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Scaling inequalities for spherical and hyperbolic eigenvalues

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Abstract. Neumann and Dirichlet eigenvalues of the Laplacian on spherical and hyperbolic domains are shown to satisfy scaling inequalities or monotonicities analogous to the $(\text{length})^{-2}$ scaling relation in Euclidean space.

For a cap of aperture Θ on the sphere $\mathbb{S}^2$, normalizing the k-th eigenvalue by the square of the Euclidean radius of the boundary circle yields that $\mu_k(\Theta)\sin^2\Theta$ is strictly decreasing, while normalizing by the stereographic radius squared gives that $\mu_k(\Theta)4\tan^2\Theta/2$ is strictly increasing. For the second Neumann eigenvalue, normalizing instead by the cap area establishes the stronger result that $\mu_2(\Theta)4\sin^2\Theta/2$ is strictly increasing.

Monotonicities of this kind are somewhat surprising, since the Neumann eigenvalues themselves can vary non-monotonically.

Cheng and Bandle-type inequalities are deduced by assuming either fixed radius or fixed area and comparing eigenvalues of disks having different curvatures.

1. Introduction and main results

Frequencies of vibration on a drum of size t scale like t^{-2}. Mathematically, $\mu_k(t\Omega)t^2$ is identically constant when μ_k is a Neumann or Dirichlet eigenvalue of the Laplacian and Ω is a Euclidean domain. This paper establishes analogous scaling results for domains in the 2-dimensional sphere and hyperbolic space, especially geodesic disks.

Inequalities rather than identities are the goal. Specifically, we aim to prove monotonicity under scaling for quantities of the form

$$(\text{eigenvalue}) \times (\text{geometric factor})^2.$$

Geometric factors include several types of radius, as well as the square root of the area, as summarized for the spherical case in Table 1.

Theorem 1 normalizes the k-th Neumann eigenvalue of the spherical cap of aperture Θ by the square of the Euclidean radius of its boundary circle and obtains that

$$\mu_k(\Theta)\sin^2\Theta$$

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scale factor	geometric meaning
Θ	aperture, or geodesic radius
$\sin\Theta$	Euclidean radius of boundary circle
$\tan\frac{\Theta}{2}$	stereographic radius
$4\pi\sin^2\frac{\Theta}{2}$	area

Table 1. Geometric factors for a spherical cap of aperture Θ.

is strictly decreasing. The same holds for Dirichlet eigenvalues $\lambda_k(\Theta)$. In the opposite direction, normalizing by the squared stereographic radius reveals

$$\mu_k(\Theta)\tan^2\frac{\Theta}{2}$$

to be strictly increasing, while for the second Neumann eigenvalue ($k=2$), after normalizing by the cap area we establish in Theorem 2 the stronger result that

$$\mu_2(\Theta)\sin^2\frac{\Theta}{2}$$

is strictly increasing. Hyperbolic analogues are provided for these theorems too. Theorem 3 and Theorem 4 then examine the first two Dirichlet eigenvalues normalized by the square of the aperture, $\lambda_1(\Theta)\Theta^2$ and $\lambda_2(\Theta)\Theta^2$, respectively.

Subsequent corollaries deduce inequalities of Bandle and Cheng type for the second Neumann and first and second Dirichlet eigenvalues on geodesic disks. In particular, Corollary 5 says that among disks having area A and constant curvature $\leq K$, the disk with constant curvature K maximizes the second Neumann eigenvalue.

The intrinsic appeal of the monotonicity properties in this paper is clear. Monotonicity could also serve practical purposes, because although the eigenvalues can be specified in terms of roots related to Legendre P-functions [3], those root conditions shed precious little light on the qualitative behavior of the eigenvalues. The monotonicity relations in the current paper might provide useful tools for authors who need robust estimates on eigenvalues for constant curvature disks.

Arbitrary surfaces. One naturally wonders: does extremality of μ_2 at the greatest curvature disk continue to hold among the larger class of all simply connected surfaces having area A and variable curvature $\leq K$? Szegő [24] handled simply connected planar domains (curvature 0). Bandle [4, 5] extended to surfaces with curvature $\leq K$ provided $K \leq 2\pi/A$, which means in the positive curvature case that the maximizing cap has area at most half that of the sphere. Recently, we breached this hemispherical barrier by handling simply connected surfaces with area up to 94% of the sphere [21, Theorem 1.1]. Getting to the conjectured 100% remains an open problem.

Prior results and literature. The monotonicity results in Theorem 1 for the full Neumann and Dirichlet spectra of spherical caps and hyperbolic disks, specifically for the cases $\mu_k(\Theta)\sin^2\Theta$ and $\mu_k(\Theta)\tan^2\Theta/2$, are new to the best of our knowledge except that we handled $\mu_k(\Theta)\sin^2\Theta$ on spherical caps in a recent paper [21].

Monotonicity of $\mu_2(\Theta)\sin^2\Theta/2$ in Theorem 2 is due to Bandle [4, 5] when $\Theta \leq \pi/2$, as explained after Corollary 5. The result is new when $\pi/2 < \Theta < \pi$, that is, for caps larger than a hemisphere. Our proof is different and more direct.

The monotonicity of $\lambda_1(\Theta)\Theta^2$ in Theorem 3 is known by work of Ashbaugh and Benguria [2] in the spherical case and Benguria and Linde [8] in the hyperbolic case, although it turns out those results for the first Dirichlet eigenvalue follow already from an older inequality by Cheng [14], as we observe in Corollary 6 and its proof.

For the second Dirichlet eigenvalue, monotonicity of $\lambda_2(\Theta)\Theta^2$ in Theorem 4 was proved by Ashbaugh and Benguria [2] in the spherical case. Our proof is different, and yields also monotonicity in the opposite direction in the hyperbolic case, which is a new result.

Along somewhat different lines, many authors have proved approximations and asymptotic formulas for the first Dirichlet eigenvalue of a small spherical cap or hyperbolic disk, notably Baginski [3], Borisov and Freitas [11], Berge [9], Kristály [19], and other authors to whom they refer.

2. Results

The eigenvalue problem. On the unit sphere $\mathbb{S}^2$ of curvature $+1$, let

$$C(\Theta) = \text{spherical cap of aperture } \Theta \text{ centered at the north pole}$$

when $\Theta \in (0, \pi)$, so that $C(\pi/2)$ is the upper hemisphere. Let $\mu_k(\Theta)$ be the k-th Neumann eigenvalue of the spherical Laplacian on the cap $C(\Theta)$, that is, the k-th eigenvalue of the problem

$$\begin{cases} -\Delta_{sph} u = \mu u & \text{on } C(\Theta), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial C(\Theta). \end{cases}$$

The coordinate expression for the spherical Laplacian is recalled in Section 5.

In the hyperbolic space $\mathbb{H}^2$ of curvature -1, when $\Theta \in (-\infty, 0)$ let

$$C(\Theta) = \text{geodesic disk of radius } |\Theta|$$

and write $\mu_k(\Theta)$ for the k-th Neumann eigenvalue of the hyperbolic Laplacian on that disk. When $\Theta < 0$, in other words, $\mu_k(\Theta)$ is the k-th eigenvalue of

$$\begin{cases} -\Delta_{hyp} u = \mu u & \text{on } C(\Theta), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial C(\Theta). \end{cases}$$

In the plane $\mathbb{R}^2$, the Euclidean Laplacian on the unit disk D has k-th Neumann eigenvalue denoted $\mu_k(D)$.

Throughout the paper, negative values of Θ correspond to the hyperbolic case and positive values to the spherical one, with the Euclidean case appearing at $\Theta = 0$. This convention unites the spherical and hyperbolic situations, for example as seen graphically in Figure 2.

Eigenvalue scaling inequalities on spherical caps and hyperbolic disks. Our first theorem holds for all eigenvalues except the first Neumann μ_1, which is identically zero. For simplicity, the theorem is stated only for spherical caps and geodesic disks, although the underlying proof in Proposition 9 applies to arbitrary domains under "stereographic dilation."

Theorem 1 (Scaling-type relations for all eigenvalues of spherical caps and hyperbolic disks). *Fix $k \geq 2$.*

(i) *The function*

$$\Theta \mapsto \begin{cases} \mu_k(\Theta)\, 4\tanh^2 \frac{\Theta}{2}, & \Theta \in (-\infty, 0), \\ \mu_k(D), & \Theta = 0, \\ \mu_k(\Theta)\, 4\tan^2 \frac{\Theta}{2}, & \Theta \in (0, \pi), \end{cases}$$

increases strictly and continuously from 0 *to* ∞.

(ii) *The function*

$$\Theta \mapsto \begin{cases} \mu_k(\Theta)\, \sinh^2 \Theta, & \Theta \in (-\infty, 0), \\ \mu_k(D), & \Theta = 0, \\ \mu_k(\Theta)\, \sin^2 \Theta, & \Theta \in (0, \pi), \end{cases}$$

decreases strictly and continuously from ∞ *to* 0.

(iii) *When $k \geq 1$, the corresponding statements hold for Dirichlet eigenvalues, except that in part* (i) *the functional increases strictly not from* 0 *to* ∞*, but from* 1 *to* ∞.

The theorem is proved in Section 6. Part (ii) for $\mu_k(\Theta)\sin^2\Theta$ on spherical caps ($0 < \Theta < \pi$) was obtained in our recent paper [21, Proposition 3.1] and we extend that method here.

Remarks. (1) Part (ii) is particularly appealing when $\Theta \in (0, \pi)$, since $\sin\Theta$ is the extrinsic Euclidean radius of the boundary circle for the cap $C(\Theta)$ and so the quantity $\mu_k(\Theta)\sin^2(\Theta)$ has the form "eigenvalue times radius squared." That form mimics the usual scaling relation for domains in Euclidean space.

(2) Since $\sin^2\Theta$ is increasing for $\Theta \in (0, \pi/2]$, it follows from the theorem that the k-th eigenvalue $\mu_k(\Theta)$ of the spherical cap is decreasing in that range. Similarly, in the hyperbolic case, part (i) of the theorem implies $\mu_k(\Theta)$ is increasing for $\Theta \in (-\infty, 0)$, since $\tanh^2\Theta/2$ is decreasing. That is, the k-th Neumann eigenvalue of a hyperbolic geodesic disk is a decreasing function of the radius $|\Theta|$.

(3) Theorem 1 implies a two-sided bound on Neumann eigenvalues of geodesic disks:

$$\begin{aligned} \frac{1}{\sinh^2\Theta} &< \frac{\mu_k(\Theta)}{\mu_k(D)} < \frac{1}{4\tanh^2\frac{\Theta}{2}}, &\quad \Theta \in (-\infty, 0), \\ \frac{1}{4\tan^2\frac{\Theta}{2}} &< \frac{\mu_k(\Theta)}{\mu_k(D)} < \frac{1}{\sin^2\Theta}, &\quad \Theta \in (0, \pi), \end{aligned}$$

for all $k \geq 2$. The estimates are sharp in the sense that both sides behave like Θ^{-2} as $\Theta \to 0$, that is, for small disks or caps.

Corresponding two-sided inequalities hold for Dirichlet eigenvalues when $k \geq 1$. For even sharper two-sided Dirichlet inequalities, see the work of Baginski [3] for all Dirichlet eigenvalues on the sphere, the results of Borisov and Freitas [11] for the first eigenvalue in both the sphere and hyperbolic space, and the extension to all Dirichlet eigenvalues by Berge [9], as well as the references in those papers to earlier work.

For the first Dirichlet eigenvalue, part (iii) of the theorem implies that $\lambda_1(\Theta) > 1/4\tanh^2\Theta/2$ when $\Theta < 0$. This inequality is stronger, in the special case of disks, than Osserman's result [23] for simply connected domains in hyperbolic space that $\lambda_1 > 1/4\tanh^2 p$ where p is the geodesic inradius.

(4) The quantities in Theorem 1 can be written more concisely as $\mu_k(\Theta)4\,\mathrm{ta}^2\,\Theta/2$ and $\mu_k(\Theta)4\,\mathrm{si}^2\,\Theta$, where the normalizing functions ta and si are defined by piecing together hyperbolic and trigonometric functions as in Figure 1:

$$\mathrm{si}\,\theta = \begin{cases} \sinh\theta, & \theta \leq 0, \\ \sin\theta, & \theta \geq 0, \end{cases} \qquad \mathrm{ta}\,\theta = \begin{cases} \tanh\theta, & \theta \leq 0, \\ \tan\theta, & \theta \geq 0. \end{cases}$$

(5) Figure 2 illustrates the above theorem, for the second Neumann and first Dirichlet eigenvalue. Some appealing bounds can be read off from the graphs, such as that $\lambda_1(\Theta)4\tan^2\Theta/2 \geq 8$ when $\Theta \in [\pi/2, \pi)$, with equality at $\Theta = \pi/2$.

The flatness of each graph near $\Theta = 0$ is best understood as a consequence of Euclidean scale invariance. When $\Theta \approx 0$, the spherical caps and hyperbolic disks are so small that their eigenvalues are approximately Euclidean and so each expression in the figure has approximately the form

$$(\text{Euclidean eigenvalue of a disk of radius } \Theta) \times \Theta^2,$$

which is constant (giving a flat graph) with respect to Θ.

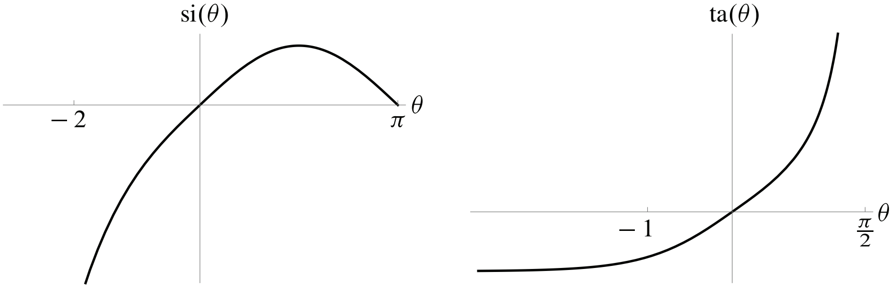

Figure 1. The piecewise trigonometric functions si and ta.

(6) The hemisphere has second Neumann eigenvalue $\mu_2(\pi/2) = 2$, with eigenfunctions the coordinate functions x and y, that is, the first spherical harmonics. Hence, by part (ii) of the theorem, $\mu_2(\Theta) > 2\csc^2\Theta$ when $\Theta \in (0, \pi/2)$. This lower bound improves by a factor of 2 on an estimate proved by Ashbaugh and Benguria [1, Lemma 3.2], although our work is in 2 dimensions whereas theirs holds in all dimensions.

(7) Monotonicity in Theorem 1 is somewhat surprising, since the Neumann eigenvalues themselves generally fail to vary monotonically. The second Neumann eigenvalue $\mu_2(\Theta)$ is graphed for $0 < \Theta < \pi$ in Figure 4, where one sees that as the cap expands to fill the sphere, the eigenvalue first decreases and then increases. This behavior was observed numerically by Dauge and Pogu [16, pp. 239–240] and proved by Dauge and Helffer [15, Section 6]. Thus, the monotonicity in our Theorem 1 must encode a subtle interplay between the eigenvalue and the normalizing geometric quantity.

Improved scaling for low Neumann and Dirichlet eigenvalues. Theorem 1 (i) says for spherical caps that $\mu_k(\Theta)4\sin^2(\Theta/2)/\cos^2(\Theta/2)$ is increasing. For the second eigenvalue ($k = 2$) we are able to strengthen that assertion by removing the cosine factor, and similarly in the hyperbolic case, as the next theorem shows.

Theorem 2 (Second Neumann eigenvalue – improved scaling). *The function*

$$\Theta \mapsto \begin{cases} \mu_2(\Theta)\,4\sinh^2\frac{\Theta}{2}, & \Theta \in (-\infty, 0), \\ \mu_2(D), & \Theta = 0, \\ \mu_2(\Theta)\,4\sin^2\frac{\Theta}{2}, & \Theta \in (0, \pi), \end{cases}$$

increases strictly from 2 *to* 8.

The theorem is proved in Section 7 and illustrated on the left side of Figure 3. The geometric interpretation is that $\mu_2 A$ is an increasing function of the cap aperture Θ, where $A = 4\pi\sin^2\Theta/2$ is the area. Numerically, $\mu_2(D) = (j'_{1,1})^2 \approx (1.84)^2$.

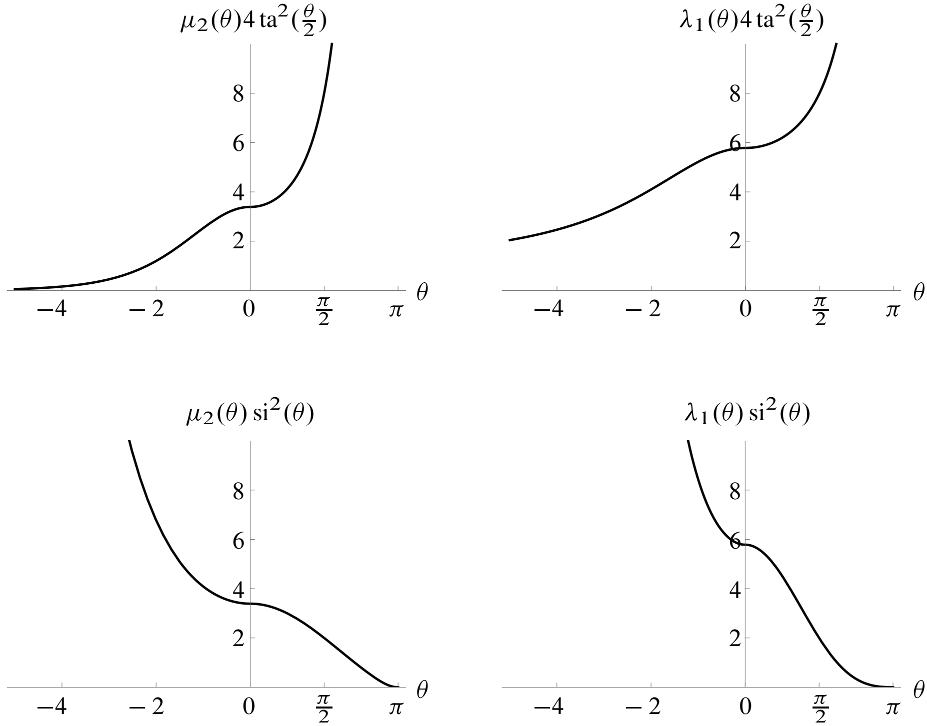

Figure 2. Monotonicity properties of the second Neumann eigenvalue (left) and first Dirichlet eigenvalue (right), for geodesic disks in hyperbolic space ($\Theta < 0$) and the sphere ($0 < \Theta < \pi$). TOP. tan/tanh from Theorem 1 (i). BOTTOM. sin/sinh from Theorem 1 (ii).

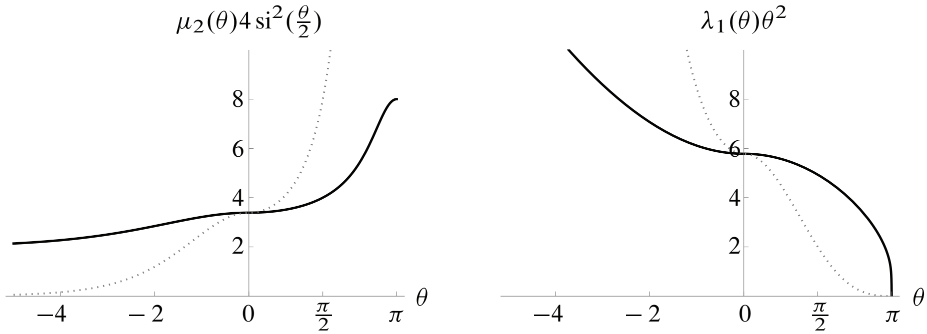

Figure 3. LEFT. Solid curve shows monotonicity of the second Neumann eigenvalue normalized by $4\,\mathrm{si}^2\,\Theta/2$, from Theorem 2, which improves on the dotted curve from Figure 2 top left where the normalization is $4\,\mathrm{ta}^2\,\Theta/2$. RIGHT. Solid curve shows monotonicity of the first Dirichlet eigenvalue normalized by Θ^2, from Theorem 3, which improves on the dotted curve from Figure 2 bottom right where the normalization is $\mathrm{si}^2\,\Theta$.

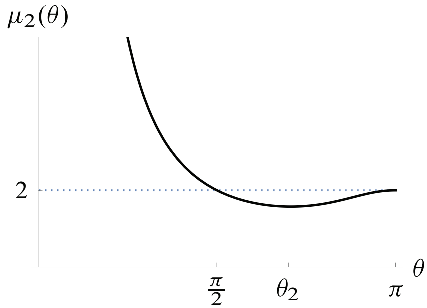

Figure 4. Second Neumann eigenvalue $\mu_2(\Theta)$ for a spherical cap, plotted against the aperture angle Θ. The eigenvalue decreases as the aperture increases, through the hemisphere ($\Theta = \pi/2$) and somewhat beyond, but then changes direction and starts increasing from $\Theta_2 \simeq 0.7\pi$, as the cap expands to fill the whole sphere ($\Theta = \pi$). The hemisphere and full sphere have eigenvalue 2 with spherical harmonic eigenfunctions $u = x = \sin\theta\cos\phi$ and $u = y = \sin\theta\sin\phi$.

The theorem combines with Theorem 1 to imply improved bounds for $k = 2$:

$$\frac{1}{\sinh^2\Theta} < \frac{\mu_2(\Theta)}{\mu_2(D)} < \frac{1}{4\sinh^2\frac{\Theta}{2}}, \qquad \Theta \in (-\infty, 0),$$
$$\frac{1}{4\sin^2\frac{\Theta}{2}} < \frac{\mu_2(\Theta)}{\mu_2(D)} < \frac{1}{\sin^2\Theta}, \qquad \Theta \in (0, \pi).$$

These inequalities are sharp to leading order as $\Theta \to 0$, although not as precise as the two-term asymptotic by Fall and Weth [17, Section 3] for the Neumann eigenvalue of a shrinking geodesic ball in a manifold with variable curvature.

The first Dirichlet eigenvalue ($k = 1$) satisfies a better scaling result than provided by Theorem 1 (ii)–(iii), because the next theorem improves the scaling factor $\mathrm{si}^2\,\Theta$ to Θ^2. By convention, $\lambda_1(\Theta)\Theta^2$ is defined at $\Theta = 0$ to equal its limiting value $\lambda_1(D)$.

Theorem 3 (First Dirichlet eigenvalue – improved scaling). *The function $\lambda_1(\Theta)\Theta^2$ decreases strictly from ∞ to 0, for $\Theta \in (-\infty, \pi)$.*

Remarks. (1) The right side of Figure 3 illustrates the theorem. Section 8 has the proof. Geometrically, the aperture Θ equals the geodesic radius of the cap or disk.

(2) Combining Theorem 3 and Theorem 1, we obtain improved two-sided bounds:

$$\frac{1}{\Theta^2} < \frac{\lambda_1(\Theta)}{\lambda_1(D)} < \frac{1}{4\tanh^2\frac{\Theta}{2}}, \qquad \Theta \in (-\infty, 0),$$
$$\frac{1}{4\tan^2\frac{\Theta}{2}} < \frac{\lambda_1(\Theta)}{\lambda_1(D)} < \frac{1}{\Theta^2}, \qquad \Theta \in (0, \pi).$$

Numerically, $\lambda_1(D) = (j_{0,1})^2 \approx (2.40)^2 = 5.76$.

(3) Theorem 3 holds in all dimensions $n \geq 2$, and will be proved in that setting. In the special case of 3 dimensions, the Dirichlet ground state of a geodesic ball of radius Θ is given explicitly in terms of the geodesic distance $\theta \in (0, \Theta)$ from the center by

$$u_1(\theta) = \begin{cases} \sin(\pi \frac{\theta}{\Theta}) / \sinh \theta, & \Theta \in (-\infty, 0), \\ \sin(\pi \frac{\theta}{\Theta}) / \sin \theta, & \Theta \in (0, \pi), \end{cases}$$

with eigenvalue $\lambda_1(\Theta) = (\pi/\Theta)^2 - \operatorname{sign} \Theta$ (see [10, Remark 3.1]). Thus, $\lambda_1(\Theta)\Theta^2 = \pi^2 - \Theta^2 \operatorname{sign} \Theta$, which is obviously decreasing. In all dimensions, the ground state can be written in terms of Legendre P or Q-functions, but there appears to be no explicit formula for the eigenvalue except in dimension 3.

(4) Cheng's theorem contains Theorem 3 and its proof as a special case, as we explain below in the proof of Corollary 6 (i). Different approaches were provided in the positive curvature case by Ashbaugh and Benguria [2, Theorem 1.2], based on differentiating $\lambda_1(\Theta)$ with respect to the aperture $\Theta \in (0, \pi)$, and in the negative curvature case by Benguria and Linde [8, Lemma 4.2], who transformed to a Schrödinger equation and obtained Corollary 6 (i). Those authors did not connect their work to Cheng's theorem.

(5) The monotonicity of $\lambda_1(\Theta)\Theta^2$ in dimension 2 follows also from a much later result by Borisov and Freitas [11, Lemma 3.1], since their function "H" is negative. Their formula could perhaps be used to prove monotonicity in dimensions $n \geq 4$, but in those dimensions the H function changes sign and so one would need to show a certain integral involving the eigenfunction is nonpositive, which seems challenging.

(6) A two-sided inequality for $\lambda_1(\Theta)$ by Borisov and Freitas [11, Theorem 3.3] is tighter than the one deduced above, although their bounding expressions are more complicated. Their Theorem 4.1 yields also a rather precise asymptotic expansion of the eigenvalue as $\Theta \to 0$.

The next theorem shows that the second Dirichlet eigenvalue ($k = 2$) satisfies a stronger inequality than Theorem 1 (i) for $\Theta \in (0, \pi)$: the factor $4\tan^2 \Theta/2$ can be improved to Θ^2. Similarly, Theorem 1 (ii) for $\Theta \in (-\infty, 0)$ can be strengthened by improving the factor $\sinh^2 \Theta$ to Θ^2.

Theorem 4 (Second Dirichlet eigenvalue – improved scaling). *For $\Theta \in (-\infty, 0)$, the function $\lambda_2(\Theta)\Theta^2$ decreases strictly from ∞ to $\lambda_2(D)$, and for $\Theta \in (0, \pi)$ it increases strictly from $\lambda_2(D)$ to $2\pi^2$.*

Remarks. (1) The theorem is proved in Section 9. The hyperbolic part, for $\Theta < 0$, will be proved in all dimensions $n \geq 2$. The spherical part, for $0 < \Theta < \pi$, holds only in 2 dimensions. See Figure 5 for a plot of the 2-dimensional case. In dimensions $n \geq 3$,

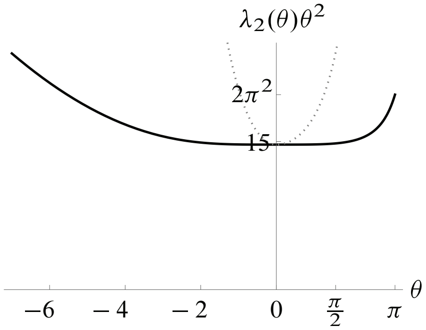

Figure 5. The solid curve shows the minimum at $\Theta = 0$ of the second Dirichlet eigenvalue normalized by Θ^2, as proved in Theorem 4. For $0 < \Theta < \pi$, this curve improves on the monotonicity of the dotted curve with normalization $4\tan^2 \Theta/2$, from Theorem 1 (i). For $\Theta < 0$, the solid curve improves on the dotted curve with normalization $\sinh^2 \Theta$, from Theorem 1 (ii).

it appears based on numerical investigations that $\lambda_2(\Theta)\Theta^2$ is strictly decreasing on $(-\infty, \Theta_*(n)]$ and strictly increasing on $[\Theta_*(n), \pi)$, for some positive number $\Theta_*(n)$, and that $\Theta_*(n)$ increases with the dimension n. In 2 dimensions, Theorem 4 shows that the minimum point occurs at $\Theta_*(2) = 0$.

(2) In conjunction with Theorem 1, the last theorem implies improved two-sided bounds on the second Dirichlet eigenvalue:

$$\frac{1}{\Theta^2} < \frac{\lambda_2(\Theta)}{\lambda_2(D)} < \frac{1}{4\tanh^2 \frac{\Theta}{2}}, \qquad \Theta \in (-\infty, 0),$$
$$\frac{1}{\Theta^2} < \frac{\lambda_2(\Theta)}{\lambda_2(D)} < \frac{1}{\sin^2 \Theta}, \qquad \Theta \in (0, \pi).$$

(3) Ashbaugh and Benguria [2, Remark on p. 1071] proved Theorem 4 in the spherical case $0 < \Theta < \pi$. Our proof in Section 9 is more direct than their perturbational approach, in that our proof uses only the Rayleigh principle and integration by parts. In the hyperbolic case $\Theta \in (-\infty, 0)$, the theorem is new as far as we know.

(4) The eigenvalue ratio $\lambda_2(\Theta)/\lambda_1(\Theta)$ is strictly increasing for $\Theta \in (0, \pi)$, by Theorem 3 and Theorem 4. This property was observed already by Ashbaugh and Benguria [2, Remark on p. 1071]. In all dimensions, they proved that the ratio is increasing up to the hemisphere [2, Theorem 1.3], that is, for $\Theta \in (0, \pi/2)$, with the 2- and 3-dimensional cases holding on the larger range $\Theta \in (0, \pi)$. Benguria and Linde [8, Theorem 1.2] extended that ratio result to the hyperbolic case $\Theta < 0$ in all dimensions (notice their θ is our $-\Theta$, which reverses the monotonicity statement), but did not investigate monotonicity of $\lambda_2(\Theta)\Theta^2$ as in Theorem 4 above. They also

proved monotonicity of the eigenvalue ratio for Schrödinger operators, under a convexity assumption on the radial derivative of the potential [7, Theorem 2.2].

Bandle-type bounds on Neumann caps with varying curvature. Until now we have considered caps and geodesic disks with varying radii in spheres and hyperbolic spaces of constant curvature ± 1. Changing viewpoint, we next examine caps with varying curvature, while fixing either the area or the geodesic radius. In particular, a corollary of Theorem 2 that we next develop strengthens an inequality of Bandle for the second Neumann eigenvalue.

Fix $A > 0$ and write $\mu_2(K; A)$ for the second Neumann eigenvalue of the Laplace–Beltrami operator on a geodesic disk of curvature K and area A. The disk sits in a hyperbolic plane if K is negative, in the Euclidean plane if K is 0, and in a sphere of radius $1/\sqrt{K}$ if $0 < K < 4\pi/A$. That last restriction on the curvature ensures $A < 4\pi/K$, so that the sphere indeed contains a cap with area A.

When $\rho > 0$ is fixed, we denote by $\mu_2(K; \rho)$ the second Neumann eigenvalue of the Laplace–Beltrami operator on a disk of geodesic radius ρ and curvature $K < (\pi/\rho)^2$. The last restriction says when K is positive that $\rho < \pi/\sqrt{K}$, which means the sphere of radius $1/\sqrt{K}$ does contain a cap of aperture ρ.

Corollary 5 (Bandle-type inequality on second Neumann eigenvalue). (i) *Fix $A > 0$. The second Neumann eigenvalue $\mu_2(K; A)$ of the Laplace–Beltrami operator on a geodesic disk of area A is a strictly increasing function of the curvature $K \in (-\infty, 4\pi/A)$.*

(ii) *Fix $\rho > 0$. The second Neumann eigenvalue $\mu_2(K; \rho)$ on a geodesic disk of radius ρ is a strictly increasing function of the curvature $K \in (-\infty, (\pi/\rho)^2)$.*

The "fixed area" result in part (i) of the corollary is stronger than the "fixed radius" result in part (ii), as the proof will make clear. Part (ii) was obtained previously in greater generality by Li, Wang, and Wu [22, Theorem 1.1] in 2 and 3 dimensions under an upper bound on the sectional curvature. That is, they allow one of the disks to have non-constant curvature and estimate its eigenvalue with that of a constant curvature disk. Their paper also provides results and references to similar inequalities for the first and second Robin eigenvalues.

For $K \leq 2\pi/A$, part (i) of the corollary is due to Bandle [4], [5, Corollary 3.9]. The extension to $K < 4\pi/A$ in Corollary 5 is new and allows us to handle spherical caps larger than a hemisphere. The proof in Section 10 is new too: Bandle derived her result from a comparison theorem for arbitrary domains whereas the approach in this paper is more direct and relies only on the caps and disks from Theorem 2.

Cheng-type bounds on Dirichlet caps with varying curvature. Theorem 3 implies a special case of Cheng's inequalities for the first Dirichlet eigenvalue.

Corollary 6 (Cheng-type inequality on first Dirichlet eigenvalue). (i) *Fix $\rho > 0$. The first Dirichlet eigenvalue $\lambda_1(K;\rho)$ of the Laplace–Beltrami operator on a geodesic disk of radius ρ is a strictly decreasing function of the curvature $K \in (-\infty, (\pi/\rho)^2)$.*

(ii) *Fix $A > 0$. The first Dirichlet eigenvalue $\lambda_1(K;A)$ on a geodesic disk of area A is a strictly decreasing function of the curvature $K \in (-\infty, 4\pi/A)$.*

Section 10 has the proof, which holds in all dimensions although we state it only for the 2-dimensional case. The Cheng-type "fixed radius" result in part (i) of the corollary is stronger than the "fixed area" result in part (ii).

Cheng [13,14] handled a much more general situation than the corollary, on manifolds with variable curvature bounded either above or below. A good exposition can be found in Chavel's book [12, Chapter III]. The negative curvature case of the corollary was proved by Benguria and Linde [8, Lemma 4.2], without them perhaps realizing it is a special case of Cheng's theorem.

For the second Dirichlet eigenvalue, we obtain in Section 10 a comparison for disks that is new as far as we know.

Corollary 7 (Cheng-type inequality on second Dirichlet eigenvalue). (i) *Fix $\rho > 0$. The second Dirichlet eigenvalue $\lambda_2(K;\rho)$ of the Laplace–Beltrami operator on a geodesic disk of radius ρ is a strictly decreasing function of negative curvature $K \in (-\infty, 0)$ and a strictly increasing function of positive curvature $K \in (0, (\pi/\rho)^2)$.*

(ii) *Fix $A > 0$. The second Dirichlet eigenvalue $\lambda_2(K;A)$ on a geodesic disk of area A is a strictly decreasing function of negative curvature $K \in (-\infty, 0)$.*

Open problems. The following conjectures are based on numerical investigations.

(a) *Second Dirichlet eigenvalue.* Is $\lambda_2(\Theta) 4\sin^2\Theta/2$ decreasing when $0 < \Theta < \pi$? If so, then the Dirichlet version of Theorem 1 (ii) would be strengthened when $\Theta \in (0,\pi)$ and hence Corollary 7 (ii) would extend to positive curvatures $K \in (-\infty, 4\pi/A)$.

This conjecture is interesting only for positive Θ values. The analogous quantity $\lambda_2(\Theta) 4\sinh^2\Theta/2$ is already known to be decreasing when $\Theta < 0$ because $\lambda_2(\Theta)\Theta^2$ is decreasing in that range by Theorem 4.

(b) *Spectral gap.* Is $(\lambda_2(\Theta) - \lambda_1(\Theta))\Theta^2$ increasing for $\Theta \in (-\infty, \pi)$? If so, then $(\lambda_2(\Theta) - \lambda_1(\Theta)) 4\,\mathrm{ta}^2\,\Theta/2$ is increasing too. These claims are known when $\Theta \in (0,\pi)$ by Ashbaugh and Benguria [2, p. 1071], i.e., by combining Theorem 3 and Theorem 4.

Is $(\lambda_2(\Theta) - \lambda_1(\Theta))\,\mathrm{si}^2\,\Theta$ decreasing for $\Theta \in (-\infty, \pi)$? When $\Theta < 0$, a stronger result seems to hold, namely that $(\lambda_2(\Theta) - \lambda_1(\Theta)) 4\sinh^2\Theta/2$ is decreasing.

(c) *Higher dimensions.* One would like analogues of Theorem 1 and Theorem 2 in higher dimensions. New ideas will be needed, because the methods in this paper are decidedly 2-dimensional: by Section 5.1, the hyperbolic and spherical eigenvalue problems transform to equations of the form $-\Delta v = \nu w v$, but in higher dimensions the transformed equation is more complicated – its left side becomes a divergence-form operator, which means that both sides of the equation involve a weight.

3. Eigenvalues tend to infinity as the weight tends pointwise to zero

This section may safely be skipped at a first reading, since it examines limiting values and not the monotonicity results that are the main focus of the paper.

Certain limits in Section 4 involve a weighted Laplacian whose weight tends to zero pointwise but not uniformly. The next lemma shows that, despite the non-uniformity of the convergence, all eigenvalues must tend to infinity.

Lemma 8. *Let Ω be a bounded domain in $\mathbb{R}^n, n \geq 1$. For each $R > 0$, suppose $w_R : \Omega \to (0, 1]$ is measurable with* $\operatorname{ess\,inf} w_R > 0$. *Write $\lambda_k(w_R)$ and $\mu_k(w_R)$ for the Dirichlet and Neumann eigenvalues of $-w_R^{-1}\Delta$, respectively, where in the Neumann case we further assume Ω has Lipschitz boundary.*

If $\lim_{R\to\infty} w_R = 0$ *a.e. then*

$$\lim_{R\to\infty} \lambda_k(w_R) = \infty \quad \textit{for each } k \geq 1$$

and

$$\lim_{R\to\infty} \mu_k(w_R) = \infty \quad \textit{for each } k \geq 2.$$

The first Neumann eigenvalue $\mu_1(w_R)$ is omitted from the theorem, as it equals 0.

Proof. Note that $w_R^{-1}\Delta$ has discrete spectrum since the weight w_R is bounded above and bounded below away from 0. Fix $0 < \delta < 1$ and define a weight

$$m_R(x) = \max(\delta, w_R(x)), \qquad x \in \Omega,$$

so that m_R is bounded away from zero. The k-th Neumann eigenvalue associated with $-m_R^{-1}\Delta$ has variational characterization

$$\mu_k(m_R) = \min_L \max_{f\in L\setminus\{0\}} \frac{\int_\Omega |\nabla f|^2\,dx}{\int_\Omega f^2 m_R\,dx}$$

where $L \subset W^{1,2}(\Omega)$ is an arbitrary k-dimensional subspace. Since $w_R \leq m_R$ by construction, we see by comparing with the corresponding characterization for $\mu_k(w_R)$

that

$$\mu_k(w_R) \geq \mu_k(m_R),$$

for each k and R.

The weight satisfies $\delta \leq m_R \leq 1$ by definition, so that

$$\delta^{-1}\mu_k(1) \geq \mu_k(m_R) \geq \mu_k(1) \tag{1}$$

by the variational characterization, where $\mu_k(1)$ is the k-th Neumann eigenvalue of the unweighted Laplacian. We will show for $k \geq 2$ that

$$\liminf_{R\to\infty} \mu_k(m_R) \geq \delta^{-1}\mu_2(1). \tag{2}$$

Since δ can be taken arbitrarily small and the second Neumann eigenvalue $\mu_2(1)$ is positive, we deduce that $\mu_k(w_R)$ must tend to infinity as $R \to \infty$.

For each R, take a k-th Neumann eigenfunction $f_R(x)$ for $-m_R^{-1}\Delta$, satisfying

$$-\Delta f_R = \mu_k(m_R) m_R f_R \tag{3}$$

weakly, meaning

$$\int_\Omega \nabla\varphi \cdot \nabla f_R \, dx = \mu_k(m_R) \int_\Omega \varphi f_R m_R \, dx$$

for all $\varphi \in W^{1,2}(\Omega)$. We choose to normalize the eigenfunction in the unweighted space $L^2(\Omega)$, so that $\int_\Omega f_R(x)^2 \, dx = 1$. Consider a sequence of R-values tending to ∞. The weak form of the eigenfunction equation (3) yields a bound on the L^2-norm of the gradient:

$$\int_\Omega |\nabla f_R|^2 \, dx = \mu_k(m_R) \int_\Omega f_R^2 m_R \, dx \leq \delta^{-1}\mu_k(1) \int_\Omega f_R^2 \, dx = \delta^{-1}\mu_k(1) < \infty$$

where the inequality uses that $m_R \leq 1$ and the final equality relies on the L^2-normalization of the eigenfunction. After passing to a subsequence of R-values, the Rellich–Kondrachov theorem yields a function $f \in W^{1,2}(\Omega)$ such that $f_R \rightharpoonup f$ weakly in $W^{1,2}(\Omega)$ and $f_R \to f$ in $L^2(\Omega)$. After passing to a further subsequence we may suppose the numbers $\mu_k(m_R)$ converge to a limit as $R \to \infty$. Hence,

$$\int_\Omega f^2 \, dx = 1$$

and f satisfies the weak eigenfunction equation

$$-\Delta f = \big(\delta \lim_{R\to\infty} \mu_k(m_R)\big) f, \tag{4}$$

as we now show. For each test function $\varphi \in W^{1,2}(\Omega)$,

$$
\begin{aligned}
&\int_\Omega \nabla f \cdot \nabla\varphi \, dx \\
&= \lim_{R\to\infty} \int_\Omega \nabla f_R \cdot \nabla\varphi \, dx && \text{(by weak convergence)} \\
&= \lim_{R\to\infty} \mu_k(m_R) \int_\Omega f_R \varphi m_R \, dx && \text{(by the weak eigenfunction equation (3))} \\
&= \lim_{R\to\infty} \mu_k(m_R) \int_\Omega f \varphi m_R \, dx && \text{(since } f_R \to f \text{ in } L^2(\Omega) \text{ and } m_R \le 1) \\
&= \lim_{R\to\infty} \mu_k(m_R) \int_\Omega f \varphi \delta \, dx
\end{aligned}
$$

by dominated convergence, since $w_R \to 0$ a.e. by hypothesis and hence $m_R \to \delta$ a.e.

The eigenfunction equation (4) says that f is an eigenfunction of the unweighted Neumann Laplacian on Ω. Its eigenvalue $\delta \lim_{R\to\infty} \mu_k(m_R)$ is positive by (1), since $k \ge 2$, and so must be greater than or equal to the second eigenvalue $\mu_2(1)$. The original sequence of R-values was arbitrary and so we conclude $\delta \liminf_{R\to\infty} \mu_k(m_R) \ge \mu_2(1)$, which proves (2) and hence finishes the Neumann case of the proof.

For the Dirichlet eigenvalues, the only difference is that for $k \ge 1$ one finds the positive number $\delta \liminf_{R\to\infty} \lambda_k(m_R)$ is greater than or equal to the *first* eigenvalue of the unweighted Dirichlet Laplacian. This Dirichlet eigenvalue is positive, unlike the first Neumann eigenvalue, and so again taking δ arbitrarily small completes the proof. ∎

4. Monotonicity for arbitrary domains

The eigenvalue problem on spherical caps and hyperbolic disks will be recast into a problem for a weighted Laplacian on Euclidean disks. We begin by studying such weighted operators. Consider a bounded Lipschitz domain Ω in the plane $\mathbb{R}^2$. Denote by $\nu_k(\Omega; w_\pm)$ the k-th Neumann eigenvalue ($k \ge 1$) of the weighted Laplacian

$$
\begin{cases}
-\Delta v = \nu w_\pm v & \text{on } \Omega, \\
\frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega,
\end{cases}
$$

where the radial weight function is

$$
w_\pm(r) = \frac{4}{(1 \pm r^2)^2}.
$$

In the w_- case, the domain Ω is assumed to be a proper subdomain of the unit disk, so that $r < 1$.

The weight corresponds to a metric of curvature ± 1, since $-(\Delta \log w_\pm)/2w_\pm = \pm 1$. (For background on curvature of surfaces, see Bandle [5, §I.3].) Note that the first eigenvalue is zero, $\nu_1 = 0$, with constant eigenfunction.

Let $\mu_k(\Omega)$ be the k-th Neumann eigenvalue of the unweighted Euclidean Laplacian.

Proposition 9 (Scaling monotonicity for weighted Laplacian eigenvalues). *Assume Ω is a bounded, planar Lipschitz domain and write $M = \max_{z \in \overline{\Omega}} |z|$, so that Ω/M is contained in the unit disk. Let $k \geq 2$.*

(i) *The functional*

$$R \mapsto \begin{cases} \nu_k(R\Omega; w_-)\, 4R^2, & R \in (-1/M, 0), \\ \mu_k(\Omega), & R = 0, \\ \nu_k(R\Omega; w_+)\, 4R^2, & R \in (0, \infty), \end{cases}$$

increases strictly and continuously. Its limit as $R \to -1$ equals $4\nu_k(\Omega; w_-)$ if Ω is a proper subdomain of the unit disk and equals 0 if $\Omega = D$. Its limit as $R \to \infty$ is ∞.

(ii) *Suppose Ω is contained in the unit disk. The functional*

$$R \mapsto \begin{cases} \nu_k(R\Omega; w_-)\, 4R^2/(1 - R^2)^2, & R \in (-1, 0), \\ \mu_k(\Omega), & R = 0, \\ \nu_k(R\Omega; w_+)\, 4R^2/(1 + R^2)^2, & R \in (0, \infty), \end{cases}$$

decreases strictly and continuously. Its limit as $R \to -1$ is ∞. Its limit as $R \to \infty$ equals

$$\begin{cases} 0 \text{ if the closure } \overline{\Omega} \text{ contains the origin,} \\ \nu_k(\Omega; |z|^{-4}) \text{ otherwise.} \end{cases}$$

(iii) *The corresponding statements hold also for Dirichlet eigenvalues when $k \geq 1$ except that in part* (i), *if $\Omega = D$ then as $R \to -1$, the limit of $\nu_k(R\Omega; w_-)\, 4R^2$ equals 1 rather than 0.*

When $R < 0$, the domain $R\Omega$ is obtained by rescaling Ω and reflecting through the origin. The reflection is harmless: it does not change the eigenvalue because the weight $w_\pm$ is radial. Thus, the eigenvalue in the proposition could be written as $\nu_k(|R|\Omega; w_\pm)$, but for notational simplicity we omit the absolute value signs.

Proof. The proposition is proved in several steps, as follows.

Rescaling the Rayleigh quotient. The Rayleigh quotient for the weighted Neumann eigenvalue $\nu_k(R) = \nu_k(R\Omega; w_\pm)$ is

$$Q_R[v] = \frac{\int_{R\Omega} |\nabla v|^2 \, dA}{\int_{R\Omega} v^2 w_\pm \, dA}$$

where dA is Euclidean area measure (2-dimensional Lebesgue measure), v belongs to the Sobolev space $W^{1,2}(R\Omega)$, and the Rayleigh quotient uses w_+ when $R > 0$ and w_- when $R < 0$. The minimax characterization of the k-th eigenvalue says

$$\nu_k(R) = \min_L \max_{v \in L \setminus \{0\}} Q_R[v] \tag{5}$$

where L ranges over all k-dimensional subspaces of $W^{1,2}(R\Omega)$.

The idea is to rescale the Rayleigh quotient from $R\Omega$ to Ω by letting $f(z) = v(Rz)$, so that $f \in W^{1,2}(\Omega)$ and

$$Q_R[v] 4R^2 = \frac{\int_\Omega |\nabla f(z)|^2 \, dA}{\int_\Omega f(z)^2 w_\pm(Rz)/4 \, dA}.$$

Dividing by $(1 \pm R^2)^2$ shows

$$Q_R[v] \frac{4R^2}{(1 \pm R^2)^2} = \frac{\int_\Omega |\nabla f(z)|^2 \, dA}{\int_\Omega f(z)^2 w_\pm(R, z) \, dA}$$

where the new weight is

$$w_\pm(R, z) = \left(\frac{1 \pm R^2}{1 \pm R^2 |z|^2} \right)^2. \tag{6}$$

Hence, the minimax characterization (5) implies

$$\nu_k(R) 4R^2 = \min_L \max_{f \in L \setminus \{0\}} \frac{\int_\Omega |\nabla f(z)|^2 \, dA}{\int_\Omega \frac{1}{4} f(z)^2 w_\pm(Rz) \, dA} \tag{7}$$

and

$$\nu_k(R) \frac{4R^2}{(1 \pm R^2)^2} = \min_L \max_{f \in L \setminus \{0\}} \frac{\int_\Omega |\nabla f(z)|^2 \, dA}{\int_\Omega f(z)^2 w_\pm(R, z) \, dA} \tag{8}$$

where L ranges over all k-dimensional subspaces of $W^{1,2}(\Omega)$. When $R = 0$, these characterizations yield the unweighted eigenvalue $\mu_k(\Omega)$ on the right side, since $w_\pm(0)/4 = 1$ and $w_\pm(0, z) = 1$.

Proof of Proposition 9 (i). The weight $w_\pm(Rz)/4 = 1/(1 \pm R^2|z|^2)^2$ in formula (7) is strictly decreasing with respect to $R \in (-1/M, \infty)$, remembering that w_- is used when $R < 0$ and w_+ when $R > 0$. Hence, it follows from (7) that $R \mapsto \nu_k(R) 4R^2$ is strictly increasing. Continuity with respect to R follows easily too.

As $R \to \infty$, we find that $\nu_k(R)4R^2 \to \infty$ by applying Lemma 8 to $w_R(z) = w_+(Rz)/4$.

If Ω is a proper subdomain of the unit disk then $M < 1$, and at $R = -1$ the quantity $\nu_k(R)4R^2$ equals $4\nu_k(\Omega; w_-)$.

It remains to show that if Ω is the unit disk D then $\nu_k(R)4R^2 \to 0$ as $R \to -1$, which we do by applying the variational characterization to a suitably chosen subspace of trial functions, as follows. Choose $h \in C^1[0,1]$ such that h is increasing, $h = 0$ on $[0, 1/4]$ and $h = 1$ on $[3/4, 1]$, and let L be the k-dimensional subspace of $W^{1,2}(D)$ spanned by $h(r)\cos j\phi$ for $j = 1, \dots, k$, where r and ϕ are the polar coordinates. Each of these functions is L^2-orthogonal to the constant and to each other, with respect to the radial weight $w_-(Rz)/4$. Considering the arbitrary linear combination $f = \sum_{j=1}^k c_j h(r)\cos j\phi$, we see from the variational characterization (7) that

$$\nu_k(R)4R^2 \leq \max_{|c|=1} \frac{\sum_{j=1}^k c_j^2 \int_D |\nabla(h(r)\cos j\phi)|^2 r\,dr\,d\phi}{\sum_{j=1}^k c_j^2 \int_D |h(r)\cos j\phi|^2 (\frac{1}{4}w_-(Rr)) r\,dr\,d\phi}$$

where $c = (c_1, \dots, c_k)$ is the coefficient vector. The numerator is independent of R, while each integral in the denominator tends to ∞ as $R \to -1$ because $h(r) = 1$ for all r near 1 and $w_-(-r)/4 = (1 - r^2)^{-2}$ is not integrable near $r = 1$. (In other words, the hyperbolic disk has infinite area.) It follows that the right side of the inequality tends to 0 as $R \to -1$, so that $\nu_k(R)4R^2 \to 0$ as claimed.

Alternatively, one could call on a result of Korevaar [18, Theorem 1.3] that estimates the Neumann eigenvalues in a general setting, under a lower curvature bound.

Proof of Proposition 9 (ii). Assume $\Omega \subset D$, so that $|z| < 1$ in what follows. The weight $w_\pm(R, z)$ in formula (6) is strictly increasing as a function of $R \in (-1, \infty)$, because

$$w_\pm(R, z)^{-1/2} = \frac{1 \pm R^2|z|^2}{1 \pm R^2} = \frac{1 - |z|^2}{1 \pm R^2} + |z|^2$$

and this expression is strictly decreasing with respect to R, remembering that the minus sign is used when $R < 0$ and the plus sign when $R > 0$.

Since $R \mapsto w_\pm(R, z)$ is strictly increasing, formula (8) ensures that the normalized eigenvalue $R \mapsto \nu_k(R)4R^2/(1 \pm R^2)^2$ is strictly decreasing.

If Ω is a proper subdomain of the unit disk then $\nu_k(-1) > 0$ while as $R \to -1$ the factor $4R^2/(1 - R^2)^2$ tends to ∞, and hence $\nu_k(R)4R^2/(1 - R^2)^2 \to \infty$. The same conclusion holds when Ω is any subdomain of the unit disk D, proper or not, as we conclude by applying Lemma 8 to the weight

$$w_-(R, z) = \left(\frac{1 - R^2}{1 - R^2|z|^2}\right)^2 \leq 1,$$

noting this weight tends to 0 as $R \to -1$, for each $z \in D$. (Although Lemma 8 is stated for the range $0 < R < \infty$ with $R \to \infty$, the same result holds for $-1 < R < 0$ with $R \to -1$, simply by relabeling.)

For the limiting value as $R \to \infty$, suppose first the origin does not lie in the closure of Ω, so that $|z|^{-4}$ is continuous and is bounded above and bounded below away from zero for $z \in \Omega$. Clearly, $w_+(R,z)/|z|^{-4} \to 1$ as $R \to \infty$, uniformly with respect to $z \in \Omega$. Letting $R \to \infty$ in the variational characterization (8) therefore implies that $\nu_k(R)4R^2/(1+R^2)^2 \to \nu_k(\Omega; |z|^{-4})$, as needed.

Suppose next that the origin does lie in the closure of Ω. Since $\partial\Omega$ is Lipschitz by hypothesis, Ω must contain some sector $\mathscr{C}$ with vertex at the origin and radius less than 1. Notice $\mathscr{C} \subset R\mathscr{C} \subset R\Omega$ for each $R \geq 1$. Consider a fixed k-dimensional subspace L of $C_0^\infty(\mathscr{C})$. Since $L \subset W_0^{1,2}(R\Omega)$, the characterization (5) implies

$$\nu_k(R) \leq \max_{v \in L\setminus\{0\}} \frac{\int_{\mathscr{C}} |\nabla v|^2 \, dA}{\int_{\mathscr{C}} v^2 w_+ \, dA}.$$

The right side is finite and independent of R, and so after multiplying by the normalizing factor $4R^2/(1+R^2)^2 = O(R^{-2})$ we see the product tends to 0 as $R \to \infty$, which completes the proof of Proposition 9 for Neumann eigenvalues.

Proof of Proposition 9 (iii) – *Dirichlet eigenvalues.* To prove analogous statements for the Dirichlet eigenvalues with $k \geq 1$, simply change the trial space from $W^{1,2}$ to $W_0^{1,2}$.

For the special case $\Omega = D$, in the Dirichlet analogue of part (i) we want the limiting value $\nu_k(RD; w_-)4R^2 \to 1$ as $R \to -1$, which means we want $\nu_k(RD; w_-)$ to converges to $1/4$. This eigenvalue equals the k-th Dirichlet eigenvalue of the hyperbolic Laplacian on a geodesic disk of hyperbolic radius $|\Theta|$, by the changes of variable in Section 5.3 below, where $\tanh \Theta/2 = R$. That is, using the notation employed elsewhere in the paper, we want $\lambda_k(\Theta) \to 1/4$ as $\Theta \to -\infty$.

The first ($k = 1$) Dirichlet eigenvalue of the geodesic disk always exceeds $1/4$, meaning $\lambda_1(\Theta) > 1/4$, by a brief argument with Cauchy–Schwarz [12, p. 47]. It is further known that $\lambda_1(\Theta) \to 1/4$ as the radius $|\Theta|$ tends to ∞, because one has $\limsup_{\Theta\to-\infty} \lambda_1(\Theta) \leq 1/4$ by a result of McKean [12, Theorem 5 on p. 46]. (A recent asymptotic formula by Kristály [19] provides a precise rate of convergence, if desired.) To handle higher eigenvalues, one may call on a result of Berge [9, Corollary 3.4], or else show as follows that $\limsup_{\Theta\to-\infty} \lambda_k(\Theta) \leq 1/4$ for each fixed $k \geq 2$.

The geodesic disk of radius $|\Theta|$ contains k disjoint geodesic disks of radius $|\Theta|/k$. Domain monotonicity for the Dirichlet spectrum ensures that the k-th eigenvalue of the disk of radius $|\Theta|$ is less than or equal to the k-th eigenvalue of the disjoint union of the smaller disks, which equals the first eigenvalue of one of the disks of radius $|\Theta|/k$.

As mentioned above, that first eigenvalue converges to $1/4$ as $|\Theta| \to \infty$, completing the proof. ■

Remark. McKean's observation that $\limsup_{\Theta\to-\infty} \lambda_1(\Theta) \le 1/4$ can be justified quickly and explicitly in polar coordinates by substituting the trial function $f(r) = (1-r^2)^q \in W_0^{1,2}(D)$ into the Dirichlet version of the Rayleigh principle (7), with $\Omega = D$ and $q > 1/2$, yielding that

$$\begin{aligned} &\nu_1(RD; w_-)4R^2 \\ &\quad \le \frac{\int_0^1 f'(r)^2 r\, dr}{\int_0^1 f(r)^2(\frac{1}{4}w_-(Rr))r\, dr} \to \frac{\int_0^1 f'(r)^2 r\, dr}{\int_0^1 f(r)^2(1-r^2)^{-2} r\, dr} \qquad \text{as } R \to -1 \\ &\quad = 2q \to 1 \qquad \text{as } q \to 1/2. \end{aligned}$$

5. Coordinate systems for the sphere and hyperbolic disk

This section summarizes coordinate expressions for the Laplace–Beltrami operators on the sphere and hyperbolic disk, and stereographically transforms those operators into Euclidean polar coordinates (r, ϕ).

5.1. Euclidean plane

The Laplacian in polar coordinates is

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}.$$

5.2. Sphere

Take $\theta \in (0, \pi)$ to be the polar angle on the 2-sphere measured from the z-axis and $\phi \in [0, 2\pi)$ to be the azimuthal (or longitudinal) angle. By stereographic projection (taking the north pole to the origin), the coordinates (θ, ϕ) transform to polar coordinates (r, ϕ) in the plane, with

$$\sin\theta = \frac{2r}{1+r^2}, \quad \cos\theta = \frac{1-r^2}{1+r^2}, \quad \tan\frac{\theta}{2} = r,$$
$$\frac{d\theta}{\sin\theta} = \frac{dr}{r}, \quad (\sin\theta)\frac{d}{d\theta} = r\frac{d}{dr}.$$

The Laplace–Beltrami operator on the sphere is

$$\Delta_{sph} u = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big((\sin\theta)\frac{\partial u}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}.$$

The eigenfunction equation $-\Delta_{sph}u = \mu u$ transforms to $-\Delta v = \nu w_+ v$, where we have written $u(\theta, \phi) = v(r, \phi)$ and $\mu = \nu$ and the weight is

$$w_+(r) = \frac{4}{(1+r^2)^2}.$$

5.3. Hyperbolic space

The polar coordinates (p, ϕ) on the 2-dimensional hyperbolic disk transform to polar coordinates (r, ϕ) on the unit disk $\{0 \le r < 1\}$ according to:

$$\sinh p = \frac{2r}{1-r^2}, \quad \cosh p = \frac{1+r^2}{1-r^2}, \quad \tanh\frac{p}{2} = r,$$
$$\frac{dp}{\sinh p} = \frac{dr}{r}, \quad (\sinh p)\frac{d}{dp} = r\frac{d}{dr}.$$

The Laplace–Beltrami operator in hyperbolic space is

$$\Delta_{hyp}u = \frac{1}{\sinh p}\frac{\partial}{\partial p}\left((\sinh p)\frac{\partial u}{\partial p}\right) + \frac{1}{\sinh^2 p}\frac{\partial^2 u}{\partial \phi^2}.$$

The eigenfunction equation $-\Delta_{hyp}u = \mu u$ transforms to $-\Delta v = \nu w_- v$ where we have written $u(p, \phi) = v(r, \phi)$ and $\mu = \nu$ and the weight is

$$w_-(r) = \frac{4}{(1-r^2)^2}.$$

6. Proof of Theorem 1

The theorem follows from Proposition 9 applied to the unit disk $\Omega = D$, in view of the transformations in Section 5 between the spherical or hyperbolic Laplacian and the Laplacian weighted by w_+ or w_-, respectively. Those transformations preserve Neumann and Dirichlet conditions on the boundary. The geodesic disk $C(\Theta)$ transforms to a disk RD where $R = \tanh \Theta/2$ if Θ is negative and $R = \tan \Theta/2$ if Θ is positive. Hence, the normalizing factors transform according to

$$4R^2 = \begin{cases} 4\tanh^2\frac{\Theta}{2}, & \Theta \in (-\infty, 0), \\ 4\tan^2\frac{\Theta}{2}, & \Theta \in (0, \pi), \end{cases}$$
$$\frac{4R^2}{(1 \pm R^2)^2} = \begin{cases} \sinh^2\Theta, & \Theta \in (-\infty, 0), \\ \sin^2\Theta, & \Theta \in (0, \pi), \end{cases}$$

where by convention, the minus sign is used in the formula when R and Θ are negative, and the plus sign is used when they are positive.

7. Proof of Theorem 2

Write $\mu_* = \mu_2(\Theta_*)$ and $\mu = \mu_2(\Theta)$, whenever $\Theta_*, \Theta < \pi$.

Spherical case. First suppose $0 < \Theta_* < \Theta < \pi$. The goal is to show

$$\mu_* \sin^2 \Theta_*/2 < \mu \sin^2 \Theta/2. \tag{9}$$

Since $\sin^2 \Theta/2$ is strictly increasing when $\Theta \in (0, \pi)$, we may assume

$$\mu_* > \mu.$$

Hence, $\Theta_* < \Theta_2$, because $\mu_2(\cdot)$ is strictly increasing on the interval (Θ_2, π) by [21, Proposition 4.2] (see Figure 4), where $\Theta_2 \simeq 0.7\pi$ is the unique aperture determined by the condition $\mu_2(\Theta) \sin^2 \Theta = 1$.

We may further suppose $\Theta \le \Theta_2$, as follows. For if not, then $\Theta \in (\Theta_2, \pi)$ and so a number $\theta_{\max} < \Theta$ exists at which $\mu_2(\theta_{\max}) = \mu_2(\Theta) = \mu$, by [21, Remark in Section 5]. Necessarily $\theta_{\max} \le \Theta_2$, because $\mu_2(\cdot)$ is strictly increasing on the interval (Θ_2, π). Clearly, it suffices to prove (9) with $\theta_{\max}$ instead of Θ on the right side. Thus, we may assume from now on that $\Theta \le \Theta_2$.

Since $0 < \Theta_* < \Theta \le \Theta_2$, by [21, Proposition 4.1] the second eigenfunction for $C(\Theta)$ has the separated form $u = g(\theta) \cos\phi$ or $g(\theta) \sin\phi$, where $g(0) = 0$ and $g' > 0$ on $(0, \Theta)$, with the Neumann condition $g'(\Theta) = 0$ at the right endpoint. Similarly, the second eigenfunction for $C(\Theta_*)$ has the form $g_*(\theta) \cos\phi$ or $g_*(\theta) \sin\phi$, where $g_*(0) = 0$ and $g_*' > 0$ on $(0, \Theta_*)$ with the Neumann condition $g_*'(\Theta_*) = 0$.

We will transform from the θ-variable to the r-variable and then rescale to obtain functions on the unit interval, after which an ODE comparison can be performed. To begin, change variable by letting $h(r) = g(\theta)$ and $h_*(r) = g_*(\theta)$ where $r = \tan\theta/2$ (as in Section 5.2), define $v = h(r)\cos\phi$ and $v_* = h_*(r)\cos\phi$, and substitute into the eigenfunction equations $-\Delta v = \mu w_+ v$ and $-\Delta v_* = \mu_* w_+ v_*$ (recall the weight is $w_+(r) = 4(1 + r^2)^{-2}$) to find that h and h_* satisfy

$$\begin{aligned} -\frac{1}{r}(rh')' + \frac{1}{r^2} h &= \mu w_+ h, & r \in (0, R), \\ -\frac{1}{r}(rh_*')' + \frac{1}{r^2} h_* &= \mu_* w_+ h_*, & r \in (0, R_*), \end{aligned} \tag{10}$$

where $R = \tan\Theta/2$ and $R_* = \tan\Theta_*/2$. Note $R_* < R$. The properties of g and g_* imply that h, h', h_*, h_*' are all positive on their respective intervals, with Neumann conditions $h'(R) = h_*'(R_*) = 0$ at the right endpoints. Now, rescale by defining

$i(r) = h(Rr)$ and $i_*(r) = h_*(R_*r)$. One calculates that i and i_* satisfy the equations

$$-\frac{1}{r}(ri')' + \frac{1}{r^2}\, i = \mu R^2 w_+(Rr) i,$$
$$-\frac{1}{r}(ri_*')' + \frac{1}{r^2}\, i_* = \mu_* R_*^2 w_+(R_*r) i_*,$$

for $r \in (0,1)$, with i, i', i_*, i_*' all positive and $i'(1) = i_*'(1) = 0$.

For the comparison step, multiply the first equation by ri_* and the second by ri and subtract and then integrate from 0 to 1:

$$\int_0^1 ((ri_*')'i - (ri')'i_*)dr = \int_0^1 (\mu R^2 w_+(Rr) - \mu_* R_*^2 w_+(R_*r)) r i i_*\, dr.$$

The left side equals $ri_*'i - ri'i_*|_0^1$, which evaluates to 0 by the Neumann conditions at $r = 1$. Thus, the right side equals 0, which is equivalent to having

$$\mu \sin^2(\Theta/2) \int_0^1 R^2 \frac{w_+(Rr)}{A_+(R)} r i i_*\, dr = \mu_* \sin^2(\Theta_*/2) \int_0^1 R_*^2 \frac{w_+(R_*r)}{A_+(R_*)} r i i_*\, dr \quad (11)$$

where $A_+(R) = 4\pi \sin^2 \Theta/2$ and $A_+(R_*) = 4\pi \sin^2 \Theta_*/2$; here we have defined

$$A_+(r) = \int_{D(r)} w_+\, dA = \frac{4\pi r^2}{1+r^2} = 4\pi \sin^2 \theta/2,$$

which is the weighted area of the disk of radius $r = \tan \theta/2$ (equivalently, the area of a spherical cap of aperture θ).

In order to deduce the desired inequality (9) from (11), we want to show

$$\int_0^1 \frac{w_+(Rr)R^2 r}{A_+(R)} i i_*\, dr < \int_0^1 \frac{w_+(R_*r)R_*^2 r}{A_+(R_*)} i i_*\, dr.$$

Since $A_+'(r) = 2\pi w(r) r$, the last inequality can be rewritten as

$$-\int_0^1 \Big(\frac{A_+(Rr)}{A_+(R)} - \frac{A_+(R_*r)}{A_+(R_*)}\Big)' i i_*\, dr > 0.$$

Integrating by parts and using $A_+(0) = 0$ reduces the task to showing

$$\int_0^1 \Big(\frac{A_+(Rr)}{A_+(R)} - \frac{A_+(R_*r)}{A_+(R_*)}\Big) (i i_*)'\, dr > 0.$$

The last inequality holds because $(i i_*)' > 0$ (remember i and i_* and their first derivatives are positive) and $R > R_*$ and

$$\frac{A_+(Rr)}{A_+(R)} = 1 - \frac{1-r^2}{1+(Rr)^2}$$

is a strictly increasing function of $R > 0$, for each $r \in (0,1)$.

The limiting value $\mu_2(\Theta)4\sin^2(\Theta/2) \to \mu_2(D)$ as $\Theta \searrow 0$ follows from the analogous limit for $\mu_2(\Theta)4\tan^2(\Theta/2)$ in Theorem 1.

Limiting value as $\Theta \to \pi$. It remains to show $\mu_2(\Theta)4\sin^2\Theta/2 \to 8$ as $\Theta \to \pi$, which means we want $\lim_{\Theta\to\pi}\mu_2(\Theta) = 2$. This limiting value is shown graphically in Figure 4 and has been established rigorously by Bandle, Kabeya, and Ninomiya [6, Theorem 1.1], who show in all dimensions that as the cap expands to fill the sphere, the Neumann spectrum of the cap converges to the spectrum of the full sphere.

Hyperbolic case. Suppose $-\infty < \Theta_* < \Theta < 0$. The aim is to show

$$\mu_* \sinh^2\frac{\Theta_*}{2} < \mu\sinh^2\frac{\Theta}{2}.$$

Let g and g_* be the radial parts of the second eigenfunctions for the geodesic disks $C(\Theta)$ and $C(\Theta_*)$, respectively. By [21, Proposition 4.4], $g(0) = 0$ and $g' > 0$ on $(0, |\Theta|)$, and $g_*(0) = 0$ and $g_*' > 0$ on $(0, |\Theta_*|)$.

The proof proceeds as for the main part of the spherical case above, and so we indicate only the necessary modifications. Under the change of variable $r = \tanh p/2$, as in Section 5.3, the geodesic disks $C(\Theta)$ and $C(\Theta_*)$ transform to disks of radii $R = \tanh|\Theta|/2$ and $R_* = \tanh|\Theta_*|/2$. Notice

$$0 < R < R_* < 1.$$

(In the current proof we find it convenient to work with R values between 0 and 1, whereas previously the hyperbolic case has corresponded to R between 0 and -1. Whether one works with R or its negative, the disk RD is the same.) The area of a disk $D(r)$ with respect to the weight $w_-(r) = 4(1-r^2)^{-2}$ is

$$A_-(r) = \int_{D(r)} w_-\,dA = \frac{4\pi r^2}{1-r^2} = 4\pi\sinh^2\frac{p}{2}. \tag{12}$$

Observe that

$$\frac{A_-(Rr)}{A_-(R)} = 1 - \frac{1-r^2}{1-(Rr)^2}$$

is a strictly decreasing function of $R < 1$, for each $r \in (0,1)$. With these adaptations, the spherical proof adapts easily to the hyperbolic situation.

The fact that $\mu_2(\Theta)4\sinh^2\Theta/2 \to \mu_2(D)$ as $\Theta \nearrow 0$ follows from the corresponding limit for $\mu_2(\Theta)4\tanh^2\Theta/2$ in Theorem 1.

Limiting value as $\Theta \to -\infty$. We still need to prove $\mu_2(\Theta) 4\sinh^2 \Theta/2 \to 2$ as $\Theta \to -\infty$, that is, as $R \to 1$. We start by showing a lower bound, that $\mu_2(\Theta)4\sinh^2\Theta/2 > 2$ for all $\Theta < 0$. That is, we want $\mu A_- > 2\pi$ when $0 < R < 1$.

Multiply the hyperbolic version of ODE (10) (with w_- instead of w_+) by r and integrate with respect to $r\,dr$ to obtain

$$\begin{aligned} -\int_0^R (rh')'r\,dr + \int_0^R h\,dr &= \mu \int_0^R w_- h r^2\,dr \\ &= \mu \int_0^R \frac{A_-'(r)}{2\pi} hr\,dr. \end{aligned}$$

Integrating by parts twice on the left and using the Neumann condition $h'(R) = 0$, and also integrating by parts once on the right, gives that

$$Rh(R) = \frac{\mu}{2\pi}\left(A_-(R)h(R)R - \int_0^R A_-(r)(hr)'\,dr \right).$$

Since $g' > 0$, we know $h' > 0$ and hence $(hr)' > 0$, so that dropping that term in the preceding equation implies the desired lower bound $2\pi < \mu A_-(R)$.

To obtain an upper bound of 2 in the limit as $\Theta \to -\infty$, we use a trial function approach. The variational characterization of the second eigenvalue (based on the ODE (10) except with w_- instead of w_+) is

$$\mu_2(\Theta) = \min_h \frac{\int_0^R (h'(r)^2 + r^{-2}h(r)^2)\,r\,dr}{\int_0^R h(r)^2 w_-(r)\,r\,dr}$$

where $R = \tanh|\Theta|/2$ and $h \in W^{1,2}(0, R)$ with $h(0) = 0$. Choosing the trial function $h(r) = r$ and substituting the definition of $w_-(r)$ yields the explicit estimate

$$\mu_2(\Theta) \le \frac{R^2/2}{\frac{R^2}{1-R^2} + \log(1 - R^2)}.$$

Multiplying by $4\sinh^2\Theta/2 = 4R^2/(1 - R^2)$, we find

$$\mu_2(\Theta)4\sinh^2\Theta/2 \le \frac{2R^4}{R^2 + (1 - R^2)\log(1 - R^2)} \to 2$$

as $R \to 1$. This limiting upper bound of 2 combines with the previous lower bound to complete the proof that $\mu_2(\Theta)4\sinh^2\Theta/2 \to 2$ as $\Theta \to -\infty$.

Remark. The Steklov spectrum underlies the limiting value 2 as $\Theta \to -\infty$, because when R is near 1, the weight $w_{-}(r)$ on $D(R)$ concentrates heavily near the boundary circle and so the eigenfunction ought to behave like the Steklov eigenfunction of the disk, which is $r \cos\phi$. That is, the radial part of the eigenfunction should behave like $h(r) = r$, which motivates the choice in the proof above. Concentration results of this kind have been developed more fully by Lamberti and Provenzano [20].

8. Proof of Theorem 3

Write $\lambda = \lambda_1(\Theta)$ and $\lambda_* = \lambda_1(\Theta_*)$ for the first Dirichlet eigenvalues, whenever $\Theta, \Theta_* \in (-\infty, \pi)$. The following proof holds in all dimensions $n \geq 2$.

Spherical case. Suppose $0 < \Theta_* < \Theta < \pi$. We want to show

$$\lambda_* \Theta_*^2 > \lambda \Theta^2. \tag{13}$$

Write g and g_* for the (radial) first Dirichlet eigenfunctions on the caps $C(\Theta)$ and $C(\Theta_*)$ of the n-dimensional sphere $\mathbb{S}^n, n \geq 2$, so that $g(\Theta) = g_*(\Theta_*) = 0$ and

$$\begin{aligned} -\frac{1}{\sin^{n-1}\theta}\big((\sin^{n-1}\theta)g'(\theta)\big)' &= \lambda g(\theta), & 0 < \theta < \Theta, \\ -\frac{1}{\sin^{n-1}\theta}\big((\sin^{n-1}\theta)g_*'(\theta)\big)' &= \lambda_* g_*(\theta), & 0 < \theta < \Theta_*, \end{aligned}$$

with $g > 0$ and $g' < 0$ on $(0, \Theta)$ and $g_* > 0$ and $g_*' < 0$ on $(0, \Theta_*)$; see Chavel [12, p. 43] for these facts, or else Ashbaugh and Benguria [2, Lemma 3.1].

Rescale g and g_* to the unit interval by defining $i(t) = g(\Theta t)$ and $i_*(t) = g_*(\Theta_* t)$. These new functions satisfy

$$\begin{aligned} -\frac{1}{\sin^{n-1}\Theta t}\big((\sin^{n-1}\Theta t)i'(t)\big)' &= \lambda\Theta^2 i(t), \\ -\frac{1}{\sin^{n-1}\Theta_* t}\big((\sin^{n-1}\Theta_* t)i_*'(t)\big)' &= \lambda_*\Theta_*^2 i_*(t), \end{aligned} \tag{14}$$

for $t \in (0, 1)$. The equation for i_* implies

$$\begin{aligned} \lambda_*\Theta_*^2 i_* &= -i_*'' - t^{-1}(n-1)\Theta_* t(\cot\Theta_* t)i_*' \\ &> -i_*'' - t^{-1}(n-1)\Theta t(\cot\Theta t)i_*' \end{aligned}$$

since $i_*' < 0$ and $s \mapsto s\cot s$ is strictly decreasing for $s \in (0, \pi)$, with $\Theta_* t < \Theta t$. Hence

$$-\frac{1}{\sin^{n-1}\Theta t}\big((\sin^{n-1}\Theta t)i_*'(t)\big)' < \lambda_*\Theta_*^2 i_*(t). \tag{15}$$

Multiply inequality (15) by $i(t)\sin^{n-1}\Theta t$ and equation (14) by $i_*(t)\sin^{n-1}\Theta t$ and then subtract and integrate from 0 to 1, obtaining that

$$
\begin{aligned}
&(\lambda_*\Theta_*^2 - \lambda\Theta^2)\int_0^1 i(t)i_*(t)\sin^{n-1}\Theta t\,dt \\
&\quad > \int_0^1 \big[\big((\sin^{n-1}\Theta t)i'(t)\big)' i_*(t) - \big((\sin^{n-1}\Theta t)i_*'(t)\big)' i(t)\big]dt = 0
\end{aligned}
$$

by parts, using the Dirichlet conditions $i(1) = i_*(1) = 0$. Conclusion (13) follows.

Hyperbolic case. Suppose $-\infty < \Theta_* < \Theta < 0$. Let g and g_*, respectively, be the (radial) first eigenfunctions for the geodesic balls of radii $|\Theta|$ and $|\Theta_*|$ in the hyperbolic space $\mathbb{H}^n$, $n \geq 2$; for properties of these eigenfunctions, see Chavel [12, p. 43] or Benguria and Linde [8, Lemma 3.1]. The goal is again to show

$$
\lambda_*\Theta_*^2 > \lambda\Theta^2.
$$

Let $i(t) = g(|\Theta|t)$ and $i_*(t) = g_*(|\Theta_*|t)$ and argue as above in the spherical case, except with sin replaced by sinh and using that $s \mapsto s\coth s$ is strictly increasing for $s \in (0, \infty)$, with $|\Theta_*|t > |\Theta|t$.

Limiting values. It is well known that $\lim_{\Theta\to-\infty}\lambda_1(\Theta) = (n-1)^2/4$ and that $\lim_{\Theta\to\pi}\lambda_1(\Theta) = 0$ (which is the first eigenvalue of the full sphere); see Chavel [12, p. 46, 50]. Hence, $\lambda_1(\Theta)\Theta^2$ tends to ∞ as $\Theta \to -\infty$, and tends to 0 as $\Theta \to \pi$.

Remark. The proof given above for Theorem 3 could be rewritten using a trial function method in which a suitable rescaling of the eigenfunction for the cap of aperture Θ_* is used as a trial function for the cap of aperture Θ. Specifically, one can use i_* as a trial function in the Rayleigh quotient for estimating $\lambda\Theta^2$: multiply inequality (15) by $i_*(t)\sin^{n-1}\Theta t$, integrate by parts on the left side, and then apply the Rayleigh principle for the eigenvalue $\lambda\Theta^2$. This approach is a special case of the proof of Cheng's theorem for geodesic balls satisfying a lower curvature bound [12, pp. 74-76]. Alternatively, one could prove Theorem 3 by adapting Cheng's theorem for geodesic balls with an upper curvature bound [12, pp. 70-71], that is, using $g((\Theta/\Theta_*)\theta)$ in Barta's inequality in order to estimate $\lambda_1(\Theta_*)$ from below.

9. Proof of Theorem 4

Write $\lambda = \lambda_2(\Theta)$ and $\lambda_* = \lambda_2(\Theta_*)$ for the second eigenvalue, when $\Theta, \Theta_* < \pi$.

Spherical case. Suppose $0 < \Theta_* < \Theta < \pi$. We want to show

$$\lambda_* \Theta_*^2 < \lambda \Theta^2.$$

Write g and g_* for the radial parts of the second Dirichlet eigenfunctions on the caps $C(\Theta)$ and $C(\Theta_*)$ of the n-dimensional sphere $\mathbb{S}^n$, $n \geq 2$. Our proof will succeed only for $n = 2$, but by treating all n we can explicate where the proof breaks down in higher dimensions.

The radial parts satisfy

$$-\frac{1}{\sin^{n-1}\theta}\big((\sin^{n-1}\theta) g'(\theta)\big)' + \frac{n-1}{\sin^2\theta} g(\theta) = \lambda g(\theta), \qquad 0 < \theta < \Theta,$$
$$-\frac{1}{\sin^{n-1}\theta}\big((\sin^{n-1}\theta) g_*'(\theta)\big)' + \frac{n-1}{\sin^2\theta} g_*(\theta) = \lambda_* g_*(\theta), \quad 0 < \theta < \Theta_*,$$

with $g > 0$ on $(0, \Theta)$ and $g_* > 0$ on $(0, \Theta_*)$, and with the Dirichlet conditions $g(0) = g(\Theta) = 0$, $g_*(0) = g_*(\Theta_*) = 0$; see Ashbaugh and Benguria [2, Lemma 3.1] for these facts.

Let $i(t) = g(\Theta t)$ and $i_*(t) = g_*(\Theta_* t)$, so that these functions on the unit interval satisfy $i(0) = i(1) = 0$, $i_*(0) = i_*(1) = 0$, and

$$-\frac{1}{\sin^{n-1}\Theta t}\big((\sin^{n-1}\Theta t) i'(t)\big)' + \frac{(n-1)\Theta^2}{\sin^2\Theta t} i(t) = \lambda \Theta^2 i(t),$$
$$-\frac{1}{\sin^{n-1}\Theta_* t}\big((\sin^{n-1}\Theta_* t) i_*'(t)\big)' + \frac{(n-1)\Theta_*^2}{\sin^2\Theta t} i_*(t) = \lambda_* \Theta_*^2 i_*(t),$$

for $0 < t < 1$. Multiplying the first equation by $i(t)\sin^{n-1}\Theta t$ and integrating yields that

$$\lambda\Theta^2 = \frac{\int_0^1 \big(i'(t)^2 \sin^{n-1}\Theta t + \frac{(n-1)\Theta^2}{\sin^2\Theta t} i(t)^2 \sin^{n-1}\Theta t\big) dt}{\int_0^1 i(t)^2 \sin^{n-1}\Theta t \, dt}. \tag{16}$$

The analogous equation holds for $\lambda_*\Theta_*^2$, which leads to a variational characterization:

$$\lambda_*\Theta_*^2 = \min_j \frac{\int_0^1 \big(j'(t)^2 \sin^{n-1}\Theta_* t + \frac{(n-1)\Theta_*^2}{\sin^2\Theta_* t} j(t)^2 \sin^{n-1}\Theta_* t\big) dt}{\int_0^1 j(t)^2 \sin^{n-1}\Theta_* t \, dt} \tag{17}$$

where $j \in W^{1,2}(0, 1)$ satisfies the Dirichlet boundary conditions $j(0) = j(1) = 0$. Define a trial function

$$j(t) = i(t)\Big(\frac{\sin\Theta t}{\sin\Theta_* t}\Big)^{(n-1)/2},$$

so that $j(t)^2 \sin^{n-1}\Theta_* t = i(t)^2 \sin^{n-1}\Theta t$ in the denominator of the Rayleigh quotient and in the second term of its numerator. Into the first term of the numerator we substitute

$$\begin{aligned} j'(t)^2 \sin^{n-1} \Theta_* t &= \Big(i'(t) + \frac{n-1}{2} i(t)(\Theta \cot \Theta t - \Theta_* \cot \Theta_* t)\Big)^2 \sin^{n-1} \Theta t \\ &= \Big(i'(t)^2 + \frac{n-1}{2}(i(t)^2)'(\Theta \cot \Theta t - \Theta_* \cot \Theta_* t) \\ &\qquad + \frac{(n-1)^2}{4} i(t)^2 (\Theta \cot \Theta t - \Theta_* \cot \Theta_* t)^2\Big) \sin^{n-1} \Theta t. \end{aligned}$$

After these substitutions, and in view of formulas (16) and (17), in order to prove $\lambda_* \Theta_*^2 < \lambda \Theta^2$ it is enough to show

$$\begin{aligned} &\int_0^1 \Big(\frac{n-1}{2}(i(t)^2)'(\Theta \cot \Theta t - \Theta_* \cot \Theta_* t) \\ &\qquad + \frac{(n-1)^2}{4} i(t)^2 (\Theta \cot \Theta t - \Theta_* \cot \Theta_* t)^2 + \frac{(n-1)\Theta_*^2}{\sin^2 \Theta_* t} i(t)^2\Big) \sin^{n-1} \Theta t \, dt \\ &\quad < \int_0^1 \frac{(n-1)\Theta^2}{\sin^2 \Theta t} i(t)^2 \sin^{n-1} \Theta t \, dt. \end{aligned}$$

After integrating by parts in the first term and then simplifying and using the identity $\csc^2 = \cot^2 + 1$, the task reduces to showing

$$\begin{aligned} \frac{n-1}{4} \int_0^1 &\big((n+1)(f(\Theta_* t)^2 - f(\Theta t)^2) \\ &\quad + 2\sigma((\Theta_* t)^2 - (\Theta t)^2)\big) t^{-2} i(t)^2 \sin^{n-1} \Theta t \, dt < 0 \end{aligned}$$

where the coefficient is $\sigma = 1$ in this spherical case, and we have defined $f(s) = s \cot s$.

It suffices to show the integrand is negative. After writing $\alpha = \Theta_* t$ and $\beta = \Theta t$, we want

$$(n+1) f(\alpha)^2 + 2\alpha^2 < (n+1) f(\beta)^2 + 2\beta^2.$$

Note $0 < \alpha < \beta < \pi$, due to the assumption that $\Theta_* < \Theta$. Thus, we wish to show $(n+1) f(s)^2 + 2s^2$ is strictly increasing for $s \in (0, \pi)$. The product expansion for sine combined with the geometric series reveals that

$$\begin{aligned} &(n+1) f(s)^2 + 2s^2 \\ &\quad = (n+1) s^2 \csc^2 s - (n-1) s^2 \\ &\quad = (n+1) \prod_{k=1}^{\infty} \Big(1 - \frac{s^2}{k^2} \pi^2\Big)^{-2} - (n-1) s^2 \\ &\quad = (n+1) \prod_{k=1}^{\infty} \Big(1 + \frac{s^2}{k^2} \pi^2 + \Big(\frac{s^2}{k^2} \pi^2\Big)^2 + \cdots\Big)^2 - (n-1) s^2 \end{aligned}$$

for $s \in (0, \pi)$. Expanding the product as a series, we see that the terms of order s^4 and higher in the expansion have positive coefficients while the coefficient of s^2 is

$$(n+1)\sum_{k=1}^{\infty} \frac{2}{k^2\pi^2} - (n-1) = \frac{n+1}{3} - (n-1) = -\frac{2}{3}(n-2),$$

which is zero when $n = 2$ and negative when $n \geq 3$. Thus, the expression

$$(n+1) f(s)^2 + 2s^2$$

is strictly increasing as a function of s when $n = 2$, which completes the proof in that case. When $n \geq 3$, the expression is instead decreasing for small s, so that the method fails in higher dimensions.

Hyperbolic case. Suppose $-\infty < \Theta < \Theta_* < 0$. The goal is to show $\lambda_* \Theta_*^2 < \lambda \Theta^2$. This part of the proof holds for all dimensions $n \geq 2$.

The argument proceeds as in the spherical case above, except we refer to Benguria and Linde [8, Lemma 3.1] for the equation satisfied by the radial part g of the second Dirichlet eigenfunction; replace sin with sinh and replace cot with coth; let $i(t) = g(|\Theta|t)$ and $i_*(t) = g_*(|\Theta_*|t)$; use the identity $\operatorname{csch}^2 = \coth^2 - 1$; take $f(s) = s \coth s$, let $\sigma = -1$, and write $\alpha = |\Theta_*|t$ and $\beta = |\Theta|t$, so that $\alpha < \beta$. Then the task is to show $(n+1) f(s)^2 - 2s^2$ is strictly increasing for $s \in (0, \infty)$, which is accomplished as follows. We have

$$\begin{aligned}(n+1) f(s)^2 - 2s^2 &= (n+1)s^2 \operatorname{csch}^2 s + (n-1)s^2 \\ &= (n+1)t \operatorname{csch}^2 \sqrt{t} + (n-1)t\end{aligned}$$

where $t = s^2$. This last expression is strictly convex as a function of t, because the product formula for sinh implies that

$$t \operatorname{csch}^2 \sqrt{t} = \prod_{k=1}^{\infty} \frac{1}{(1 + t/k^2\pi^2)^2}$$

is a product of positive, decreasing, strictly convex factors and hence is strictly convex. The series expansion

$$(n+1)t \operatorname{csch}^2 \sqrt{t} + (n-1)t = (n+1) + \frac{2}{3}(n-2)t + O(t^2)$$

guarantees that the first derivative at $t = 0$ is $(2/3)(n-2) \geq 0$, and so the strictly convex function $(n+1)t \operatorname{csch}^2 \sqrt{t} + (n-1)t$ must be strictly increasing for $t > 0$.

Note. We could have argued like this with the t-variable in the spherical part of the proof too, but it was quicker there simply to expand the product as a series having positive coefficients.

Limiting values. In Theorem 3 we noted $\lambda_1(\Theta)\Theta^2$ tends to ∞ as $\Theta \to -\infty$. Hence, $\lambda_2(\Theta)\Theta^2$ also tends to ∞. As $\Theta \to 0$, we know $\lambda_2(\Theta)\Theta^2 \to \lambda_2(D)$ by Theorem 1, since $\sinh^2 \Theta$ and $\sin^2 \Theta$ are asymptotic to Θ. As $\Theta \to \pi$, the second Dirichlet eigenvalue $\lambda_2(\Theta)$ of the spherical cap tends to the second eigenvalue (first positive eigenvalue) of the full sphere, which equals n; see the discussion and references in Chavel [12, p. 53]. Thus, in 2 dimensions, $\lambda_2(\Theta)\Theta^2$ tends to $2\pi^2$ as $\Theta \to \pi$.

10. Proofs of the corollaries

Proof of Corollary 5. (i) First, suppose $0 < K < 4\pi/A$, so that we are considering caps on the sphere. Multiplying the metric by K produces a new metric on the disk, with curvature 1, area KA and eigenvalue $\mu_2(1; KA) = \mu_2(K; A)/K$. Hence

$$\mu_2(K; A)A = \mu_2(1; KA)KA = \mu_2(\Theta)4\pi \sin^2 \frac{\Theta}{2}$$

where $\Theta \in (0, \pi)$ is the aperture of the spherical cap having curvature 1 and area $KA = 4\pi \sin^2 \Theta/2$.

Next, suppose $K = 0$, meaning the disk is Euclidean. Scale invariance ensures that $\mu_2(0; A)A = \mu_2(D)\pi$.

Suppose lastly that $K < 0$, meaning the geodesic disk lies in a hyperbolic space. Multiplying the metric by $|K|$ gives a metric having curvature -1, area $|K|A$, and eigenvalue $\mu_2(-1; |K|A) = \mu_2(K; A)/|K|$. Hence

$$\mu_2(K; A)A = \mu_2(-1; |K|A)|K|A = \mu_2(\Theta)4\pi \sinh^2 \frac{\Theta}{2}$$

where $\Theta < 0$ is determined by requiring $|K|A = 4\pi \sinh^2 \Theta/2$; that is, $|\Theta|$ is the geodesic radius of the hyperbolic disk having curvature -1 and area $|K|A$ (recall formula (12) for the area).

In each case, Θ is an increasing function of K, and so Corollary 5 (i) follows from Theorem 2, which says $\mu_2(\Theta)4\,\mathrm{si}^2\,\Theta/2$ is strictly increasing and tends to $\mu_2(D)$ at $\Theta = 0$.

(ii) Suppose $0 < K < (\pi/\rho)^2$. Multiplying the metric by K yields a metric on the disk with curvature 1, geodesic radius $K^{1/2}\rho$ and eigenvalue $\mu_2(1; K^{1/2}\rho) = \mu_2(K; \rho)/K$. Hence

$$\mu_2(K; \rho)\rho^2 = \mu_2(1; K^{1/2}\rho)(K^{1/2}\rho)^2 = \mu_2(\Theta)\Theta^2$$

where we have chosen $\Theta = K^{1/2}\rho$.

If $K = 0$ then $\mu_2(0; \rho)\rho^2 = \mu_2(D)$ by Euclidean scale invariance.

Now, suppose $K < 0$. After multiplying the metric by $|K|$ one obtains a metric with curvature -1, geodesic radius $|K|^{1/2}\rho$, and eigenvalue $\mu_2(-1; |K|^{1/2}\rho) = \mu_2(K;\rho)/|K|$. Hence,

$$\mu_2(K;\rho)\rho^2 = \mu_2(-1; |K|^{1/2}\rho)(|K|^{1/2}\rho)^2 = \mu_2(\Theta)\Theta^2$$

where $\Theta = -|K|^{1/2}\rho$.

Noting Θ is an increasing function of K in each case, we deduce Corollary 5 (ii) from Theorem 2 since

$$\mu_2(\Theta)\Theta^2 = \Big(\mu_2(\Theta) 4\,\mathrm{si}^2 \frac{\Theta}{2}\Big)\Big(\frac{\Theta/2}{\mathrm{si}\,\Theta/2}\Big)^2$$

is a product of strictly increasing factors when $\Theta < \pi$.

Proof of Corollary 6. By rescaling the metric as in part (ii) of the preceding proof, the task for Corollary 6 (i) reduces to showing $\lambda_1(\Theta)\Theta^2$ is strictly decreasing for $\Theta < \pi$, with limiting value $\lambda_1(D)$ at $\Theta = 0$. This fact is known already by Theorem 3. Part (ii) of the corollary follows similarly, since

$$\lambda_1(\Theta) 4\,\mathrm{si}^2\,\Theta/2 = \lambda_1(\Theta)\Theta^2\Big(\frac{\mathrm{si}\,\Theta/2}{\Theta/2}\Big)^2$$

is a product of strictly decreasing factors for $\Theta < \pi$.

Proof of Corollary 7. Like in the previous proof, the task reduces to showing that $\lambda_2(\Theta)\Theta^2$ is strictly decreasing for $\Theta < 0$ and strictly increasing for $0 < \Theta < \pi$. Those monotonicities are provided by Theorem 4. Part (ii) of the corollary is similar, since

$$\lambda_2(\Theta) 4 \sinh^2 \Theta/2 = \lambda_2(\Theta)\Theta^2\Big(\frac{\sinh \Theta/2}{\Theta/2}\Big)^2$$

is a product of strictly decreasing factors when $\Theta < 0$.

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