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Lower bound of Schrödinger operators on Riemannian manifolds

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Abstract. We show that a complete weighted manifold which satisfies to a relative Faber–Krahn inequality admits a trace inequality for the measure with density V, with the constant depending on a Morrey norm of V. From this, we obtain estimates on the lower bound of the spectrum of the Schrödinger operators with potential V and positivity conditions for such operators. It also yields a L^2 Hardy inequality.

1. Introduction

In [8, 9], Fefferman and Phong established the inequality, for $p > 1$,

$$\int_{\mathbf{R}^n} V(x)\psi(x)^2 \, \mathrm{d}x \leq C_{n,p} N_p(V) \int_{\mathbf{R}^n} |\nabla\psi(x)|^2 \, \mathrm{d}x, \tag{1.1}$$

for any ψ smooth with compact support, where V is a non negative and locally integrable function, $C_{n,p}$ is a constant depending only on the dimension and p, and N_p is the Morrey norm

$$N_p(V) = \sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \Big(r^{2p-n} \int_{B(x,r)} |V(y)|^p \, \mathrm{d}y \Big)^{1/p}.$$

Such an inequality yields a positivity condition for the Schrödinger operator $H = \Delta - V$ (with $\Delta = -\sum_{i=1}^n \partial_i^2$), namely that if $N_p(V) \leq 1/C_{n,p}$, then H is a positive operator. In fact, they also gave the following estimates on the lower bound of the spectrum of H, $\lambda_1(H)$:

$$\sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \Big(C_1 r^{-n} \int_{B(x,r)} V \, \mathrm{d}y - r^{-2} \Big) \leq \sup_{\substack{x\in\mathbf{R}^n \\ r>0}} \Big(C_p \Big(r^{-n} \int_{B(x,r)} V^p \, \mathrm{d}y \Big)^{1/p} - r^{-2} \Big). \tag{1.2}$$

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Conditions for inequalities such as (1.1) (though with a constant that does not necessarily depends on the Morrey norm) to hold in $\mathbf{R}^n$ has been studied extensively, see for example in [4, 15, 18]. In [19], Maz'ya and Verbitsky establish necessary and sufficient conditions for an inequality analog to (1.1) to hold with complex valued V. That being the case, it seems interesting to study to what extent, and under which geometrical hypotheses, those results extend on other spaces, such as Riemannian manifolds.

The first aim of this article is to generalise the results of Fefferman and Phong to a weighted Riemannian manifold M. A natural way to do that would be to use the Poincaré inequality: for any $\kappa > 1$, there is a constant $C > 0$, such that for all $x \in M$, $r > 0$, and for any $f \in \mathcal{C}^\infty(B(x, \kappa r))$,

$$\int_{B(x,r)} |f - f_{B(x,r)}| \, \mathrm{d}\mu \leq C r \int_{B(x,\kappa r)} |\nabla f| \, \mathrm{d}\mu,$$

where $f_B = \frac{1}{\mu(B)} \int_B f \, \mathrm{d}\mu$. It turns out that the result still holds under some weaker hypothesis. Our proof will follow the general idea used by Schechter in [26], that (1.1) follows from the inequality (which holds in $\mathbf{R}^n$ following a result of Muckenhoupt and Wheeden [21]):

$$\|I_1 f\|_{L^2} \leq C \|M_1 f\|_{L^2},$$

with

$$I_1 f(x) = c_n \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-1}} \, \mathrm{d}\mu(y), \quad M_1 f(x) = \sup_{r>0} r^{1-n} \int_{B(x,r)} |f(y)| \, \mathrm{d}y,$$

and that (1.2) is proved using similar estimates, with $(\Delta + \lambda^2)^{-1/2}$ replacing I_1.

The proof of the generalisation of (1.2) will naturally yields weak versions of (1.1), which holds under weaker hypothesis.

1.1. Definitions and notations

A *weighted Riemannian manifold* (M, g, μ), or simply a *weighted manifold,* is the data of a smooth manifold M, g a smooth Riemannian metric on M, and a Borel measure $\mathrm{d}\mu = \sigma^2 \, \mathrm{d}v_g$ on M, with σ a smooth positive function on M and v_g is the Riemannian volume measure associated with the metric g. We define the *(weighted) Dirichlet Laplace operator* as the Friedrichs extension of the operator on $\mathcal{C}_0^\infty(M)$ given by

$$\Delta_\mu f = -\sigma^{-2} \operatorname{div}(\sigma^2 \nabla f),$$

with associated quadratic form $Q(\psi) = \int_M |\nabla \psi|^2 \, \mathrm{d}\mu$. We will usually write the Dirichlet Laplace operator as simply Δ.

On a metric space (X, d), for $x \in X$, $r > 0$, the ball of center x and radius r is the set $B(x,r) = \{y: d(x,y) < r\}$. If $B = B(x,r)$ is the ball, $\theta \in \mathbf{R}$, then θB refers to the set $B(x, \theta r)$.

For $p \geq 1$, we let $\|\cdot\|_p$ be the L^p norm on (M, μ). We define

$$\|f\|_p = \left(\int_M |f|^p \,\mathrm{d}\mu\right)^{1/p}.$$

For T a bounded operator on L^p, we use $\|T\|_{L^p \to L^p}$, or $\|T\|_p$ when there is no confusion, to refer to its operator norm

$$\|T\|_p = \sup_{\substack{\psi \in L^p \\ \psi \neq 0}} \frac{\|T\psi\|_p}{\|\psi\|_p}.$$

For an open set $U \subset M$, $\lambda_1(U)$ refers to lower bound of the spectrum of Δ_μ on U,

$$\lambda_1(U) = \inf_{\substack{\psi \in \mathscr{C}_0^\infty(U) \\ \psi \neq 0}} \frac{\|\nabla \psi\|_2^2}{\|\psi\|_2^2}.$$

When H is a symmetric operator defined on smooth functions with compact support, $\lambda_1(H)$ is similarly defined to be

$$\lambda_1(H) = \inf_{\substack{\psi \in \mathscr{C}_0^\infty(M) \\ \psi \neq 0}} \frac{\langle H\psi, \psi\rangle}{\|\psi\|_2^2}.$$

On a weighted manifold (M, g, μ), we define the *Morrey norms* N_p, $p \geq 0$, as follows:

$$N_p(f) = \sup_{\substack{x \in M \\ r > 0}} \left(r^{2p} \fint_{B(x,r)} |f|^p \,\mathrm{d}\mu\right)^{1/p} \quad \text{for all } f \in L^1_{\mathrm{loc}}(M), \tag{1.3}$$

where $\fint_B f \,\mathrm{d}\mu = \frac{1}{\mu(B)} \int_B f \,\mathrm{d}\mu$ is the mean of f over B. We also define the *Morrey norm* taken on balls of radius less than $R > 0$,

$$N_{p,R}(f) = \sup_{\substack{x \in M \\ 0 < r < R}} \left(r^{2p} \fint_{B(x,r)} |f|^p \,\mathrm{d}\mu\right)^{1/p}. \tag{1.4}$$

For our generalization to hold, it is important that (M, g, μ) must admits a *relative Faber–Krahn inequality* (property $(\mathbf{RFK})^\eta$) defined as follows:

Definition 1.1. A weighted Riemannian manifold (M, g, μ) *admits a relative Faber–Krahn inequality* if there exist constants $b, \eta > 0$, such that for all $x \in M$, $r > 0$, and for any relatively compact open set $U \subset B(x, r)$, the following inequality holds:

$$\lambda_1(U) \geq \frac{b}{r^2}\Big(\frac{\mu(B(x,r))}{\mu(U)}\Big)^{\frac{2}{\eta}}. \tag{1.5}$$

We say that M *admits a relative Faber–Krahn inequality at scale* R (*property* $(\mathbf{RFK})^{\eta}_{R}$) if (1.5) holds only for $0 \leq r \leq R$.

In what follows, we refer to the constants b, η in (1.5) as the *Faber–Krahn constants* of the manifold.

1.2. Statements of the results

Theorem 1.1. *Let (M, g, μ) be a weighted complete Riemannian manifold satisfying $(\mathbf{RFK})^{\eta}$, then for any $p > 1$, there is a constant C_p depending only on the Faber–Krahn constants and on p, such that for any $V \in L^1_{\mathrm{loc}}(M)$, $V \geq 0$, and any $\psi \in \mathcal{C}_0^{\infty}(M)$, the following inequality holds:*

$$\int_M V\psi^2 \,\mathrm{d}\mu \leq C_p N_p(V) \int_M |\nabla\psi|^2 \,\mathrm{d}\mu. \tag{1.6}$$

If only $(\mathbf{RFK})^{\eta}_{R}$ holds, then we can prove the following localised inequality:

Theorem 1.2. *Let (M, g, μ) be a complete weighted Riemannian manifold, such that, for some $R > 0$, $(\mathbf{RFK})^{\eta}_{R}$ holds. Then, for any $p > 1$, there is a constant $C_p > 0$ depending only on the Faber–Krahn constant and on p, such that for any $V \in L^1_{\mathrm{loc}}(M)$, $V \geq 0$, and any $\psi \in \mathcal{C}_0^{\infty}(M)$, the following inequality holds:*

$$\int_M V\psi^2 \,\mathrm{d}\mu \leq C_p N_{p,R}(V)\Big(\int_M |\nabla\psi|^2 \,\mathrm{d}\mu + \frac{1}{R^2}\int_M \psi^2 \,\mathrm{d}\mu\Big). \tag{1.7}$$

From this inequality, we can generalise the Fefferman–Phong estimate on the lower bound of the spectrum of the operator $H = \Delta - V$. Indeed, if $(\mathbf{RFK})^{\eta}$ holds, then for any $R > 0$, $(\mathbf{RFK})^{\eta}_{R}$ is satisfied. Thus, (1.7) is true for any R. Then the following theorem follows easily:

Theorem 1.3. *Let (M, g, μ) be a complete weighted Riemannian manifold satisfying $(\mathbf{RFK})^{\eta}$. Then, for any $p > 1$, there exist constants $C_1, C_p > 0$ depending only on the Faber–Krahn constants (and C_p depending also on p) such that, for any $V \in L^1_{\mathrm{loc}}(M)$,*

$V \geq 0$, and for the operator $H = \Delta_\mu - V$ the following inequalities hold:

$$\sup_{\substack{x\in M\\ \delta>0}}\Big(C_1 \fint_{B(x,\delta)} V \,\mathrm{d}\mu - \delta^{-2}\Big) \leq -\lambda_1(H) \leq \sup_{\substack{x\in M\\ \delta>0}}\Big(C_p\Big(\fint_{B(x,\delta)} V^p \,\mathrm{d}\mu\Big)^{1/p} - \delta^2\Big).$$

In addition, if $\lambda_1(M) > 0$, then we can strengthen (1.7), and obtain the following result, giving a condition for $\Delta - V$ to be positive:

Theorem 1.4. *Let (M, g, μ) be a complete weighted Riemannian manifold, such that $(\mathbf{RFK})^\eta_R$ holds for $R > 0$. If, in addition, $\lambda_1(M) > 0$, then, for any $p > 1$, there is a constant $C_p > 0$ depending only on the Faber–Krahn constants such that, for $V \in L^1_{\mathrm{loc}}(M)$, $V \geq 0$, and any $\psi \in \mathcal{C}_0^\infty(M)$, the following inequality holds:*

$$\int_M V\psi^2 \,\mathrm{d}\mu \leq C_p N_{p,R}(V)\frac{1+\lambda_1(M)R^2}{\lambda_1(M)R^2}\Big(\int_M |\nabla\psi|^2 \,\mathrm{d}\mu + \frac{\lambda_1(M)}{2}\int_M \psi^2 \,\mathrm{d}\mu\Big). \tag{1.8}$$

1.3. L^2 Hardy inequality

Notice that the inequality (1.6) is, for potentials V with $N_p(V) < +\infty$, nothing more than the generalized L^2 Hardy inequality:

$$\int_M \frac{\psi^2}{\rho^2} \,\mathrm{d}\mu \leq C\int_M |\nabla\psi|^2 \,\mathrm{d}\mu, \quad \text{for all } \psi \in \mathcal{C}_0^\infty(M),$$

with $\rho = V^{-1/2}$. Thus, on manifolds for which Theorem 1.1 holds, the “classical” Hardy inequality, where ρ is the distance to a point, is true whenever $N_p(d(o,\cdot)^{-2})$ is finite. For this to hold, we must make an additional assumption on the measure μ.

Definition 1.2. A metric measure space (X, d, μ) *satisfies the reverse doubling property of order ν (property* $(\mathbf{RD})^\nu$ for short), or *μ is ν-reverse doubling* if there is some constant $a > 0$ such that, for all $x \in M$, $0 < r \leq r'$, the following inequality holds:

$$a\Big(\frac{r'}{r}\Big)^\nu \leq \frac{\mu(B(x,r'))}{\mu(B(x,r))}.$$

Theorem 1.5. *Let (M, g, μ) be a weighted Riemannian manifold. Assume that M satisfies $(\mathbf{RFK})^\eta$, and that μ satisfies $(\mathbf{RD})^\nu$ with $\nu > 2$. There is some constant $C > 0$ depending only on the Faber–Krahn and reverse doubling constants, such that, for any $o \in M$, then for any $\psi \in \mathcal{C}_0^\infty(M)$ the following inequality holds:*

$$\int_M \frac{\psi(x)^2}{\rho(x)^2} \,\mathrm{d}\mu(x) \leq C\int_M |\nabla\psi|^2 \,\mathrm{d}\mu,$$

with $\rho(x) = d(o, x)$.

We can compare this to the results of V. Minerbe [20] or G. Grillo [13], who proved L^p Hardy inequalities assuming a Poincaré inequalities and a doubling measure. While we only get a L^2 inequality, it holds true under the weaker hypothesis of a relative Faber–Krahn inequality.

A recent work by Cao, Grigor'yan, and Liu [2] proved Hardy inequalities as a consequence of volume doubling, reverse doubling, and certain estimates on either the Green function or the heat kernel. Their results are far more general than what we prove on Hardy inequality here.

1.4. Examples

We give various cases of manifolds which will satisfy a relative Faber–Krahn inequality (or a relative Faber–Krahn inequality at scale R). Then, Theorem 1.1 (respectively, Theorem 1.2) holds.

1.4.1. Complete manifolds with Ricci curvature bounded from below. From Li and Yau [17], the heat kernel of a complete manifold (M, g, μ) of dimension n, with μ here being the Riemannian volume measure, with Ricci curvature bounded from below by $-K$, for a constant $K \geq 0$, admits the following diagonal estimate

$$p_t(x,x) \leq \frac{C_0}{\mu(B(x,\sqrt{t}))} e^{C_1 K t}.$$

Also, as a consequence of the Bishop–Gromov volume comparison theorem, we get that (see [5,6,23] for example), for any $0 < r \leq r'$,

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \leq \left(\frac{r'}{r}\right)^n \exp(\sqrt{(n-1)K}r').$$

Those two conditions implies, (see for example [14,23], or Proposition 3.1 later), that there is some $R > 0$ such that M satisfies $(\mathbf{RFK})^n_R$. If the Ricci curvature is non-negative, then we also have $(\mathbf{RFK})^n$.

1.4.2. Manifolds satisfying Faber–Krahn inequalities outside a compact set. We consider a complete weighted manifold M, and remove from it a compact set with smooth boundary K. We let $E_1, \dots, E_k$ be the connected components of $M \setminus K$, and suppose that each E_i is the exterior of a compact set with smooth boundary in a complete manifold M_i.

A simple example of such manifold is the connected sum of two (or more) copies of $\mathbf{R}^n$. It admits $(\mathbf{RFK})^n$, but it is known that such manifold does not satisfy a Poincaré inequality (see for example [1]).

Using [12], we get that if each M_i satisfies $(\mathbf{RFK})^\eta$, then there is some $R > 0$ such that M satisfies $(\mathbf{RFK})^\eta_R$.

2. Some techniques of harmonic analysis

Remark. The letters c, C will usually be used for generic constants, the values of which might change from line to line. When the dependence on some parameter is judged important and non obvious, it will be made clear when it appears, before being folded into the generic constants on subsequent lines.

2.1. Dydadic cubes

In $\mathbf{R}^n$, the natural decomposition of the space into cubes of length 2^k, $k \in \mathbf{Z}$ is a very powerful tool. It turns out that families of open sets satisfying similar properties to those of the dyadic cubes in the euclidean space can be constructed in a more general setting.

We will use the construction of such "dyadic cubes" given by E. Sawyer and R. L. Wheeden in [24] (though other such constructions, such as the one given in [7], could also be used without major changes). Though it remains true in a more general setting, for our purposes it can be stated as:

Theorem 2.1. *Let (X, d) be a separable metric space, then there is a constant $\rho > 1$ ($\rho = 8$ works), such that for any (large negative) integer m, there are points $\{x_\alpha^k\}$ and a family $\mathcal{D}_m = \{\mathcal{E}_\alpha^k\}$ of Borel sets for $k = m, m+1, \ldots$, $\alpha = 1, 2, \ldots$, which satisfy the following properties:*

- $B(x_\alpha^k, \rho^k) \subset \mathcal{E}_\alpha^k \subset B(x_\alpha^k, \rho^{k+1})$*;*
- *for each $k = m, m+1, \ldots$, the family $\{\mathcal{E}_\alpha^k\}_\alpha$ is pairwise disjoint in α and $X = \bigcup_\alpha \mathcal{E}_\alpha^k$;*
- *df $m \leq k < l$, then either $\mathcal{E}_\alpha^k \cap \mathcal{E}_\beta^l = \emptyset$ or $\mathcal{E}_\alpha^k \subset \mathcal{E}_\beta^l$.*

Given such a family $\mathcal{D}_m$, the sets $\mathcal{E}_\alpha^k$ will be called *dyadic cubes* of M, or simply *cubes*. The ball $B(x_\alpha^k, \rho^{k+1})$ is called the *containing ball* of the cube $\mathcal{E}_\alpha^k$. For any cube Q the containing ball is denoted by $B(Q)$. ρ will be called the *sidelength constant* of dyadic cubes.

The length of a cube Q is the radius of $\rho^{-1}B(Q)$, written $\ell(Q)$.

2.2. Properties of doubling measures

We start by recalling the definitions and some standard properties of doubling measures. Most of the proofs are classical, but are rarely explicitly done for the R doubling case, and we thus give them for completeness sake, without claiming originality.

Definition 2.1. A metric measure space (X, d, μ) *satisfies the doubling property* $(\mathbf{D})^\eta$ *of order* η if there is some constant $A > 0$ such that for all $x \in M$, $0 < r \leq r'$, the following inequality holds:

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \le A\Big(\frac{r'}{r}\Big)^{\eta}. \tag{2.1}$$

We call A the *doubling constant,* and η the *doubling order.* We will also say "the doubling constant" to refer to both A and η at the same time. The property $(\mathbf{D})^{\eta}$ is equivalent to the fact that for some constant $A > 0$, for any ball $B \subset M$,

$$\mu(2B) \le A\mu(B). \tag{2.2}$$

The proof of the equivalence is the same as that of the R-doubling case given after Definition 2.3, (with $R = \infty$).

A note on the constants: (2.2) implies (2.1) with $\eta = \log_2 A$ (and A the same in both inequalities), while, conversely, (2.1) implies that the constant in (2.2) be $2^{\eta}A$. By increasing A and η if necessary, we can always assume that $A = 2^{\eta}$.

We repeat, for completeness, the definition of the reverse doubling property:

Definition 2.2. A metric measure space (X, d, μ) *satisfies the reverse doubling property* $(\mathbf{RD})^{\nu}$ *of order* ν if there is some constant $a > 0$ such that for all $x \in M$, $0 < r \le r'$, the following inequality holds:

$$a\Big(\frac{r'}{r}\Big)^{\nu} \le \frac{\mu(B(x,r'))}{\mu(B(x,r))}. \tag{2.3}$$

We call a the *reverse doubling constant,* and ν the *reverse doubling order.* The property $(\mathbf{RD})^{\nu}$ is equivalent to the fact that for some constant $a \in (0,1)$, for any ball $B \subset M$,

$$\mu(B) \le a\mu(2B). \tag{2.4}$$

Proof of (2.4) *implies* (2.3). We can assume that $a \le 1$. Let $x \in X$, $0 < r \le r'$. Writing $\lfloor t \rfloor$ for the integer part of $t \in \mathbf{R}$, let $k = \lfloor \log_2 \frac{r'}{r} \rfloor$. Then

$$\begin{aligned}\mu(B(x,r)) &\le a^k \mu(B(x,2^k r)) \\ &\le a^k \mu(B(x,r')) \\ &\le a^{-1+\log_2 \frac{r'}{r}} \mu(B(x,r')) \quad (a \le 1) \\ &\le \frac{1}{a}\Big(\frac{r'}{r}\Big)^{-\nu} \mu(B(x,r')),\end{aligned}$$

with $\nu = -\log_2 a$. Thus,

$$a\Big(\frac{r'}{r}\Big)^{\nu} \le \frac{\mu(B(x,r'))}{\mu(B(x,r))}. \qquad \blacksquare$$

Proposition 2.1. *Let (X, d, μ) satisfies $(\mathbf{D})^{\eta}$. Then for any $x, y \in M$, $r, r' > 0$ such that $B(y, r) \subset B(x, r')$, we have*

$$\frac{\mu(B(x,r'))}{\mu(B(y,r))} \leq A^2\left(\frac{r'}{r}\right)^{\eta}. \tag{2.5}$$

This is a classical result. The proof is similar to what we will do to prove Proposition 2.2.

Definition 2.3. A metric measure space (X, d, μ) *satisfies the R-doubling property* $(\mathbf{D})^{\eta}_{R}$ if there is some constant $A > 0$ such that (2.1) holds for all $x \in M$ and $0 < r \leq r' \leq 2R$. This is equivalent to (2.2) being true for all ball B with radius less than R.

X satisfies the *R-reverse doubling property* $(\mathbf{RD})^{\nu}_{R}$ if (2.4) holds for all balls of radius less than R (this is equivalent to (2.3) being true for all $x \in X$ and $0 < r \leq r' \leq 2R$).

We will write A_R for the doubling constant when will be important to precise which R the constant is associated with.

Some care is needed to get precisely those maximal radius. That (2.2) follows from (2.1) is immediate.

Proof of (2.2) *implies* (2.1). Suppose that there is some constant A such that for all ball B of radius less than R. Then, $\mu(2B) \leq A\mu(B)$. Let $r \leq r' \leq 2R$, $k = \lfloor \log_2 \frac{r'}{r} \rfloor$.

We have

$$2^{-k-1}r' < r \leq 2^{-k}r',$$

and, using repeatedly the doubling inequality $\mu(B(x,\rho)) \leq A\mu(B(x,\rho/2))$, valid for all $\rho \leq 2R$, we have

$$\begin{aligned}
\mu(B(x,r')) &\leq A^{k+1}\mu(B(x, 2^{-k-1}r')) \\
&\leq A^{k+1}\mu(B(x,r)) \\
&\leq Ae^{(\log A \log \frac{r'}{r})/\log 2}\mu(B(x,r)) \\
&\leq A\left(\frac{r'}{r}\right)^{\eta}\mu(B(x,r)),
\end{aligned}$$

with $\eta = \log_2 A$. ■

Proposition 2.2. *Let X satisfies $(\mathbf{D})^{\eta}_{R}$, then for all $x, y \in X$, $r, r' > 0$ such that $B(y, r) \subset B(x, r')$ and with $r' < R$, then for $\eta = \log_2 A$,*

$$\frac{\mu(B(x,r'))}{\mu(B(y,r))} \leq A^2\left(\frac{r'}{r}\right)^{\eta}.$$

If, in addition, X satisfies $(\mathbf{RD})_R^\nu$*, then we also have for some constant* $c > 0$*, that for all* $0 < r, r' < R$ *and* $B(y, r) \subset B(x, r')$*,*

$$c\left(\frac{r'}{r}\right)^\nu \leq \frac{\mu(B(x,r'))}{\mu(B(y,r))}.$$

Proof. For the first part, we simply use $B(x, r) \subset B(y, 2r)$ then applies (2.1).

For the second part, since $B(x, r') \subset B(y, 2r')$, we can use (2.5) and we get

$$\begin{aligned} \frac{\mu(B(x,r'))}{\mu(B(y,r))} &= \frac{\mu(B(y,r'))}{\mu(B(y,r))}\frac{\mu(B(x,r'))}{\mu(B(y,r'))} \\ &\geq a\left(\frac{r'}{r}\right)^\nu \frac{\mu(B(x,r'))}{\mu(B(y,2r'))} \\ &\geq aA^{-2}2^{-\eta}\left(\frac{r'}{r}\right)^\nu. \end{aligned}$$

■

We now suppose that (X, d) is a *path metric space*, i.e. that the distance $d(x, y)$ is realised as the infimum of the length of continuous path with end points x and y. We will keep making this assumption in everything that follows. (Most results are still true in a more general setting, but this simplify some proofs and is sufficient for our purposes.)

Proposition 2.3. *Let X be a metric space satisfying* $(\mathbf{D})_R^\eta$*. Assume that the annuli* $B(x, r') \setminus B(x, r)$*, for any* r, r' *with* $0 \leq r < r' \leq R$*, are all non empty. Then, there is some* $\nu > 0$ *such that X satisfies* $(\mathbf{RD})_{R/2}^\nu$*.*

Proof. Let $x \in X$, $r < R/2$. Take $y \in B(x, 7r/4) \setminus B(x, 5r/4)$ (which is non empty as $7r/4 \leq R$). Then,

$$B(y, r/4) \subset B(x, 2r) \setminus B(x, r).$$

Therefore,

$$\begin{aligned} \mu(B(x,2r)) &\leq A^2 8^\eta \mu(B(y,r/4)), \\ \mu(B(y,r/4)) &\leq \mu(B(x,2r)) - \mu(B(x,r)). \end{aligned}$$

So, with $C = A^2 8^\eta$,

$$(1 + C^{-1})\mu(B(x,r)) \leq \mu(B(x,2r)).$$

Thus, the measure satisfies the R-reverse doubling property. ■

The R-doubling also implies some upper bound on the volume of balls of large radius. The two following propositions, and their proof, are taken from [14].

Proposition 2.4. *If (X, d, μ) is a path metric space satisfying $(\mathbf{D})_R^\eta$, then there is some $C > 0$ that depends only on the doubling constant and order, such that we have, for any $r > 0$, $R' \leq R$,*

$$\mu(B(x, r + R'/4)) \leq C\mu(B(x, r)).$$

Proof. The case $r \leq R$ is obvious by the doubling property. For $r > R$, then let $\{x_i\}_i$ be a maximal family in $B(x, r - R/4)$ such that for any $i \neq j$, $d(x_i, x_j) > R'/2$. Then the balls $B(x_i, R'/4) \subset B(x, r)$ are disjoints, and the balls $B(x_i, R')$ cover $B(x, r + R'/4)$, since a point of $B(x, r + R'/4)$ is at distance at most $R'/2$ of $B(x, r - R'/4)$ (this because (X, d) is a path-metric space). Thus,

$$\begin{aligned}\mu(B(x, r + R'/4)) &\leq \sum_i \mu(B(x_i, R')) \leq A^2 \sum_i \mu(B(x_i, R'/4)) \\ &\leq A^2 \mu(B(x, r)).\end{aligned}$$

■

Proposition 2.5. *If (X, d, μ) satisfies $(\mathbf{D})_R^\eta$, then there is a $D > 0$, that depends only on the doubling constants, such that, for any $r > 0$,*

$$\mu(B(x, r)) \leq e^{D\frac{r}{R}} \mu(B(x, R)). \tag{2.6}$$

Proof. Let $r > R$, $k = \lfloor 4\frac{r-R}{R} \rfloor$. We have

$$\mu(B(x, r)) \leq \mu(B(x, R + (k + 1)R/4)).$$

Thus, by Proposition 2.4, $\mu(B(x, r)) \leq C^{k+1}\mu(B(x, R))$. Moreover, $k + 1 \leq 4\frac{r}{R} - 3 \leq 4\frac{r}{R}$, and so

$$\mu(B(x, r)) \leq \exp\Big(4 \ln(C) \frac{r}{R}\Big) \mu(B(x, R)).$$

Therefore, we get (2.6) with $D = 4 \ln(C)$.

If $r \leq R$, then

$$\mu(B(x, r)) \leq \mu(B(x, R)) \leq e^{D\frac{r}{R}} \mu(B(x, R)),$$

and thus (2.6) still holds. ■

Similarly to how we always use A for the doubling constant, D will always be used for this constant $D = 8 \log A$.

Proposition 2.6. *Let X satisfies $(\mathbf{D})_R^\eta$ and let $r \leq R$. There exists a constant $C > 0$, that depends only on the doubling constant and order, such that, for any $x, y \in X$, $\mu(B(x, r)) \leq Ce^{D\frac{d(x,y)}{r}} \mu(B(y, r))$.*

Proof. We have the inclusion $B(x,r) \subset B(y, r + d(x,y)) \subset B(y, R + d(x,y))$. Then, by Proposition 2.4,

$$\mu(B(x,r)) \leq A^8 \mu(B(y, d(x,y))),$$

and so, using Proposition 2.5,

$$\mu(B(x,r)) \leq C e^{D \frac{d(x,y)}{R}} \mu(B(y,r)) \leq C e^{D \frac{d(x,y)}{r}} \mu(B(y,r)). \qquad \blacksquare$$

Proposition 2.7. *If (X,d,μ) satisfies $(\mathbf{D})^\eta_R$, then it also satisfies $(\mathbf{D})^\eta_{R'}$ for any $R' > 0$, with a doubling constant $A_{R'} = A_R$ if $R' \leq R$, and $A_{R'} = e^{2D\frac{R'}{R}}$ if $R' > R$.*

Proof. The case $R' \leq R$ is obvious. Thus, assume $R > R'$. Let $r \leq R'$. If $r \leq R$, then the result is trivial since $A_R \leq A_{R'}$. If $r > R$, then, by Proposition 2.5,

$$\mu(B(x,2r)) \leq e^{2D\frac{r}{R}} \mu(B(x,r))$$

Since $e^{2D\frac{r}{R}} \leq e^{2D\frac{R'}{R}}$, we conclude that μ is R'-doubling, with a doubling constant $A_{R'} = e^{2D\frac{R'}{R}}$. $\blacksquare$

With this, we can generalise Proposition 2.6 for any $r > 0$: if $r > R$, we can use the r-doubling and apply Proposition 2.6 for it. The constants are $A_r = e^{2D\frac{r}{R}}$, $D_r = 4\log(A_r^2) = 16D\frac{r}{R}$, $A_r^8 = e^{16D\frac{r}{R}}$. Then we have, for any $x, y \in X$, $r > 0$,

$$\mu(B(x,r)) \leq e^{16D\frac{r+d(x,y)}{R}} \mu(B(y,r)).$$

Proposition 2.8. *Let (X,d,μ) be a metric measure space that satisfies $(\mathbf{D})^\eta_R$. If it also satisfies $(\mathbf{RD})^\nu_R$, then for any $\kappa > 1$, it satisfies $(\mathbf{RD})^\nu_{\kappa R}$ with a different reverse doubling constant, that depends only on the doubling and reverse doubling constants, and on κ.*

The notable part of this proposition is that the reverse doubling order is the same.

Proof. By Proposition 2.7, μ is κR-doubling for all κ, with some doubling order $\eta = \eta(\kappa)$. We take a point $x \in M$, and r, r' with $0 < r \leq r' \leq \kappa R$. We want to prove that there is some constant a_κ such that, for any such x, r, r',

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \geq a_\kappa \left(\frac{r'}{r}\right)^\nu.$$

If $0 < r \leq r' \leq R$, then there is nothing to do but apply $(\mathbf{RD})^\nu_R$. If $0 < r \leq R < r' \leq \kappa R$, then

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \geq \frac{\mu(B(x,R))}{\mu(B(x,r))} \geq a\left(\frac{R}{r}\right)^\nu \geq a\kappa^{-\nu}\left(\frac{r'}{r}\right)^\nu.$$

Finally, when $R < r \leq r' < \kappa R$, then

$$\frac{\mu(B(x,r'))}{\mu(B(x,r))} \geq \frac{\mu(B(x,\frac{r'}{\kappa}))}{A\kappa^{\eta}\mu(B(x,\frac{r}{\kappa}))} \geq \frac{a}{A\kappa^{\eta}}\Big(\frac{r'}{r}\Big)^{\nu}$$

Thus, (2.2) holds for $a_\kappa = \min(a, a\kappa^{-\nu}, aA^{-1}\kappa^{-\eta}) = aA^{-1}\kappa^{-\eta}$. ∎

Proposition 2.9. *Let (X,d,μ) satisfies $(\mathbf{D})_R^{\eta}$. Take $x \in X$, $r > 0$, and let $B = B(x,r)$. Let δ be such that $0 < \delta \leq \min(r, R)$, and $\{x_i\}_i \subset B$ be a family of points such that the balls $B_i = B(x_i,\delta)$ form a covering of B and that for any $i \neq j$, $\frac{1}{2}B_i \cap \frac{1}{2}B_j = \emptyset$.*

Then, there are constants C, c, depending only on the doubling constant, such that

$$\operatorname{card}(I) \leq Ce^{c\frac{r}{\delta}}.$$

Proof. For any i, $B_i \subset B(x, r+\delta)$, and since $\delta \leq R$, then we can use Proposition 2.4 to get

$$\mu(B(x,r+\delta)) \leq C\mu(B(x,r)).$$

Now, if $r > R$, then by Proposition 2.5, since $\delta \leq R$ then μ is δ doubling with the same doubling constant as that of the R-doubling, and

$$\mu(B(x,r)) \leq e^{D\frac{r}{\delta}}\mu(B(x,\delta))$$

Moreover, by Proposition 2.6,

$$\mu(B(x,\delta)) \leq Ce^{D\frac{d(x,x_i)}{\delta}}\mu(B(x_i,\delta)) \leq Ce^{D\frac{r}{\delta}}\mu(B_i),$$

using that, since $x_i \in B$, then $d(x,x_i) \leq r$. Thus, we have $\mu(B(x,r)) \leq Ce^{2D\frac{r}{\delta}}\mu(B_i)$, and the constant C depends only on the doubling constants. We then have

$$\begin{aligned}(\operatorname{card} I)\mu(B(x,r+\delta)) &\leq Ce^{2D\frac{r}{\delta}}\sum_{i\in I}\mu(B_i)\\ &\leq ACe^{2D\frac{r}{\delta}}\sum_{i}\mu\Big(\frac{1}{2}B_i\Big)\\ &\leq Ce^{2D\frac{r}{\delta}}\mu(B(x,r+\delta)).\end{aligned}$$

Thus, $\operatorname{card}(I) \leq Ce^{2D\frac{r}{\delta}}$ and the constant C depends only on the doubling constants. ∎

Remark. For any ball B, such a covering always exists: take for $\{x_i\}_i \subset B$ a maximal family with $d(x_i,x_j) \geq \delta$ for any $i \neq j$.

Proposition 2.10. *Let M_R be the centered maximal function defined by*

$$M_R f(x) = \sup_{r<R} \fint_{B(x,r)} |f| \, \mathrm{d}\mu \quad \text{for all } f \in L^1_{\mathrm{loc}}(M).$$

If μ satisfies $(\mathbf{D})^\eta_R$, then $M_{R/2}$ is bounded on L^p for all $p \in (1, +\infty]$, and the operator norm is bounded by a constant that only depends on the doubling constant A and on p.

We will use the following classical results:

Lemma 2.1 (Vitali's covering lemma). *Let (X, d) be a separable metric space, and $\{B_j\}_{j\in J}$ a collection of balls, such that $\sup_j r(B_j) < \infty$. For any $c > 3$, there exists a subcollection $\{B_{j_n}\}_{n\in\mathbf{N}} \subset \{B_j\}_{j\in J}$ such that the B_{j_n} are pairwise disjoint and $\bigcup_{j\in J} B_j \subset \bigcup_{n\in\mathbf{N}} cB_{j_n}$.*

Theorem 2.2 (Marcinkiewicz interpolation theorem). *Let (X, μ) be a measure space, and let T be a sublinear operator acting on functions, i.e., there is a $\kappa > 0$ such that for any f, g measurable. Then, Tf and Tg are measurable and $T(f + g)(x) \le \kappa(Tf(x) + Tg(x))$ for almost every $x \in X$.*

Let $1 \le p < r \le \infty$. If $r < \infty$, assume that

$$\mu\{x \in X : Tf(x) > \lambda\} \le \frac{A}{\lambda^p} \|f\|_p^p \quad \text{for all } f \in L^p;$$

$$\mu\{x \in X : Tf(x) > \lambda\} \le \frac{B}{\lambda^r} \|f\|_r^r \quad \text{for all } f \in L^r.$$

If $r = \infty$, then assume instead that

$$\mu\{x \in X : Tf(x) > \lambda\} \le \frac{A}{\lambda^p} \|f\|_p^p \text{ for all } f \in L^p;$$

$$|Tf(x)| \le B|f(x)|, \quad a.e.\ x \in X \text{ for all } f \in L^\infty.$$

Then, for every $s \in (p, r)$, for all $f \in L^s$, $Tf \in L^s$ and

$$\|Tf\|_s \le C(A, B, p, r, s, \kappa)\|f\|_s.$$

Proof of the Proposition 2.10. We have, for any $f \in L^\infty(M)$, $\|M_R f\|_\infty \le \|f\|_\infty$. If $f \in L^1(M)$, then, for any $\lambda > 0$, define

$$E_\lambda = \{x \in M : M_{R/2} f(x) > \lambda\}.$$

If $x \in E_\lambda$, then there is some $r_x > 0$ such that $\lambda < \fint_{B(x,r_x)} |f| \, \mathrm{d}\mu$, and $2r_x \le R$. Therefore,

$$\mu(B(x, r_x)) \le \lambda^{-1} \int_{B(x,r)} |f| \, \mathrm{d}\mu.$$

We have $E_\lambda \subset \bigcup_x B(x, r_x)$, thus, by Vitali's covering lemma, there is a subcollection $\{x_n\}$ such that the $B(x_n, r_n)$ are pairwise disjoint and $E_\lambda \subset \bigcup_n B(x_n, 4r_n)$. Also, since $r_n < R/2$, and μ is R-doubling, we have $\mu(B(x_n, 4r_n)) \leq A^2 \mu(B(x_n, r_n))$. Then,

$$\begin{aligned}\mu(E_\lambda) &\leq \sum_n \mu(B(x_n, 4r_n)) \leq A^2 \sum_n \mu(B(x_n, r_n)) \\ &\leq A^2 \lambda^{-1} \sum_n \int_{B(x_n, r_n)} |f| \,\mathrm{d}\mu \leq A^2 \frac{\|f\|_1}{\lambda}.\end{aligned}$$

So, by the Marcinkiewicz interpolation theorem, for any $p \in (1, +\infty)$, $M_{R/2}$ is bounded on L^p with an operator norm $\|M_{R/2}\|_{p\to p} \leq C_p$, with C_p depending only on A and p. ∎

Remark. Of course, $(\mathbf{D})_R^\eta$ implies $(\mathbf{D})_{R'}^\eta$ for all $R' > R$. Then, M_R itself is also bounded, but with the constant C_p depending on the constant for $(\mathbf{D})_{2R}^\eta$. And so are all the $M_{R'}$ with $R' > R$, with the constant C_p depending on p, the R-doubling constant, and the ratio R'/R.

Proposition 2.11. *Let $\widetilde{M}_R$ the uncentered maximal function defined by*

$$\widetilde{M}_R f(x) = \sup_{\substack{x \in B,\\ r(B) \leq R}} \fint_B |f| \,\mathrm{d}\mu \quad \text{for all } f \in L^1_{\mathrm{loc}}(M),$$

with this supremum to be interpreted as being over all balls B satisfying the given condition, and $r(B)$ being the radius of B.

Then, if μ is R-doubling, there exist some constant $C > 0$ such that $M_R \leq \widetilde{M}_R \leq C M_{2R}$.

Proof. Since a ball centered at x is a ball containing x, $M_R \leq \widetilde{M}_R$ is obvious. Now, for some balls $B = B(y, r)$ containing x, with radius less than R, we have $B \subset B(x, 2r)$ and

$$\fint_B |f| \,\mathrm{d}\mu \leq \frac{\mu(B(x, 2r))}{\mu(B)} \fint_{B(x,2r)} |f| \,\mathrm{d}\mu \leq C M_{2R} f(x).$$ ∎

Proposition 2.12. *Let (X, d, μ) be a separable metric measure space, and $\mathfrak{D}_m$ be a chosen construction of dyadic cubes on X. Define the associated dyadic maximal function $M_{d,m}$ by*

$$M_{d,m} f(x) = \sup_{\substack{Q \in \mathfrak{D}_m\\ x \in Q}} \fint_Q |f| \,\mathrm{d}\mu.$$

Then, there is a constant C_p such that for any $p > 1$, for any $f \in L^p$, $\|M_{d,m} f\|_p \leq C_p \|f\|_p$.

As a consequence, $M_{d,m,l}$, the maximal function defined the same way, but with the cubes in the supremum being only those of length less than l, is also bounded on L^p for all $p > 1$.

Proof. Let $f \in L^1(X)$, $\lambda > 0$. We define

$$E_\lambda = \{x \in X : M_{d,m} f(x) > \lambda\}.$$

If $x \in E_\lambda$, then there is a cube $Q \in \mathcal{D}_m$ such that $⨍_Q |f| \,\mathrm{d}\mu > \lambda$, and so $Q \subset E_\lambda$. Then there are two possibilities.

- If there is a maximal dyadic cube P containing x such that $⨍_P |f| \,\mathrm{d}\mu > \lambda$. then this cube satisfies $P \subset E_\lambda$.
- If there is no such cube (in which case, x is in a region of space with infinite diameter but finite measure), then define $\Omega = \bigcup_{Q \in \mathcal{D}_m,\, x \in Q} Q$. We can always find an arbitrarily large cube containing x which is a subset of E_λ, and so $\Omega \subset E_\lambda$, and $\mu(\Omega) \leq \lambda^{-1} \int_\Omega |f| \,\mathrm{d}\mu < \infty$.

Le $\{Q_i\}_i$ be the family of all the maximal dyadic cubes such that $⨍_{Q_i} |f| \,\mathrm{d}\mu > \lambda$ and $\{\Omega_j\}_j$ be the family of all the regions $\Omega_j = \bigcup_k Q_k^j$, where $\{Q_k^j\}$ is an infinite increasing sequence of cubes with $⨍_{Q_j^k} |f| \,\mathrm{d}\mu > \lambda$. The Q_i, Ω_j are pairwise disjoints. First it is clear by maximality that the Q_i are, Then, if, for a cube Q, we have $Q \cap \Omega_j \neq \emptyset$, then there is a cube $P \subset \Omega_j$ such that $P \cap Q \neq \emptyset$, thus we have either $P \subset Q$ or $Q \subset P$. In both case, $Q \subset \Omega_j$ since Ω_j is the union of all cubes containing P. This mean both that $Q_i \cap \Omega_j = \emptyset$ for all i, j, and that $\Omega_j \cap \Omega_l = \emptyset$ for $j \neq l$.

Thus, we have the disjoint union

$$E_\lambda = \bigcup_i Q_i \cup \bigcup_j \Omega_j,$$

Then $\mu(Q_i) < \lambda^{-1} \int_{Q_i} |f| \,\mathrm{d}\mu$, and $\mu(\Omega_j) \leq \lambda^{-1} \int_{\Omega_j} |f| \,\mathrm{d}\mu$. Summing on all cubes and all regions, $\mu(E_\lambda) \leq \lambda^{-1} \int_{E_\lambda} |f| \,\mathrm{d}\mu \leq \lambda^{-1} \|f\|_1$. Thus,

$$\mu(\{x \in X : M_{d,m} f(x) > \lambda\}) \leq \frac{\|f\|_1}{\lambda}.$$

Moreover, for $f \in L^\infty(X)$, we clearly have $M_{d,m} f(x) \leq \|f\|_\infty$. Then, by Marcinkiewicz interpolation theorem, for any $p > 1$ there is a constant $C_p > 1$ such that $\|M_{d,m} f\|_p \leq C_p \|f\|_p$. ■

2.3. Estimates of operator norms by that of a maximal function

We refers to the works of C. Pérez and R. L. Wheeden [22] for a more general approach. We will first describe one of their result in the more specific context that is of interest to us; then, will give a generalization of this result that holds on a R-doubling space.

In what follows, we let (X, d) be a separable R-doubling metric space. We take T an operator given by a kernel $K\colon X \times X \setminus \mathrm{Diag} \to \mathbf{R}$, i.e.,

$$Tf(x) = \int_X f(y)K(x,y)\,\mathrm{d}\mu(y). \tag{2.7}$$

We say that the operator T, or its kernel K, *satisfies the condition* (**K**) if K is non negative and if there are constants $C_1, C_2 > 1$ such that

$$\begin{aligned} d(x',y) \le C_2 d(x,y) &\implies K(x,y) \le C_1 K(x',y), \\ d(x,y') \le C_2 d(x,y) &\implies K(x,y) \le C_1 K(x,y'). \end{aligned} \tag{2.8}$$

We take $\rho > 1$ such as, by Theorem 2.1, for any integer $m \in \mathbf{Z}$, we have a decomposition of X in dyadic cubes $\mathcal{D}_m$ of lengths ρ^ℓ, $\ell \ge m$. We define φ as the following functional on balls:

$$\varphi(B) = \sup_{\substack{x,y \in B \\ d(x,y) \ge \frac{1}{2\rho} r(B)}} K(x,y);$$

and M_φ to be the following maximal functions:

$$M_\varphi f(x) = \sup_{x \in B} \varphi(B) \int_B |f|\,\mathrm{d}\mu.$$

We want to establish an inequality of the type $\|Tf\|_p \le C_p \|M_\varphi f\|_p$, as the later can be more convenient to estimate.

For T satisfying (**K**), it is shown in [25, (4.3)] that φ is decreasing in the following sense:

Proposition 2.13. *There is a constant α, depending only on C_1, C_2, ρ such that for any balls $B \subset B'$, $\varphi(B') \le \alpha\varphi(B)$*

Proof. First, we want to prove that if (2.8) holds, then, for any $C_2 > 1$, there exist a corresponding C_1 such that (2.8) holds with those new constants. We can of course replace C_2 by a smaller constant. To replace it with a smaller, we show that, for any integer $k \ge 1$,

$$d(x',y) \le C_2^k d(x,y) \implies K(x,y) \le C_1^k K(x',y),$$

and that the same holds with (x, y') replacing (x', y).

We proceed by induction. The case $k = 1$ is simply (2.8).

Let $k > 2$. Take $x, x', y \in X$ such that $d(x', y) \le C_2^k d(x, y)$, and suppose that

$$d(x', y) \le C_2^{k-1} d(x, y) \implies K(x, y) \le C_1^{k-1} K(x' y).$$

If $d(x', y) \le C_2^{k-1} d(x, y)$, then the result holds and there is nothing to prove. If $d(x', y) > C_2^{k-1} d(x, y)$, then X is a path metric space, so there is a path from y to x' of length $d(x', y)$, and on this path is a point z such that $d(y, z) = C_2^{k-1} d(x, y)$. But then

$$d(x', y) \le C_2^k d(x, y) = C_2 d(z, y);$$

thus $K(z, y) \le C_1 K(x', y)$.

By induction, we proved that $K(x, y) \le C_1^k K(x', y)$ for all x, x', y with $d(x', y) \le C_2^k d(x, y)$. It follows that, if (2.8) holds, then for any $C_2 > 1$ there exist a $C_1 > 1$ such that (2.8) holds.

Now, we can prove the proposition proper. Take $x', y' \in B'$, $x, y \in B$ such that

$$d(x', y') \ge c r(B'), \qquad d(x, y) \ge c r(B),$$

with $c = \frac{1}{2\rho}$. By exchanging x' and y' if necessary, we can suppose that $d(x, y') \ge d(x, x')$. Then

$$c r(B') \le d(x', y') \le d(x', x) + d(x, y') \le 2d(x, y').$$

Moreover, since $B \subset B'$, we have $d(x, y') \le 2r(B')$, and thus

$$d(x, y') \le \frac{2}{c} d(x', y'),$$

So, by (2.8), there is a constant $c_1 > 1$ such that $K(x', y') \le c_1 K(x, y')$.

Moreover,

$$d(x, y) \le d(x, y') + d(y', y) \le d(x, y') + 2r(B') \le (1 + 4/c) d(x, y').$$

Therefore, by (2.8), there is a constant $c_2 > 1$ such that $K(x, y') \le c_2 K(x, y)$. Thus,

$$K(x', y') \le c_1 c_2 K(x, y),$$

and we have $\varphi(B') \le c_1 c_2 \varphi(B)$. ■

We further assume that φ satisfies the following condition: there is some $\varepsilon > 0$ and some constant $L > 0$ such that, for any balls B_1, B_2, with $B_1 \subset B_2$, we have

$$\varphi(B_1)\mu(B_1) \le L \left(\frac{r(B_1)}{r(B_2)} \right)^{\varepsilon} \varphi(B_2)\mu(B_2). \tag{2.9}$$

Theorem 2.3 (C. Pérez and R. L. Wheeden [22]). *Let (X, d, μ) be a metric space with a doubling measure μ. Let T be an operator defined by (2.7) and satisfying* (**K**), *with φ satisfying (2.9). Then there is a constant C, depending only on the doubling constant and p, such that, for any measurable $f: X \to \mathbf{R}$,*

$$\|Tf\|_p \leq C \|M_\varphi f\|_p.$$

In addition, for the operator $Tf(x) = \int_M \frac{d(x,y)^s}{\mu(B(x,d(x,y)))} f(y) \, \mathrm{d}\mu(y)$, we can replace M_φ by the maximal function defined by $M_s f(x) = \sup_{r>0} r^s ⨍_{B(x,r)} |f| \, \mathrm{d}\mu$. See Corollary 2.1 for the justification.

This theorem is useful, but cannot be applied to spaces that are only R-doubling. We will now prove a version that we can use in R-doubling spaces.

We consider the operator T_δ, $\delta < R$, with kernel $K_\delta(x, y) = K(x, y)\chi_{\{d(x,y)<\delta\}}$, and we want to compare its L^p norm to that of the maximal function $M_{\varphi,\delta}$ defined by

$$M_{\varphi,\delta} f(x) = \sup_{\substack{x \in B \\ r(B)<\delta}} \varphi(B) \int_B |f| \, \mathrm{d}\mu.$$

The idea of the proof of this comparison will be essentially the same as that of Theorem 2.3 given in [22], but some care must be taken to account for the different hypotheses properly, and thus we will give the details in what follows.

The hypothesis to prove $\|Tf\|_p \leq C\|M_{\varphi,\delta} f\|_p$ can be weakened compared to those of Theorem 2.3. A key point is that Proposition 2.13 has to hold at least for balls of radius at most 2δ. Looking at the proof of the proposition, this is true as long as (2.8) holds for $C_2 \leq (1 + 8\rho)$ and $d(x, y) \leq 4\delta$.

Then we take (X, d, μ) a R-doubling space. T an operator defined by a kernel K. We say that T, or K, *verifies the condition* $(\mathbf{K})_\delta$, if there exist constants $C_1 > 1$, $C_2 \geq 1 + 8\rho$, such that for any x, y such that $d(x, y) \leq 4\delta$, we have

$$d(x', y) \leq C_2 d(x, y), \quad K(x, y) \leq C_1 K(x', y) \quad \text{for all } x' \in X;$$
$$d(x, y') \leq C_2 d(x, y), \quad K(x, y) \leq C_1 K(x, y') \quad \text{for all } x' \in X.$$

Property $(\mathbf{K})_\delta$ ensure that 2.13 holds for balls of radius less than 2δ.

Since we will end up considering balls of a radius slightly larger than δ, the following proposition will be useful.

Proposition 2.14. *Let (X, d, μ) satisfies $(\mathbf{D})^\eta_{2(2\kappa+1)\delta}$ for $\delta > 0$, $\kappa > 1$, T an operator satisfying $(\mathbf{K})_{4(2\kappa+1)\delta}$, and such that the associated functional φ satisfies (2.9) when $r(B_1), r(B_2) \leq 2(2\kappa + 1)\delta$. Then, for any $p \in (1, \infty]$, there is some constant C depending only on p, κ, the doubling constants, and the constants α, L, ε, in Proposition 2.13 and in (2.9), such that for any non negative f, $\|M_{\varphi,\kappa\delta} f\|_p \leq C\|M_{\varphi,\delta} f\|_p$.*

Proof. We have

$$\begin{aligned} M_{\varphi,\kappa\delta} f(x) &= M_{\varphi,\delta} f(x) + \sup_{\substack{x\in B,\\ \delta<r(B)\le\kappa\delta}} \varphi(B) \int_B |f| \,\mathrm{d}\mu \\ &\le M_{\varphi,\delta} f(x) + C \sup_{\substack{x\in B,\\ r(B)=\kappa\delta}} \varphi(B) \int_{B(x,2\kappa\delta)} |f| \,\mathrm{d}\mu. \end{aligned}$$

Using that for $x \in B$, $B \subset B(x, 2r(B)) \subset B(x, 2\kappa\delta)$ and that for any ball B with radius greater than δ, by (2.9) (on balls with radius at most $\kappa\delta$), we have

$$\varphi(B) \le AL\kappa^{\eta}\varphi\Big(\frac{\kappa\delta}{r(B)}B\Big).$$

Now, for any ball B containing x with radius equal to $\kappa\delta$, let $B_x = B(x,\delta)$. For $y \in 2\kappa B_x$, consider the ball $Q(y) = B(y,\delta)$. We have $Q(y) \subset (2\kappa+1)B_x$; thus, using $(\mathbf{D})^{\eta}_{(2\kappa+1)\delta}$, we have that

$$\mu(2\kappa B_x) \le A^2(2\kappa+1)^{\eta}\mu(Q(y)).$$

For $y \in (2\kappa+1)B_x$, we also have that $B \subset B(z, 2(2\kappa+1)\delta)$; thus, using (2.9) (for balls with radius at most $2(2\kappa+1)\delta$, $(\mathbf{D})^{\eta}_{2(2\kappa+1)}$, and $(\mathbf{K})_{4(2\kappa+1)\delta}$), we get that

$$\varphi(B) \le A^2\Big(\frac{2(2\kappa+1)}{2\kappa}\Big)^{\eta}\alpha\varphi(Q(y)).$$

Putting all this together, we get

$$\begin{aligned} \varphi(B)\int_{B(x,2\kappa\delta)} |f| \,\mathrm{d}\mu &= \varphi(B) \fint_{2\kappa B_x} \mu(2\kappa B_x)|f| \,\mathrm{d}\mu \\ &\le C\varphi(B) \fint_{2\kappa B_x} \mu(Q(y))|f(y)| \,\mathrm{d}\mu(y) \\ &\le C \fint_{2\kappa B_x} \varphi(B) \int_{Q(y)} \mathrm{d}\mu(z)|f(y)| \,\mathrm{d}\mu(y) \\ &\le C\frac{1}{\mu(B(x,2\kappa\delta))} \int_{(2\kappa+1)B_x} \varphi(B) \int_{2\kappa B_x\cap B(z,\delta)} |f(y)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \\ &\le CA\Big(\frac{2\kappa+1}{2\kappa}\Big)^{\eta} \fint_{(2\kappa+1)B_x} \varphi(B(z,\delta)) \int_{B(z,\delta)} |f(y)| \,\mathrm{d}\mu(y)\,\mathrm{d}\mu(z) \\ &\le C \fint_{(2\kappa+1)B_x} M_{\varphi,\delta} f \,\mathrm{d}\mu, \end{aligned}$$

and the constant C depends only on the doubling constants, L, α and κ. So,

$$M_{\varphi,\kappa\delta} f(x) \leq M_{\varphi,\delta} f(x) + C M_{(2\kappa+1)\delta}(M_{\varphi,\delta} f)(x).$$

The theorem follows then from the boundedness of the classical maximal function $M_{(2\kappa+1)\delta}$ on any L^p, $p > 1$, under $(\mathbf{D})^{\eta}_{2(2\kappa+1)\delta}$. ∎

Theorem 2.4. *Let $\delta > 0$. Let $\rho > 0$ be the sidelength constant of dyadic cubes. Suppose that (X, d, μ) satisfies $(\mathbf{D})^{\eta}_{2(6\rho+1)\delta}$. Assume that K satisfies $(\mathbf{K})_{4(6\rho+1)\delta}$, and that φ satisfies (2.9) for balls with radius at most $2(6\rho+1)\delta$. Let $p \geq 1$. Then, there is a constant $C > 0$, depending only on the doubling constants, ρ, p and of the constants in (2.9) and (2.8), such that*

$$\int_X |T_\delta f|^p \, \mathrm{d}\mu \leq C \int_X (M_{\varphi,\delta} f)^p \, \mathrm{d}\mu. \tag{2.11}$$

Proof. We will show that there exist some constant $C > 0$ such that for any non negative function f, we have $\int_X |T_\delta f|^p \, \mathrm{d}\mu \leq C \int_X (M_{\varphi,3\rho\delta} f)^p \, \mathrm{d}\mu$. The theorem will follows by Proposition 2.14.

To prove this, we define, for any $m \in \mathbf{Z}$, the operator T_m by

$$T_m f(x) = \int_{d(x,y)>\rho^m} K_\delta(x, y) f(y) \, \mathrm{d}\mu(y).$$

If, for any $m \in \mathbf{Z}$, and for any non negative measurable functions f, g,

$$\int_X T_m f g \, \mathrm{d}\mu = \int_{d(x,y)>\rho^m} K_\delta(x, y) f(y) g(x) \, \mathrm{d}\mu(x, y) \leq C \|M_{\varphi,3\delta} f\|_p \|g\|_{p'},$$

then, by the monotone convergence theorem, taking $m \to -\infty$, the same inequality holds but with T_m replaced by T, and by duality, (2.11) is true.

Take $m \in \mathbf{Z}$. Let f, g be non negative measurable functions. Let $\mathcal{D}_m = \{\mathcal{E}^k_\alpha\}^{k\geq m}_{\alpha\in\mathbf{N}^*}$ be a decomposition of X in dyadic cubes given by Theorem 2.1 with sidelengths ρ^k. If $(x, y) \in X$ are such that $d(x, y) > \rho^m$, we take the integer $l \geq m$ such that

$$\rho^l < d(x, y) \leq \rho^{l+1}.$$

Let Q be the cube of length ρ^l containing x, $B(Q) = B(c_Q, \rho^{l+1})$ the containing ball. We recall that $\rho^{-1} B(Q) \subset Q \subset B(Q)$. We have

$$d(c_Q, y) \leq d(c_Q, x) + d(x, y) \leq 2\rho^{l+1},$$

thus $y \in 2B(Q)$. Since $d(x, y) > \rho^l = \frac{1}{2\rho} r(2B(Q))$, by definition of φ and by Proposition 2.13,

$$K(x, y) \leq \varphi(2B(Q)) \leq \alpha\varphi(B(Q)).$$

To apply Proposition 2.13, we need $(\mathbf{K})_{4\rho\delta}$. If we suppose that $\delta \le \rho^l = \ell(Q)$, then $d(x,y) \ge \delta$ and $K_\delta(x,y) = 0$. We have proved that if Q is the cube of length comparable with $d(x,y)$, containing x, we have $y \in 2B(Q)$ and

$$K_\delta(x,y) \le C\varphi(B(Q))\chi_{\{R\in\mathcal{D}_m,\ell(R)<\delta\}}(Q)\chi_Q(x)\chi_{2B(Q)}(y).$$

If r is the largest integer such that $\rho^r < \delta$, define $\mathcal{D}_m^r = \{\mathcal{E}_\alpha^k : m \le k \le r\}$. For any $x,y \in X$ with $d(x,y) > \rho^m$, there is at least one cube $Q \in \mathcal{D}_m$ such that the previous inequality holds, and since both sides of it are zero if $\ell(Q) \ge \delta$, we have, for any $x,y \in X$,

$$K_\delta(x,y) \le \sum_{Q\in\mathcal{D}_m^r} C\varphi(B(Q))\chi_Q(x)\chi_{2B(Q)}(y),$$

and so, for any $f, g \ge 0$,

$$\int_X T_m fg \,\mathrm{d}\mu \le C\sum_{Q\in\mathcal{D}_m^r}\varphi(B(Q))\int_{2B(Q)} f\,\mathrm{d}\mu\int_Q g\,\mathrm{d}\mu.$$

But, for any fixed integer $k \ge m$, the cubes of length of length ρ^k, $\{\mathcal{E}_\alpha^k\}$ are pairwise disjoints, and $X = \bigcup_\alpha \mathcal{E}_\alpha^k$. Then, using this decomposition for $k = r$,

$$\int_X T_m fg\,\mathrm{d}\mu \le C\sum_{\alpha\ge 1}\sum_{\substack{Q\in\mathcal{D}_m^r\\ Q\subset\mathcal{E}_\alpha^r}}\varphi(B(Q))\int_{2B(Q)} f\,\mathrm{d}\mu\int_Q g\,\mathrm{d}\mu.$$

Then, for a constant $\gamma \ge 1$ to be determined, for any $\alpha \ge 1$, and $n \in \mathbf{Z}$, define

$$\mathcal{C}_\alpha^n = \left\{Q \in \mathcal{D}_m^r, Q \subset \mathcal{E}_\alpha^r : \gamma^n < \frac{1}{\mu(B(Q))}\int_Q g\,\mathrm{d}\mu \le \gamma^{n+1}\right\}.$$

We let n_α be the unique integer such that $\mathcal{E}_\alpha^r \in \mathcal{C}_\alpha^{n_\alpha}$. Notice that $\{\mathcal{C}_\alpha^n\}_{n\in\mathbf{Z}}$ is a partition of $\{Q \in \mathcal{D}_m^r : Q \subset \mathcal{E}_\alpha^r\}$. We have

$$\int_X T_m fg\,\mathrm{d}\mu \le C\sum_{\alpha\ge 1}\sum_{n\in\mathbf{Z}}\gamma^{n+1}\sum_{Q\in\mathcal{C}_\alpha^n}\varphi(B(Q))\mu(B(Q))\int_{2B(Q)} f\,\mathrm{d}\mu.$$

For any $\alpha \ge 1$, we let $\{Q_{j,\alpha}^n\}_{j\in J_n}$, for some index set J_n, be the collection of the maximal dyadic cubes subset of $\mathcal{E}_\alpha^r$ such that

$$\gamma^n < \frac{1}{\mu(B(Q_{j,\alpha}^n))}\int_{Q_{j,\alpha}^n} g\,\mathrm{d}\mu.$$

If $n \leq n_\alpha$, then there is exactly one such maximal cube: $\mathcal{E}^r_\alpha$. Also, the function $(n, Q) \mapsto Q$ is an injection from the set of the couples (n, Q) with $n \leq n_\alpha$, $Q \in \mathcal{C}^n_\alpha$ to $\{Q \in D^r_m : Q \subset \mathcal{E}^r_\alpha\}$; thus,

$$\sum_{n \leq n_\alpha} \sum_{Q \in \mathcal{C}^n_\alpha} \gamma^{n+1} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu \leq \gamma^{n_\alpha+1} \sum_{\substack{Q \in \mathcal{D}^r_m \\ Q \subset \mathcal{E}^r_\alpha}} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu.$$

If $n > n_\alpha$, then any $Q^n_{j,\alpha}$ is a strict subset of $\mathcal{E}^r_\alpha$. For such a maximal cube $\mathcal{F}$, we let P be his dyadic parent i.e., the only cube of length $\rho\ell(\mathcal{F})$ containing P. We have $P \subset \mathcal{E}^r_\alpha$, and, by using the maximality of $\mathcal{F}$, and that $B(\mathcal{F}) \subset 2B(P)$, and, using the $\rho\delta$-doubling ($B(P)$ has radius less than $\rho\delta$),

$$\gamma^n < \frac{1}{\mu(B(\mathcal{F}))} \int_{\mathcal{F}} g \, \mathrm{d}\mu \leq \frac{\mu(B(P))}{\mu(B(\mathcal{F}))} \frac{1}{\mu(B(P))} \int_P g \, \mathrm{d}\mu \leq C\rho^\eta \gamma^n = \kappa \gamma^n, \quad (2.12)$$

with the constant κ depending only on ρ and on the doubling constant. Choosing $\gamma > \kappa$, we have

$$\frac{1}{\mu(B(\mathcal{F}))} \int_{\mathcal{F}} g \, \mathrm{d}\mu \leq \gamma^{n+1},$$

thus $\mathcal{F} \in \mathcal{C}^n_\alpha$. Therefore, for a fixed $n > n_\alpha$, every cube in $\mathcal{C}^n_\alpha$ is in a (unique) $Q^n_{j,\alpha}$, which are disjoint in j by maximality. So, writing $Q^{n_\alpha}_{j,\alpha}$ for $\mathcal{E}^r_\alpha$,

$$\int_X (T_m f) g \, \mathrm{d}\mu \leq C \sum_{\alpha \geq 1} \sum_{n \geq n_\alpha} \gamma^{n+1} \sum_{j \in J_n} \sum_{\substack{Q \in \mathcal{D}^m_\alpha \\ Q \subset Q^n_{j,\alpha}}} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu.$$

Now, we use the following lemma (see [22, Lemma 6.1]):

Lemma 2.2. *Let (X, d, μ) satisfies $(\mathbf{D})^\eta_\delta$. Let φ be a functional on balls that satisfies (2.9) for balls of radius at most $\rho\delta$. Then there is a constant C depending only on the constant L of (2.9) and on the doubling constant such that, for any $f \geq 0$ and any dyadic cube $Q_0 \in \mathcal{D}^r_m$, with $\rho^r \leq \delta$,*

$$\sum_{\substack{Q \in \mathcal{D}_m \\ Q \subset Q_0}} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q)} f \, \mathrm{d}\mu \leq C\varphi(B(Q_0))\mu(B(Q_0)) \int_{3B(Q_0)} f \, \mathrm{d}\mu.$$

Proof. By (2.9), we have

$$\begin{aligned}
&\sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0}} \varphi(B(Q))\mu(B(Q)) \int_{2B(Q)} f \,\mathrm{d}\mu \\
&\quad \le L\varphi(B(Q_0))\mu(B(Q_0)) \sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0}} \Big(\frac{\ell(Q)}{\ell(Q_0)}\Big)^{\varepsilon} \int_{2B(Q)} f \,\mathrm{d}\mu \\
&\quad \le L\varphi(B(Q_0))\mu(B(Q_0)) \sum_{l=0}^{+\infty} \rho^{-\varepsilon l} \sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0\\ \ell(Q)=\rho^{-l}\ell(Q_0)}} \int_{2B(Q)} f \,\mathrm{d}\mu. \qquad (2.13)
\end{aligned}$$

For $Q \in \mathcal{D}_m$, $Q \subset Q_0$, and $\ell(Q) \le \ell(Q_0)$ we have $2B(Q) \subset 3B(Q_0)$. Indeed, if $y \in 2B(Q)$, then

$$d(y, x_{Q_0}) \le d(y, x_Q) + d(x_Q, x_{Q_0}) \le 2r(B(Q)) + r(B(Q_0)) \le 3r(B(Q_0)).$$

Thus, the left-hand side of (2.13) is less than

$$L\varphi(B(Q_0))\mu(B(Q_0)) \int_{3B(Q_0)} f(x) \sum_{l=0}^{\infty} \rho^{-\varepsilon l} \sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0\\ \ell(Q)=\rho^{-l}\ell(Q_0)}} \chi_{2B(Q)}(x) \,\mathrm{d}\mu(x).$$

Then, it suffices to show that, for each l, any x of $3B(Q_0)$ is in at most N of the $2B(Q)$, with $\ell(Q) = \rho^{-l}\ell(Q_0)$, with N independent of the choices of x and Q_0. For $l = 0$, there is only one Q: Q_0 itself, and thus it is true.

Now, fix $l > 1$, let $x \in M$, and Q be a cube of sidelength $\rho^{-l}\ell(Q_0)$ such that $x \in 2B(Q)$. We write $\ell = \ell(Q) \le \rho^{-1}\delta$. Then, for $y \in Q$,

$$d(x, y) \le d(x, x_Q) + d(y, x_Q) \le 3\rho\ell \le 3\delta,$$

and so we have $B(x_Q, \ell) \subset Q \subset B(x, 3\rho\ell)$. By the Proposition 2.9, there can be at most N disjoint balls of radius $\ell \le \delta$ with center in a ball of radius $3\rho\ell$, with the constant N depending only on ρ and on the δ-doubling constant. Thus,

$$\sum_{l=0}^{\infty} \rho^{-\varepsilon l} \sum_{\substack{Q\in\mathcal{D}_m\\ Q\subset Q_0\\ \ell(Q)=\rho^{-l}\ell(Q_0)}} 1 \le N \frac{1}{1-\rho^{-\varepsilon}},$$

and the lemma follows. ∎

Applying the lemma, we get

$$\int_X (T_m f) g \, \mathrm{d}\mu \le C \sum_{\alpha\ge 1}\sum_{n\ge n_\alpha} \gamma^{n+1} \sum_{j\in J_n} \varphi(B(Q^n_{j,\alpha}))\mu(B(Q^n_{j,\alpha})) \int_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu,$$

and, thus, since $Q^n_{j,\alpha} \in \mathcal{C}^n_\alpha$, we have $\gamma^n \le \frac{1}{\mu(B(Q^n_{j,\alpha}))}\int_{Q^\alpha_{j,n}} g \, \mathrm{d}\mu$, and so

$$\int_X (T_m f) g \, \mathrm{d}\mu \le C\gamma \sum_{\alpha\ge 1}\sum_{n\ge n_\alpha}\sum_{j\in J_n} \varphi(B((Q^n_{j,\alpha}))) \int_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu,$$

and

$$\int_X (T_m f) g \, \mathrm{d}\mu \le c \sum_{\alpha,n,j} \varphi(B(Q^n_{j,\alpha}))\mu(Q^n_{j,\alpha}) \int_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu \frac{1}{\mu(Q^n_{j\alpha})} \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu.$$

Using Hölder's inequality, and that by (2.9) there is some constant c depending only on $\alpha, A, L, \varepsilon$ such that $\varphi(B) \le c\varphi(3B)$ (ball of radius $3\rho\delta$), we get

$$\int_X (T_m f) g \, \mathrm{d}\mu \le C\Big(\sum_{\alpha,n,j} \mu(Q^n_{j,\alpha})\Big(\varphi(B(3Q^n_{j,\alpha})) \int_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu\Big)^p\Big)^{\frac{1}{p}}$$
$$\times \Big(\sum_{\alpha,n,j} \mu(Q^n_{j,\alpha})\Big(\frac{1}{\mu(Q^n_{j\alpha})} \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu\Big)^{p'}\Big)^{\frac{1}{p'}}.$$

Now, we just need to establish a majoration of $\mu(Q^n_{j,\alpha})$ by a constant time the measure of a set $E^n_{j,\alpha}$, with the $E^n_{j,\alpha}$ being pairwise disjoint in j, n, α. For this, define Ω^n_α by

$$\Omega^n_\alpha = \Big\{ x \in \mathcal{E}^r_\alpha \colon \sup_{\substack{Q\in\mathcal{D}^r_m \\ x\in Q}} \frac{1}{\mu(B(Q))} \int_Q g \, \mathrm{d}\mu > \gamma^n \Big\} = \bigcup_{j\in J_n} Q^n_{j,\alpha},$$

and define the set $E^n_{j,\alpha} = Q^n_{j,\alpha} \setminus \Omega^{n+1}_\alpha$. We have that $E^n_{j,\alpha} \subset \Omega^n_\alpha \setminus \Omega^{n+1}_\alpha$, and the $E^n_{j,\alpha}$ are pairwise disjoints in j, n, α.

We want to show that for γ chosen large enough, $\mu(Q^n_{j,\alpha}) \le 2\mu(E^n_{j,\alpha})$. First,

$$Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha = \bigcup_i (Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha}),$$

but we have

$$\frac{1}{\mu(B(Q^{n+1}_{i,\alpha}))} \int_{Q^{n+1}_{i,\alpha}} g \, \mathrm{d}\mu > \gamma^{n+1} > \gamma^n.$$

Thus, by maximality of $Q^n_{j,\alpha}$ and by the properties of dyadic cubes, etiher $Q^{n+1}_{i,\alpha} \subset Q^n_{j,\alpha}$ or $Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha} = \emptyset$. Hence,

$$\mu(Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha) = \sum_{i:Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha} = \emptyset} \mu(Q^n_{j,\alpha} \cap Q^{n+1}_{i,\alpha}) = \sum_{i:Q^{n+1}_{i,\alpha} \subset Q^n_{j,\alpha}} \mu(Q^{n+1}_{i,\alpha}),$$

but

$$\mu(Q^{n+1}_{i,\alpha}) \leq \mu(B(Q^{n+1}_{i,\alpha})) \leq \gamma^{-n-1} \int_{Q^{n+1}_{i,\alpha}} g \, \mathrm{d}\mu,$$

and, since the $Q^{n+1}_{i,\alpha}$ considered are disjoints and subsets of $Q^n_{j,\alpha}$, we have

$$\mu(Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha) \leq \gamma^{-n-1} \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu \leq \kappa\gamma^{-1} \mu(B(Q^n_{j,\alpha})),$$

where κ is the constant in (2.12). But

$$\mu(Q^n_{j,\alpha}) = \mu(E^n_{j,\alpha}) + \mu(Q^n_{j,\alpha} \cap \Omega^{n+1}_\alpha),$$

and so, choosing $\gamma = 2\kappa$, it follows that

$$\mu(Q^n_{j,\alpha}) \leq \frac{\gamma}{\gamma - \kappa} \mu(E^n_{j,\alpha}) = 2\mu(E^n_{j,\alpha}).$$

Consequently,

$$\int_X (T_m f) g \, \mathrm{d}\mu \leq 2C \Bigg(\sum_{\alpha,n,j} \mu(E^n_{j,\alpha}) \Bigg(\varphi(B(3Q^n_{j,\alpha})) \int_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu \Bigg)^p \Bigg)^{\frac{1}{p}}$$
$$\times \Bigg(\sum_{\alpha,n,j} \mu(E^n_{j,\alpha}) \Bigg(\frac{1}{\mu(Q^n_{j\alpha})} \int_{Q^n_{j,\alpha}} g \, \mathrm{d}\mu \Bigg)^{p'} \Bigg)^{\frac{1}{p'}}.$$

Since $E^n_{j,\alpha} \subset Q^n_{j,\alpha}$, it follows that

$$\mu(E^n_{j,\alpha}) \Bigg(\varphi(B(3Q^n_{j,\alpha})) \fint_{3B(Q^n_{j,\alpha})} f \, \mathrm{d}\mu \Bigg)^p \leq \int_{E^n_{j,\alpha}} (M_{\varphi,3\rho^{r+1}} f)^p \, \mathrm{d}\mu,$$

and a similar inequality for the integral on g. In addition, using that the $E^n_{j,\alpha}$ are pairwise disjoint, and that $\rho^r < \delta$, we get

$$\int_X (T_m f) g \, \mathrm{d}\mu \leq 2C \Bigg(\int_X (M_{\varphi,3\rho\delta} f)^p \, \mathrm{d}\mu \Bigg)^{\frac{1}{p}} \Bigg(\int_X (M_{d,\delta} g)^{p'} \, \mathrm{d}\mu \Bigg)^{\frac{1}{p'}}.$$

Now, using Proposition 2.12, for all $f, g \geq 0$, there is a constant C depending only on $p, A, \alpha, \varepsilon$ (specifically it depends on the constants for the $\rho\delta$-doubling) such that

$$\int_X (T_m f)g \, \mathrm{d}\mu \leq C \|M_{\varphi,3\rho\delta} f\|_p \|g\|_{p'}.$$

This holds under $(\mathbf{D})^{\eta}_{r\delta}$, $(\mathbf{K})_{2\rho\delta}$, and the fact that (2.9) holds for balls of radius at most $3\rho\delta$. The stronger hypotheses are what we need to apply Proposition 2.14 which gives us

$$\int_X (T_m f)g \, \mathrm{d}\mu \leq C \|M_{\varphi,\delta} f\|_p \|g\|_{p'},$$

which proves the theorem. ■

Finally, we have the theorem applied to the operators which will be of interest to us:

Corollary 2.1. *Let μ be a measure satisfying $(\mathbf{D})^{\eta}_{R}$ and $(\mathbf{RD})^{\nu}_{R}$, with $R > 0$, $\eta \geq \nu > 0$ ($\eta \geq \nu$ is automatic). Let $s \leq \nu$. Let $\delta \leq R$. If $K(x, y) = \frac{d(x,y)^s}{\mu(B(x,d(x,y)))}$, then the associated operator T_δ satisfies the hypotheses of Theorem 2.4. Moreover, the theorem still holds with $M_{\varphi,\delta} f$ replaced by the following maximal function:*

$$M_{s,\delta} f(x) = \sup_{0<r<\delta} r^s \fint_{B(x,r)} |f| \, \mathrm{d}\mu.$$

Proof. First, take some $b > 1$, by Proposition 2.8, μ is bR-reverse doubling of order ν. Then, we must verify that K satisfies the hypotheses of Theorem 2.4. Let $d(x, y) \leq R$ and $d(x, y') \leq bd(x, y)$, then, by doubling and reverse doubling,

$$\begin{aligned}\frac{1}{\mu(B(x, d(x, y)))} &\leq \frac{1}{\mu(B(x, d(x, y')))} \frac{\mu(B(x, bd(x, y)))}{\mu(B(x, d(x, y)))} \frac{\mu(B(x, d(x, y')))}{\mu(B(x, bd(x, y)))} \\ &\leq Cb^{\eta-\nu} \Big(\frac{d(x, y')}{d(x, y)}\Big)^{\nu} \frac{1}{\mu(B(x, d(x, y')))}.\end{aligned}$$

Thus, provided that $s \leq \nu$,

$$K(x, y) \leq Cb^{\eta-\nu} \Big(\frac{d(x, y')}{d(x, y)}\Big)^{\nu-s} K(x, y') \leq Cb^{\eta-s} K(x, y').$$

Furthermore, if $d(x', y) \leq \alpha d(x, y)$, using the doubling property, there are c, C such that $c\mu(B(y, d(x', y))) \leq \mu(B(x', d(x', y))) \leq C\mu(B(y, d(x', y)))$, and so doing the same calculations we have

$$K(x, y) \leq Cb^{\nu-s} K(x', y),$$

and there are $C_1, C_2 > 1$ such that (2.8) is satisfied.

Then, using the definition of φ and doubling,

$$c\frac{r(B)^s}{\mu(B)} \leq \varphi(B) \leq C\frac{r(B)^s}{\mu(B)},$$

for some constants that depends only on s, ρ and the doubling constant. Since, for $B_1 \subset B_2$,

$$r(B_1)^s \leq 2^s r(B_2)^s,$$

we easily verify that φ satisfies (2.9) with $\varepsilon = s$.

Then, it is enough to prove that the centered and uncentered version of the maximal function $M_{s,\delta}$ are equivalent in L^p norms. This follow from the same argument as that of Proposition 2.11. ∎

3. Relative Faber–Krahn inequality and estimates on the heat kernel and the Riesz and Bessels potentials

3.1. Faber–Krahn and doubling

The results from this subsection are due to A. A. Grigor'yan [10, 11], or are slight adaptations of his results to the R-doubling case.

Theorem 3.1 ([11]). *Let (M, g, μ) be a weighted manifold, and let $\{B(x_i, r_i)\}_{i\in I}$ be a family of relatively compact balls in M, where I is an arbitrary index set. Assume that, for any $i \in I$, $U \subset B(x_i, r_i)$, there is a constant $a_i > 0$ such that the following Faber–Krahn inequality holds*

$$\lambda_1(U) \geq a_i \mu(U)^{-2/\eta}.$$

Let $\Omega = \bigcup_{i\in I} B(x_i, \frac{r_i}{2})$. Then, for all $x, y \in \Omega$ and $t \geq t_0 > 0$, we have

$$p_t(x,y) \leq \frac{C(\eta)(1+\frac{d(x,y)^2}{t})^{\eta/2}\exp(-\frac{d(x,y)^2}{4t} - \lambda_1(M)(t-t_0))}{(a_i a_j \min(t_0, r_i^2)\min(t_0, r_j^2))^{\eta/4}},$$

where i, j are the indices such that $x \in B(x_i, \frac{r_i}{2})$ and $y \in B(x_j, \frac{r_j}{2})$.

On a manifold which admits $(\mathbf{RFK})_R^\eta$, applying this theorem with the family of all balls of radius less than R, $\{B(x,r)\}_{x\in M, 0<r\leq R}$, with $a_{x,r} = \frac{b}{r^2}\mu(B(x,r))^{2/\eta}$, $t_0 = t$, and $r = \sqrt{t}$, when $t \leq R^2$ we get

$$\begin{aligned} p_t(x,y) &\leq C(\eta)\frac{(1+\frac{d(x,y)^2}{t})^{\eta/2}e^{-\frac{d(x,y)^2}{4t}}}{(a_{x,\sqrt{t}}b_{y,\sqrt{t}}t^2)^{\eta/4}}, \\ &\leq \frac{C(\eta)}{b^{\eta/2}}\frac{e^{-\frac{d(x,y)^2}{ct}}}{\mu(B(x,\sqrt{t}))^{1/2}\mu(B(y,\sqrt{t}))^{1/2}}. \end{aligned}$$

If $t > R^2$, then we do the same thing, but with $r = R$, and we obtain the following:

Theorem 3.2. *Let (M, g, μ) be a weighted Riemannian manifold, suppose that there is $R > 0$ such that M satisfies $(\mathbf{RFK})^\eta_R$. Then μ satisfies $(\mathbf{D})^\eta_R$, and for any $c > 4$ there is some constant $K > 0$ such that the heat kernel has the following upper bounds:*

$$p_t(x,y) \le \frac{K}{\mu(B(x,\sqrt{t}))^{1/2}\mu(B(y,\sqrt{t}))^{1/2}} e^{-\frac{d(x,y)^2}{ct}}, \quad t \le R^2$$

$$p_t(x,y) \le \frac{K}{\mu(B(x,R))^{1/2}\mu(B(y,R))^{1/2}} e^{-\frac{d(x,y)^2}{ct}}, \qquad t > R^2.$$

The constant K depends only on b and η in the Faber–Krahn inequality and on the $c > 4$ chosen.

The estimate on the heat kernel follows from [10, Theorem 5.2]. The R-doubling follow from the proof of [10, Proposition 5.2].

Conversely, we have:

Proposition 3.1 ([10]). *Let (M, g, μ) be a complete, weighted Riemannian manifold. If μ satisfies $(\mathbf{D})^\eta_R$, if for any $x \in M$, the annuli $B(x,r') \setminus B(x,r)$, for $0 \le r < r' \le R$ are non-empty, and if there is some constant B such the heat kernel satisfies*

$$p_t(x,x) \le \frac{B}{\mu(B(x,\sqrt{t}))},$$

for all $x \in M$, and for all $0 < t \le R^2$, then there is some constant $\kappa \in (0,1)$, depending only on the doubling and reverse doubling constants, such that M admits a relative Faber–Krahn inequality at scale κR, with η being the doubling order and b depending only on A, B, and κ depends only on the doubling constants and on B.

Proof. This is a modification of the proof in [10], to take into account the R doubling case.

Fix a ball $B(x,r)$, with $r < R$, and let U be an open relatively compact subset of $B(x,r)$. Using the doubling volume property, if $t \le r^2$, then

$$e^{-\lambda_1(U)t} \le \int_U p_t(y,y)\, \mathrm{d}\mu(y) \le B \int_U \frac{\mathrm{d}\mu(y)}{\mu(B(y,\sqrt{t}))} \le AB \frac{\mu(U)}{\mu(B(x,r))} \Big(\frac{r}{\sqrt{t}}\Big)^\eta;$$

thus,

$$\lambda_1(U) \ge \frac{1}{t} \log\Big(\frac{1}{AB} \frac{\mu(B(x,r))}{\mu(U)} \Big(\frac{\sqrt{t}}{r}\Big)^\eta\Big).$$

Choose t such that the logarithm in the above inequality is equal to 1, i.e.,

$$t = r^2 \Big(eAB \frac{\mu(U)}{\mu(B(x,r))}\Big)^{2/\eta};$$

the condition $t \leq r^2$ then impose $\mu(U) \leq \frac{1}{eAB}\mu(B(x,r))$. For such U, we have

$$\lambda_1(U) \geq \frac{(eAB)^{-2/\eta}}{r^2}\Big(\frac{\mu(B(x,r))}{\mu(U)}\Big)^{2/\eta}. \tag{3.1}$$

Now, since the measure μ satisfies $(\mathbf{D})^{\eta}_{R}$, and since the annuli of radius less than R are non empty, it satisfies $(\mathbf{RD})^{\nu}_{R}$ for some $\nu > 0$. There is some constant $a \in (0,1)$ such that for any $0 < r < r' \leq R$ we have

$$\mu(B(x,r)) \leq a\Big(\frac{r}{r'}\Big)^{\nu}\mu(B(x,r')),$$

with $\nu = -\log_2 a$.

Then, for $\kappa = (aeAB)^{-1/\nu}$, if $r \leq \kappa R$, choose $r' = \kappa^{-1}r$. We have, for all U relatively compact open subset of $B(x,r)$,

$$\mu(U) \leq \frac{1}{eAB}\mu(B(x,r'));$$

thus, we can apply (3.1). Using by R-reverse doubling

$$\mu(B(x,r')) \geq a^{-1}\kappa^{-\nu}\mu(B(x,r)),$$

we have that

$$\lambda_1(U) \geq \frac{b}{r^2}\Big(\frac{\mu(B(x,r))}{\mu(U)}\Big)^{2/\eta},$$

with $b = \kappa^2$. ∎

3.2. An estimate on the heat kernel

Proposition 3.2. *Let (M,g,μ) be a complete weighted manifold satisfying $(\mathbf{RFK})^{\eta}_{R}$ for $R > 0$. For any $c > 4$, there are constants $K_1, K_2, K_3 > 0$ and $\alpha > 0$ such that the following estimates on the heat kernel hold.*

If $0 \leq t \leq R^2$, then

$$p_t(x,y) \leq \frac{K_1}{\mu(B(x,\sqrt{t}))}e^{-\frac{d(x,y)^2}{ct}}.$$

If $t > R^2$ and $d(x,y) \leq R$, then

$$p_t(x,y) \leq \frac{K_2}{\mu(B(x,R))}e^{-\frac{d(x,y)^2}{ct}}.$$

If $t > R^2$ and $d(x,y) > R$, then

$$p_t(x,y) \leq \frac{K_3}{\mu(B(x,R))}e^{\alpha\frac{t}{R^2}}e^{-\frac{d(x,y)^2}{ct}}.$$

Proof. Using the R-doubling, we have that, for any $t > 0$,

$$\mu(B(x, \sqrt{t})) \leq C\mu(B(y, \sqrt{t}))e^{D\frac{d(x,y)}{\sqrt{t}}},$$

and so, for $t \leq R^2$, we get, for any $c' > 4$,

$$p_t(x, y) \leq \frac{CK}{\mu(B(x, \sqrt{t}))} e^{\frac{D}{2}\frac{d(x,y)}{\sqrt{t}} - \frac{d(x,y)^2}{c't}};$$

so, taking $c' < c$, there is some constant K_1 such that

$$p_t(x, y) \leq \frac{K_1}{\mu(B(x, \sqrt{t}))} e^{-\frac{d(x,y)^2}{ct}}.$$

When $t > R^2$ and $d(x, y) \leq R$, the R-doubling property for small balls immediately lead to the desired result. When $d(x, y) > R$, by the R-doubling we obtain, for any $c' > 4$,

$$p_t(x, y) \leq \frac{CK}{\mu(B(x, R))} e^{\frac{D}{2}\frac{d(x,y)}{R} - \frac{d(x,y)^2}{c't}}.$$

We have that $\frac{Dd(x,y)}{2R} - \frac{d(x,y)^2}{c't} \leq \frac{c'cD^2 t}{16R^2(c-c')} - \frac{d(x,y)^2}{ct}$, thus there is some constants K_3, α which depend on the doubling constant and the choice of c, c', such that

$$p_t(x, y) \leq \frac{K_3}{\mu(B(x, R))} e^{\alpha\frac{t}{R^2} - \frac{d(x,y)^2}{ct}}. \qquad \blacksquare$$

3.3. Estimation of the Riesz potential

Let $s > 0$, and define the Riesz potential to be the operator $I_s = \Delta^{-s/2}$ on $L^2(M, \mu)$. Define $i_s(x, y)$ by

$$i_s(x, y) = \frac{1}{\Gamma(\frac{s}{2})} \int_0^{+\infty} t^{s/2-1} p_t(x, y) dt.$$

Whenever i_s is finite for all $x, y \in M$, it is the Schwartz kernel of the Riesz potential: in such case, for any $f \in \mathcal{C}_0^\infty(M)$, f is in the domain of I_s and

$$I_s f(x) = \int_M i_s(x, y) f(y) d\mu(y);$$

we thus call i_s the *Riesz kernel*. A sufficient condition for the Riesz kernel to be defined is given in the following proposition, which also yields an estimate on it:

Proposition 3.3. *Let* (M, g, μ) *be a manifold satisfying* $(\mathbf{RFK})^\eta$ *and* $(\mathbf{RD})^\nu$, $\nu > 0$. *Then, for any* $s < \nu$, *there is a constant* C *depending only on the Faber–Krahn and reverse doubling constants, such that the following inequality holds:*

$$i_s(x, y) \leq C \frac{d(x, y)^s}{\mu(B(x, d(x, y)))}.$$

Proof. Using Proposition 3.2 when the manifold satisfies $(\mathbf{RFK})^\eta$, there is $C > 0$ such that for all $x, y \in M$, $t > 0$,

$$p_t(x, y) \leq \frac{K_1}{\mu(B(x, \sqrt{t}))} \exp\Big(-\frac{d(x, y)^2}{5t}\Big),$$

and so

$$i_s(x, y) \leq C_s \int_0^\infty \frac{t^{s/2-1}}{\mu(B(x, \sqrt{t}))} e^{-\frac{d(x,y)^2}{5t}} dt.$$

Using the doubling and reverse property, with η the doubling order and ν the reverse doubling order, we obtain, writing $d = d(x, y)$,

$$\begin{cases} \frac{1}{\mu(B(x,\sqrt{t}))} \leq c \frac{1}{\mu(B(x,d))} (\frac{d}{\sqrt{t}})^\eta, & 0 < t \leq d^2, \\ \frac{1}{\mu(B(x,\sqrt{t}))} \leq c \frac{1}{\mu(B(x,d))} (\frac{d}{\sqrt{t}})^\nu, & t > d^2, \end{cases}$$

and so

$$i_s(x, y) \leq C \frac{1}{\mu(B(x, d))} \Bigg(d^\eta \int_0^{d^2} t^{\frac{s-\eta}{2}-1} e^{-\frac{d^2}{5t}} dt + d^\nu \int_{d^2}^\infty t^{\frac{s-\nu}{2}-1} e^{-\frac{d^2}{5t}} dt \Bigg).$$

Then, provided $\nu > s$,

$$\int_{d^2}^\infty t^{\frac{s-\nu}{2}-1} e^{-\frac{d^2}{ct}} dt \leq \int_{d^2}^\infty t^{\frac{s-\nu}{2}-1} dt = \frac{2}{s-\nu} d^{s-\nu}.$$

For the other integral, we make the change of variable $t = d^2/u$, obtaining

$$\int_0^{d^2} t^{\frac{s-\eta}{2}-1} e^{-\frac{d^2}{ct}} dt = d^{s-\eta} \int_1^\infty u^{\frac{\eta-s}{2}-1} e^{-\frac{u}{c}} du.$$

This integral is convergent and equal to a constant that depends only on s and c. Then for every $x, y \in M$,

$$i_s(x, y) \leq C \frac{d(x, y)^s}{\mu(B(x, d(x, y)))}.$$

∎

3.4. Estimation of the Bessel potential

Define the Bessel potential for $\lambda > 0$, $s > 0$ to be the operator $G_{s,\lambda} = (\Delta + \lambda^2)^{-s/2}$ on $L^2(M, \mu)$. It is, by the spectral theorem, a bounded operator, and, similarly to the case of the Riesz potential, admits for kernel

$$g_s^\lambda(x, y) = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty t^{s/2-1} e^{-\lambda^2 t} p_t(x, y) \, \mathrm{d}t,$$

provided that g_s^λ is finite for all $x, y \in M$.

Proposition 3.4. *Let (M, g, μ) be a complete weighted manifold satisfying $(\mathbf{RFK})_R^\eta$ and $(\mathbf{RD})_R^\nu$. If $\lambda > 0$ is such that $\lambda R \geq 1$, then for any $s < \nu$, there is a constant $C > 0$, depending only on s and on the Faber–Krahn constants, such that for all $x, y \in X$ with $d(x, y) \leq R$, we have*

$$g_s^\lambda(x, y) \leq C \frac{d(x, y)^s}{\mu(B(x, d(x, y)))}.$$

Proof. We have

$$g_s^\lambda(x, y) = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x, y) dt,$$

and we split this integral into three, integrating on $(0, d^2)$, (d^2, R^2) and $(R^2, +\infty)$.

We use Proposition 3.2. The same calculations as in the proof for the Riesz potential yields the estimate

$$\int_0^{d^2} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x, y) dt \leq C \frac{d(x, y)^s}{\mu(B(x, d(x, y)))}$$

When $d \leq \sqrt{t} \leq R$, we have by the R-reverse doubling that

$$\mu(B(x, d)) \leq a \Big(\frac{d}{\sqrt{t}}\Big)^\nu \mu(B(x, \sqrt{t}));$$

thus,

$$\int_{d^2}^{R^2} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x, y) dt \leq C \int_{d^2}^{R^2} t^{\frac{s}{2}-1} e^{-\lambda^2 t} \frac{1}{\mu(B(x, \sqrt{t}))} e^{-\frac{d^2}{5t}} dt$$

$$\leq C \frac{d^\nu}{\mu(B(x, d))} \int_{d^2}^{R^2} t^{\frac{s-\nu}{2}-1} e^{-\lambda^2 t} e^{-\frac{d^2}{5t}} dt$$

$$\leq C \frac{d^\nu}{\mu(B(x,d))} \int_{d^2}^{R^2} t^{\frac{s-\nu}{2}-1} e^{-\lambda^2 t} dt$$
$$\leq C \frac{d^\nu}{\mu(B(x,d))} \frac{2}{s-\nu} (R^{s-\nu} - d^{s-\nu}),$$

and, since $s - \nu < 0$,

$$\int_{d^2}^{R^2} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x,y) dt \leq C \frac{d(x,y)^s}{\mu(B(x,d(x,y)))}.$$

Now, for $t \geq R^2$, we simply have $\mu(B(x,R)) \leq \mu(B(x,\sqrt{t}))$. Thus,

$$\int_{R^2}^{\infty} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x,y) dt \leq C \frac{1}{\mu(B(x,R))} \int_{R^2}^{\infty} t^{\frac{s}{2}-1} e^{-\lambda^2 t} e^{-\frac{d^2}{5t}} dt;$$

then, since $d \leq R$, by using the reverse doubling, we obtain

$$\mu(B(x,d)) \leq \Big(\frac{d}{R}\Big)^\nu \mu(B(x,R)).$$

Moreover, we have $t^{\frac{s}{2}-1} e^{-\lambda^2 t} \leq c_s \lambda^{2-s} e^{-\frac{\lambda^2}{2} t}$; thus,

$$\int_{R^2}^{\infty} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x,y) dt \leq C \Big(\frac{d}{R}\Big)^\nu \frac{\lambda^{2-s}}{\mu(B(x,d))} \int_{R^2}^{\infty} e^{-\frac{\lambda^2}{2} t} dt$$
$$\leq C \Big(\frac{d}{R}\Big)^\nu \frac{\lambda^{-s}}{\mu(B(x,d))} e^{-\frac{(\lambda R)^2}{2}};$$

then, since $\lambda R \geq 1$, and $f(t) = t^{-s} e^{-\frac{t^2}{2}}$ is decreasing, we have $\lambda^{-s} e^{-\frac{(\lambda R)^2}{2}} \leq R^s e^{-\frac{1}{2}}$, which leads to

$$\int_{R^2}^{\infty} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x,y) dt \leq C \Big(\frac{R}{d}\Big)^{s-\nu} \frac{d^s}{\mu(B(x,d))}.$$

Then, $s - \nu < 0$ and $d < R$. Thus,

$$\int_{R^2}^{\infty} t^{\frac{s}{2}-1} e^{-\lambda^2 t} p_t(x,y) dt \leq C \frac{d(x,y)^s}{\mu(B(x,d(x,y)))}. \qquad \blacksquare$$

4. Proof of the main results

Let (M, g, μ) be a weighted Riemannian manifold and $V \in L^1_{\text{loc}}(M, \mathrm{d}\mu)$, $V \geq 0$. For any $R > 0$ and $p \geq 1$, we define $N_p(V)$ and $N_{p,R}(V)$ as in (1.3) and (1.4). Notice that $N_p(V) = M_{2p}(V^p)^{1/p}$.

Though we can deduce Theorem 1.1 as a special case of Theorem 1.2, we start by giving a separate, simpler proof of it. The general idea behind the proof of both theorems remains the same, but in the case of Theorem 1.2, much more care will be required in establishing the bounds on the norm of certain operators.

4.1. Proof the global inequality (Theorem 1.1)

We assume here that μ is reverse doubling of order ν, with $\nu > 1$, and we will show later on that this implies the general result.

Given $\varphi \in L^2(M)$, we first estimate $\|\Delta^{-1/2}(V^{1/2}\varphi)\|_2$. By Proposition 3.3, for any non-negative, measurable function f,

$$\Delta^{-\frac{1}{2}} f(x) \leq C \int_M \frac{d(x,y)}{\mu(B(x,d(x,y)))} f(y) d\mu(y).$$

Let T be the operator defined by the kernel $K(x,y) = \frac{d(x,y)}{\mu(B(x,d(x,y)))}$. Since M is a doubling space, applying Corollary 2.1, we have that

$$\|Tf\|_2 \leq C\|M_1 f\|_2,$$

and so

$$\|\Delta^{-\frac{1}{2}} f\| \leq \|M_1 f\|_2.$$

It follows that

$$\|\Delta^{-1/2}(V^{1/2}\varphi)\|_2 \leq C\|M_1(V^{1/2}\varphi)\|_2.$$

Then, using the Hölder inequality, we have, with $q = 2p$, $\frac{1}{q} + \frac{1}{q'} = 1$,

$$M_1(V^{1/2}\varphi) \leq M_q(V^{q/2})^{\frac{1}{q}} M_0(|\varphi|^{q'})^{\frac{1}{q'}} \leq N_p(V)^{\frac{1}{2}} M_0(|\varphi|^{q'})^{\frac{1}{q'}},$$

since $N_p(V) = M_{2p}(V^p)^{\frac{1}{p}}$. By the $L^{2/q'}$-boundedness of the Hardy–Littlewood maximal function, we obtain that

$$\begin{aligned}
\|M_1(V^{1/2}\varphi)\|_2 &\leq N_p(V)^{\frac{1}{2}} \|M_0(|\varphi|^{q'})^{\frac{1}{q'}}\|_2 \\
&\leq N_p(V)^{\frac{1}{2}} \|M_0(|\varphi|^{q'})\|_{\frac{2}{q'}}^{\frac{1}{q'}} \\
&\leq C_p N_p(V)^{\frac{1}{2}} \|\varphi\|_2,
\end{aligned}$$

and so

$$\|\Delta^{-1/2}(V^{1/2}\varphi)\|_2 \le C_p N_p(V)^{\frac{1}{2}} \|\varphi\|_2, \tag{4.1}$$

and $\Delta^{-1/2}(V^{1/2}\cdot)$ is a bounded linear operator on L^2. Its adjoint is $V^{1/2}\Delta^{-1/2}$, and, for any $\psi \in \mathcal{C}_0^\infty(M)$, if we let $\varphi = \Delta^{1/2}\psi$, $\varphi \in L^2$ and

$$\int_M V\psi^2 d\mu = \int_M |V^{1/2}\Delta^{-1/2}\varphi|^2 d\mu,$$

by (4.1), we get

$$\int_M V\psi^2 d\mu \le C_p N_p(V)^{1/2} \|\varphi\|_2^2 = C_p N_p(V) \|\nabla\psi\|_2^2.$$

4.2. Proof of the local inequality (Theorem 1.2)

We again make a technical hypothesis on the reverse doubling order, proving the following result:

Theorem 4.1. *Let (M, g, μ) be a complete weighted Riemannian manifold satisfying* $(\mathbf{RFK})_R^\eta$ *for some $R > 0$, and* $(\mathbf{RD})_R^\nu$ *for some $\nu > 1$. Then, for any $p > 1$, there is some constant C_p depending only on the Faber–Krahn constants and p, such that for any non-negative, locally integrable V, and any $\psi \in \mathcal{C}_0^\infty(M)$, the following inequality holds:*

$$\int_M V\psi^2 d\mu \le C_p N_{p,R}(V)\Big(\int_M |\nabla\psi|^2 d\mu + \frac{1}{R^2}\int_M \psi^2 d\mu\Big).$$

We will show afterwards how to remove this hypothesis to obtain Theorem 1.2.

4.2.1. Proof of Theorem 4.1. Given $\lambda > 0$ such that $\lambda R \ge 1$, we let $g^\lambda = g_1^\lambda$ be the kernel of the Bessel potential $G^\lambda = (\Delta + \lambda^2)^{-\frac{1}{2}}$. By Proposition 3.4, we have $g^\lambda(x, y) \le \frac{d(x,y)}{\mu(B(x,d(x,y)))}$, for $\lambda d(x, y) < 1$. We let

$$T_1\psi(x) = \int_{d(x,y)\le R} g^\lambda(x, y)V^{\frac{1}{2}}(y)\psi(y)d\mu(y),$$

$$T_2\psi(x) = \int_{d(x,y)>R} g^\lambda(x, y)V^{\frac{1}{2}}\psi(y)d\mu(y).$$

By Corollary 2.1, we have $\|T_1\psi\|_p \le C_p\|M_{1,R}V^{\frac{1}{2}}\psi\|_p$, and the rest follows as in the global case. To estimate $\|T_2\|$, we can study the operator $T_2T_2^*$, with kernel $a(x, z)$ defined as

$$a(x, z) = \int_M g^\lambda(x, y)\chi_{\{d(x,y)>R\}}|V(y)|\chi_{\{d(y,z)>R\}}g^\lambda(y, z)d\mu(y),$$

where we recall χ_E to be the characteristic function of the set E. We then apply the Schur test to $T_2 T_2^*$: being a symmetric operator, it will be bounded on L^2 if the integral

$$\int_M |a(x,z)| d\mu(z)$$

is uniformly bounded with respect to x. Given that we have

$$g^\lambda(y,z) = \int_0^\infty \frac{e^{-\lambda^2 t}}{\sqrt{\pi t}} p_t(y,z) dt,$$

as well as

$$\int_M p_t(y,z) d\mu(z) \leq 1,$$

we calculate

$$\int_M g^\lambda(y,z) d\mu(z) \leq \int_0^\infty \frac{e^{-\lambda^2 t}}{\sqrt{\pi t}} dt = \frac{1}{\lambda};$$

but then

$$\begin{aligned}
&\int_M |a(x,z)| d\mu(z) \\
&\quad \leq \int_M \int_M g^\lambda(x,y) \chi_{\{d(x,y)>R\}} |V(y)| \chi_{\{d(y,z)>R\}} g^\lambda(y,z) d\mu(y) d\mu(z) \\
&\quad \leq \int_{M\setminus B(x,R)} g^\lambda(x,y) \int_{M\setminus B(z,R)} g^\lambda(y,z) |V(y)| d\mu(y) d\mu(z),
\end{aligned}$$

and so we get

$$\int_M |a(x,z)| d\mu(z) \leq \frac{1}{\lambda} \int_{M\setminus B(x,R)} g^\lambda(x,y) |V(y)| d\mu(y),$$

or

$$\int_M |a(x,z)| d\mu(z) \leq \frac{1}{\lambda} \int_0^{+\infty} \frac{e^{-\lambda^2 t}}{\sqrt{\pi t}} \int_{M\setminus B(x,R)} p_t(x,y) |V(y)| d\mu(y) dt \tag{4.3}$$

To estimate this integral, we estimate $\int_{M\setminus B(x,R)} p_t(x,y)|V(y)|d\mu(y)$ by distinguishing the cases $t \geq R^2$ and $t < R^2$. For any $x \in M$, $r \geq R$, $p \geq 1$,

$$\int_{B(x,r)} |V(y)| d\mu(y) \leq \frac{C}{R^2} \mu(B(x,r)) N_{p,R}(V) \tag{4.4}$$

Indeed, we cover $B(x,r)$ by a family B_i of balls of radius R with center in $B(x,r)$ such that the balls with half the radius are pairwise disjoints. Then,

$$\begin{aligned}\int_{B(x,r)} |V(y)|d\mu(y) &\le \sum_i \int_{B_i} |V(y)|d\mu(y) \\ &\le C\sum_i \mu\Big(\frac{1}{2}B_i\Big) \fint_{B_i} |V(y)|d\mu(y) \\ &\le \frac{C}{R^2}\sum_i \mu\Big(\frac{1}{2}B_i\Big)\Big(R^2 \fint_{B_i} |V(y)|^p d\mu(y)\Big)^{\frac{1}{p}} \\ &\le \frac{C}{R^2}\mu\Big(B\Big(x, r+\frac{R}{2}\Big)\Big)N_{p,R}(V) \\ &\le \frac{C}{R^2}\mu(B(x,r))N_{p,R}(V).\end{aligned}$$

For all $t \ge R^2$, we use the corresponding estimate of Proposition 3.2, and get a constant $\alpha > 0$ such that

$$\int_{M\setminus B(x,R)} p_t(x,y)|V(y)|d\mu(y) \le \frac{Ce^{\alpha\frac{t}{R^2}}}{\mu(B(x,R))}\int_{M\setminus B(x,R)} e^{-\frac{d(x,y)^2}{5t}}|V(y)|d\mu(y).$$

By writing $e^{-\frac{d(x,y)^2}{5t}} = \int_{d(x,y)}^{+\infty} \frac{2r}{5t}e^{-\frac{r^2}{5t}}dr$, we have

$$\int_{M\setminus B(x,R)} e^{-\frac{d(x,y)^2}{5t}}|V(y)|d\mu(y) = \int_R^{\infty} e^{-\frac{r^2}{5t}}\frac{2r}{5t}\Big(\int_{B(x,r)\setminus B(x,R)} |V|d\mu\Big)dr.$$

Using (4.4), we obtain

$$\int_{M\setminus B(x,R)} e^{-\frac{d(x,y)^2}{5t}}|V(y)|d\mu(y) \le \frac{C}{R^2}N_{p,R}(V)\int_R^{\infty} e^{-\frac{r^2}{5t}}\frac{2r}{5t}\mu(B(x,r))dr.$$

By the R-doubling, using (2.6), there is a constant $\beta > 0$ that depends on the doubling constant such that $\mu(B(x,r)) \le \mu(B(x,R))e^{\beta\frac{r}{R}}$, thus

$$\int_{M\setminus B(x,R)} e^{-\frac{d(x,y)^2}{5t}}|V(y)|d\mu(y) \le \frac{C}{R^2}N_{p,R}(V)\mu(B(x,R))\int_R^{\infty} \frac{2r}{5t}e^{-\frac{r^2}{5t}+\beta\frac{r}{R}}dr,$$

and we then can find a constant $\gamma > 0$ such that $e^{-\frac{r^2}{5t}+\beta\frac{r}{R}} \le e^{-\frac{r^2}{10t}+\gamma\frac{t}{R^2}}$. As a result, we get that

$$\int_R^\infty \frac{2r}{5t} e^{-\frac{r^2}{10t}} dr = 2e^{-\frac{R^2}{10t}}.$$

To conclude, we obtain that for all $t \ge R^2$,

$$\int_{M\setminus B(x,r)} p_t(x,y)|V(y)|d\mu(y) \le C\frac{N_{p,R}(V)}{R^2} e^{-\frac{R^2}{10t}+2\gamma\frac{t}{R^2}}.$$

For $t \le R^2$, we obtain in the same way

$$\begin{aligned}
&\int_{M\setminus B(x,R)} p_t(x,y)|V(y)|d\mu(y) \\
&\quad \le \frac{c}{\mu(B(x,\sqrt{t}))}\frac{N_{p,R}(V)}{R^2}\int_R^\infty e^{-\frac{r^2}{5t}}\frac{2r}{5t}\mu(B(x,r))dr \\
&\quad \le \frac{c}{\mu(B(x,\sqrt{t}))}\mu(B(x,R))\frac{N_{p,R}(V)}{R^2}\int_R^\infty e^{-\frac{r^2}{5t}}\frac{2r}{5t}e^{\beta\frac{r}{R}}dr \\
&\quad \le c\Big(\frac{R}{\sqrt{t}}\Big)^\nu \frac{N_{p,R}(V)}{R^2} e^{-\frac{R^2}{10t}+\gamma\frac{t}{R^2}},
\end{aligned}$$

and, finally,

$$\int_{M\setminus B(x,R)} p_t(x,y)|V(y)|d\mu(y) \le C\Big(\max\Big(\frac{R}{\sqrt{t}},1\Big)\Big)^\nu \frac{N_{p,R}(V)}{R^2} e^{-\frac{R^2}{10t}+2\gamma\frac{t}{R^2}}.$$

Thus, we get the majoration

$$\int_M |a(x,z)|d\mu(z) \le \frac{C}{\lambda}\frac{N_{p,R}(V)}{R^2}\int_0^\infty \Big(\max\Big(\frac{R}{\sqrt{t}},1\Big)\Big)^\nu e^{-\frac{R^2}{10t}+2\gamma\frac{t}{R^2}} e^{-\lambda^2 t}\frac{dt}{\sqrt{\pi t}},$$

which by a change of variable $t = R^2 u$, transform into

$$\int_M |a(x,z)|d\mu(z) \le \frac{C}{\lambda}\frac{N_{p,R}(V)}{R}\int_0^\infty \Big(\max\Big(\frac{1}{\sqrt{u}},1\Big)\Big)^\nu e^{-\frac{1}{10u}+(2\gamma-\lambda^2R^2)u}\frac{du}{\sqrt{\pi u}},$$

and if $\lambda R \ge \sqrt{3\gamma} = \kappa > 1$ we obtain

$$\int_M a(x,z)dz \le CN_{p,R}(V).$$

Thus, by the Schur test,

$$\|T_2 T_2^*\|_{L^2\to L^2} \le C N_{p,R}(V),$$

and

$$\|T_2\|_{L^2\to L^2} \le C N_{p,R}(V)^{\frac{1}{2}}.$$

Then, for all $\lambda \ge \frac{\kappa}{R}$,

$$\int_M V\psi^2 d\mu \le C\Big(\int_M |\nabla\psi|^2 d\mu + \lambda^2 \int_M \psi^2 d\mu\Big),$$

and, in particular,

$$\int_M V\psi^2 d\mu \le C\kappa^2\Big(\int_M |\nabla\psi|^2 d\mu + \frac{1}{R^2}\int_M \psi^2 d\mu\Big).$$

4.2.2. Proof of Theorem 1.4. We now suppose that $\lambda_1(M) > 0$. Then the previous results can be strenghtened to prove Theorem 1.4.

Proof. We apply Theorem 1.2, and use that $\lambda_1(M)\int_M \psi^2 \,\mathrm{d}\mu \le \int_M |\nabla\psi|^2 \,\mathrm{d}\mu$. Then,

$$\langle V\psi,\psi\rangle \le C_p N_{p,R}(V)\Big(1 + \frac{1}{\lambda_1(M)R^2}\Big)\int_M |\nabla\psi|^2 \,\mathrm{d}\mu,$$

which gives

$$\frac{\lambda_1(M)R^2}{C_p N_{p,R}(V)(1+\lambda_1(M)R^2)}\int_M V\psi^2 \,\mathrm{d}\mu \le \int_M |\nabla\psi|^2 \,\mathrm{d}\mu,$$

and

$$\frac{\lambda_1(M)R^2}{2C_p N_{p,R}(V)(1+\lambda_1(M)R^2)}\int_M V\psi^2 \,\mathrm{d}\mu + \frac{\lambda_1(M)}{2}\int_M \psi^2 \,\mathrm{d}\mu \le \int_M |\nabla\psi|^2 \,\mathrm{d}\mu.$$

Therefore, for any V,

$$\langle V\psi,\psi\rangle \le \frac{C_p N_{p,R}(V)(1+\lambda_1(M)R^2)}{\lambda_1(M)R^2}\Big(\|\nabla\psi\|^2 - \frac{\lambda_1(M)}{2}\|\psi\|^2\Big),$$

which is (1.8). ∎

4.3. Proof of Theorem 1.3

Let C_p be the constant of Theorem 1.2. We let

$$L = \sup_{x,\delta}\left(2C_p\left(\fint_{B(x,\delta)} V^p \,\mathrm{d}\mu\right)^{1/p} - \delta^{-2}\right).$$

Then,

$$\left(\fint_{B(x,\delta)} V^p \,\mathrm{d}\mu\right)^{1/p} \le \frac{L+\delta^{-2}}{2C_p}, \quad (M_{2p,\delta}(V^p)(x))^{1/p} \le \frac{\delta^2 L + 1}{2C_p}.$$

Take $\delta = L^{-1/2}$. Then, $N_{p,\delta}(V) \le \frac{1}{C_p}$. By Theorem 1.2, we have

$$\langle V\psi, \psi\rangle - \|\nabla\psi\|_2^2 \le L\|\psi\|^2;$$

thus,

$$-\lambda_1(\Delta - V) \le \sup_{x,\delta}\left(2C_p\left(\fint_{B(x,\delta)} V^p \,\mathrm{d}\mu\right)^{1/p} - \delta^{-2}\right).$$

Meanwhile, let $r < \lambda^{-1} \le R$, and define $f_r\colon [0,\infty) \to [0,+\infty)$ by $f(t) = r$ if $t \le r$, $f(t) = 2r - t$ if $t \in (r, 2r]$, and $f_r(t) = 0$ if $t > 2r$. Then for $o \in M$, $\psi = f_r(d(o,x))$. ψ is a Lipschitz function with compact support, and, by $(\mathbf{D})_R^\eta$,

$$\begin{aligned}\lambda_1(\Delta - V) &\le \frac{\|\nabla\psi\|^2 - \int_M V\psi^2 \,\mathrm{d}\mu}{\|\psi\|^2} \le \frac{\mu(B(x,2r))}{r^2\mu(B(x,r))} - \fint_{B(x,r)} V \,\mathrm{d}\mu \\ &\le Ar^{-2} - \fint_{B(x,r)} V \,\mathrm{d}\mu \le (r/\sqrt{A})^{-2} - A^{-1-\eta/2}\fint_{B(x,r/\sqrt{A})} V \,\mathrm{d}\mu,\end{aligned}$$

for all $r > 0$. Thus,

$$-\lambda_1(\Delta - V) \ge \sup_{x,\delta}\left(A^{-1-\eta/2}\fint_{B(x,\delta)} V \,\mathrm{d}\mu - \delta^{-2}\right).$$

4.4. Removing the dependancy on reverse doubling

Let M be a manifold satisfying $(\mathbf{RFK})^\eta$. We consider $\widetilde{M} = \mathbf{R} \times M$, $(\widetilde{M}, \tilde{g}, \tilde{\mu})$ the product Riemannian manifold: $\tilde{g} = \mathrm{d}x^2 + g$, $\mathrm{d}\tilde{\mu} = \mathrm{d}x\,\mathrm{d}\mu$. For $V \in L^1_{\mathrm{loc}}(M)$, we define $\widetilde{V}(x,m) = V(m)$. We write $\widetilde{\Delta}$ for the laplacian on $(\widetilde{M}, \tilde{g}, \tilde{\mu})$, and Δ for the laplacian on (M, g, μ). The Morrey norm in $\widetilde{M}$ is written $\widetilde{N}_{p,R}$.

We have:

Proposition 4.1. $(\widetilde{M}, \tilde{g}, \tilde{\mu})$ *satisfies the following properties:*

(1) *if μ is R-doubling, then $\tilde{\mu}$ is R-doubling, and R-reverse doubling with order $\nu > 1$;*

(2) *the heat kernel of $\widetilde{M}$ is $\tilde{p}_t((x,m),(y,n)) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} p_t(m,n)$;*

(3) *if M satisfies $(\mathbf{RFK})^{\eta}_{R}$, then there is some $\theta \in (0,1)$ such that $\widetilde{M}$ satisfies $(\mathbf{RFK})^{\eta}_{\theta R}$. θ depends only on the Faber–Krahn constants;*

(4) $\lambda_1(\widetilde{\Delta} - \widetilde{V}) = \lambda_1(\Delta - V)$*;*

(5) *if μ is R-doubling, then there are two constants c, C, which depends only on the doubling constant, such that $cN_{p,R}(V) \leq \widetilde{N}_{p,R}(\widetilde{V}) \leq CN_{p,R}(V)$.*

Proof. (1) For $E \subset \mathbf{R}$ measurable, we denote $|E|$ the usual Lebesgue measure of E. We have

$$|(-r/2, r/2)|\mu(B(m, r/2)) \leq \tilde{\mu}(\widetilde{B}((x,m), r)) \leq |(-r,r)|\mu(B(m,r)). \tag{4.5}$$

From this, with $r \leq R$ we immediately get

$$\tilde{\mu}(\widetilde{B}((x,m), 2r)) \leq 4A^2 \tilde{\mu}(\widetilde{B}((x,m), r)),$$

with A the R-doubling constant of μ. Moreover, since μ is R-doubling, it is R-reverse doubling, with reverse doubling order $\nu > 0$. Then, for $r < r' < \theta R$,

$$\begin{aligned}
\frac{\tilde{\mu}(\widetilde{B}((x,m), r'))}{\tilde{\mu}(\widetilde{B}((x,m), r))} &\geq \frac{r'}{2r} \frac{\mu(B(m, r'/2))}{\mu(B(m,r))} \\
&\geq \frac{1}{2A} \frac{r'}{r} \frac{\mu(B(m,r'))}{\mu(B(m,r))} \\
&\geq \frac{a}{2A} \left(\frac{r'}{r}\right)^{1+\nu}
\end{aligned}$$

Thus, $\tilde{\mu}$ is reverse doubling of order $\tilde{\nu} = 1 + \nu > 1$.

(2) and (4) We have $\widetilde{\Delta} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Delta$. Thus,

$$\tilde{p}_t((x,m),(y,n)) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} p_t(m,n),$$

and the spectrum of $\widetilde{\Delta} - \widetilde{V}$ is

$$\mathrm{Sp}(\widetilde{\Delta} - \widetilde{V}) = \{\lambda + \lambda' : \lambda \in \mathrm{Sp}(\Delta - V), \lambda' \geq 0\}.$$

Thus, the infimum of the spectrum of $\widetilde{\Delta} - \widetilde{V}$ is the infimum of the spectrum of $\Delta - V$.

(3) We use Proposition 3.1.

(5) We use (4.5). Using that $\int_{\widetilde{B}} \widetilde{V} \, \mathrm{d}\tilde{\mu} \leq 2r \int_B V \, \mathrm{d}\mu$, we have

$$\frac{r^{2p}}{\tilde{\mu}(\widetilde{B}((x,m),r))} \int_{\widetilde{B}} \widetilde{V}^p \, \mathrm{d}\tilde{\mu} \leq \frac{r^{2p}}{(r/2)\mu(B(m,r/2))} 2r \int_B V^p \, \mathrm{d}\mu.$$

Then, by R-doubling, $\widetilde{N}_{p,R}(\widetilde{V}) \leq 4AN_{p,R}(V)$. The other inequality is obtained in a similar same way. ■

Proof of Theorem 1.2. From (1) and (3) of the above proposition, if (M,g,μ) is a manifold satisfying $(\mathbf{RFK})^\eta_R$, then there is some $\theta \in (0,1)$, depending only on the Faber–Krahn constants, such that $(\widetilde{M},\tilde{g},\tilde{\mu})$ satisfies $(\mathbf{RFK})^\eta_{\theta R}$ and $(\mathbf{RD})^\nu_R$, with $\nu > 1$. So, we can apply Theorem 1.2 to $\widetilde{M}$: there is a constant $\widetilde{C}_p$ such that if $\widetilde{V}$ is satisfies $\widetilde{C}_p \widetilde{N}_{p,R}(\widetilde{V}) \leq 1$, then $\lambda_1(\widetilde{\Delta} - \widetilde{V}) \geq -\frac{1}{\theta^2 R^2}$.

Using (5), then there is a constant $C_p > 0$ such that $C_p N_{p,R}(V) \geq \widetilde{C}_p \widetilde{N}_{p,R}(\widetilde{V})$. Since $\lambda_1(\Delta - V) = \lambda_1(\widetilde{\Delta} - \widetilde{V})$, if $\mathbf{C}_p N_{p,R}(V) \leq 1$, then $\lambda_1(\Delta - V) \geq -\frac{1}{\theta^2 R^2}$. For an arbitrary $V \geq 0$, locally integrable, with $N_{p,R}(V) < +\infty$, we can apply the above to $V/C_p N_{p,R}(V)$; then, for any $\psi \in \mathcal{C}_0^\infty(M)$,

$$\frac{1}{C_p N_{p,R}(V)} \int_M V\psi^2 \, \mathrm{d}\mu \leq \frac{1}{\theta^2} \int_M \left(|\nabla\psi|^2 + \frac{1}{R^2}\psi^2 \right) \mathrm{d}\mu,$$

which is (1.7). ■

5. Hardy inequality

For some point $o \in M$, the L^2 Hardy inequality

$$\int_M \frac{\psi(x)^2}{d(o,x)^2} \, \mathrm{d}\mu(x) \leq C \int_M |\nabla\psi(x)|^2 \, \mathrm{d}\mu(x), \quad \text{for all } \psi \in \mathcal{C}_0^\infty(M),$$

is equivalent to the positivity of the operator $\Delta - V$, with $V(x) = \frac{1}{C} d(o,x)^{-2}$. Moreover, we have:

Proposition 5.1. *Let (M,g,μ) be a weighted Riemannian manifold, $R \in (0,\infty]$. If μ satisfies $(\mathbf{D})^\eta_R$ and $(\mathbf{RD})^\nu_R$, with $\nu > 1$, then for any $p \in (1,\nu/2)$, there is a constant $K_p < \infty$ such that for all $r < R$ we have*

$$r^2 \left(\fint_{B(x,r)} d(o,y)^{-2p} \, \mathrm{d}\mu \right)^{1/p} \leq K_p.$$

Proof. We let $\rho(y) = d(o, y)$, $B = B(x, r)$, for $r < R$. If $r \le \rho(x)/2$, then for $y \in B(x, r)$, we have $\rho(y) \ge \rho(x) - r \ge \rho(x)/2 \ge r$. Therefore,

$$\int_B \rho(y)^{-2p}\, \mathrm{d}\mu \le r^{-2p}\mu(B).$$

If $r > \rho(x)/2$, then $B(x, r) \subset B(o, 3r)$, and

$$\begin{aligned}\int_B \rho^{-2p}\, \mathrm{d}\mu &\le \int_{B(o,3r)} \rho^{-2p}\, \mathrm{d}\mu \\ &\le \int_0^\infty (2p-1)t^{-2p-1}\mu(B(o, \min(t, 3r)))\, \mathrm{d}t \\ &\le \int_0^{3r} a^{-1}(2p-1)t^{\nu-2p-1}(3r)^{-\nu}\mu(B(o, 3r))\, \mathrm{d}t + r^{-2p}\mu(B(o, 3r)) \\ &\le \Big(\frac{1}{3^{3p}a}\frac{2p-1}{\nu-2p} + 1\Big) r^{-2p}\mu(B(o, 3r)) \le C_p r^{-2p}\mu(B(x, r)),\end{aligned}$$

since $\nu > 2p$, with the constant C_p depending uniquely on p and the doubling and reverse doubling constants. ■

Applying Theorems 1.2 and 1.1, we immediately obtain:

Corollary 5.1. *If (M, g, μ) satisfies $\mathbf{(RFK)}_R^\eta$ and $\mathbf{(RD)}_R^\nu$ with $\nu > 2$, then there is a constant C such that for any $\psi \in \mathscr{C}_0^\infty(M)$, $o \in M$,*

$$\int_M \frac{\psi(x)^2}{d(o, x)^2}\, \mathrm{d}\mu(x) \le C\Big(\|\nabla\psi\|_2^2 + \frac{1}{R^2}\|\psi\|_2^2\Big).$$

Corollary 5.2. *If (M, g, μ) satisfies $\mathbf{(RFK)}^\eta$, $\mathbf{(RD)}^\nu$ with $\nu > 2$, then there is a constant C such that*

$$\int_M \frac{\psi(x)^2}{d(o, x)^2}\, \mathrm{d}\mu(x) \le C \int_M |\nabla\psi|^2\, \mathrm{d}\mu \quad \text{for all } \psi \in \mathscr{C}_0^\infty(M). \tag{5.1}$$

The second corollary follows from Theorem 1.5.

This time the condition on the reverse doubling order is not merely a technical hypothesis. It is, in fact, a necessary condition for the Hardy inequality to holds if we assume the measure μ to be doubling:

Proposition 5.2. *Let (M, g, μ) be a weighted Riemannian manifold, with μ a doubling measure, assume that there is a constant $\nu > 2$ such that for any $o \in M$, $\psi \in \mathcal{C}_0^\infty(M)$, M admits the Hardy inequality,*

$$\Big(\frac{\nu-2}{2}\Big)^2 \int_M \frac{\psi(x)^2}{d(o,x)^2} \, \mathrm{d}\mu(x) \le \int_M |\nabla\psi|^2 \, \mathrm{d}\mu; \tag{5.2}$$

then μ satisfies $(\mathbf{RD})^\nu$.

Note that we can always write a Hardy inequality (5.1) in the form (5.2) simply by choosing $\nu = 2 + 2\sqrt{1/C}$.

Using a method from [3, 16], we have:

Proof. Take $0 < r < R$. Define $f(t) = r^{-\frac{\nu-2}{2}}$ for $0 \le t \le r$, $f(t) = t^{-\frac{\nu-2}{2}}$ for $r \le t \le R$, $f(t) = 2R^{-\frac{\nu-2}{2}} - R^{-\frac{\nu}{2}} t$ for $R \le t \le 2R$ and $f(t) = 0$ for $t \ge 2R$. When $r \le t \le R$, we have $f'(t)^2 = (\frac{\nu-2}{2})^2 \frac{f(t)^2}{t^2}$. For some point $o \in M$ choose $\varphi(x) = f(d(o,x))$. The Hardy inequality applied to φ leads to

$$\Big(\frac{\nu-2}{2}\Big)^2 \int_{B(o,r)} \frac{\varphi(x)^2}{d(o,x)^2} \, \mathrm{d}\mu(x) \le \int_{B(o,2R)\setminus B(o,R)} |\nabla\varphi|^2 \, \mathrm{d}\mu(x).$$

So,

$$\Big(\frac{\nu-2}{2}\Big)^2 r^{-\nu} \mu(B(o,r)) \le R^{-\nu} \mu(B(o,2R) \setminus B(o,R)) \le A R^{-\nu} \mu(B(o,R)),$$

using that μ is doubling. Thus, there is some constant $a > 0$ such that

$$a\Big(\frac{R}{r}\Big)^\nu \le \frac{\mu(B(o,R))}{\mu(B(o,r))},$$

and μ is reverse doubling of order $\nu > 2$. ∎

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