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Deformation of the spectrum for Darboux–Treibich–Verdier potential along $\operatorname{Re}\tau = \frac{1}{2}$

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Abstract. In this paper, we study the spectrum $\sigma(L)$ of the complex Hill operator with Darboux–Treibich–Verdier potential

$$L = \frac{d^2}{dx^2} - 6\wp(x + z_0; \tau) - 2\wp\Big(x + \frac{1}{2} + z_0; \tau\Big) \quad \text{in } L^2(\mathbb{R}, \mathbb{C}),$$

where $\wp(z;\tau)$ is the Weierstraß elliptic function with periods 1 and τ, and $z_0 \in \mathbb{C}$ is chosen such that L has no singularities on $\mathbb{R}$. We give a complete picture of the deformation of the spectrum with $\tau = \frac{1}{2} + ib$ as $b > 0$ varies. A new idea of the proof is to apply the result of the mean field equation and its connection with this operator.

1. Introduction

Let $T_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be a flat torus with $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$, and $\wp(z;\tau)$ be the Weierstraß elliptic function with basic periods $\omega_1 = 1$ and $\omega_2 = \tau$. Denote by $\omega_0 = 0$, $\omega_3 = \omega_1 + \omega_2$ and $\mathbf{n} = (n_0, n_1, n_2, n_3) \in \mathbb{N}^4$ satisfying $\mathbf{n} \neq (0,0,0,0)$. The Darboux–Treibich–Verdier (DTV for short) potential [33]

$$q^{\mathbf{n}}(z;\tau) := -\sum_{k=0}^{3} n_k(n_k+1)\wp\Big(z + \frac{\omega_k}{2}; \tau\Big), \quad z \in \mathbb{C},$$

is famous as an *algebro-geometric finite-gap potential* associated with the stationary KdV hierarchy, which means that $q^{\mathbf{n}}(z;\tau)$ is a solution of stationary KdV hierarchy equations (cf. [17, 20]). Specifically, there is an odd-order differential operator

$$P_{2k+1} = \Big(\frac{d}{dz}\Big)^{2k+1} + \sum_{j=0}^{2k-1} b_j(z)\Big(\frac{d}{dz}\Big)^{2k-1-j} \tag{1.1}$$

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such that

$$\left[P_{2k+1}, \frac{d^2}{dz^2} + q^{\mathbf{n}}(z;\tau)\right] = 0. \tag{1.2}$$

We refer the reader to [4, 9, 10, 19, 28–33, 35] and references therein for historical reviews and subsequent developments.

Let P_{2g+1} be the unique operator of the form (1.1) satisfying (1.2) such that its order $2g+1$ is *smallest*. Then a celebrated theorem by Burchnall and Chaundy [3] implies the existence of the so-called *spectral polynomial* $Q_\tau^{\mathbf{n}}(\lambda)$ of degree $2g+1$ in λ associated to $q^{\mathbf{n}}(z;\tau)$ such that

$$P_{2g+1}^2 = Q_\tau^{\mathbf{n}}\Big(\frac{d^2}{dz^2} + q^{\mathbf{n}}(z;\tau)\Big).$$

In this paper, we study the spectrum $\sigma(L_\tau^{\mathbf{n}})$ of the complex Hill operator with the DTV potential

$$L_\tau^{\mathbf{n}} = \frac{d^2}{dx^2} + q^{\mathbf{n}}(x+z_0;\tau), \quad x \in \mathbb{R},$$

in $L^2(\mathbb{R},\mathbb{C})$, where $z_0 \in \mathbb{C}$ is chosen such that $q^{\mathbf{n}}(x+z_0;\tau)$ has no singularities on $\mathbb{R}$. The spectral theory of the complex Hill operator has been studied widely in the literature; see e.g., [1,2,18,20,21,27] and references therein. In particular, it is known [27] that

$$\sigma(L_\tau^{\mathbf{n}}) = \Delta^{-1}([-2,2]) = \{\lambda \in \mathbb{C} \mid -2 \le \Delta(\lambda) \le 2\},$$

where $\Delta(\lambda)$ is the so-called Hill's discriminant which is the trace of the monodromy matrix for $L_\tau^{\mathbf{n}} y = \lambda y$ with respect to $x \to x+1$. Furthermore, it was proved in [20] that $\sigma(L_\tau^{\mathbf{n}})$ consists of finitely many bounded simple analytic arcs and one semi-infinite simple analytic arc with the finite endpoints of such arcs being those zeros of the spectral polynomial $Q_\tau^{\mathbf{n}}(\lambda)$ with odd orders.

Let $\tau \in i\mathbb{R}_{>0}$. We introduce two relations:

$$(n_1+n_2)-(n_0+n_3) \ge 2, \quad n_1 \ge 1, n_2 \ge 1, \tag{1.3}$$

$$(n_0+n_3)-(n_1+n_2) \ge 2, \quad n_0 \ge 1, n_3 \ge 1. \tag{1.4}$$

Recently, we proved in [8, Theorem 1.1] that $\sigma(L_\tau^{\mathbf{n}}) \subset \mathbb{R}$ if and only if $\mathbf{n}$ satisfies neither (1.3) nor (1.4), and in this case $\sigma(L_\tau^{\mathbf{n}})$ is completely determined by the spectral polynomial $Q_\tau^{\mathbf{n}}(\lambda)$ as follows. If $\mathbf{n}$ satisfies neither (1.3) nor (1.4), then all the roots of $Q_\tau^{\mathbf{n}}(\lambda)$ are real and distinct, denoted by $\lambda_{2g} < \lambda_{2g-1} < \cdots < \lambda_1 < \lambda_0$, and consequently,

$$\sigma(L_\tau^{\mathbf{n}}) = (-\infty, \lambda_{2g}] \cup [\lambda_{2g-1}, \lambda_{2g-2}] \cup \cdots \cup [\lambda_1, \lambda_0] \subseteq \mathbb{R}.$$

Naturally, people would ask what $\sigma(L_\tau^{\mathbf{n}})$ is if $\mathbf{n}$ satisfies either (1.3) or (1.4). This question is very difficult to study because $\sigma(L_\tau^{\mathbf{n}}) \not\subset \mathbb{R}$! So, we start from some special cases. Note that if $\mathbf{n} = (g, 0, 0, g)$ (or $(0, g, g, 0)$) with $g \geq 1$, the spectrum of the Hill operator $L^{\mathbf{n}}_{\tau}$ with $\mathrm{Re}\tau = 0$ is a horizontal translation of the spectrum of the classical Lamé operator $L^{g}_{\tilde{\tau}}$ with $\mathrm{Re}\tilde{\tau} = 1/2$ (cf. [7, Lemma 4.1]), where

$$L^{g}_{\tilde{\tau}} := L^{\mathbf{n}}_{\tilde{\tau}} = \frac{d^2}{dx^2} - g(g+1)\wp(x + z_0; \tilde{\tau}), \quad x \in \mathbb{R}.$$

denotes the Lamé operator [22] which corresponds to $\mathbf{n} = (g, 0, 0, 0)$. If $g = 1$, this question has been solved even for all $\tau \in \mathbb{H}$ (see [1,18,21]). In this case, the spectrum $\sigma(L^1_\tau)$ consists of two regular analytic arcs and so there are totally three different types of graphs for different τ's. It was pointed out in [21, Section 5] that the rigorous analysis of $g \geq 2$ cases seems to be difficult since the related explicit formulae quickly become quite complicated as g grows. The case $g = 2$ already becomes very complicated and was studied recently in [7], where we proved that the spectrum $\sigma(L^2_\tau)$ has exactly 9 different types of graphs for different b's. Furthermore, the $g = 3$ case is much more difficult and only some partial results were given in [16], where we discovered 7 different types of graphs for the spectrum as b varies around the double zeros of the spectral polynomial.

In this paper, we will focus on the operator $L^{(2,1,1,2)}_{\tau}$ with $\mathrm{Re}\tau = 0$. Compared with previous cases, we cannot relate the spectrum $\sigma(L^{(2,1,1,2)}_{\tau})$ with a Lamé operator and there is no explicit description of the spectrum for any DTV potential which cannot convert to the Lamé case in the literature. Fortunately, Lemma 2.3 in Section 2.1 tells us that $\sigma(L^{(2,1,1,2)}_{\tau})$ is a horizontal translation of the spectrum $L^{(2,1,0,0)}_{\tilde{\tau}}$ for some $\tilde{\tau} \in \mathbb{H}$ with $\mathrm{Re}\tilde{\tau} = 1/2$, which is symmetric with respect to $\mathbb{R}$. Let $\tau = \frac{1}{2} + bi$ with $b > 0$ in what follows and consider the spectrum of

$$L_b := L^{(2,1,0,0)}_{b} = \frac{d^2}{dx^2} - 6\wp(x + z_0; \tau) - 2\wp\Big(x + \frac{1}{2} + z_0; \tau\Big) \quad \text{in } L^2(\mathbb{R}, \mathbb{C}).$$

In order to emphasize $\tau = \frac{1}{2} + bi$, we use b instead of τ in notations. Sometimes, we omit the notation τ freely to simplify notations when no confusion arises.

Let $e_k := e_k(b) = \wp(\frac{\omega_k}{2}; b)$, $k = 1, 2, 3$ be the well-known invariants of the elliptic curve. It is well known (see [31, p.394]) that the spectral polynomial $Q_b(\lambda)$ of L_b is given by

$$Q_b(\lambda) = (\lambda - 4e_1)R_1(\lambda)R_2(\lambda), \tag{1.5}$$

where

$$R_1(\lambda) = \lambda^2 - 2(3e_2 + 4e_3)\lambda - 31e_2^2 - 52e_2e_3 - 12e_3^2,$$
$$R_2(\lambda) = \lambda^2 - 2(3e_3 + 4e_2)\lambda - 31e_3^2 - 52e_2e_3 - 12e_2^2.$$

By applying [20, Theorem 4.1], we see that *the spectrum $\sigma(L_b)$ consists of $\tilde{g} \leq 2$ bounded simple analytic arcs σ_k and* one *semi-infinite simple analytic arc σ_∞ which* *tends to* $-\infty + \langle q\rangle$, *with* $\langle q\rangle = \int_{x_0}^{x_0+1} q(x)dx$, *i.e.,*

$$\sigma(L) = \sigma_\infty \cup \bigcup_{k=1}^{\tilde{g}} \sigma_k, \quad \tilde{g} \leq 2,$$

where the finite endpoints of such arcs must be those roots of the spectral polynomial $Q_b(\lambda)$ *with odd order.*

In order to study the geometry of $\sigma(L_b)$, we first need to determine all finite endpoints of $\sigma(L_b)$. For this purpose, we have to analyze the roots of $Q_b(\lambda)$ and the number of semi-arcs met at each root, which are described in the following theorem.

Theorem 1.1. *Let* $\tau = \frac{1}{2} + bi$ *with* $b > 0$ *and* $d(\lambda)$ *be the number of semi-arcs met at* λ. *Then all zeros of the spectral polynomial* $Q_b(\lambda)$ *are distinct and listed as follows:*

$$4e_1, \quad \mu, \quad \bar{\mu}, \quad \nu, \quad \bar{\nu}.$$

Furthermore, $d(\mu) = d(\bar{\mu}) = d(\nu) = d(\bar{\nu}) = 1$, *and there exist* $b_1 \approx 0.2716572$ *and* $b_2 \approx 0.596803$ *such that*

$$d(4e_1) \begin{cases} \geq 3 & \text{if } b \in \{b_1, b_2\}, \\ = 1 & \text{otherwise}. \end{cases}$$

This theorem tells us that the spectrum $\sigma(L_b)$ has exactly 5 finite endpoints and thus has exactly 3 spectral arcs. The main result of this paper is as follows, which says that there are totally 5 different patterns for the spectrum $\sigma(L_b)$ during the deformation as $b > 0$ deforms.

Theorem 1.2. *Let* $\tau = \frac{1}{2} + ib$ *with* $b > 0$. *Then*

$$\sigma(L_b) = (-\infty, 4e_1] \cup \sigma_1 \cup \sigma_2,$$

where the notations σ_i *with* $i = 1, 2$ *denote simple arcs symmetric with respect to* $\mathbb{R}$ *and they are disjoint with each other. Denote by* λ_-, λ_+ *the roots of*

$$f(\lambda) := \lambda^2 + (5e_1 + 4\eta_1)\lambda - 27e_1^2 + 2e_1\eta_1 + \frac{3}{4}g_2.$$

The deformation of $\sigma(L_b)$ *as* $b > 0$ *increases are described in the following graphs and statements (see Figure 1.*

(1) *If* $0 < b < b_1$, *then* $\sigma_i \cap \mathbb{R} = \{\lambda_i\}$, $i = 1, 2$, *and* $4e_1 < \lambda_1 < \lambda_2$.

(2) *If* $b = b_1$, *then* $\sigma_1 \cap \mathbb{R} = \{4e_1\}$ *and* $\sigma_2 \cap \mathbb{R} = \{\lambda_0\}$ *for some* $\lambda_0 > 4e_1$.

(3) *If* $b_1 < b < b_2$, *then* $\sigma_1 \cap \mathbb{R} = \{\lambda_-\}$ *with* $\lambda_- < 4e_1$ *and* $\sigma_2 \cap \mathbb{R} = \{\lambda_0\}$ *for some* $\lambda_0 > 4e_1$.

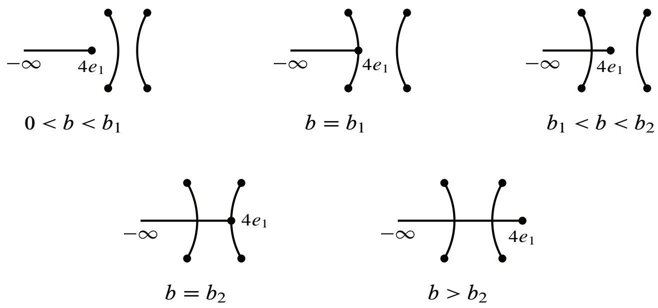


Figure 1

(4) *If* $b = b_2$*, then* $\sigma_1 \cap \mathbb{R} = \{\lambda_-\}$ *and* $\sigma_2 \cap \mathbb{R} = \{4e_1\}$ *with* $\lambda_- < \lambda_+ = 4e_1$.

(5) *If* $b > b_2$*, then* $\sigma_1 \cap \mathbb{R} = \{\lambda_-\}$ *and* $\sigma_2 \cap \mathbb{R} = \{\lambda_+\}$ *with* $\lambda_- < \lambda_+ < 4e_1$.

Note that $\sigma(L_b)$ is symmetric with respect to $\mathbb{R}$ (see Lemma 2.3 in Section 2) and the complement $\mathbb{C} \setminus \sigma(L_b)$ is path-connected (cf. [18, Theorem 2.2]). In order to prove this main theorem, we need to determine the intersection points of σ_i with $\mathbb{R}$ for $i = 1, 2$ as $b > 0$ varies. There are three kinds of intersection points: the one is less than $4e_1$, so it is met by $2k$ semi-arcs for some $k \geq 2$, and we call this kind of intersection point as *an inner intersection point*; the one is equal to $4e_1$, so it is the endpoint of the spectrum; the third one is bigger than $4e_1$, which could be an inner intersection point if σ_1 and σ_2 intersect. Therefore, there are two questions we need to solve:

Question 1. *If* $b \in \{b_1, b_2\}$*, what is* $d(4e_1)$*?*

Question 2. *Could we determine all inner intersection points of the spectrum for all* $b > 0$*? In particular, how to prove* $\sigma_1 \cap \sigma_2 = \emptyset$*?*

For example, $d(4e_1) \geq 3$ at $b = b_1$ indicates that there are at least two possible diagrams for $\sigma(L_{b_1})$ (Figure 2, where (S4a) (resp. (S4b)) corresponds to $d(4e_1) = 3$ (resp. $d(4e_1) = 5$), and how to rule out (S4b) is not easy. To overcome this difficulty, it is not a good way to compute $d(4e_1)$ directly for $b \in \{b_1, b_2\}$, but we can use other ideas to get the rough graph of $\sigma(L_b)$ without solving this question apriori. Taking the $b = b_2$ case for example, if we know there is an inner intersection point, then it follows from $d(4e_1) \geq 3$ at $b = b_2$ that the rough graph of $\sigma(L_{b_2})$ must be the one stated in Theorem 1.2, and so $d(4e_1) = 3$ at $b = b_2$ at a consequence. Inspired by this observation, we only consider Question 2, which is challenging because the computation is huge. We overcome this difficulty by some technique and obtain a complete and nice result.

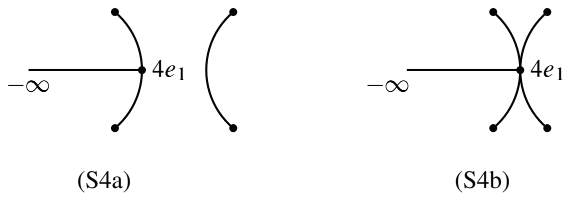


Figure 2

Theorem 1.3. *Let $\tau = \frac{1}{2} + bi$ with $b > 0$ and $\lambda_0 \in \sigma(L_b)$ with $Q_b(\lambda_0) \neq 0$. Then λ_0 is an inner intersection point if and only if λ_0 is a root of $f(\lambda)$. Moreover, the two roots of f denoted by λ_+, λ_- are real and $\lambda_- < 0 < \lambda_+$.*

Thanks to Theorem 1.3, the rough figure of $\sigma(L_{b_2})$ can be determined as stated in Theorem 1.2. Unfortunately, Theorem 1.3 is not enough for us to determine the rough graphs of $\sigma(L_b)$ for $b \leq b_1$. For example, we still cannot rule out the figure (S4b) via Theorem 1.3. However, the graphs for $b > b_1$ give us a new surprising idea: we should consider the spectrum along the imaginary axis. Specifically, we define

$$\hat{\sigma}(L_b) = \frac{1}{-4b^2}\sigma(L_{\frac{1}{4b}})$$

which plays the same role as the spectrum of L_b along the imaginary axis. As an auxiliary tool, we consider the mean field equation

$$\Delta u + e^u = 16\pi\delta_0 + 8\pi\delta_{\frac{1}{2}} \quad \text{on } T_b, \tag{1.6}$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator and δ_p denotes the Dirac measure at point p. Geometrically, a solution u to (1.6) leads to a metric $\frac{1}{2}e^u|dz|^2$ with constant curvature $+1$ acquiring two conic singularities with angles 10π and 6π. Physically, (1.6) appears in statistical physics as the *mean field limit* of the Euler flow, hence the name. It is also related to the self-dual condensates of the Chern–Simons–Higgs model in superconductivity. See [4, 6, 13, 15, 23, 24, 26] and references therein.

The solvability of (1.6) depends on the moduli b in a sophisticated manner and has been studied in [9, 10, 15]. In particular, a solution $u(z)$ is called *even* if $u(z) = u(-z)$ and is called *axisymmetric* if $u(z) = u(\bar{z})$. The number of even axisymmetric solutions of (1.6) has been calculated and reviewed here.

Lemma 1.4 ([15, Example 4.3]). *Let $\tau = \frac{1}{2} + bi$ with $b > 0$. Then there are $k_1 < k_2 < \frac{1}{2}$ such that*

(1) *If $b \in (0, k_1) \cup (k_2, \frac{1}{4k_2}) \cup (\frac{1}{4k_1}, +\infty)$, then (1.6) has exactly two even axisymmetric solutions.*

(2) *If* $b \in [k_1, k_2] \cup [\frac{1}{4k_2}, \frac{1}{4k_1}]$, *then* (1.6) *has a unique even axisymmetric solution.*

In fact, we will see $k_1 = b_1$ and $\frac{1}{4k_2} = b_2$ in the proof of Theorem 1.2. We prove that the number of even axisymmetric solutions of (1.6), which is computed in Lemma 1.4, is the same as the number of real points in the set

$$\Xi_b := (\hat{\sigma}(L_b) \cap \sigma(L_b)) \setminus Z(Q_b),$$

where $Z(Q_b)$ denotes the set of roots of $Q_b(\lambda)$.

Theorem 1.5. *Let* $\tau = \frac{1}{2} + bi$ *with* $b > 0$. *The number of even solutions of the mean field equation* (1.6) *equals* $\#\Xi_b$. *Furthermore, the number of even axisymmetric solutions equals* $\#(\Xi_b \cap \mathbb{R})$.

Thanks to Theorem 1.5, we can eliminate all impossible graphs and then uniquely determine the rough graphs of $\sigma(L_b)$ for $b \leq b_1$.

Remark 1.6. This idea might be used to study the spectrum for general DVT potentials $q^{\mathbf{n}}(z;\tau)$ with $\tau = bi$ when $b > 0$ approaches to 0 or ∞.

Indeed, let $\tau = bi$ with $b > 0$. If $\mathbf{n} = (n_0, n_1, n_2, n_3)$ satisfies either (1.3) or (1.4), the spectrum $\sigma(L_b^{\mathbf{n}})$ for the DTV potential $q^{\mathbf{n}}(z;\tau)$ does not lie on the real axis. On the other hand, Eremenko and Gabrielov [15] described the number of even axisymmetric solutions of the corresponding mean field equation:

$$\Delta u + e^u = 8\pi \sum_{k=0}^{3} n_k \delta_{\frac{\omega_k}{2}} \quad \text{on } T_b, \tag{1.7}$$

for $b > 0$ sufficiently small or large. Specifically, let $\varepsilon = (n_1 + n_2) - (n_0 + n_3)$. The quadruple (n_0, n_1, n_2, n_3) is called *special* if $\varepsilon/2$ is an odd integer and one of the following holds: either $\min\{2n_1, 2n_2\} \geq \varepsilon > 0$ or $\min\{2n_0, 2n_3\} \geq -\varepsilon > 0$. Denote by

$$M_0 = \min\Big\{n_0 + \frac{2+\varepsilon}{4}, n_1 + \frac{2-\varepsilon}{4}, n_2 + \frac{2-\varepsilon}{4}, n_3 + \frac{2+\varepsilon}{4}\Big\},$$
$$M_1 = \Big[\min\Big\{\frac{2+\varepsilon}{4}, \frac{1+n_1}{2}, \frac{1+n_2}{2}\Big\}\Big],$$
$$M_2 = \Big[\min\Big\{\frac{2-\varepsilon}{4}, \frac{1+n_0}{2}, \frac{1+n_3}{2}\Big\}\Big].$$

Eremenko and Gabrielov's results can be translated in the language of the mean field equation as follows.

Theorem 1.7 ([15, Theorem 1.5]). *Let $\tau = bi$ with $b > 0$. If b is sufficiently small or large, the number of even axisymmetric solutions of* (1.7) *is*

$$\begin{cases} M_0 & \text{if } (n_0, n_1, n_2, n_3) \text{ is special and satisfies either (1.3) or (1.4)}, \\ M_1 & \text{if } (n_0, n_1, n_2, n_3) \text{ is not special and satisfies (1.3)}, \\ M_2 & \text{if } (n_0, n_1, n_2, n_3) \text{ is not special and satisfies (1.4)}. \end{cases}$$

Now, consider the spectrum $\hat{\sigma}(L_b^{\mathbf{n}})$ along the imaginary axis (i.e., in τ direction). By a similar calculation, we have

$$\hat{\sigma}(L_b^{\mathbf{n}}) = -\frac{1}{b^2}\sigma(L_{\frac{1}{b}}^{\mathbf{n}}).$$

From the proof of Theorem 1.5, we can obtain that the number of even axisymmetric solutions of the mean field equation (1.7) equals $\#(\Xi_b^{\mathbf{n}} \cap \mathbb{R})$, where $\Xi_b^{\mathbf{n}} = (\hat{\sigma}(L_b^{\mathbf{n}}) \cap \sigma(L_b^{\mathbf{n}})) \setminus Z(Q_b^{\mathbf{n}})$. This fact and Theorem 1.7 should be useful in studying the spectrum $\sigma(L_b^{\mathbf{n}})$ for $b > 0$ sufficiently small and large.

The rest of this paper is organized as follows. In Section 2, we review the spectral theory of generalized Lamé equation from [9, 10]. In Section 3, we compute the monodromy at the unique real root of the spectral polynomial and then prove Theorem 1.1. We prove Theorem 1.3 in Section 4. In Section 5, we study the connection between the spectrum and the mean field equation and prove Theorem 1.5. In the last section, we give the proof of Theorem 1.2.

2. Preliminaries

In this section, we briefly review some preliminary results about the spectral theory of the complex Hill operator with DTV potential

$$L_\tau = \frac{d^2}{dx^2} - 6\wp(x + z_0; \tau) - 2\wp\Big(x + \frac{1}{2} + z_0; \tau\Big) \quad \text{in } L^2(\mathbb{R}, \mathbb{C}),$$

that are needed in later sections.

2.1. Spectrum of L_τ

Let $y_1(x)$ and $y_2(x)$ be any two linearly independent solutions of

$$L_\tau y = \lambda y. \tag{2.1}$$

Then so do $y_1(x+1)$ and $y_2(x+1)$ and hence there is a monodromy matrix $M(\lambda) \in \mathrm{SL}(2, \mathbb{C})$ such that

$$(y_1(x+1), y_2(x+1)) = (y_1(x), y_2(x))M(\lambda).$$

Define the *Hill's discriminant* $\Delta(\lambda)$ by

$$\Delta(\lambda) := \operatorname{tr} M(\lambda),$$

which is clearly an invariant of (2.1), i.e., does not depend on the choice of linearly independent solutions. This entire function $\Delta(\lambda)$ encodes all information of the spectrum $\sigma(L_\tau)$; see e.g. [18] and references therein. Indeed, Rofe and Beketov [27] proved that the spectrum $\sigma(L_\tau)$ can be described as

$$\sigma(L_\tau) = \Delta^{-1}([-2,2]) = \{\lambda \in \mathbb{C} \mid -2 \le \Delta(\lambda) \le 2\}.$$

This important fact plays a key role in this paper.

Clearly, λ is a (anti)periodic eigenvalue if and only if $\Delta(\lambda) = \pm 2$. Define

$$d(\lambda) := \operatorname{ord}_\lambda(\Delta(\cdot)^2 - 4).$$

Then it is well known (cf. [34, Section 2.3]) that $d(\lambda)$ equals the algebraic multiplicity of (anti)periodic eigenvalues. Let $c(\lambda, x, x_0)$ and $s(\lambda, x, x_0)$ be the special fundamental system of solutions of (2.1) satisfying the initial values

$$c(\lambda, x_0, x_0) = s'(\lambda, x_0, x_0) = 1,\ c'(\lambda, x_0, x_0) = s(\lambda, x_0, x_0) = 0.$$

Then we have

$$\Delta(\lambda) = c(\lambda, x_0 + 1, x_0) + s'(\lambda, x_0 + 1, x_0).$$

Define

$$p(\lambda, x_0) := \operatorname{ord}_\lambda s(\cdot, x_0 + 1, x_0),$$
$$p_i(\lambda) := \min\{p(\lambda, x_0) \colon x_0 \in \mathbb{R}\}.$$

It is known that $p(\lambda, x_0)$ is the algebraic multiplicity of a Dirichlet eigenvalue on the interval $[x_0, x_0 + 1]$, and $p_i(\lambda)$ denotes the immovable part of $p(\lambda, x_0)$ (cf. [20]). It was proved in [20, Theorem 3.2] that $d(\lambda) - 2p_i(\lambda) \ge 0$.

Note that $\deg Q_\tau(\lambda) = 5$. Apply the general result [20, Theorem 4.1] to L_τ, we obtain

Theorem 2.1 ([20, Theorem 4.1]). *Let $\tau \in \mathbb{H}$, the spectrum $\sigma(L_\tau)$ consists of finitely many bounded simple analytic arcs σ_k, $1 \le k \le \tilde{g}$ for some $\tilde{g} \le 2$ and one semi-infinite simple analytic arc σ_∞ which tends to $-\infty + \langle q \rangle$, with $\langle q \rangle = \int_{x_0}^{x_0+1} q(x)dx$, i.e.,*

$$\sigma(L_\tau) = \sigma_\infty \cup \bigcup_{k=1}^{\tilde{g}} \sigma_k.$$

Furthermore,

(1) *the finite endpoints of such arcs are exactly zeros of $Q_\tau(\cdot)$ with odd order;*

(2) *there are exactly $d(\lambda)$'s semi-arcs of $\sigma(L_\tau)$ meeting at each zero λ of $Q_\tau(\cdot)$ and*

$$d(\lambda) = \mathrm{ord}_\lambda\, Q_\tau(\cdot) + 2p_i(\lambda). \tag{2.2}$$

Furthermore, we need the following conclusions about $\sigma(L_\tau)$.

Theorem 2.2 ([18, Theorem 2.2]). *The complement $\mathbb{C} \setminus \sigma(L_\tau)$ is path-connected.*

In addition, the spectrum $\sigma(L_\tau)$ is symmetric with respect to $\mathbb{R}$ if $\mathrm{Re}\tau = \frac{1}{2}$. More general, we have the following conclusion.

Lemma 2.3. *Let $\tau = \frac{1}{2} + bi$ with $b > 0$. The spectrum of*

$$L_\tau^{(m,n)} = \frac{d^2}{dx^2} - m(m+1)\wp(x + z_0; \tau) - n(n+1)\wp\Big(x + \frac{1}{2} + z_0; \tau\Big)$$

is symmetric with respect to the real line $\mathbb{R}$.

Proof. Let $\tilde{\tau} = 2ib$ and consider

$$\begin{aligned} L_{\tilde{\tau}}^{(m,n)} := \frac{d^2}{dx^2} &- m(m+1)\Big(\wp(x + z_0; \tilde{\tau}) + \wp\Big(x + z_0 + \frac{1+\tilde{\tau}}{2}; \tilde{\tau}\Big)\Big) \\ &- n(n+1)\Big(\wp\Big(x + z_0 + \frac{1}{2}; \tilde{\tau}\Big) + \wp\Big(x + z_0 + \frac{\tilde{\tau}}{2}; \tilde{\tau}\Big)\Big). \end{aligned}$$

Since $\tilde{\tau} \in i\mathbb{R}_{>0}$, it was proved in [13, Lemma 3.5] that the spectrum $\sigma(L_{\tilde{\tau}}^{(m,n)})$ is symmetric with respect to $\mathbb{R}$. Since $\frac{1+\tilde{\tau}}{2} = \frac{1}{2} + ib = \tau$, we can rewrite the elliptic functions in the potential of $L_{\tilde{\tau}}^{(m,n)}$ as

$$\wp(z; \tilde{\tau}) + \wp\Big(z + \frac{1+\tilde{\tau}}{2}; \tilde{\tau}\Big) = \wp(z; \tau) + e_3(\tilde{\tau}),$$

and then

$$\wp\Big(z + \frac{1}{2}; \tilde{\tau}\Big) + \wp\Big(z + \frac{\tilde{\tau}}{2}; \tilde{\tau}\Big) = \wp\Big(z + \frac{1}{2}; \tau\Big) + e_3(\tilde{\tau}),$$

which implies $\sigma(L_{\tilde{\tau}}^{(m,n)}) = \sigma(L_\tau^{(m,n)}) - (m(m+1) + n(n+1))e_3(\tilde{\tau})$. From here and $e_3(\tilde{\tau}) \in \mathbb{R}$, we conclude that $\sigma(L_\tau^{(m,n)})$ is also symmetric with respect to $\mathbb{R}$. ∎

2.2. The spectral polynomial

First of all, recall that

$$\wp(z; \tau) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \Big(\frac{1}{(z - m - n\tau)^2} - \frac{1}{(m + n\tau)^2}\Big),$$

and it is well known that

$$\wp'(z;\tau)^2 = 4\prod_{k=1}^{3}(\wp(z;\tau) - e_k(\tau)) = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau), \quad (2.3)$$

where $e_k(\tau) = \wp(\frac{\omega_k}{2};\tau)$, $k = 1, 2, 3$, and $g_2(\tau), g_3(\tau)$ are well-known invariants of the elliptic curve. The Weierstraß zeta function is defined by

$$\zeta(z) = \zeta(z;\tau) := -\int^{z} \wp(\xi;\tau)d\xi$$

with two quasi-periods $\eta_j = \eta_j(\tau)$, $j = 1, 2$,

$$\eta_j(\tau) = 2\zeta\Big(\frac{\omega_j}{2};\tau\Big) = \zeta(z + \omega_j;\tau) - \zeta(z;\tau), \quad j = 1, 2,$$

and the Weierstraß sigma function is defined by

$$\sigma(z) = \sigma(z;\tau) := \exp\int^{z} \zeta(\xi)d\xi.$$

It is well known that $\zeta(z)$ is an odd meromorphic function with simple poles at $\mathbb{Z} + \mathbb{Z}\tau$ and $\sigma(z)$ is an odd entire function with simple zeros at $\mathbb{Z} + \mathbb{Z}\tau$.

Recall that (see [31, p. 394]) the spectral polynomial $Q_\tau(\lambda)$ of L_τ is given by

$$Q_\tau(\lambda) = (\lambda - 4e_1)R_1(\lambda)R_2(\lambda), \quad (2.4)$$

where

$$R_1(\lambda) = \lambda^2 - 2(3e_2 + 4e_3)\lambda - 31e_2^2 - 52e_2e_3 - 12e_3^2,$$
$$R_2(\lambda) = \lambda^2 - 2(3e_3 + 4e_2)\lambda - 31e_3^2 - 52e_2e_3 - 12e_2^2.$$

Consider the associated hyperelliptic curve

$$\Gamma_\tau := \{(\lambda, W) \,|\, W^2 = Q_\tau(\lambda)\},$$

which is of genus 2. There is an embedding $i\colon \Gamma_\tau \hookrightarrow \mathrm{Sym}^3 T_\tau$ such that the image of Γ_τ in $\mathrm{Sym}^3 T_\tau$ is defined by (cf. [9, 10])

$$Y_\tau := \left\{ \{a_1, a_2, a_3\} \in \Sigma_\tau \;\middle|\; \begin{array}{l} 2\sum_{j\neq i}\big(\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)\big) \\ = \zeta(a_i + \frac{1}{2}) + \zeta(a_i - \frac{1}{2}) - 2\zeta(a_i), \\ \text{for } i = 1, 2, 3. \end{array} \right\} \cup \left\{\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\}\right\},$$

where

$$\Sigma_\tau = \left\{\{a_1, a_2, a_3\} \in \mathrm{Sym}^3\Big(T_\tau \setminus \Big\{0, \frac{1}{2}\Big\}\Big) \;\Big|\; a_i \neq a_j \text{ for } i \neq j\right\}.$$

Then $Y_\tau = i(\Gamma_\tau) \cong \Gamma_\tau$ is a hyperelliptic curve of genus 2. Clearly, if $\mathbf{a} := \{a_1, a_2, a_3\} \in Y_\tau$, then $-\mathbf{a} := \{-a_1, -a_2, -a_3\} \in Y_\tau$. In fact, we have a branched covering map of degree 2 (See [4, Theorem 7.4])

$$\lambda.: Y_\tau \to \mathbb{C}, \quad \mathbf{a} := \{a_1, a_2, a_3\} \mapsto \lambda_{\mathbf{a}} = 3\left(\wp(a_1) + \wp(a_2) + \wp(a_3)\right) - 5e_1. \tag{2.5}$$

Note that $\lambda_{\mathbf{a}} = \lambda_{-\mathbf{a}}$, then $\mathbf{a} \in Y_\tau$ is a *branch point*, i.e., $Q_\tau(\lambda_{\mathbf{a}}) = 0$, if and only if $\mathbf{a} = -\mathbf{a}$. Equivalently, we have

$$\{\mathbf{a} \in Y_\tau \mid \mathbf{a} = -\mathbf{a}\} = \{\mathbf{a} \in Y_\tau \mid Q_\tau(\lambda_{\mathbf{a}}) = 0\}. \tag{2.6}$$

Lemma 2.4. *Let $\tau = \frac{1}{2} + bi$ with $b > 0$. Then all roots of the spectral polynomial $Q_b(\lambda)$ are distinct and listed as follows:*

$$4e_1, \quad \mu, \quad \bar{\mu}, \quad \nu, \quad \bar{\nu}.$$

Moreover, $d(\mu) = d(\bar{\mu}) = d(\nu) = d(\bar{\nu}) = 1$.

Proof. Clearly, $4e_1 \in \mathbb{R}$ is a root of $Q_b(\lambda)$. Note that the discriminant of $R_1(\lambda)$:

$$\Delta_{R_1} = 16(2e_2 + e_3)(5e_2 + 7e_3) \neq 0$$

because $e_3 = \overline{e_2}$ and $e_2 \notin \mathbb{R}$. Denote the roots of $R_1(\lambda)$ by μ, ν, then $\mu \neq \nu$. Clearly, $\bar{\mu} \neq \bar{\nu}$ and $\bar{\mu}, \bar{\nu}$ are roots of $R_2(\lambda)$. Since

$$\mu + \nu = 2(3e_2 + 4e_3) \notin \mathbb{R},$$

we have $\nu \neq \bar{\mu}$. So, if we could show $\mu, \nu \notin \mathbb{R}$, then $4e_1, \mu, \bar{\mu}, \nu, \bar{\nu}$ are distinct.

Let $\mathbf{a} \in Y_b$. If $\mathbf{a} = -\mathbf{a}$, then one of the following cases holds:

(1) $\mathbf{a} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$,

(2) $\mathbf{a} = \mathbf{a}_a^1 := \{\frac{\tau}{2}, a, -a\}$ for some $a \in T_b \setminus \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$,

(3) $\mathbf{a} = \mathbf{a}_a^2 := \{\frac{1+\tau}{2}, a, -a\}$ for some $a \in T_b \setminus \{0, \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$.

In case (1), we have $\lambda_{\{\frac{1}{2},\frac{1}{2},\frac{1}{2}\}} = 4e_1$ which is a real root of $Q_b(\lambda)$. In case (2) and (3), by direct computation, we obtain that $\mathbf{a}_a^1 = \{\frac{\tau}{2}, a, -a\} \in Y_b$ if and only if $\mathbf{a}_{\bar{a}}^2 = \{\frac{1+\tau}{2}, \bar{a}, -\bar{a}\} \in Y_b$. Suppose that $\lambda_{\mathbf{a}_a^1} = 6\wp(a) + 3e_2 - 5e_1$ is a root of $Q_b(\lambda_{\mathbf{a}_a^1})$. Since $Q_b(\lambda_{\mathbf{a}_a^1})$ is of real coefficient, $\overline{\lambda_{\mathbf{a}_a^1}} = 6\wp(\bar{a}) + 3e_3 - 5e_1 = \lambda_{\mathbf{a}_{\bar{a}}^2}$ is also a root of $Q_b(\lambda_{\mathbf{a}_a^1})$. From $\mathbf{a}_{\bar{a}}^2 \neq \pm\mathbf{a}_a^1$ and (2.5), we have $\lambda_{\mathbf{a}_a^1} \neq \lambda_{\mathbf{a}_{\bar{a}}^2}$, so $\lambda_{\mathbf{a}_a^1} \notin \mathbb{R}$. Therefore, $\mu, \nu \notin \mathbb{R}$, which is desired.

Finally, note that all roots of $Q_b(\lambda)$ are distinct, it was proved in [12, Theorem 1.3] that the spectrum $\sigma(L_b)$ has at most one endpoint with $d(\lambda) \geq 3$. Therefore, $d(\mu) = d(\bar{\mu}) = d(\nu) = d(\bar{\nu}) = 1$ by Lemma 2.3. ∎

2.3. Generalized Lamé equation

In this section, we study the generalized Lamé equation

$$\mathcal{L}_\lambda: \quad y''(z) = \Big(6\wp(z;\tau) + 2\wp\Big(z + \frac{1}{2};\tau\Big) + \lambda\Big)y(z), \quad z \in \mathbb{C}.$$

Let $y(z)$ be a solution of $\mathcal{L}_\lambda$. Consider the Laurent expansion of $y(z)$ at $z = z_0$. We obtain that the local exponent at $z_0 = 0$ is -2 or 3, the local exponent at $z_0 = \frac{1}{2}$ is -1 or 2, and the local exponent at any other point is 0 or 1. Furthermore, note that $6\wp(z) + 2\wp(z + \frac{1}{2}) + \lambda$ is even elliptic, it is easily seen (cf. [19, 28]) that $y(z)$ is meromorphic in $\mathbb{C}$. Hence, the monodromy representation of $\mathcal{L}_\lambda$ is a group homomorphism $\rho_\tau: \pi_1(T_\tau) \to SL(2, \mathbb{C})$, which is abelian and thus reducible. Then there is a common eigenfunction $y(z)$ of ρ_τ, i.e., $y(z + \omega_j) = \lambda_j y(z)$ for some $\lambda_j \neq 0$, $j = 1, 2$, so $y(z)$ is elliptic of the second kind. From the theory of elliptic functions, we conclude that up to a constant, the common eigenfunction $y(z)$ can be written as (cf. [4, 36]):

$$y(z) = y_{\mathbf{a}}(z) := e^{c_{\mathbf{a}} z} \frac{\sigma(z - a_1)\sigma(z - a_2)\sigma(z - a_3)}{\sigma(z)^2 \sigma(z - \frac{1}{2})}$$

where $\mathbf{a} \in \{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}\} \cup \Sigma_\tau$ and $c_{\mathbf{a}} \in \mathbb{C}$ is a constant related to $\mathbf{a}$. The following theorem tells us that Y_τ could parametrize the solutions of $\mathcal{L}_\lambda$.

Theorem 2.5 ([4, 36]). *Let* $\mathbf{a} = \{a_1, a_2, a_3\} \in \{\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}\} \cup \Sigma_\tau$. *Then* $y_{\mathbf{a}}(z)$ *is a solution of* $\mathcal{L}_\lambda$ *for some* λ *if and only if* $\mathbf{a} \in Y_\tau$, $\lambda = \lambda_{\mathbf{a}}$ *and*

$$c_{\mathbf{a}} = \zeta(a_1) + \zeta(a_2) + \zeta(a_3) - \frac{1}{2}\eta_1. \tag{2.7}$$

Proof. Note that

$$\frac{y_{\mathbf{a}}'(z)}{y_{\mathbf{a}}(z)} = c_{\mathbf{a}} + \zeta(z - a_1) + \zeta(z - a_2) + \zeta(z - a_3) - 2\zeta(z) - \zeta\Big(z - \frac{1}{2}\Big),$$
$$\Big(\frac{y_{\mathbf{a}}'(z)}{y_{\mathbf{a}}(z)}\Big)' = -\wp(z - a_1) - \wp(z - a_2) - \wp(z - a_3) + 2\wp(z) + \wp\Big(z - \frac{1}{2}\Big)$$

are both elliptic functions. Consider elliptic function

$$\begin{aligned} g(z) := {} & \Big(\frac{y_{\mathbf{a}}'(z)}{y_{\mathbf{a}}(z)}\Big)' + \Big(\frac{y_{\mathbf{a}}'(z)}{y_{\mathbf{a}}(z)}\Big)^2 - 6\wp(z) - 2\wp\Big(z + \frac{1}{2}\Big) - \lambda, \\ = {} & -\wp(z - a_1) - \wp(z - a_2) - \wp(z - a_3) - 4\wp(z) - \wp\Big(z - \frac{1}{2}\Big) - \lambda \\ & + \Big(c_{\mathbf{a}} + \zeta(z - a_1) + \zeta(z - a_2) + \zeta(z - a_3) - 2\zeta(z) - \zeta\Big(z - \frac{1}{2}\Big)\Big)^2, \end{aligned}$$

where $\lambda \in \mathbb{C}$. Clearly, $y_{\mathbf{a}}(z)$ is a solution of $\mathcal{L}_\lambda$ if and only if $g(z) \equiv 0$ if and only if none of $0, \frac{1}{2}, a_1, a_2$ and a_3 are poles of $g(z)$ and the constant term of the Laurent expansion at $z = 0$ is 0.

Case 1. Assume that $\mathbf{a} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$. Note that

$$g(z) = \left(c_{\mathbf{a}} + 2\zeta\left(z - \frac{1}{2}\right) - 2\zeta(z)\right)^2 - 4\wp(z) - 4\wp\left(z - \frac{1}{2}\right) - \lambda.$$

First, 0 is not a pole of $g(z)$ if and only if

$$c_{\mathbf{a}} = 2\zeta\left(\frac{1}{2}\right) = \eta_1, \tag{2.8}$$

if and only if $\frac{1}{2}$ is not a pole of $g(z)$. Second, the constant term of the Laurent expansion of $g(z)$ at $z = 0$ is 0 if and only if

$$\lambda = -4\left(-2\wp\left(\frac{1}{2}\right)\right) - 4\wp\left(\frac{1}{2}\right) = 4e_1 = \lambda_{\{\frac{1}{2},\frac{1}{2},\frac{1}{2}\}}. \tag{2.9}$$

Case 2. Assume that $\mathbf{a} \in \Sigma_\tau$. First, 0 is not a pole of $g(z)$ if and only if

$$c_{\mathbf{a}} = \zeta(a_1) + \zeta(a_2) + \zeta(a_3) - \zeta\left(\frac{1}{2}\right). \tag{2.10}$$

Second, $\frac{1}{2}$ is not a pole of $g(z)$ if and only if

$$c_{\mathbf{a}} = \zeta\left(a_1 - \frac{1}{2}\right) + \zeta\left(a_2 - \frac{1}{2}\right) + \zeta\left(a_3 - \frac{1}{2}\right) + 2\zeta\left(\frac{1}{2}\right). \tag{2.11}$$

Third, a_i is not a pole of $g(z)$ if and only if

$$c_{\mathbf{a}} = \zeta(a_j - a_i) + \zeta(a_k - a_i) + 2\zeta(a_i) + \zeta\left(a_i - \frac{1}{2}\right), \tag{2.12}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

The system of equations (2.10), (2.11), and (2.12) are equivalent to (2.10) and

$$\zeta\left(a_i + \frac{1}{2}\right) + \zeta\left(a_i - \frac{1}{2}\right) - 2\zeta(a_i) = 2\sum_{j \neq i}^{3}\big(\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)\big)$$

for $i = 1, 2, 3$, i.e., $\mathbf{a} \in Y_\tau$. Furthermore, the constant term of the Laurent expansion of $g(z)$ at $z = 0$ is 0 if and only if

$$\lambda = 3\big(\wp(a_1) + \wp(a_2) + \wp(a_3)\big) - 5e_1 = \lambda_{\mathbf{a}}. \qquad \blacksquare$$

In what follows, we always assume that $c_{\mathbf{a}}$ is defined by (2.7) in $y_{\mathbf{a}}(z)$.

Corollary 2.6 ([12, p. 464]). *Let* $\mathbf{a} \in Y_\tau$, *then either* $\mathbf{a} = -\mathbf{a}$ *or* $\mathbf{a} \cap -\mathbf{a} = \emptyset$. *Moreover,* $y_{\mathbf{a}}(z)$ *and* $y_{-\mathbf{a}}(z)$ *are linearly independent if and only if* $\mathbf{a} \cap -\mathbf{a} = \emptyset$, *equivalently, if and only if* $Q_\tau(\lambda_{\mathbf{a}}) \neq 0$.

Proof. First of all, $y_{\pm\mathbf{a}}(z)$ are solutions of $\mathcal{L}_{\lambda_{\mathbf{a}}}$ because $\mathbf{a} \in Y_\tau$, then

$$\left(y_{\mathbf{a}}(z)y'_{-\mathbf{a}}(z) - y'_{\mathbf{a}}(z)y_{-\mathbf{a}}(z)\right)' \equiv 0.$$

Hence, the Wronskian of $y_{\mathbf{a}}(z)$ and $y_{-\mathbf{a}}(z)$:

$$W(y_{\mathbf{a}}, y_{-\mathbf{a}}) = \begin{vmatrix} y_{\mathbf{a}}(z) & y_{-\mathbf{a}}(z) \\ y'_{\mathbf{a}}(z) & y'_{-\mathbf{a}}(z) \end{vmatrix}$$

is constant. Note that the zero set of $y_{\mathbf{a}}(z)$ is $\mathbf{a}$. If $\mathbf{a} \cap -\mathbf{a} \neq \emptyset$, we have $W(y_{\mathbf{a}}, y_{-\mathbf{a}}) = 0$, thus $y_{\mathbf{a}}(z)$ and $y_{-\mathbf{a}}(z)$ are linearly dependent, which forces $\mathbf{a} = -\mathbf{a}$. On the other hand, if $\mathbf{a} = -\mathbf{a}$, then $y_{\mathbf{a}}(z)$ and $y_{-\mathbf{a}}(z)$ are linearly dependent by the transformation law (denote by $\eta_3 = 2\zeta(\frac{\omega_3}{2}) = \eta_1 + \eta_2$)

$$\sigma(z + \omega_k) = -e^{\eta_k(z + \frac{\omega_k}{2})}\sigma(z), \quad k = 1, 2, 3. \qquad \blacksquare$$

2.4. Monodromy and Hill's discriminant

Let $\mathbf{a} \in Y_\tau$ in the following discussion, then $y_{\pm\mathbf{a}}(z)$ are solutions of the same generalized Lamé equation $\mathcal{L}_{\lambda_{\mathbf{a}}}$. The Legendre relation $\tau\eta_1 - \eta_2 = 2\pi i$ implies that there is a unique $(r, s) \in \mathbb{C}^2$ satisfying

$$r + s\tau = a_1 + a_2 + a_3 - \frac{1}{2}$$

and

$$r\eta_1 + s\eta_2 = \zeta(a_1) + \zeta(a_2) + \zeta(a_3) - \frac{1}{2}\eta_1,$$

which is equivalent to

$$\zeta(a_1) + \zeta(a_2) + \zeta(a_3) - \eta_1(a_1 + a_2 + a_3) = -2\pi i s,$$
$$\tau(\zeta(a_1) + \zeta(a_2) + \zeta(a_3)) - \eta_2(a_1 + a_2 + a_3) - \pi i = 2\pi i r.$$

Furthermore, the transformation law $\sigma(z + \omega_j) = -e^{(z + \frac{\omega_j}{2})\eta_j}\sigma(z)$ with $j = 1, 2$ implies

$$y_{\pm\mathbf{a}}(z + 1) = e^{\pm\sum_{j=1}^{3}(\zeta(a_j) - \eta_1 a_j)}y_{\pm\mathbf{a}}(z) = e^{\mp 2\pi i s}y_{\pm\mathbf{a}}(z), \qquad (2.13a)$$
$$y_{\pm\mathbf{a}}(z + \tau) = -e^{\pm\tau\sum_{j=1}^{g}(\zeta(a_j) - \eta_2 a_j)}y_{\pm\mathbf{a}}(z) = e^{\pm 2\pi i r}y_{\pm\mathbf{a}}(z), \qquad (2.13b)$$

namely $y_{\pm\boldsymbol{a}}(z)$ are elliptic of the second kind. Since $y_{\pm\mathbf{a}}(z)$ are solutions of $\mathcal{L}_{\lambda_{\mathbf{a}}}$, thus $y_{\pm\mathbf{a}}(x + z_0)$ are solutions of $L_\tau y = \lambda_{\mathbf{a}} y$.

Case 1. If $\mathbf{a}$ is not a branch point, i.e., $\mathbf{a} \cap -\mathbf{a} = \emptyset$, then $y_{\mathbf{a}}(x+z_0)$ and $y_{-\mathbf{a}}(x+z_0)$ are linearly independent solutions of $L_\tau y = \lambda_{\mathbf{a}} y$ and satisfy

$$y_{\pm\mathbf{a}}(x+z_0+1) = e^{\mp 2\pi i s} y_{\pm\mathbf{a}}(x+z_0). \tag{2.14}$$

Case 2. If $\mathbf{a}$ is a branch point, i.e., $\mathbf{a} = -\mathbf{a}$, then $y_{\mathbf{a}}(z)$ and $y_{-\mathbf{a}}(z)$ are linearly dependent. By (2.13), we get $2r, 2s \in \mathbb{Z}$. Note that $Q_\tau(\lambda_{\mathbf{a}}) = 0$, it was proved in [9, Theorem 2.7] that the monodromy of $\mathscr{L}_{\lambda_{\mathbf{a}}}$ is not completely reducible and there is a solution $y_2(z)$ linearly independent with $y_{\mathbf{a}}(z)$ such that (note $e^{2\pi i s} = e^{-2\pi i s} = \pm 1$)

$$y_{\mathbf{a}}(z+1) = e^{-2\pi i s} y_{\mathbf{a}}(z), \quad y_2(z+1) = e^{2\pi i s} y_2(z) + e^{2\pi i s}\chi_{\mathbf{a}} y_{\mathbf{a}}(z), \tag{2.15}$$

where $\chi_{\mathbf{a}} \in \mathbb{C}$ is a constant.

From (2.14) and (2.15), the Hill's discriminant in any case is given by

$$\Delta(\lambda_{\mathbf{a}}) = e^{-2\pi i s} + e^{2\pi i s} = e^{\sum_{j=1}^{g}(\zeta(a_j)-\eta_1 a_j)} + e^{-\sum_{j=1}^{g}(\zeta(a_j)-\eta_1 a_j)}. \tag{2.16}$$

Clearly, $\lambda_{\mathbf{a}} \in \sigma(L_\tau)$ if and only if $s \in \mathbb{R}$, i.e., $\sum_{j=1}^{g}(\zeta(a_j) - \eta_1 a_j) \in i\mathbb{R}$.

3. Monodromy at the real endpoint

In this section, we always assume $\tau = \frac{1}{2} + bi$ with $b > 0$. We will first recall some basic properties for the quantities e_1, e_2, e_3, g_2, g_3 and η_1 associated with the Weierstraß elliptic function $\wp(z;\tau)$, which will also be frequently used in the following sections.

First of all, $e_1, \eta_1 \in \mathbb{R}$, $e_3 = \overline{e_2} \notin \mathbb{R}$ and the second equality in (2.3) gives us

$$e_1 + e_2 + e_3 = 0, \tag{3.1}$$
$$g_2 = 2(e_1^2 + e_2^2 + e_3^2) \in \mathbb{R}, \tag{3.2}$$
$$g_3 = 4e_1 e_2 e_3 \in \mathbb{R}. \tag{3.3}$$

Note that e_1, $\overline{e_2} = e_3 \notin \mathbb{R}$ and (3.1), in what follows, we set

$$e_1 = 2x, \quad e_2 = -x + iy, \quad e_3 = -x - iy \quad \text{with } x, y \in \mathbb{R} \text{ and } y \neq 0, \tag{3.4}$$

and then

$$g_2 = 4(3x^2 - y^2), \tag{3.5a}$$
$$g_3 = 8x(x^2 + y^2) = 4e_1^3 - e_1 g_2. \tag{3.5b}$$

Since $e_1 \neq e_2 \neq e_3 \neq e_1$, it is easy to see that

$$g_2 - 3e_k^2 = (e_i - e_j)^2 \neq 0, \quad \text{for } \{i, j, k\} = \{1, 2, 3\}. \tag{3.6}$$

In particular,

$$g_2 - 3e_1^2 = (e_2 - e_3)^2 = -4y^2 < 0, \quad \text{i.e.,} \quad g_2 < 3e_1^2. \tag{3.7}$$

The derivatives of e_1, g_2 and η_1 with respect to b are listed as follows:

$$e_1'(b) = \frac{1}{\pi}\Big(e_1^2 - \eta_1 e_1 - \frac{1}{6}g_2\Big) \quad \text{(see [11, (2.15)])}, \tag{3.8a}$$

$$g_2'(b) = \frac{1}{\pi}(3g_3 - 2\eta_1 g_2) = \frac{1}{\pi}(12e_1^3 - 3e_1 g_2 - 2\eta_1 g_2) \quad \text{(see [5])}, \tag{3.8b}$$

$$\eta_1'(b) = \frac{1}{24\pi}(g_2 - 12\eta_1^2) \quad \text{(see [12, (1.5)])}. \tag{3.8c}$$

Moreover, we have the following conclusions.

Proposition 3.1 ([23, Theorem 1.7]). *We have $e_1(\frac{1}{2}) = 0$ and*

$$e_1'(b) > 0 \quad \textit{for all } b > 0.$$

Proposition 3.2 ([5, Corollary 4.4]). *There exists $b_g \approx 0.47 \in (\frac{1}{2\sqrt{3}}, \frac{1}{2})$ such that*

$$g_2'(b) \begin{cases} < 0 & \textit{for } b \in (0, b_g), \\ = 0 & \textit{for } b = b_g, \\ > 0 & \textit{for } b \in (b_g, \infty). \end{cases}$$

And $g_2(b) = 0$ if and only if $b \in \{\frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}\}$.

Proposition 3.3 ([23, Theorem 1.7]). *There exists $b_\eta \approx 0.24108 < \frac{1}{2\sqrt{3}}$ such that*

$$\eta_1'(b) \begin{cases} > 0 & \textit{for } b \in (0, b_\eta), \\ = 0 & \textit{for } b = b_\eta, \\ < 0 & \textit{for } b \in (b_\eta, +\infty). \end{cases}$$

Proposition 3.4. [23, Theorem 1.7] *Both $e_1 + \eta_1$ and $\frac{1}{2}e_1 - \eta_1$ increase in b, and*

$$\frac{1}{2}e_1 < \eta_1. \tag{3.9}$$

Moreover, there exists $\tilde{b} \in (0.3, 0.4)$ such that $e_1(\tilde{b}) + \eta_1(\tilde{b}) = 0$.

Remark 3.5. All numerical computations in this paper are based on the $q = e^{2\pi i\tau} = -e^{-2\pi b}$ expansions of e_1, g_2, η_1 which are recalled here for readers' convenience:

$$e_1(b) = 16\pi^2\Big(\frac{1}{24} + \sum_{k=1}^{\infty}(-1)^k \sigma_o^k e^{-2k\pi b}\Big), \qquad \text{where } \sigma_o^k = \sum_{1\le d|k,\, d \text{ is odd}} d,$$

$$g_2(b) = 320\pi^4\Big(\frac{1}{240} + \sum_{k=1}^{\infty}(-1)^k\sigma_3^k e^{-2k\pi b}\Big), \quad \text{where } \sigma_3^k = \sum_{1\le d|k} d^3,$$
$$\eta_1(b) = 8\pi^2\Big(\frac{1}{24} - \sum_{k=1}^{\infty}(-1)^k\sigma_1^k e^{-2k\pi b}\Big), \quad \text{where } \sigma_1^k = \sum_{1\le d|k} d.$$

It was proved in [12] that there is a unique $\hat{b} > 0$ such that $\eta_1(\frac{1}{2} + \hat{b}i) = 0$. Furthermore, a numerical computation shows $\hat{b} \approx 0.13094$.

3.1. Monodromy data at the real root

Note that $4e_1$ is the only real root of $Q_b(\lambda)$, we will calculate the degree $d(4e_1)$ using the monodromy data in this section.

Note that we may rewrite (see, e.g., [25, Lemma 3.2])

$$\frac{1}{\wp(z) - \wp(\frac{1}{2})} = c_0 + c_1\wp\Big(z - \frac{1}{2}\Big) \tag{3.10}$$

with

$$c_1 = \lim_{z\to\frac{1}{2}} \frac{(z-\frac{1}{2})^2}{\wp(z) - \wp(\frac{1}{2})} = \frac{2}{\wp''(\frac{1}{2})} = \frac{2}{6e_1^2 - \frac{1}{2}g_2} = \frac{4}{12e_1^2 - g_2} \neq 0, \tag{3.11}$$
$$c_0 = -c_1\wp\Big(0 - \frac{1}{2}\Big) = -e_1c_1 = -\frac{4e_1}{12e_1^2 - g_2}, \tag{3.12}$$

where (3.12) is due to 0 is a second order pole of $\wp(z) - \wp(\frac{1}{2})$ and so a second order zero of $1/\big(\wp(z) - \wp(\frac{1}{2})\big)$.

Define

$$\chi(z) := \int_0^z \frac{d\xi}{(\wp(\xi) - \wp(\frac{1}{2}))^2}.$$

Recall that $\zeta(z) = -\int^z \wp(\xi)d\xi$ and $\wp'' = 6\wp^2 - \frac{1}{2}g_2$. We obtain, from (3.10), that

$$\begin{aligned}\chi(z) &= -2c_0c_1\Big(\zeta\Big(z - \frac{1}{2}\Big) + \zeta\Big(\frac{1}{2}\Big)\Big) + \frac{1}{6}c_1^2\wp'\Big(z - \frac{1}{2}\Big) + \Big(c_0^2 + \frac{1}{12}g_2c_1^2\Big)z \\ &= -2c_0c_1\zeta\Big(z - \frac{1}{2}\Big) + \frac{1}{6}c_1^2\wp'\Big(z - \frac{1}{2}\Big) + \Big(c_0^2 + \frac{1}{12}g_2c_1^2\Big)z - c_0c_1\eta_1.\end{aligned}$$

Note that $\eta_1 = 2\zeta(\frac{1}{2}) = \zeta(z + \frac{1}{2}) - \zeta(z - \frac{1}{2})$, then χ is odd and has two quasi-periods:

$$\chi_1 = \chi(z+1) - \chi(z) = c_0^2 + \frac{1}{12}g_2c_1^2 - 2c_0c_1\eta_1,$$
$$\chi_2 = \chi(z+\tau) - \chi(z) = \Big(c_0^2 + \frac{1}{12}g_2c_1^2\Big)\tau - 2c_0c_1\eta_2.$$

Let $\mathbf{a} = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$, then $\lambda_{\mathbf{a}} = 4e_1$. By the transformation law

$$\sigma(z+\omega_j) = -e^{\eta_j(z+\frac{\omega_j}{2})}\sigma(z),$$

we have

$$y_{\mathbf{a}}(z) = e^{\eta_1 z}\frac{\sigma(z-\frac{1}{2})^2}{\sigma(z)^2} = e^{-\eta_1 z}\frac{\sigma(z+\frac{1}{2})^2}{\sigma(z)^2}.$$

Note that

$$\frac{\sigma(z-\frac{1}{2})\sigma(z+\frac{1}{2})}{\sigma(z)^2} = -\sigma(\frac{1}{2})^2\Big(\wp(z) - \wp\Big(\frac{1}{2}\Big)\Big);$$

then, a direct computation shows that

$$y_2(z) := \chi(z)y_{\mathbf{a}}(z) = \sigma\Big(\frac{1}{2}\Big)^4 y_{\mathbf{a}}(z)\int_0^z \frac{1}{y_{\mathbf{a}}(\xi)^2}d\xi$$

is also a solution of the $\mathcal{L}_{4e_1}$ Note that $y_{\mathbf{a}}(z+1) = y_{\mathbf{a}}(z)$. Then

$$y_2(z+1) = (\chi_1 + \chi(z))\, y_{\mathbf{a}}(z) = \chi_1 y_{\mathbf{a}}(z) + y_2(z);$$

thus the monodromy matrix is

$$M(4e_1) = \begin{pmatrix} 1 & \chi_1 \\ 0 & 1 \end{pmatrix}.$$

Note that $d(4e_1) = 1 + 2p_i(4e_1)$ by (2.2) and [20, Proposition 3.1] proved that $p_i(\lambda) \geq 1$ if and only if all solutions of $L_b y = \lambda y$ are (anti)periodic. Then $d(4e_1) \geq 3$ if and only if

$$\chi_1 = \Big(e_1^2 + 2e_1\eta_1 + \frac{1}{12}g_2\Big)c_1^2 = 0$$

if and only if

$$e_1^2 + 2e_1\eta_1 + \frac{1}{12}g_2 = 0.$$

Let $h(b) = e_1^2 + 2e_1\eta_1 + \frac{1}{12}g_2$, we have

$$h'(b) = 3e_1'(e_1 + \eta_1).$$

Note that

$$\begin{aligned} &e_1' > 0, && e_1\Big(\frac{1}{2}\Big) = 0, \\ &g_2(b_\eta) = 12(\eta_1(b_\eta))^2, && e_1(b_\eta) + \eta_1(b_\eta) < 0, \\ &g_2\Big(\frac{1}{2\sqrt{3}}\Big) = g_2\Big(\frac{\sqrt{3}}{2}\Big) = 0, && e_1\Big(\frac{1}{2\sqrt{3}}\Big) + 2\eta_1\Big(\frac{1}{2\sqrt{3}}\Big) > 0; \end{aligned}$$

then

$$h(b_\eta) = (e_1 + \eta_1)^2 > 0, \quad h\Big(\frac{1}{2\sqrt{3}}\Big) = e_1(e_1 + 2\eta_1) < 0,$$
$$h\Big(\frac{1}{2}\Big) = \frac{1}{12}g_2 < 0, \qquad h\Big(\frac{\sqrt{3}}{2}\Big) = e_1^2 + 2e_1\eta_1 > 0,$$

so there exist $b_1 \approx 0.2716572 \in (b_\eta, \frac{1}{2\sqrt{3}})$ and $b_2 \approx 0.596803 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ such that

$$h(b) \begin{cases} > 0 & \text{for } b \in (0, b_1) \cup (b_2, +\infty), \\ = 0 & \text{for } b \in \{b_1, b_2\}, \\ < 0 & \text{for } b \in (b_1, b_2). \end{cases}$$

Therefore, $d(4e_1) \geq 3$ if and only if $b \in \{b_1, b_2\}$. From here and Lemma 2.4, we proved Theorem 1.1.

4. Inner intersection points

In this section, we study the inner intersection points of the spectrum $\sigma(L_\tau)$ with $\mathrm{Re}\tau = \frac{1}{2}$ and prove Theorem 1.3 recalled here.

Theorem 4.1 (= Theorem 1.3). *Let $\tau = \frac{1}{2} + bi$ with $b > 0$ and $\lambda_0 \in \sigma(L_\tau)$ with $Q_\tau(\lambda_0) \neq 0$. Then λ_0 is an inner intersection point if and only if λ_0 is a root of*

$$f(\lambda) := \lambda^2 + (5e_1 + 4\eta_1)\lambda - 27e_1^2 + 2e_1\eta_1 + \frac{3}{4}g_2.$$

Moreover, the two roots of f denoted by λ_+, λ_- are real and $\lambda_- < 0 < \lambda_+$.

Proof. Let $\lambda_0 \in \mathbb{C}$ with $Q_\tau(\lambda_0) \neq 0$. From $Y_\tau \cong \Gamma_\tau$ and (2.5), there is a small neighborhood $U \subset \mathbb{C}$ of λ_0 such that $Q_\tau(\lambda) \neq 0$ for $\lambda \in U$ and $\lambda \in U$ can be a local coordinate for the hyperelliptic curve Y_τ, namely $a_1 = a_1(\lambda), a_2 = a_2(\lambda)$ and $a_3 = a_3(\lambda)$ are holomorphic for $\lambda \in U$. For all $\lambda \in U$, (2.5) tells us that

$$\lambda = \lambda_{\mathbf{a}} = 3\left(\wp(a_1) + \wp(a_2) + \wp(a_3)\right) - 5e_1 \tag{4.1}$$

and then

$$\wp'(a_1)a_1'(\lambda) + \wp'(a_2)a_2'(\lambda) + \wp'(a_3)a_3'(\lambda) = \frac{1}{3} \quad \text{for } \lambda \in U$$

and so

$$(a_1'(\lambda_0), a_2'(\lambda_0), a_3'(\lambda_0)) \neq (0, 0, 0). \tag{4.2}$$

Next, note that $Q_\tau(\lambda) \neq 0$ for $\lambda \in U$ implies

$$\{a_1(\lambda), a_2(\lambda), a_3(\lambda)\} \cap \{-a_1(\lambda), -a_2(\lambda), -a_3(\lambda)\} = \emptyset \quad \text{for } \lambda \in U,$$

i.e., $\mathbf{a}(\lambda) = \{a_1(\lambda), a_2(\lambda), a_3(\lambda)\}$ is not a branch point of Y_τ for all $\lambda \in U$. Hence

$$\wp(a_i(\lambda)) \neq \wp(a_j(\lambda)) \quad \text{for all } \lambda \in U,\ 1 \leq i < j \leq 3, \tag{4.3}$$

and, for $\lambda \in U$, we have

$$2\sum_{j\neq i}(\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)) = \zeta\Big(a_i + \frac{1}{2}\Big) + \zeta\Big(a_i - \frac{1}{2}\Big) - 2\zeta(a_i), \quad i = 1, 2, 3,$$

which is equivalent to (cf. [10, Theorem A.2])

$$\begin{cases} \wp'(a_1) + \wp'(a_2) + \wp'(a_3) = 0, \\ \displaystyle\sum_{i=1}^{3} \wp'(a_i) \prod_{j\neq i}(\wp(a_j) - e_1) = 0. \end{cases} \tag{4.4}$$

Taking derivative with respect to λ in (4.4) and evaluating at λ_0, we obtain from $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ and $\wp'' = 6\wp^2 - \frac{g_2}{2}$ that

$$\sum_{i=1}^{3}\Big(6\wp_i^2 - \frac{g_2}{2}\Big)a_i'(\lambda_0) = 0, \tag{4.5}$$

$$\sum_{i=1}^{3}\varphi_i a_i'(\lambda_0) = 0, \tag{4.6}$$

where $\wp_i := \wp\,(a_i(\lambda_0))$ for $i = 1, 2, 3$, and

$$\begin{aligned} \varphi_i &= \wp''(a_i)(\wp_j - e_1)(\wp_k - e_1) + \wp'(a_i)\wp'(a_j)(\wp_k - e_1) \\ &\quad + \wp'(a_i)\wp'(a_k)(\wp_j - e_1) \\ &= \Big(6\wp_i^2 - \frac{g_2}{2}\Big)(\wp_j - e_1)(\wp_k - e_1) + \Big(2\wp_j^2 + 2\wp_j\wp_k + 2\wp_k^2 - \frac{1}{2}g_2\Big)(\wp_j - \wp_k)^2 \\ &\quad - \frac{1}{2}(4\wp_i^3 - g_2\wp_i - g_3)(\wp_j + \wp_k - 2e_1) \end{aligned}$$

with $\{i, j, k\} = \{1, 2, 3\}$.

By $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ and (4.4), we easily obtain

$$\frac{4\wp_1^3 - g_2\wp_1 - g_3}{(\wp_1 - e_1)^2(\wp_2 - \wp_3)^2} = \frac{4\wp_2^3 - g_2\wp_2 - g_3}{(\wp_2 - e_1)^2(\wp_1 - \wp_3)^2} = \frac{4\wp_3^3 - g_2\wp_3 - g_3}{(\wp_3 - e_1)^2(\wp_1 - \wp_2)^2} =: \mho,$$

which is equivalent to

$$\begin{cases} 4\wp_1^3 - g_2\wp_1 - g_3 = \mho(\wp_1 - e_1)^2(\wp_2 - \wp_3)^2, \\ 4\wp_2^3 - g_2\wp_2 - g_3 = \mho(\wp_2 - e_1)^2(\wp_1 - \wp_3)^2, \\ 4\wp_3^3 - g_2\wp_3 - g_3 = \mho(\wp_3 - e_1)^2(\wp_1 - \wp_2)^2. \end{cases} \tag{4.7}$$

Note that $\mho \neq 0$, otherwise, $a_1, a_2, a_3 \in \{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Denote by

$$\begin{cases} s_1 & := s_1(\lambda_0) = \wp_1 + \wp_2 + \wp_3, \\ s_2 & := s_2(\lambda_0) = \wp_1\wp_2 + \wp_1\wp_3 + \wp_2\wp_3, \\ s_3 & := s_3(\lambda_0) = \wp_1\wp_2\wp_3, \end{cases}$$

we have

$$(x - \wp_1)(x - \wp_2)(x - \wp_3) = x^3 - s_1x^2 + s_2x - s_3, \tag{4.8}$$

then (4.7) is equivalent to

$$\begin{cases} 4s_1\wp_1^2 - (4s_2 + g_2)\wp_1 + 4s_3 - g_3 = \mho(\wp_1 - e_1)^2(\wp_2 - \wp_3)^2, & (4.9) \\ 4s_1\wp_2^2 - (4s_2 + g_2)\wp_2 + 4s_3 - g_3 = \mho(\wp_2 - e_1)^2(\wp_1 - \wp_3)^2, & (4.10) \\ 4s_1\wp_3^2 - (4s_2 + g_2)\wp_3 + 4s_3 - g_3 = \mho(\wp_3 - e_1)^2(\wp_1 - \wp_2)^2. & (4.11) \end{cases}$$

First, (4.9), (4.10), and (4.11) lead to

$$\begin{aligned} &4s_1^3 - 12s_1s_2 + 12s_3 - g_2s_1 - 3g_3 \\ &\quad = 2\mho(s_2^2 - 3s_1s_3 - e_1s_1s_2 + 9e_1s_3 + e_1^2s_1^2 - 3e_1^2s_2). \end{aligned} \tag{4.12}$$

Note that $\wp_i \neq \wp_j$ for $i \neq j$, then (4.9)–(4.10), (4.9)–(4.11), and (4.10)–(4.11) yield

$$4s_2 + g_2 - 4s_1(\wp_1 + \wp_2) = \mho(\wp_3 - e_1)(3\wp_1\wp_2 + 3e_1\wp_3 - e_1s_1 - s_2), \tag{4.13}$$

$$4s_2 + g_2 - 4s_1(\wp_1 + \wp_3) = \mho(\wp_2 - e_1)(3\wp_1\wp_3 + 3e_1\wp_2 - e_1s_1 - s_2), \tag{4.14}$$

$$4s_2 + g_2 - 4s_1(\wp_2 + \wp_3) = \mho(\wp_1 - e_1)(3\wp_2\wp_3 + 3e_1\wp_1 - e_1s_1 - s_2). \tag{4.15}$$

Next, (4.13)–(4.15) gives us

$$8s_1^2 - 12s_2 - 3g_2 = \mho(6e_1s_2 - 2e_1s_1^2 + s_1s_2 - 9s_3), \tag{4.16}$$

and (4.13)–(4.14) gives us

$$4s_1 = \mho(2e_1s_1 - s_2 - 3e_1^2). \tag{4.17}$$

Combine (4.12), (4.16), and (4.17). We obtain that

$$\frac{4s_1}{2e_1s_1 - s_2 - 3e_1^2} = \frac{8s_1^2 - 12s_2 - 3g_2}{6e_1s_2 - 2e_1s_1^2 + s_1s_2 - 9s_3}, \tag{4.18}$$

$$\frac{4s_1}{2e_1s_1 - s_2 - 3e_1^2} = \frac{4(s_1^3 - 3s_1s_2 + 3s_3) - g_2s_1 - 3g_3}{2(s_2^2 - 3s_1s_3 - e_1s_1s_2 + 9e_1s_3 + e_1^2s_1^2 - 3e_1^2s_2)}. \tag{4.19}$$

If $s_1 = 0$, then $2e_1 s_1 - s_2 - 3e_1^2 = 0$, i.e., $s_2 = -3e_1^2$, because $\mho \neq 0$. Since $8s_1^2 - 12s_2 - 3g_2 = 3(12e_1^2 - g_2) \neq 0$, by (4.16), $6e_1 s_2 - 2e_1 s_1^2 + s_1 s_2 - 9s_3 \neq 0$, i.e., $s_3 \neq -2e_1^3$. From (4.12) and (4.16), we have $s_3 = \frac{1}{4}(g_3 + 2e_1 g_2 - 24e_1^3)$.

If $s_1 \neq 0$, then $2e_1 s_1 - s_2 - 3e_1^2 \neq 0$ by (4.17). From (4.18), we have

$$
\begin{aligned}
s_3 = \frac{1}{12s_1}\big(& -3e_1^2 g_2 + 2e_1 g_2 s_1 + 8e_1^2 s_1^2 - 8e_1 s_1^3 - 12e_1^2 s_2 \\
& - g_2 s_2 + 16e_1 s_1 s_2 + 4s_1^2 s_2 - 4s_2^2\big).
\end{aligned} \tag{4.20}
$$

Plug (4.20) into (4.19). We have

$$
\begin{aligned}
& \frac{1}{s_1}(3e_1^2 - 2e_1 s_1 + s_2) \\
& \quad \times (3e_1^2 g_2 + 4e_1 g_2 s_1 + 3g_3 s_1 - 8e_1^2 s_1^2 - g_2 s_1^2 - 8e_1 s_1^3 + 4s_1^4 + 12e_1^2 s_2 \\
& \qquad + g_2 s_2 + 8e_1 s_1 s_2 - 8s_1^2 s_2 + 4s_2^2) = 0.
\end{aligned} \tag{4.21}
$$

Note that $s_1 \neq 0$, $2e_1 s_1 - s_2 - 3e_1^2 \neq 0$ and $g_3 = 4e_1^3 - e_1 g_2$, we have

$$
\begin{aligned}
& 4s_2^2 + (12e_1^2 + g_2 + 8e_1 s_1 - 8s_1^2) s_2 \\
& + 3e_1^2 g_2 + 4e_1 g_2 s_1 + 3g_3 s_1 - 8e_1^2 s_1^2 - g_2 s_1^2 - 8e_1 s_1^3 + 4s_1^4 \\
& = 4(s_2 - s_1^2 + e_1 s_1 + 3e_1^2)(s_2 - s_1^2 + e_1 s_1 + \frac{1}{4} g_2) = 0.
\end{aligned} \tag{4.22}
$$

From (4.20) and (4.22), we obtain that

$$
\begin{cases}
s_2 = s_1^2 - e_1 s_1 - 3e_1^2, \\
s_3 = e_1 s_1^2 - \Big(e_1^2 + \frac{1}{12} g_2\Big) s_1 - 5e_1^3 + \frac{1}{4} e_1 g_2,
\end{cases} \tag{4.23}
$$

or

$$
\begin{cases}
s_2 = s_1^2 - e_1 s_1 - \frac{1}{4} g_2 \\
s_3 = e_1 s_1^2 - 2e_1^2 s_1 + e_1^3 - \frac{1}{4} e_1 g_2,
\end{cases} \tag{4.24}
$$

where the $s_1 = 0$ case is only included in (4.23).

Let $V = \{\lambda \in \mathbb{C} \mid Q(\lambda) \neq 0\}$ which is a connected open subset of $\mathbb{C}$. Let $A = \{\lambda \in V \mid \lambda$ satisfies (4.23)$\}$ and $B = \{\lambda \in U \mid \lambda$ satisfies (4.24)$\}$. By the above analysis, we have $V = A \cup B$. Note that $-5e_1 \in V$ and $s_1(-5e_1) = (-5e_1 + 5e_1)/3 = 0$ by (4.1), thus $-5e_1 \in A$. By definition, both A and B are closed subsets of V. Since V is connected and $A \neq \emptyset$, we have $A = V$ and $B = \emptyset$. Therefore, (4.23) always holds for all $\lambda \in \mathbb{C}$ combining with the continuity.

On the other hand, for any $\lambda \in U$, denote by $A(\lambda) := \sum_{j=1}^{3}(\zeta(a_j) - \eta_1 a_j)$. Since $\mathbf{a}(\lambda) \cap -\mathbf{a}(\lambda) = \emptyset$, by (2.16), we have that for $\lambda \in U$,

$$\begin{aligned} \Delta(\lambda) &= e^A + e^{-A}, \\ \Delta'(\lambda) &= (e^A - e^{-A})A', \\ \Delta''(\lambda) &= (e^A - e^{-A})A'' + \Delta(A')^2, \\ \Delta'''(\lambda) &= (e^A - e^{-A})(A''' + (A')^3) + 3\Delta A' A''. \end{aligned} \tag{4.25}$$

Sufficiency. Let $\lambda_0 \in \sigma(L_\tau)$ with $Q_\tau(\lambda_0) \neq 0$ be an inner intersection point, then λ_0 is met by $2k \geq 4 (k \in \mathbb{Z})$ semi-arcs of the spectrum.

Consider the local behavior of the spectrum at $\lambda_0 \in \sigma(L_\tau)$:

$$\Delta(\lambda) - \Delta(\lambda_0) = c(\lambda - \lambda_0)^k (1 + O(|\lambda - \lambda_0|)), \quad k \geq 1, \; c \neq 0. \tag{4.26}$$

If $\Delta(\lambda_0) \in (-2, 2)$, it follows from (4.26) and $\sigma(L_\tau) = \{\lambda | -2 \leq \Delta(\lambda) \leq 2\}$ that there are precisely $2k$ semi-arcs of $\sigma(L_\tau)$ meeting at λ_0. If $\Delta(\lambda_0) = \pm 2$, then there are precisely k semi-arcs of $\sigma(L_\tau)$ meeting at λ_0.

If $\Delta(\lambda_0) = \pm 2$, then our assumption implies $k \geq 4$, i.e., $\Delta'(\lambda_0) = \Delta''(\lambda_0) = \Delta'''(\lambda_0) = 0$. Since $\Delta(\lambda_0) = \pm 2$ implies $e^A = \pm 1$ at λ_0, we obtain $A'(\lambda_0) = 0$.

If $\Delta(\lambda_0) \in (-2, 2)$, then our assumption implies $2k \geq 4$, i.e. $k \geq 2$ and so $\Delta'(\lambda_0) = 0$. Since $\Delta(\lambda_0) \neq \pm 2$ implies $e^A \neq \pm 1$ at λ_0, again we obtain $A'(\lambda_0) = 0$.

Therefore, we always have $A'(\lambda_0) = 0$, i.e.,

$$(\wp_1 + \eta_1)\, a_1'(\lambda_0) + (\wp_2 + \eta_1)\, a_2'(\lambda_0) + (\wp_3 + \eta_1)\, a_3'(\lambda_0) = 0. \tag{4.27}$$

Noting from (4.2), we conclude from (4.5, 4.6, 4.27) that the determinant of the matrix

$$\Omega := \begin{pmatrix} \wp_1 + \eta_1 & \wp_2 + \eta_1 & \wp_3 + \eta_1 \\ 6\wp_1^2 - \frac{g_2}{2} & 6\wp_2^2 - \frac{g_2}{2} & 6\wp_3^2 - \frac{g_2}{2} \\ \varphi_1 & \varphi_2 & \varphi_3 \end{pmatrix}$$

vanishes, i.e.,

$$\begin{aligned} \det \Omega = \frac{1}{4}&(\wp_1 - \wp_2)(\wp_2 - \wp_3)(\wp_3 - \wp_1) \\ &\times \big(g_2^2 - 24e_1 g_3 + 36e_1 g_2 \eta_1 + 12 g_3 \eta_1 + 4(5e_1 g_2 + 3g_3)s_1 - 20 g_2 s_1^2 \\ &\quad - 48\eta_1 s_1^3 + 48 s_1^4 + 4(11 g_2 + 60 e_1 \eta_1) s_2 + 48 \eta_1 s_1 s_2 - 240 s_1^2 s_2 \\ &\quad + 192 s_2^2 + 48(5e_1 - 4\eta_1)s_3 + 144 s_1 s_3\big) = 0 \end{aligned}$$

By (4.3), we obtain that

$$\begin{aligned} &g_2^2 - 24e_1 g_3 + 36e_1 g_2 \eta_1 + 12 g_3 \eta_1 + 4(5e_1 g_2 + 3g_3)s_1 - 20 g_2 s_1^2 \\ &\quad - 48\eta_1 s_1^3 + 48 s_1^4 + 4(11 g_2 + 60 e_1 \eta_1) s_2 + 48 \eta_1 s_1 s_2 - 240 s_1^2 s_2 \\ &\quad + 192 s_2^2 + 48(5e_1 - 4\eta_1)s_3 + 144 s_1 s_3 = 0. \end{aligned} \tag{4.28}$$

Plug (4.23) into (4.28). We get

$$4(g_2 - 12e_1^2)\Big(3s_1^2 + (4\eta_1 - 5e_1)s_1 - 9e_1^2 - 6e_1\eta_1 + \frac{1}{4}g_2\Big) = 0. \tag{4.29}$$

Since

$$\lambda_0 = 3(\wp_1 + \wp_2 + \wp_3) - 5e_1 = 3s_1 - 5e_1, \tag{4.30}$$

we have that

$$\lambda_0^2 + (5e_1 + 4\eta_1)\lambda_0 - 27e_1^2 + 2e_1\eta_1 + \frac{3}{4}g_2 = 0.$$

Necessity. Suppose $\lambda_0 \in \sigma(L_\tau)$ satisfies $Q_\tau(\lambda_0) \neq 0$ and

$$\lambda_0^2 + (5e_1 + 4\eta_1)\lambda_0 - 27e_1^2 + 2e_1\eta_1 + \frac{3}{4}g_2 = 0.$$

This, together with (4.30), (4.29), and (4.28) implies $\det\Omega = 0$. Since $\wp_i \neq \wp_j$ for $i \neq j$, the second row of Ω is nonzero. Suppose that the last two rows of Ω are linearly dependent. There is $c \in \mathbb{C}$ such that

$$\varphi_i = c\Big(6\wp_i^2 - \frac{g_2}{2}\Big), \quad i = 1, 2, 3.$$

Let

$$\begin{aligned} r_i &= \varphi_i - c\Big(6\wp_i^2 - \frac{g_2}{2}\Big) \\ &= \Big(6\wp_i^2 - \frac{g_2}{2}\Big)((\wp_j - e_1)(\wp_k - e_1) - c) \\ &\quad + \Big(2\wp_j^2 + 2\wp_j\wp_k + 2\wp_k^2 - \frac{1}{2}g_2\Big)(\wp_j - \wp_k)^2 \\ &\quad - \frac{1}{2}(4\wp_i^3 - g_2\wp_i - g_3)(\wp_j + \wp_k - 2e_1) \end{aligned}$$

with $\{i, j, k\} = \{1, 2, 3\}$, then $r_1 = r_2 = r_3 = 0$. Note that $\wp_1 \neq \wp_2 \neq \wp_3 \neq \wp_1$, simplify $r_1 - r_2$, $r_2 - r_3$ and $r_3 - r_1$ gives us $p_{12} = p_{23} = p_{31} = 0$, where

$$r_i - r_j = -\frac{1}{2}(x_i - x_j)p_{ij}.$$

Furthermore,

$$\begin{aligned} p_{12} - p_{23} &= (\wp_1 - \wp_3)(12c - 12e_1^2 - g_2 + 8s_1^2 - 8s_2 - 8e_1s_1) = 0, \\ p_{23} - p_{13} &= (\wp_2 - \wp_1)(12c - 12e_1^2 - g_2 + 8s_1^2 - 8s_2 - 8e_1s_1) = 0, \\ p_{13} - p_{12} &= (\wp_3 - \wp_2)(12c - 12e_1^2 - g_2 + 8s_1^2 - 8s_2 - 8e_1s_1) = 0, \end{aligned}$$

which gives us

$$12c - 12e_1^2 - g_2 + 8s_1^2 - 8s_2 - 8e_1s_1 = 0.$$

Since $s_2 = s_1^2 - e_1 s_1 - 3e_1^2$, we have $c = \frac{1}{12}g_2 - e_1^2$. From here, (4.23) and $p_{12} + p_{23} + p_{31} = 0$, we have

$$s_1 = \frac{3e_1}{2}.$$

By (4.23), and $r_1 + r_2 + r_3 = 0$, we have

$$(12e_1^2 - g_2)(39e_1^2 - g_2) = 0,$$

which is a contradiction!

Hence, the last two rows of Ω are linearly independent and then the first row can be linearly spanned by the last two rows. So, (4.5) and (4.6) yields (4.27).

If $\Delta(\lambda_0) \in (-2, 2)$, then we see from (4.27) and (4.25) that $\Delta'(\lambda_0) = 0$, i.e., $k \geq 2$ in (4.26) and so there are $2k \geq 4$ semi-arcs of $\sigma(L_\tau)$ meeting at this λ_0. If $\Delta(\lambda_0) = \pm 2$, then $e^A = \pm 1$ at λ_0. From here and (4.27) and (4.25), we see that $\Delta'(\lambda_0) = \Delta''(\lambda_0) = \Delta'''(\lambda_0) = 0$. This means $k \geq 4$ in (4.26) and so there are $k \geq 4$ semi-arcs of $\sigma(L_\tau)$ meeting at this λ_0. Therefore, λ_0 is an inner intersection point.

Finally, by direct computation, we have

$$\begin{aligned}\Delta_f &= 133e_1^2 - 3g_2 + 32e_1\eta_1 + 16\eta_1^2 \\ &= 115e_1^2 + 18\Big(e_1^2 - e_1\eta_1 - \frac{1}{6}g_2\Big) + 50e_1\eta_1 + 16\eta_1^2 \\ &= \Big(\frac{25}{4}e_1 + 4\eta_1\Big)^2 + \frac{1215}{16}e_1^2 + 18\pi e_1'(b) > 0, \quad \text{for all } b > 0,\end{aligned}$$

and

$$\begin{aligned}f(0) &= -27e_1^2 + 2e_1\eta_1 + \frac{3}{4}g_2 \\ &= -2\Big(e_1^2 - e_1\eta_1 - \frac{1}{6}g_2\Big) - 5\Big(5e_1^2 - \frac{1}{12}g_2\Big) < 0, \quad \text{for all } b > 0.\end{aligned}$$

Hence, f has two real roots λ_-, λ_+ for all $b > 0$ and $\lambda_- < 0 < \lambda_+$. ■

5. Mean field equation

The purpose of this section is to study the relation between the spectrum $\sigma(L_\tau)$ and the number of even axisymmetric solutions of the mean field equation

$$\Delta u + e^u = 16\pi\delta_0 + 8\pi\delta_{\frac{1}{2}} \quad \text{on } T_\tau. \tag{5.1}$$

First of all, we recall the connection between (5.1) and the generalized Lamé equation $\mathcal{L}_\lambda$ which was studied in [4].

Theorem 5.1 ([10, Theorem 3.1]). *The mean field equation (5.1) has an even solution if and only if there exists $\lambda \in \mathbb{C}$ such that the monodromy of $\mathcal{L}_\lambda$ is unitary.*

Furthermore, the number of even solutions equals the number of those λ's such that the monodromy of $\mathcal{L}_\lambda$ is unitary.

Now, we always assume $\tau = \frac{1}{2} + ib$ with $b > 0$. For any $\lambda \in \mathbb{C}$, there exists $\mathbf{a} \in Y_b$ such that $\lambda = \lambda_{\mathbf{a}}$ by the covering map (2.5). Recall the monodromy theory of the generalized Lamé equation

$$\mathcal{L}_{\lambda_{\mathbf{a}}}\colon \quad y''(z) = \Big(6\wp(z;\tau) + 2\wp\Big(z + \frac{1}{2};\tau\Big) + \lambda_{\mathbf{a}}\Big)y(z) \tag{5.2}$$

stated in Section 2.3 and

$$y_{\pm\mathbf{a}}(z + 2\tau - 1) = e^{\pm 2\pi i(2r+s)} y_{\pm\mathbf{a}}(z),$$
$$\Delta(\lambda_{\mathbf{a}}) = e^{2\pi i s} + e^{-2\pi i s}.$$

Define

$$\widehat{\Delta}(\lambda_{\mathbf{a}}) := e^{2\pi i(2r+s)} + e^{-2\pi i(2r+s)},$$

and

$$\hat{\sigma}(L_b) := \{\lambda \in \mathbb{C} \mid -2 \le \widehat{\Delta}(\lambda) \le 2\}. \tag{5.3}$$

The geometry of $\hat{\sigma}(L_b)$ can be described as the following lemma.

Lemma 5.2. *Let $\tau = \frac{1}{2} + bi$ with $b > 0$. We have*

$$\hat{\sigma}(L_b) = \frac{1}{-4b^2}\sigma(L_{\frac{1}{4b}})$$

and the endpoints of $\hat{\sigma}(L_b)$ are exactly the endpoints of $\sigma(L_b)$.

Proof. Note that

$$\frac{\tau - 1}{2\tau - 1} = \frac{1}{2} + i\frac{1}{4b} \quad \text{for } \tau = \frac{1}{2} + ib.$$

Since $y_{\pm\mathbf{a}}(z)$ are solutions of (5.2), then $\hat{y}_{\pm\mathbf{a}}(z) := y_{\pm\mathbf{a}}((2\tau - 1)z)$ satisfies

$$\begin{aligned}
\hat{y}''_{\pm\mathbf{a}}(z) &= (2\tau - 1)^2\Big(6\wp((2\tau - 1)z;\tau) + 2\wp\Big((2\tau - 1)z + \frac{1}{2};\tau\Big) + \lambda_{\mathbf{a}}\Big)\hat{y}_{\pm\mathbf{a}}(z) \\
&= \Big(6\wp\Big(z;\frac{\tau - 1}{2\tau - 1}\Big) + 2\wp\Big(z + \frac{1}{2};\frac{\tau - 1}{2\tau - 1}\Big) + (2\tau - 1)^2\lambda_{\mathbf{a}}\Big)\hat{y}_{\pm\mathbf{a}}(z) \\
&= \Big(6\wp\Big(z;\frac{1}{2} + i\frac{1}{4b}\Big) + 2\wp\Big(z + \frac{1}{2};\frac{1}{2} + i\frac{1}{4b}\Big) - 4b^2\lambda_{\mathbf{a}}\Big)\hat{y}_{\pm\mathbf{a}}(z),
\end{aligned}$$

and

$$\hat{y}_{\pm\mathbf{a}}(z + 1) = y_{\pm\mathbf{a}}\left((2\tau - 1)z + (2\tau - 1)\right) = e^{\pm 2\pi i(2r+s)}\hat{y}_{\pm\mathbf{a}}(z).$$

Therefore, the analysis in Section 2.3 tells us that

$$\Delta(-4b^2\lambda_{\mathbf{a}};\frac{1}{4b}) = e^{2\pi i(2r+s)} + e^{-2\pi i(2r+s)} = \widehat{\Delta}(\lambda_{\mathbf{a}};b). \tag{5.4}$$

Consequently, we conclude from (5.3) that

$$\begin{aligned}\hat{\sigma}(L_b) &= \left\{\lambda \in \mathbb{C} \,\middle|\, -2 \le \Delta\Big(-4b^2\lambda;\frac{1}{4b}\Big) \le 2\right\} \\ &= \left\{\lambda \in \mathbb{C} \,\middle|\, -4b^2\lambda \in \sigma(L_{\frac{1}{4b}})\right\} = \frac{1}{-4b^2}\sigma(L_{\frac{1}{4b}}).\end{aligned} \tag{5.5}$$

Note that $\left(\begin{smallmatrix}1 & -1\\ 2 & -1\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, by the modular properties of $e_1(b)$, $e_2(b)$, we have (cf. [14])

$$\frac{1}{-4b^2}e_1\Big(\frac{1}{4b}\Big) = e_1(b), \quad \frac{1}{-4b^2}e_2\Big(\frac{1}{4b}\Big) = e_3(b), \quad \frac{1}{-4b^2}e_3\Big(\frac{1}{4b}\Big) = e_2(b).$$

Consider (5.5) and the expression (2.4) of $Q_b(\lambda)$, the finite endpoints of arcs of $\hat{\sigma}(L_b)$ is also

$$Z(Q_b) := \{\lambda \in \mathbb{C} \mid Q_b(\lambda) = 0\},$$

which is the set of finite endpoints of $\sigma(L_b)$. ■

By the proof of Lemma 5.2, we

$$Z(Q_b) \subset \sigma(L_b) \cap \hat{\sigma}(L_b),$$

which is the set of finite endpoints of both $\sigma(L_b)$ and $\hat{\sigma}(L_b)$. Denote by

$$\Xi_b := (\sigma(L_b) \cap \hat{\sigma}(L_b)) \setminus Z(Q_b).$$

The following theorem establishes the precise connection between even solutions of the mean field equation and the spectrum.

Theorem 5.3 (= Theorem 1.5). *Let $\tau = \frac{1}{2} + bi$ with $b > 0$. The number of even solutions of the mean field equation (5.1) equals $\#\Xi_b$. Furthermore, the number of even axisymmetric solutions equals $\#(\Xi_b \cap \mathbb{R})$.*

Proof. It was proved in [9, Theorem 2.7] that the monodromy of $\mathcal{L}_{\lambda_{\mathbf{a}}}$ is completely reducible if and only if $Q_b(\lambda_{\mathbf{a}}) \neq 0$. Hence, the monodromy of $\mathcal{L}_{\lambda_{\mathbf{a}}}$ is unitary if and only if $Q_b(\lambda_{\mathbf{a}}) \neq 0$ and the corresponding (r,s) of this $\lambda_{\mathbf{a}}$ satisfies $(r,s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, and so if and only if $\lambda_{\mathbf{a}} \in \Xi_b$ (note $(r,s) \notin \frac{1}{2}\mathbb{Z}^2$ follows from $Q_b(\lambda_{\mathbf{a}}) \neq 0$). Together with Theorem 5.1, we conclude that the number of even solutions of (5.1) equals $\#\Xi_b$.

In order to compute the number of even axisymmetric solutions, we need to apply the precise connection between an even solution $u(z) = u(x, y)$ (here we use complex variable $z = x + iy$) and the corresponding $\lambda_{\mathbf{a}} \in \Xi_b$ proved in [4]:

$$\left(u_{zz} - \frac{1}{2}u_z^2\right)(z) = -2\left(6\wp(z;\tau) + 2\wp\left(z + \frac{1}{2};\tau\right) + \lambda_{\mathbf{a}}\right),$$

and in Theorem 5.1 the developing map $f(z) = y_{\mathbf{a}}(z)/y_{-\mathbf{a}}(z)$, where $y_{\pm\mathbf{a}}(z)$ are solutions of (5.2) stated in Section 2.3.

Clearly, $\tilde{u}(z) = \tilde{u}(x, y) := u(x, -y) = u(\bar{z})$ is also an even solution of (5.1) and satisfies (note that $u(z)$ is real-valued as a solution of (5.1))

$$\begin{aligned}\left(\tilde{u}_{zz} - \frac{1}{2}\tilde{u}_z^2\right)(z) &= \overline{\left(u_{zz} - \frac{1}{2}u_z^2\right)(\bar{z})} \\ &= \overline{-2\left(6\wp(\bar{z};\tau) + 2\wp\left(\bar{z} + \frac{1}{2};\tau\right) + \lambda_{\mathbf{a}}\right)} \\ &= -2\left[6\wp(z;\tau) + 2\wp\left(z + \frac{1}{2};\tau\right) + \overline{\lambda_{\mathbf{a}}}\right],\end{aligned}$$

i.e., $\overline{\lambda_{\mathbf{a}}} \in \Xi_b$ if $\lambda_{\mathbf{a}} \in \Xi_b$. From here and the fact stated in Theorem 5.1 that there is a one-to-one correspondence between $\lambda \in \Xi_b$ and even solutions of (5.1), we conclude that $\lambda_{\mathbf{a}} = \overline{\lambda_{\mathbf{a}}}$ if and only if $u(z) = \tilde{u}(z)$, i.e., $u(z) = u(\bar{z})$ is axisymmetric. Therefore, the number of even axisymmetric solutions equals $\#(\Xi_b \cap \mathbb{R})$. ∎

From this theorem and Lemma 1.4, we have the following corollary.

Corollary 5.4. *Let $\tau = \frac{1}{2} + bi$ with $b > 0$. Then*

$$\#(\Xi_b \cap \mathbb{R}) = \begin{cases} 2 & \text{if } b \in (0, k_1) \cup \left(k_2, \frac{1}{4k_2}\right) \cup \left(\frac{1}{4k_1}, +\infty\right), \\ 1 & \text{if } b \in [k_1, k_2] \cup \left[\frac{1}{4k_2}, \frac{1}{4k_1}\right]. \end{cases} \tag{5.6}$$

6. Proof of the main theorem

In this section, we prove the main Theorem 1.2.

Proof of Theorem 1.2. Let $\tau = \frac{1}{2} + bi$ with $b > 0$. First of all, by Theorem 1.1, the zeros of the spectral polynomial $Q_b(\lambda)$ are

$$4e_1, \quad \mu, \quad \bar{\mu}, \quad \nu, \quad \bar{\nu}$$

with $4e_1 \in \mathbb{R}$ and $\mu, \nu \notin \mathbb{R}$. Moreover,

$$d(\mu) = d(\bar{\mu}) = d(\nu) = d(\bar{\nu}) = 1$$

and $d(4e_1) \geq 3$ if and only if $b \in \{b_1, b_2\}$, otherwise, $d(4e_1) = 1$. From Theorem 2.1 and Lemma 2.3, the spectrum can be expressed as

$$\sigma(L_b) = (-\infty, 4e_1] \cup \sigma_1 \cup \sigma_2, \tag{6.1}$$

where σ_1, σ_2 denote simple arcs and $\sigma_1 \cup \sigma_2$ is symmetric with respect to $\mathbb{R}$. Since the complement $\mathbb{C} \setminus \sigma(L_b)$ is path connected (cf. [18, Theorem 2.2]), there is at most one intersection point for any two spectral arcs among $(-\infty, 4e_1]$, σ_1 and σ_2. Furthermore, all intersection points are real by Theorem 1.3. In particular, $\lambda \in \sigma(L_b)$ is an intersection point if and only if $\lambda \in \{4e_1, \lambda_-, \lambda_+\} \cap \sigma(L_b)$ and $4e_1$ is an intersection point if and only if $b \in \{b_1, b_2\}$. Here, $\lambda_- < \lambda_+$ are roots of

$$f(\lambda) = \lambda^2 + (5e_1 + 4\eta_1)\lambda - 27e_1^2 + 2e_1\eta_1 + \frac{3}{4}g_2.$$

Note that

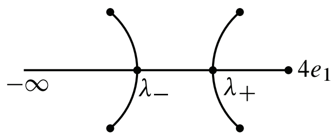
$$f(4e_1) = 9h(b) \begin{cases} > 0 & \text{for } b \in (0, b_1) \cup (b_2, +\infty), \\ = 0 & \text{for } b \in \{b_1, b_2\}, \\ < 0 & \text{for } b \in (b_1, b_2), \end{cases}$$

we split the proof into the following steps.

Step 1: *the spectrum for* $b > b_2$. Since $f(4e_1) > 0$ and

$$-\frac{5e_1 + 4\eta_1}{2} < 0 < 4e_1,$$

we obtain that $\lambda_- < \lambda_+ < 4e_1$. Therefore, both λ_- and λ_+ are inner intersection points and then the rough graph of spectrum $\sigma(L_b)$ for $b > b_2$ must be the one given in Figure 3.



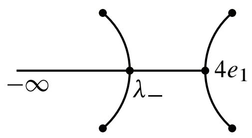
(S1)

Figure 3

Step 2: *the spectrum at* $b = b_2 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Note that $4e_1 > 0$ is a root of f, then the other root of f is

$$\lambda_- = -(5e_1 + 4\eta_1) - 4e_1 = -9e_1 - 4\eta_1 < 0 < 4e_1, \tag{6.2}$$

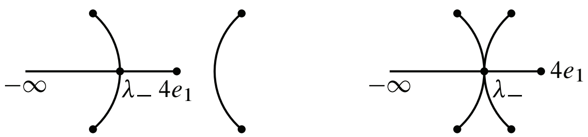
so λ_- is an inner intersection point of $\sigma(L_{b_2})$. Since $d(4e_1) \geq 3$, the rough graph of $\sigma(L_{b_2})$ is the one given in Figure 4.



(S2)

Figure 4

Step 3: *the spectrum for* $b \in (b_1, b_2)$. Since $f(4e_1) < 0$, we have $\lambda_- < 4e_1 < \lambda_+$, then λ_- is the one and only one inner intersection point of $\sigma(L_b)$ for any $b \in (b_1, b_2)$. Hence, there are two choices for the rough graph of $\sigma(L_b)$, see Figure 5. In order to transform from one to the other in the above two graphs, we have to pass $4e_1$, but $d(4e_1) = 1$ for all $b \in (b_1, b_2)$, which means the spectral arc σ_i cannot pass through $4e_1$. So, the rough graph of $\sigma(L_b)$ is either (S3a) for all $b \in (b_1, b_2)$ or (S3b) for all $b \in (b_1, b_2)$. Since the rough graph (S2) of $\sigma(L_{b_2})$ cannot be continuously deformed to (S3b), the rough graph of $\sigma(L_b)$ must be (S3a) for all $b \in (b_1, b_2)$.



(S3a) (S3b)

Figure 5

Before we move on to next steps. Let $b \in (0, \frac{1}{4b_2})$. By Lemma 5.2,

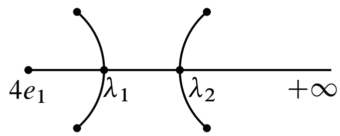
$$\tilde{\sigma}(L_b) = \frac{1}{-4b^2}\sigma(L_{\frac{1}{4b}}), \tag{6.3}$$

and $\tilde{\sigma}(L_b)$ has the same endpoints as $\sigma(L_b)$. Moreover, $\frac{1}{4b} \in (b_2, +\infty)$, by reflecting the rough graph (S1) with respect to y-axis and with some stretch, we obtain the rough graph of $\tilde{\sigma}(L_b)$ for $b \in (0, \frac{1}{4b_2})$ given in Figure 6.

Step 4: *the spectrum for* $b = b_1 \in (b_\eta, \frac{1}{2\sqrt{3}})$. Note that $4e_1 < 0$ is a root of f, then the other root of f is

$$\lambda_+ = -(5e_1 + 4\eta_1) - 4e_1 = -5e_1 - 4(e_1 + \eta_1) > 0 > 4e_1. \tag{6.4}$$

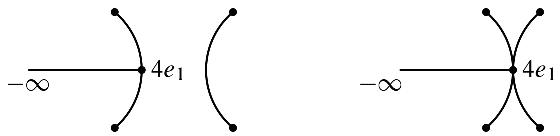
So, there is no inner intersection points on $\sigma(L_{b_1})$. Note that $d(4e_1) \geq 3$, there are two possible rough graphs of $\sigma(L_{b_1})$, see Figure 7. It is clear that there is no real



(T1)

Figure 6

intersection point for (T1) and (S4b), then $\#(\Xi_{b_1} \cap \mathbb{R}) = 0$ which contradicts with Corollary 5.4. So, the rough graph of $\sigma(L_{b_1})$ must be (S4a) and then $\#(\Xi_{b_1} \cap \mathbb{R}) = 1$.



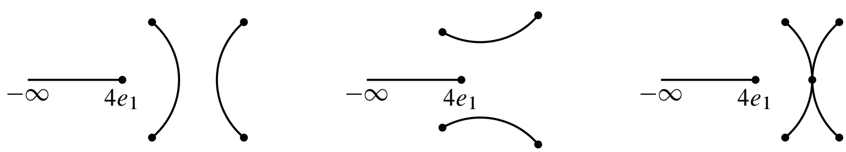
(S4a) (S4b)

Figure 7

Step 5: *the spectrum for* $b \in (0, b_1)$. Since $f(4e_1) > 0$ and

$$-\frac{5e_1 + 4\eta_1}{2} = -\frac{1}{2}e_1 - 2(e_1 + \eta_1) > 0 > 4e_1,$$

we have $\lambda_+ > \lambda_- > 4e_1$. Then $(-\infty, 4e_1] \cap \sigma_i = \emptyset$ with $i = 1, 2$ and $\sigma_1 \cap \sigma_2$ has at most one point, so there are three possible rough graphs of $\sigma(L_b)$, see Figure 8. Note that the rough graph of $\sigma(L_{b_1})$ is (S4a) and (S5a) is the only graph among these three possible graphs which can be continuously deformed from (S4a), then the rough graph of $\sigma(L_b)$ with $b_1 - b > 0$ very small must be (S5a).

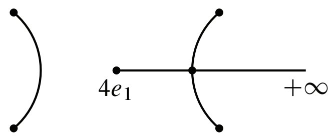


(S5a) (S5b) (S5c)

Figure 8

Next, by direct computation, we have $\#(\Xi_{b_1} \cap \mathbb{R}) = 1$ and $\#(\Xi_b \cap \mathbb{R}) = 2$ for $b_1 - b > 0$ very small. By Corollary 5.4, we obtain that $b_1 = k_1$ and then $\#(\Xi_b \cap \mathbb{R}) = 2$ for all $b \in (0, b_1)$. Hence, except for the real endpoint $4e_1$, there are exactly 2 real intersection points of (T1) with $\sigma(L_b)$ for all $b \in (0, b_1)$. So, the rough graph of $\sigma(L_b)$ is (S5a) for all $b \in (0, b_1)$. The proof is complete. ∎

Remark 6.1. In fact, we also have $b_2 = \frac{1}{4k_2}$. Indeed, let $b \in (\frac{1}{4b_2}, \frac{1}{4b_1})$, then $\frac{1}{4b} \in (b_1, b_2)$. Similar to $b \in (0, \frac{1}{4b_2})$ case, we obtain the rough graph of $\tilde{\sigma}(L_b)$ for $b \in (\frac{1}{4b_2}, \frac{1}{4b_1})$ by reflecting and stretching the graph (S3a) of $\sigma(L_{\frac{1}{4b}})$ as in Figure 9. Consider the rough graph of $\sigma(L_b)$ for $b \in (\frac{1}{4b_2}, \frac{1}{4b_1})$, we get $\#(\Xi_b \cap \mathbb{R}) = 2$ for $b \in (\frac{1}{4b_2}, b_2)$ and $\#(\Xi_{b_2} \cap \mathbb{R}) = 1$ by direct computations. Hence, $b_2 = \frac{1}{4k_2}$ by Corollary 5.4.



(T2)

Figure 9

References

[1] V. Batchenko and F. Gesztesy, On the spectrum of Schrödinger operators with quasi-periodic algebro-geometric KdV potentials. *J. Anal. Math.* **95** (2005), 333–387 Zbl 1088.34071 MR 2145569

[2] B. Birnir, Complex Hill's equation and the complex periodic Korteweg–de Vries equations. *Comm. Pure Appl. Math.* **39** (1986), no. 1, 1–49 Zbl 0592.47004 MR 820337

[3] J. L. Burchnall and T. W. Chaundy, Commutative Ordinary Differential Operators. *Proc. London Math. Soc. (2)* **21** (1923), 420–440 Zbl 49.0311.03 MR 1575373

[4] C.-L. Chai, C.-S. Lin, and C.-L. Wang, Mean field equations, hyperelliptic curves and modular forms: I. *Camb. J. Math.* **3** (2015), no. 1-2, 127–274 Zbl 1327.35116 MR 3356357

[5] Z. Chen and C. S. Lin, Distribution of critical points of the classical Eisenstein series E_4. Preprint

[6] C.-C. Chen, C.-S. Lin, and G. Wang, Concentration phenomena of two-vortex solutions in a Chern–Simons model. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **3** (2004), no. 2, 367–397 Zbl 1170.35413 MR 2075988

[7] Z. Chen, E. Fu, and C.-S. Lin, Spectrum of the Lamé operator and application, I: Deformation along $\operatorname{Re}\tau = \frac{1}{2}$. *Adv. Math.* **383** (2021), article no. 107699 Zbl 1468.34116 MR 4233282

[8] Z. Chen, E. Fu, and C.-S. Lin, A necessary and sufficient condition for the Darboux–Treibich–Verdier potential with its spectrum contained in $\mathbb{R}$. *Amer. J. Math.* **144** (2022), no. 3, 851–872 Zbl 1498.81074 MR 4436146

[9] Z. Chen, T.-J. Kuo, and C.-S. Lin, The geometry of generalized Lamé equation, I. *J. Math. Pures Appl. (9)* **127** (2019), 89–120 Zbl 1416.33027 MR 3960139

[10] Z. Chen, T.-J. Kuo, and C.-S. Lin, The geometry of generalized Lamé equation, II: Existence of pre-modular forms and application. *J. Math. Pures Appl. (9)* **132** (2019), 251–272 Zbl 1428.33038 MR 4030254

[11] Z. Chen and C.-S. Lin, On algebraic multiplicity of (anti)periodic eigenvalues of Hill's equations. *Proc. Amer. Math. Soc.* **146** (2018), no. 7, 3039–3047 Zbl 1387.34043 MR 3787364

[12] Z. Chen and C.-S. Lin, Critical points of the classical Eisenstein series of weight two. *J. Differential Geom.* **113** (2019), no. 2, 189–226 Zbl 1471.11138 MR 4023291

[13] Z. Chen and C.-S. Lin, Sharp nonexistence results for curvature equations with four singular sources on rectangular tori. *Amer. J. Math.* **142** (2020), no. 4, 1269–1300 Zbl 1448.35244 MR 4124120

[14] Z. Chen and C.-S. Lin, Spectrum of the Lamé operator and application, II: When an endpoint is a cusp. *Comm. Math. Phys.* **378** (2020), no. 1, 335–368 Zbl 1451.34108 MR 4124990

[15] A. Eremenko and A. Gabrielov, Spherical rectangles. *Arnold Math. J.* **2** (2016), no. 4, 463–486 Zbl 1365.30006 MR 3564884

[16] E. Fu, Spectrum of the Lamé operator along $\operatorname{Re}\tau = 1/2$: The genus 3 case. 2021, arXiv:2111.15059

[17] F. Gesztesy and H. Holden, *Soliton equations and their algebro-geometric solutions. Vol. I*. Camb. Stud. Adv. Math. 79, Cambridge University Press, Cambridge, 2003 Zbl 1061.37056 MR 1992536

[18] F. Gesztesy and R. Weikard, Floquet theory revisited. In *Differential equations and mathematical physics (Birmingham, AL, 1994)*, pp. 67–84, International Press, Boston, MA, 1995 Zbl 0946.47031 MR 1703573

[19] F. Gesztesy and R. Weikard, Treibich–Verdier potentials and the stationary (m)KdV hierarchy. *Math. Z.* **219** (1995), no. 3, 451–476 Zbl 0830.35119 MR 1339715

[20] F. Gesztesy and R. Weikard, Picard potentials and Hill's equation on a torus. *Acta Math.* **176** (1996), no. 1, 73–107 Zbl 0927.37040 MR 1395670

[21] W. A. Haese-Hill, M. A. Hallnäs, and A. P. Veselov, On the spectra of real and complex Lamé operators. *SIGMA Symmetry Integrability Geom. Methods Appl.* **13** (2017), article no. 049 Zbl 1385.34061 MR 3667223

[22] G. Lamé, Sur les surfaces isothermes dans les corps homogènes en équilibre de température. *J. Math. Pures Appl.* **2** (1837), 147–188

[23] C.-S. Lin and C.-L. Wang, Elliptic functions, Green functions and the mean field equations on tori. *Ann. of Math. (2)* **172** (2010), no. 2, 911–954 Zbl 1207.35011 MR 2680484

[24] C.-S. Lin and C.-L. Wang, Mean field equations, hyperelliptic curves and modular forms: II. *J. Éc. polytech. Math.* **4** (2017), 557–593 Zbl 1376.33022 MR 3665608

[25] C.-S. Lin and C.-L. Wang, Geometric quantities arising from bubbling analysis of mean field equations. *Comm. Anal. Geom.* **28** (2020), no. 6, 1289–1313 Zbl 1458.35203 MR 4184820

[26] C.-S. Lin and S. Yan, Existence of bubbling solutions for Chern-Simons model on a torus. *Arch. Ration. Mech. Anal.* **207** (2013), no. 2, 353–392 Zbl 1260.35155 MR 3005320

[27] F. S. Rofe-Beketov, On the spectrum of non-selfadjoint differential operators with periodic coefficients. *Dokl. Akad. Nauk SSSR* **152** (1963), 1312–1315 Zbl 0199.14002 MR 0157033

[28] K. Takemura, The Heun equation and the Calogero-Moser-Sutherland system. I. The Bethe Ansatz method. *Comm. Math. Phys.* **235** (2003), no. 3, 467–494 Zbl 1086.81051 MR 1974511

[29] K. Takemura, The Heun equation and the Calogero–Moser–Sutherland system. II. Perturbation and algebraic solution. *Electron. J. Differential Equations* (2004), article no. 15 Zbl 1087.81029 MR 2036199

[30] K. Takemura, The Heun equation and the Calogero-Moser-Sutherland system. III. The finite-gap property and the monodromy. *J. Nonlinear Math. Phys.* **11** (2004), no. 1, 21–46 Zbl 1087.81028 MR 2031210

[31] K. Takemura, The Heun equation and the Calogero-Moser-Sutherland system. IV. The Hermite-Krichever ansatz. *Comm. Math. Phys.* **258** (2005), no. 2, 367–403 Zbl 1088.81056 MR 2171700

[32] K. Takemura, The Heun equation and the Calogero–Moser–Sutherland system. V. Generalized Darboux transformations. *J. Nonlinear Math. Phys.* **13** (2006), no. 4, 584–611 Zbl 1122.34069 MR 2263090

[33] A. Treibich and J.-L. Verdier, Revêtements exceptionnels et sommes de 4 nombres triangulaires. *Duke Math. J.* **68** (1992), no. 2, 217–236 Zbl 0806.14013 MR 1191559

[34] B. L. van der Waerden, *Algebra.* Vol 1. Frederick Ungar, New York, 1970 Zbl 0219.34001 MR 0263582

[35] A. P. Veselov, On Darboux-Treibich-Verdier potentials. *Lett. Math. Phys.* **96** (2011), no. 1-3, 209–216 Zbl 1242.34152 MR 2788911

[36] E. T. Whittaker and G. N. Watson, *A course of modern analysis.* Cambridge University Press, Cambridge, 1927 JFM 53.0180.04

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