1. The 1D model

The 1D symmetric model $H^{\mathrm{s}}_{0,h}$ is defined in (3.8) and its symbol $G^{\mathrm{s}}_{0,h}(\xi)$ in (3.9). The symbol for the continuous Dirac operator $G_0(\xi)$ is defined in (3.1).

Lemma 6.1. *There exists $C > 0$ such that*

$$\big\|((G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G_0(\xi) - \mathrm{i}\mathbf{1})^{-1})(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\big\|_{\mathcal{B}(\mathbf{C}^2)} \le Ch$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$.

Proof. Note that Lemma 2.3 and (3.10) imply the estimate

$$\|(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \le (1 + m^2)^{-1}$$

and that this estimate cannot be improved for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$. We have

$$\begin{aligned}((G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} &- (G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1})(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1} \\ &= (G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}(G^{\mathrm{s}}_{0,h}(\xi) - G_0(\xi))(G_0(\xi) - \mathrm{i}\mathbf{1})^{-2}.\end{aligned}$$

Now

$$G^{\mathrm{s}}_{0,h}(\xi) - G_0(\xi) = \begin{bmatrix} 0 & \dfrac{1}{h}\sin(h\xi) - \xi \\ \dfrac{1}{h}\sin(h\xi) - \xi & 0 \end{bmatrix}.$$

Using Taylor's formula (4.14) together with the estimates

$$\|(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \le \frac{1}{|\xi|} \quad \text{and} \quad \|(G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)} \le 1,$$

we get

$$\big\|((G^{\mathrm{s}}_{0,h}(\xi) - \mathrm{i}\mathbf{1})^{-1} - (G_0(\xi) - \mathrm{i}\mathbf{1})^{-1})(G_0(\xi) - \mathrm{i}\mathbf{1})^{-1}\big\|_{\mathcal{B}(\mathbf{C}^2)} \le Ch^2|\xi| \le Ch$$

for $h\xi \in [-\frac{3\pi}{2}, \frac{3\pi}{2}]$. ∎

Proposition 6.2. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\|J_h(H^{\mathrm{s}}_{0,h} - zI_h)^{-1}K_h - (H_0 - zI)^{-1}\|_{\mathcal{B}(H^1(\mathbf{R}) \otimes \mathbf{C}^2, \mathcal{H}^1)} \le Ch \tag{6.1}$$

for all $z \in K$ and $h \in (0, 1]$.

Remark 6.3. The estimate (6.1) implies

$$\operatorname*{s-lim}_{h\to 0} J_h(H^{\mathrm{s}}_{0,h}-zI_h)^{-1}K_h=(H_0-zI)^{-1}$$

uniformly in $z\in K$.

Proof. The result follows if we prove the estimate

$$\|(J_h(H^{\mathrm{s}}_{0,h}-zI_h)^{-1}K_h-(H_0-zI)^{-1})(H_0-\mathrm{i}I)^{-1}\|_{\mathcal{B}(\mathcal{H}^1)}\le Ch.$$

The proof is very similar to the proof of Theorem 4.4. In the arguments one replaces $\mathcal{F}^*u$ by $\mathcal{F}^*(H_0-\mathrm{i}I)^{-1}u$ and uses Lemma 6.1. Further details are omitted. ■

6.2. The 2D model

Consider first the symmetric difference model. The continuous 2D free Dirac operator is denoted by H_0 and its symbol by $G_0(\xi)$; see (4.2). The symmetric difference model is denoted by $H^{\mathrm{s}}_{0,h}$; see (4.3). Its symbol is denoted by $G^{\mathrm{s}}_{0,h}(\xi)$; see (4.4).

Lemma 6.4. *There exists $C>0$ such that*

$$\|((G^{\mathrm{s}}_{0,h}(\xi)-\mathrm{i}\mathbf{1})^{-1}-(G_0(\xi)-\mathrm{i}\mathbf{1})^{-1})(G_0(\xi)-\mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)}\le Ch$$

for $h\xi\in[-\frac{3\pi}{2},\frac{3\pi}{2}]^2$.

Proof. The proof is almost the same as the proof of Lemma 6.1. It follows from Lemma 2.3 and (4.8) that we have $\|(G^{\mathrm{s}}_{0,h}(\xi)-\mathrm{i}\mathbf{1})^{-1}\|_{\mathcal{B}(\mathbf{C}^2)}\le 1$ for $h\xi\in[-\frac{3\pi}{2},\frac{3\pi}{2}]^2$. We have

$$G^{\mathrm{s}}_{0,h}(\xi)-G_0(\xi)=\begin{bmatrix}0 & (\frac{1}{h}\sin(h\xi_1)-\xi_1)-\mathrm{i}(\frac{1}{h}\sin(h\xi_2)-\xi_2)\\ (\frac{1}{h}\sin(h\xi_1)-\xi_1)+\mathrm{i}(\frac{1}{h}\sin(h\xi_2)-\xi_2) & 0\end{bmatrix}.$$

Then (4.14) implies $\|G^{\mathrm{s}}_{0,h}(\xi)-G_0(\xi)\|_{\mathcal{B}(\mathbf{C}^2)}\le Ch^2|\xi|^3$. The remaining parts of the argument in the proof of Lemma 6.1 can then be repeated. ■

The proof of the next result is almost identical to the proof of Proposition 6.2 and is omitted.

Proposition 6.5. *Let $K\subset(\mathbf{C}\setminus\mathbf{R})\cup(-m,m)$ be compact. Then there exists $C>0$ such that*

$$\|J_h(H^{\mathrm{s}}_{0,h}-zI_h)^{-1}K_h-(H_0-zI)^{-1}\|_{\mathcal{B}(H^1(\mathbf{R}^2)\otimes\mathbf{C}^2,\mathcal{H}^2)}\le Ch$$

for all $z\in K$ and $h\in(0,1]$.

Next we consider the forward-backward difference model. The arguments are almost identical to those for the symmetric model. We state the result without proof. The discretization $H_{0,h}^{\text{fb}}$ is defined in (4.20).

Proposition 6.6. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\| J_h (H_{0,h}^{\text{fb}} - zI_h)^{-1} K_h - (H_0 - zI)^{-1} \|_{\mathcal{B}(H^1(\mathbf{R}^2)\otimes\mathbf{C}^2, \mathcal{H}^2)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

6.3. The 3D model

In this section H_0 denotes the free 3D Dirac operator. The symmetric difference model $H_{0,h}^{\text{s}}$ is given by (5.6) and the forward-backward difference model $H_{0,h}^{\text{fb}}$ is given by (5.9).

We state the following results without proofs, since they are very similar to the proofs of Propositions 6.2, 6.5, and 6.6.

Proposition 6.7. *Let $K \subset (\mathbf{C} \setminus \mathbf{R}) \cup (-m, m)$ be compact. Then there exists $C > 0$ such that*

$$\| J_h (H_{0,h}^{\text{s}} - zI_h)^{-1} K_h - (H_0 - zI)^{-1} \|_{\mathcal{B}(H^1(\mathbf{R}^3)\otimes\mathbf{C}^4, \mathcal{H}^3)} \leq Ch$$

and

$$\| J_h (H_{0,h}^{\text{fb}} - zI_h)^{-1} K_h - (H_0 - zI)^{-1} \|_{\mathcal{B}(H^1(\mathbf{R}^3)\otimes\mathbf{C}^4, \mathcal{H}^3)} \leq Ch$$

for all $z \in K$ and $h \in (0, 1]$.

7. Perturbed Dirac operators

In this section we state results on perturbed Dirac operators and their discretizations, with respect to norm resolvent convergence. We use the following condition on the perturbation.

Assumption 7.1. Assume that $V : \mathbf{R}^d \to \mathcal{B}(\mathbf{C}^{\nu(d)})$ is bounded and Hölder continuous with exponent $\theta \in (0, 1]$. Assume $(V(x))^* = V(x)$, $x \in \mathbf{R}^d$.

We require another assumption on ψ_0 in addition to Assumption 2.2. We emphasize that concrete examples of ψ_0 satisfying these assumptions are given in [2, Section 2.1].

Assumption 7.2. Assume there exists $\tau > d$ such that

$$|\psi_0(x)| \leq (1 + |x|)^{-\tau}, \quad x \in \mathbf{R}^d.$$

Define a discretization of V by

$$V_h(k) = V(hk), \quad k \in \mathbf{Z}^d. \tag{7.1}$$

Let $H_{0,h}$ be one of the discretizations of H_0 from Sections 3–5. Then we define self-adjoint operators $H = H_0 + V$ and $H_h = H_{0,h} + V_h$.

The following result is an adaptation of [2, Proposition 4.3] to the present framework.

Lemma 7.3. *Let V satisfy Assumption 7.1 and let ψ_0 satisfy Assumption 7.2. Define*

$$\frac{1}{\theta'} = \frac{1}{\theta} + \frac{1}{\tau - d}. \tag{7.2}$$

Then

$$\|V_h K_h - K_h V\|_{\mathcal{B}(\mathcal{H}^d, \mathcal{H}_h^d)} \leq C h^{\theta'}.$$

Proof. The proof in [2] can be directly adapted to the current framework. We omit the details. Note that $\psi_0(x)$ is a scalar, such that $\psi_0(x)V(x)f(x) = V(x)\psi_0(x)f(x)$, $f \in \mathcal{H}^d$. ∎

We can then state our main result on the perturbed Dirac operators, which follows from Lemma 7.3 and a direct adaptation of the proof of [2, Theorem 4.4].

Theorem 7.4. *Let J_h and K_h be the operators defined in Section 2, and let ψ_0 satisfy Assumption 7.2. Let V satisfy Assumption 7.1 and define V_h by (7.1). Let $H_{0,h}$ equal either of*

$$\begin{cases} H_{0,h}^{\mathrm{fb}}, & d = 1, \\ \widetilde{H}_{0,h}^{\mathrm{fb}}, & d = 2, 3, \\ \widetilde{H}_{0,h}^{\mathrm{s}}, & d = 1, 2, 3. \end{cases}$$

Let $H_h = H_{0,h} + V_h$. Let $H = H_0 + V$, where H_0 is the free Dirac operator in the relevant dimension. Assume $V \not\equiv 0$ and let θ' be given by (7.2). Then the following result holds.

Let $K \subset \mathbf{C} \setminus \mathbf{R}$ be compact. Then there exist $C > 0$ and $h_0 > 0$ such that

$$\|J_h(H_h - zI_h)^{-1}K_h - (H - zI)^{-1}\|_{\mathcal{B}(\mathcal{H}^d)} \leq C h^{\theta'}$$

for all $z \in K$ and $h \in (0, h_0]$.

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