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# Regularity of the scattering matrix for nonlinear Helmholtz eigenfunctions

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**Abstract.** We study the nonlinear Helmholtz equation $(\Delta - \lambda^2)u = \pm|u|^{p-1}u$ on $\mathbb{R}^n$, $\lambda > 0$, $p \in \mathbb{N}$ odd, and more generally $(\Delta_g + V - \lambda^2)u = N[u]$, where $\Delta_g$ is the (positive) Laplace–Beltrami operator on an asymptotically Euclidean or conic manifold, $V$ is a short range potential, and $N[u]$ is a more general polynomial nonlinearity. Under the conditions $(p-1)(n-1)/2 > 2$ and $k > (n-1)/2$, for every $f \in H^k(\mathbb{S}^{n-1}_\omega)$ of sufficiently small norm, we show there is a nonlinear Helmholtz eigenfunction taking the form

$$u(r,\omega) = r^{-(n-1)/2}(e^{-i\lambda r} f(\omega) + e^{+i\lambda r} b(\omega) + O(r^{-\varepsilon})), \quad \text{as } r \to \infty,$$

for some $b \in H^k(\mathbb{S}^{n-1}_\omega)$ and $\varepsilon > 0$. That is, the nonlinear scattering matrix $f \mapsto b$ preserves Sobolev regularity, which is an improvement over the authors' previous work (2020) with Zhang, that proved a similar result with a loss of four derivatives.

## 1. Introduction and statement of results

We consider a Hamiltonian $H = \Delta_g + V$ defined on $\mathbb{R}^n$, where $g$ is an asymptotically Euclidean Riemannian metric in the sense defined below (an example is any smooth, compactly supported perturbation of the flat metric), and $V \in C^\infty(\mathbb{R}^n)$ is a real valued potential function which is short range and satisfies symbolic estimates in the sense that

$$|D_z^\alpha V(z)| \le C\langle z\rangle^{-\gamma-|\alpha|}$$

for some $\gamma > 1$. We study the scattering problem for the nonlinear Helmholtz equation

$$(H - \lambda^2)u = N[u], \tag{1.1}$$

for certain polynomial nonlinearities $N$. The admissible nonlinearities are defined below, but for now we note that examples include

$$N[u] = (c_1|u|^{p-1} + c_2|\nabla u|^{p-1})u,$$

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where $p \geq 3$ is an odd integer. For prescribed, sufficiently small data $f \in H^k(\mathbb{S}^{n-1})$, we seek $u$ solving equation (1.1), such that

$$u(r, y) \sim r^{-(n-1)/2}(e^{-i\lambda r} f(y) + e^{+i\lambda r} b(y)), \quad b \in H^{k'}(\mathbb{S}^{n-1}). \tag{1.2}$$

Here $f$ is the "incoming" data and $b$ is the "outgoing" data. We refer to the association $f \mapsto b$ as the "nonlinear scattering matrix." In the linear setting, where $N[u] \equiv 0$, the map $f \mapsto b$ is called the *linear scattering matrix* and denoted $S_{\mathrm{lin}}(\lambda)$. In this setting, it is a pseudodifferential operator of order zero on $\mathbb{S}^{n-1}$ composed with the antipodal map.

The results from [1] include the following.

**Theorem.** *Assume that* $(p-1)(n-1)/2 > 2$ *and suppose that* $k - 4 > (n-1)/2$, $k \in \mathbb{N}$. *Then there is* $c > 0$ *sufficiently small, such that for all* $f \in H^k(\mathbb{S}^{n-1})$ *with* $\|f\|_{H^k} < c$, *there is a solution* $u$ *to* (1.1) *satisfying* (1.2) *with* $k' = k - 4$.

The purpose of this article is to prove, as is expected from the inherent symmetry in the determination of $b$ from $f$ and vice-versa, that the value of $k'$ can be taken equal to $k$, i.e., that the nonlinear scattering matrix preserves Sobolev regularity.

**Theorem 1.** *Assume that* $p$ *is an odd integer and* $k$ *is an integer satisfying*

$$(p-1)\frac{n-1}{2} > 2 \quad \text{and} \quad k > \max\Big(1, \frac{n-1}{2}\Big). \tag{1.3}$$

*There is a* $c > 0$ *such that for all* $f \in H^k(\mathbb{S}^{n-1})$ *with* $\|f\|_{H^k} < c$, *there is a solution* $u$ *to* (1.1) *satisfying* (1.2) *with* $b \in H^k(\mathbb{S}^{n-1})$.

*Moreover,* $u$ *is unique in the sense of the main theorem of* [1], *described in detail in Section* 4, *and the error term in the asymptotic expansion* (1.2),

$$E_r := u - r^{-(n-1)/2}(e^{-i\lambda r} f + e^{+i\lambda r} b)$$

*satisfies*

$$\|E_r\|_{H^{k-2}(\mathbb{S}^{n-1})} = O(r^{-(n-1)/2-\varepsilon}) \quad \text{for some } \varepsilon > 0. \tag{1.4}$$

*Assume further that the stricter inequality* $(p-1)(n-1)/2 > 3$ *holds. Then we have a decomposition* $b = S_{\mathrm{lin}}(\lambda) f + b_1$ *where* $S_{\mathrm{lin}}(\lambda)$ *is as defined above and* $b_1 \in H^{k+1}$. *Moreover, still for this stricter condition, for* $j \in \mathbb{N}$, *if* $f \in H^{k+j}(\mathbb{S}^{n-1})$ *(in addition to the smallness condition in* $H^k$*) then* $b_1 \in H^{k+j+1}(\mathbb{S}^{n-1})$, *in particular,* $f \in C^\infty(\mathbb{S}^{n-1}) \implies b \in C^\infty(\mathbb{S}^{n-1})$.

To elaborate on the uniqueness statement, we show that, for an appropriate microlocalizing pseudodifferential operator $A_-$ (see (2.6)) to the incoming radial set, with $u_- = A_- u_0$ and $u_0 = P(\lambda) f$ the linear generalized eigenfunction with incoming data $f$, then $u - u_-$ is uniquely determined in a small ball around the origin in the Hilbert space $H_+^{s,-1/2-\delta;1,k-1}$ defined in Section 2.2.

Note the convergence in (1.4) is in $H^{k-2}(\mathbb{S}^{n-1})$. This reflects the well-known phenomenon from the linear setting whereby an asymptotic expansion for an incoming (or outgoing) approximate eigenfunction is produced by computing successive terms in a formal expansion in negative powers of $r$. This process in general produces a "distributional" expansion, in which coefficients of higher order terms have decreasing regularity. For example, in flat Euclidean space, an incoming approximate eigenfunction $(\Delta_0 - \lambda^2)u_- \in \mathcal{S}(\mathbb{R}^n)$ with incoming data $f \in C^\infty(\mathbb{S}^{n-1})$ admits an asymptotic expansion

$$u_- \sim r^{-(n-1)/2} e^{i\lambda r} \sum_{j=0}^{\infty} r^{-j} v_j(y),$$

$$v_0 = f, \quad v_{j+1} = \frac{1}{2i(j+1)\lambda}\Big(\Delta_{\mathbb{S}^{n-1}} + \frac{(n-1)(n-3)}{4} - j(j+1)\Big) v_j.$$

From this, one sees immediately that with $f \in H^k$ one can obtain a partial expansion of an approximate eigenfunction in which each subsequent term has a coefficient two orders rougher than the previous one. It therefore seems very natural that the convergence in our theorem takes place in $H^{k-2}(\mathbb{S}^{n-1})$. See also Remark 3.8.

Our methods extend to prove a generalization of this result in the setting of asymptotically conic manifolds. These are Riemannian manifolds $(M^\circ, g)$ where $M^\circ$ is the interior of a compact manifold $M$ with boundary $\partial M$ and $g$ is a so-called *scattering metric,* meaning it takes the form

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, dy)}{x^2},$$

in a neighborhood of $\partial M$ where $x$ is a boundary defining function, i.e., $\partial M = \{x = 0\}$ and $x \geq 0$ has that $dx$ is nonvanishing over $\partial M$, and $y$ are coordinates on $\partial M$. Here $h$ is a smooth $(0, 2)$-tensor that restricts to a metric on $\partial M$. Flat Euclidean space is an example of an asymptotically conic space; write $M^\circ = \mathbb{R}^n$ and include $\mathbb{R}^n \hookrightarrow \overline{\mathbb{R}^n} = \{w \in \mathbb{R}^n : |w| \leq 1\} =: M$ where the inclusion can be realized by the map $z \mapsto z/(1 + \langle z \rangle)$ and note that the metric form is realized by writing the flat metric in polar coordinates and setting $x = 1/r$.

In the Euclidean case, $\partial M$ is the sphere $\mathbb{S}^{n-1}$ at infinity with its standard metric, and $h$ is independent of $x$. In general, if $(\partial M, h(0))$ is the sphere with its standard metric, then we call $(M^\circ, g)$ an asymptotically Euclidean metric.

On a general asymptotically conic manifold, writing $r = 1/x$ one obtains an analogue of the radial variable in this more general context, and the metric then takes the form near infinity

$$g = dr^2 + r^2 h\Big(\frac{1}{r}, y, dy\Big).$$

The admissible nonlinearities are those $N[u] := N(u, \bar{u}, \nabla u, \nabla \bar{u}, \nabla^{(2)} u, \nabla^{(2)} \bar{u})$ which are a sum of monomial terms, of degree not less than $p$, in $u$ and $\bar{u}$ and their derivatives up to order two, with coefficients smooth on $M$. Moreover, we require $p$ to satisfy the first condition in (1.3).

**Theorem 2** (Main theorem, asymptotically conic case). *Let $(M^\circ, g)$ be an asymptotically conic manifold of dimension $n$, and let $V$ be a conormal short range potential, that is, a smooth potential on $M^\circ$ satisfying estimates near infinity of the form*

$$|(rD_r)^j D_y^\alpha V(r, y)| \le C\langle r\rangle^{-\gamma} \quad \textit{for all } j \ge 0,\ \alpha \in \mathbb{N}^{n-1} \tag{1.5}$$

*for some $\gamma > 1$. Let $H = \Delta_g + V$ where $\Delta_g$ is the Laplace–Beltrami operator on $(M^\circ, g)$. Let $N[u]$ be an admissible nonlinearity, and let $p$ and $k$ be integers satisfying (1.3). There exists $c > 0$ sufficiently small, such that for every $f \in H^k(\partial M)$ with $\|f\|_{H^k(\partial M)} < c$, there is a solution $u$ to*

$$(H - \lambda^2)u = N[u]$$

*on $M^\circ$ satisfying*

$$u(r, y) = r^{-(n-1)/2}(e^{-i\lambda r} f(y) + e^{+i\lambda r} b(y) + O_{H^{k-2}}(r^{-\varepsilon})) \tag{1.6}$$

*for some $b \in H^k(\partial M)$ and some $\varepsilon > 0$.*

*Assume further that the stricter inequality $(p-1)(n-1)/2 > 3$ holds and that the nonlinearity $N[u]$ involves derivatives up to order one (instead of two as allowed above). Then we again have the decomposition $b = S_{\mathrm{lin}}(\lambda) f + b_1$ with $S_{\mathrm{lin}}(\lambda) f$ again the linear scattering matrix (now an FIO associated to geodesic flow for time $\pi$ [8]) and $b_1 \in H^{k+1}(\partial M)$. Again, if $f \in H^{k+j}(\partial M)$ (in addition to the smallness condition in $H^k$), then $b_1 \in H^{k+j+1}(\partial M)$.*

For ease of exposition, we return to the Euclidean case in the remainder of this introduction. Given $f \in L^2(\mathbb{S}^{n-1})$, the linear solution $u_0$ to $(H - \lambda^2)u_0 = 0$ with incoming data $f$ is the image of $f$ under the *incoming Poisson operator*, $P(\lambda)$. This $u_0$ can be written (non-uniquely) as a sum of incoming and outgoing terms $u_0 = u_- + u_+$, where, roughly speaking, $u_\pm \sim r^{-(n-1)/2} e^{\pm ir\lambda} f_\pm$ as $r$ goes to infinity. For adequate decompositions $u_\pm$, solutions $u$ to (1.1) satisfying (1.2) can be constructed by a contraction mapping argument in which one writes

$$u = u_- + w = u_0 + (w - u_+)$$

where $w$ will be outgoing (in a sense to be made precise below) and a fixed point of the mapping

$$\Phi(w) = u_+ + R(\lambda + i0)N[u_- + w],$$

where $R(\lambda + i0)$ is the outgoing resolvent, that is, $(H - (\lambda + i0)^2)^{-1}$. The true nonlinear solution therefore satisfies

$$u = u_0 + R(\lambda + i0)N[u].$$

At issue in this paper is the regularity of the outgoing data of $w$, and how this can be understood in terms of the mapping properties of the Poisson operator, the decomposition $u_\pm$, and the resolvent.

The construction of the nonlinear eigenfunction $u$ requires a decomposition $u_0 = u_- + u_+$ for the free solution, and we improve on the results in [1] by using the Schwartz kernel of the Poisson operator $P(\lambda)$ (see Section 3) to obtain a decomposition with optimal regularity. Indeed, the Poisson operator is given by the action of a well-understood oscillatory integral kernel, due to [8] in the asymptotically conic case, with many antecedents in the Euclidean case, e.g., [5, 6]. Using this we prove a relationship between the regularity of the incoming data $f \in H^k(\mathbb{S}^{n-1})$ for $k \geq 0$ and the corresponding linear generalized eigenfunction $u_0$. This is best understood using the theory of scattering pseudodifferential operators [9]. It is well known that $u_0$ lies in weighted Sobolev spaces $H^{s,-1/2-\varepsilon}(\mathbb{R}^n)$, where $s \in \mathbb{R}$ and $\varepsilon > 0$ are arbitrary. This means that

$$\langle z\rangle^{-1/2-\varepsilon} u_0 \in H^s(\mathbb{R}^n),$$

where $H^s(\mathbb{R}^n)$ are the standard $L^2$-based Sobolev spaces on $\mathbb{R}^n$. (In particular, $u_0$ is a smooth function in the interior, a consequence of elliptic regularity.) What we prove below is that $u_0$ can be decomposed into $u_0 = u_- + u_+$ where each $u_\pm$ has $k$ additional order of "module regularity," specifically each remains in $H^{s,-1/2-\varepsilon}$ after application of $k-$fold combinations of angular derivatives and the radial annihilators $r(D_r \mp \lambda)$ of the oscillatory factors $e^{\pm i\lambda r}$. As we describe below, these operators are determined directly by the microlocal structure of the problem; they comprise the modules $\mathcal{M}_\pm$ of scattering pseudodifferential operators in $\Psi^{1,1}_{\mathrm{sc}}$ which are characteristic on the incoming/outgoing $(-/+)$ *radial* sets $\mathcal{R}_\pm$ of the operators $H - \lambda^2$, thought of as a *non-elliptic* scattering operator with non-degenerate characteristic set over spatial infinity.

The paper is organized as follows. In Section 2 we recall the basic definitions and structures that will be used in the paper, including scattering pseudodifferential operators and the weighted Sobolev spaces between which they act. We also recall there the definitions of the module regularity spaces used in [1] and the crucial mapping properties of the resolvent between such spaces. In Section 3 we discuss the mapping properties of the Poisson operator. These properties are very closely related to mapping properties of the incoming and outgoing resolvents, due to formula (3.4). The key result is Proposition 3.4 which shows that the resolvent applied to certain functions admit asymptotic expansions corresponding to one of the terms in (1.6). The PDE-type argument, however, does not give the optimal regularity for the leading coefficient. The optimal regularity is obtained in Propositions 3.6 and 3.9, which relates the map taking a function $F$ to the leading expansion of $R(\lambda \pm i0)F$ to the *adjoint* of the Poisson operator, $P(\mp\lambda)^*$, applied to $F$. In Section 4 we put these results together to prove the main theorem.

Our analysis of the Poisson operator is based on the description of its Schwartz kernel as an oscillatory integral, which is due to Melrose and Zworski [8]. The powerful tools from microlocal scattering theory we employ were developed in [9] (radial point estimates), [2] (test modules), [11] (Fredholm theory for real principal type operators on anisotropic Sobolev spaces) and [12] (radial point estimates and Fredholm theory in the scattering calculus), and we refer the reader to [1] for the detailed discussion of related literature.

## 2. Microlocal analysis of the resolvents $R(\lambda \pm i0)$

We review the relevant objects here only briefly as they are discussed in detail in other work. A detailed introduction to scattering differential operators on $\mathbb{R}^n$ can be found in Vasy's minicourse notes [12], while the more general development on scattering manifolds is due to Melrose [9]. See also [1, Sections 2 and 3].

### 2.1. Weighted Sobolev spaces and scattering pseudodifferential operators

We confine most of our introductory discussion to the case of Euclidean space. Letting $\mathcal{S}(\mathbb{R}^n)$ denote the space of Schwartz functions and $\mathcal{S}'$ the tempered distributions, each $u \in \mathcal{S}'$ lies in some weighted $L^2$-based Sobolev space

$$H^{s,l}(\mathbb{R}^n) = \langle z \rangle^{-l} H^m(\mathbb{R}^n),$$

for $s, l \in \mathbb{R}$.

Recall the scattering symbols and scattering pseudodifferential operators, defined for $m, l \in \mathbb{R}$ by

$$S^{s,l}(\mathbb{R}^n) = \{a(z,\zeta) \in C^\infty(\mathbb{R}^n_z \times \mathbb{R}^n_\zeta) \colon \|a\|_{s,l;N} < \infty \text{ for all } N \in \mathbb{N}_0\},$$
$$\Psi^{s,l}_{\mathrm{sc}}(\mathbb{R}^n) = \{\mathrm{Op}(a) \colon a \in S^{s,l}(\mathbb{R}^n)\},$$

where

$$\|a\|_{s,l;N} = \sum_{|\alpha|+|\beta| \le N} \sup_{z,\zeta} |\langle z \rangle^{-l+|\alpha|} \langle \zeta \rangle^{-s+|\beta|} D_z^\alpha D_\zeta^\beta a(z,\zeta)|,$$

Here $\mathrm{Op}(a)$ denotes the operator with integral kernel $\int e^{-i(z-z')}a(z,\zeta)d\zeta$. Thus, the scattering pseudodifferential operators are by definition (here) the left quantizations of scattering symbols.

There is a notion of principal symbol attached to scattering PsiDO's which includes their leading order behavior at spatial infinity. The (filtered) algebra of scattering PsiDO's admits a natural mapping to the graded algebra of scattering symbols,

$$\Psi^{s,l}_{\mathrm{sc}} \to S^{[s,l]} := S^{s,l}/S^{s-1,l-1}.$$

Given $A \in \Psi^{s,l}_{\mathrm{sc}}$ its principal symbol will be denoted by $\sigma_{s,l}(A)$.

We work here exclusively with classical scattering symbols, which are functions $a(z,\zeta)$ with joint asymptotic expansions in $z,\zeta$ as $|z|,|\zeta| \to \infty$. We will need to construct explicit symbols whose corresponding operators will function as microlocal cutoffs. In general, classical symbols can be written

$$a \equiv \langle z\rangle^l \langle\zeta\rangle^s a_0(z,\zeta),$$

where $a_0(z,\zeta)$ is bounded and is smooth in $r^{-1} = |z|^{-1}$ and $|\zeta|^{-1}$. This characterization of the asymptotic behavior of $a_0$ is equivalent to the statement that $a_0$ extends to an element in $C^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n})$ where $\overline{\mathbb{R}^n}$ is the radial compactification of $\mathbb{R}^n$ in which the map $z \mapsto z/(1+\langle z\rangle)$ realizes $\mathbb{R}^n$ as the interior of the unit ball $\mathbb{B}^n$, and the overline notation denotes this entire structure (the map from Euclidean space into the ball as opposed to just the ball.)

For $[a] \in S^{[s,l]}$, i.e., $[a]$ the principal symbol of a scattering PsiDO, the value of $a_0$ is determined here only to the addition of elements in $S^{s-1,l-1}$, so in particular, in sets of bounded frequency, $|\zeta| < C$, with $\mathrm{supp}(\chi\colon \mathbb{R}_t \to \mathbb{R}) \subset \{|t| \leq C\}$,

$$\chi(\zeta)(a_0(z,\zeta) - a(\hat{z},\zeta)) \equiv 0 \pmod{S^{s-1,l-1}},$$

with a similar expression in the $|z| < C$ regions. An $A \in \Psi^{s,l}_{\mathrm{sc}}$ is by definition *scattering elliptic* if its principal symbol is invertible in the graded algebra $\bigcup_{s,l} S^{[s,l]}$. The microlocal notion of scattering ellipticity will be used as follows. On regions of large frequency $|\zeta| > C$, scattering ellipticity is implied by the uniform estimate

$$\sigma_{s,l}(A)(z,\zeta) \geq C\langle z\rangle^l \langle\zeta\rangle^s \quad \text{for } |\zeta| > C.$$

Below, the operators of interest are (scattering) elliptic on such large frequency regions but are not globally scattering elliptic, hence the more detailed estimates coming from propagation phenomena will arise from analysis on sets of bounded frequency, whence we will typically need only to discuss the "spatial" principal symbol of $A \in \Psi^{m,l}_{\mathrm{sc}}$, defined for classical scattering symbols by

$$\sigma_{\mathrm{base},\,l}(A)(\hat{z},\zeta) = \lim_{r\to\infty} \langle z\rangle^{-l} a(z,\zeta).$$

Note that the behavior in $\zeta$ of this function is in general symbolic but not homogeneous. In this region, (scattering) ellipticity is the same familiar ellipticity but in the $z$ directions; $(\hat{z}_0, \zeta_0)$ is in the elliptic set if $\sigma_{s,l}(A)(z, \zeta) \geq C\langle z\rangle^l$ in a region $|z| > C, |\hat{z} - \hat{z}_0|, |\zeta - \zeta_0| < \varepsilon$. It is straightforward to check that this is equivalent to $\sigma_{\mathrm{base},\, l}(A)(\hat{z}_0, \zeta_0) \neq 0$.

The basic boundedness property of scattering pseudodifferential operators is that for $A \in \Psi_{\mathrm{sc}}^{s',l'}$,

$$A\colon H^{s,l} \to H^{s-s',l-l'} \text{ is bounded.} \tag{2.1}$$

The residual operators are

$$\Psi^{-\infty,-\infty} = \bigcap_{m,l} \Psi_{\mathrm{sc}}^{m,l},$$

which have Schwartz kernels in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, and in particular $A \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}$ is bounded between *any* two weighted $L^2$-based Sobolev spaces; in particular since $H^{s,l} \subset H^{s',l'}$ for any $s > s', l > l'$, residual $A$'s define compact operators on $L^2$. If $A$ is globally scattering elliptic then the map in (2.1) is Fredholm, since in that case there is an approximate inverse $B \in \Psi_{\mathrm{sc}}^{-s,-l}$ such that $\mathrm{Id} - AB, \mathrm{Id} - BA \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}$.

All of the foregoing material extends directly to the general asymptotically conical case; see [1, Section 2.2] for further details. In the general case, one has the fiberwise radial compactification of the scattering cotangent bundle ${}^{\mathrm{sc}}\overline{T}^*M$ which is a manifold with corners of codimension 2; there are boundary-defining functions $x$ for spatial infinity and $\rho$ for fiber infinity (which in the Euclidean case can be taken to be $\langle z\rangle^{-1}$ and $\langle \zeta\rangle^{-1}$, respectively) and the symbol estimates are essentially the same as the Euclidean space symbol estimates written in terms of $x$ and $\rho$. The scattering pseudodifferential operators $\Psi_{\mathrm{sc}}^{s,l}(M)$ are the quantizations of these symbols, and the scattering Sobolev spaces, denoted $H_{\mathrm{sc}}^{s,l}(M)$, consist of distributions $u$ with $Au \in L^2(M)$ for all $A \in \Psi_{\mathrm{sc}}^{s,l}(M)$.

## 2.2. Mapping properties of $H - \lambda^2$ and the resolvent

Analysis of $H - \lambda^2$ as a scattering differential operator was first carried out by Melrose in [9]. We review the relevant material again on $\mathbb{R}^n$. We have $H - \lambda^2 \in \Psi_{\mathrm{sc}}^{2,0}$, with

$$\sigma_{2,0}(H - \lambda^2) = |\zeta|^2 + V - \lambda^2 \geq C\langle \zeta\rangle^2, \quad \text{for } |\zeta| > C.$$

In the general asymptotically Euclidean case, we get that the spatial principal symbol is rather simple:

$$\sigma_{\mathrm{base},\,2}(H - \lambda^2)(\hat{z}, \zeta) = |\zeta|^2 - \lambda^2.$$

*In particular, $H - \lambda^2$ is not globally scattering elliptic.* Its characteristic set $\Sigma$, which is a subset of $\mathbb{S}^{n-1}_{\hat{z}} \times \mathbb{R}^n_\zeta$, is the vanishing locus of the fiber principal symbol:

$$\Sigma = \{(\hat{z}, \zeta) \colon |\zeta|^2 - \lambda^2 = 0\}.$$

To understand the action of the flow of the Hamilton vector field on $\Sigma$, we can work in polar coordinates $(r, y)$, where $y$ are arbitrary coordinates in a coordinate patch on the sphere $\mathbb{S}^{n-1}$; these automatically give rise to dual coordinates $(\nu, \eta)$, and we rescale the angular dual variable $\eta$ by writing $r\mu = \eta$. Via this localization and rescaling we effectively pass to the case of an arbitrary asymptotically conic manifold.

Noting that in these variables we have

$$|\zeta|^2 = \nu^2 + |\mu|_h^2,$$

where $|\mu|_h$ denotes the norm with respect to the dual of metric $h$ in the definition of the asymptotically conic metric $g$. Apart from giving the obvious rewriting of the characteristic set as $\Sigma = \{\nu^2 + |\mu|_h^2 = \lambda^2\}$, it clarifies the behavior of the Hamilton vector field, which, recall, is defined with respect to arbitrary coordinates and their canonical dual coordinates to be

$$H_p := \frac{\partial p}{\partial \nu}\frac{\partial}{\partial r} - \frac{\partial p}{\partial r}\frac{\partial}{\partial \nu} + \sum_{j=1}^{n-1}\Big(\frac{\partial p}{\partial \tilde{\mu}_j}\frac{\partial}{\partial y_j} - \frac{\partial p}{\partial y_j}\frac{\partial}{\partial \tilde{\mu}_j}\Big).$$

Scattering calculus propagation results are phrased in terms of the natural conformal rescaling (or reweighting) of this vector field, namely $\mathsf{H}_p := \langle z \rangle H_p$, and with $p = \sigma_{2,0}(H - \lambda^2)$; we obtain

$$\mathsf{H}_p = -2\nu(x\partial_x + R_\mu) + 2|\mu|_h^2\partial_\nu + H_{\partial M,h}, \tag{2.2}$$

where $R_\mu$ is the radial vector field in $\mu$ and $H_{\partial M,h}$ is the Hamilton vector field on $\partial M$ for the metric $h$. (Note that the apparent discrepancy in sign between (2.2) and [9, equation (8.19)] is due to our usage of $\nu$, the dual variable to $r$, in contrast with Melrose's use of $\tau = -\nu$.)

From this one deduces that the two submanifolds

$$\mathcal{R}_\pm := \{\mu = 0 = x,\ \nu = \pm\lambda\}$$

are sinks $(+)$ or sources $(-)$ for the (rescaled) Hamilton flow, i.e., the flow of $\mathsf{H}_p$, on the characteristic set $\Sigma$. The well-known formula for the principal symbol of a commutator of pseudodifferential operators also extends to the scattering setting, namely if $A \in \Psi_{\mathrm{sc}}^{s,l}$, $B \in \Psi_{\mathrm{sc}}^{s',l'}$ with $a = \sigma_{s,l}(A), b = \sigma_{s',l'}(B)$, then $[A, B] \in \Psi_{\mathrm{sc}}^{s+s'-1,l+l'-1}$ and

$$\sigma_{s+s'-1,l+l'-1}(i[A, B]) = \{a, b\} = H_a b,$$

where here $\{\cdot, \cdot\}$ denote the Poisson bracket.

The most important example of the use of commutators for us will be those of the form $[H, Q]$ where $H$ is the Hamiltonian and $Q \in \Psi_{\mathrm{sc}}^{0,0}$, in particular we will take $Q$ to have various microsupport properties (discussed below) which will allow them to act as microlocal cutoffs to various portions of scattering phase space. Particularly simple examples include symbols of the form

$$q(x, y, \nu, \mu) = \chi(x)\chi\Big(\frac{|\mu|_h^2 + \nu^2}{2}\lambda^2\Big)\phi(\nu), \quad \phi \in C_c^\infty(\mathbb{R}^n),$$

where $\chi$ is a smooth function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. (This $q$ is then a function whose support lies in a neighborhood of $\Sigma$ and whose restriction to $\Sigma$ depends only on $\nu$.) Then in fact $Q \in \Psi_{\mathrm{sc}}^{-\infty,0}$ and $[H, Q] \in \Psi_{\mathrm{sc}}^{-\infty,-1}$, whence

$$\langle z\rangle\sigma_{-\infty,-1}(i[H, Q]) = \mathsf{H}_p q = 2|\mu|_h^2\phi'(\nu) \tag{2.3a}$$

in the region $|x| < 1$, $|\mu|_h^2 + \nu^2 < 2\lambda^2$, and

$$\langle z\rangle\sigma_{-\infty,-1}(i[H, Q]) = 0 \quad \text{for } |\mu|_h^2 + \nu^2 \geq 2\lambda. \tag{2.3b}$$

We can now define the modules of pseudodifferential operators $\mathcal{M}_\pm$ used to measure the regularity of distributions in the domain and range of our formulation of the resolvent mapping. Specifically,

$$\mathcal{M}_\pm := \{A \in \Psi_{\mathrm{sc}}^{1,1} : \mathcal{R}_\pm \subset \Sigma_{1,1}(A)\},$$

or, in words, $\mathcal{M}_+$ is the vector space of scattering pseudodifferential operators of order $(1, 1)$ which are characteristic on the "outgoing" radial set $\mathcal{R}_+$ and $\mathcal{M}_-$ is the same for $\mathcal{R}_-$. Locally, an element $A \in \Psi_{\mathrm{sc}}^{1,1}$ lies in $\mathcal{M}_\pm$ if and only if it can be written

$$A = B_0 r(D_r \mp \lambda) + \sum_{i=1}^{n-1} B_i D_{y_i} + B'$$

with $B_0, B_i, B' \in \Psi_{\mathrm{sc}}^{0,0}$. Note that $\mathcal{M}_+$ and $\mathcal{M}_-$ both contain the identity operator. Also, $\mathcal{M}_\pm$ contain an elliptic element of order $(1, 1)$, namely the radial operator $r(D_r \mp \lambda)$, at the opposite radial set $\mathcal{R}_\mp$.

We also need the "small" module of angular derivatives

$$\mathcal{N} = \Big\{\sum_{i=1}^{n-1} B_i D_{y_i} + B' : B_i, B' \in \Psi_{\mathrm{sc}}^{0,0}\Big\} = \mathcal{M}_+ \cap \mathcal{M}_-.$$

[1, Theorem 2.7] then gives the following mapping property for the resolvent. Let $s, l \in \mathbb{R}, \kappa, k \in \mathbb{N}_0$, and define

$$\mathcal{Y}_\pm^{s,l;\kappa,k} := H_\pm^{s,l;\kappa,k} = \{u \in H^{s,l} : \mathcal{N}^k \mathcal{M}_\pm^\kappa u \subset H^{s,l}\},$$
$$\mathcal{X}_\pm^{s,l;\kappa,k} := \{u \in H_\pm^{s,l;\kappa,k} : (H - \lambda^2)u \in H_\pm^{s-2,l+1;\kappa,k}\}.$$

Then

$$R(\lambda \pm i0)\colon \mathcal{Y}_{\pm}^{s,l+1;\kappa,k} \to \mathcal{X}_{\pm}^{s+2,l;\kappa,k} \tag{2.4}$$

provided $k \geq 1$, $l \in (-\frac{1}{2} - k, -\frac{1}{2})$. Here, the value $l = -1/2$ is referred to as the "threshold" value; it is the critical spatial order for which we need different "radial propagation estimates" for $l$ greater than vs. less than $-1/2$. See [1, Section 3]. The condition in (2.4) is that $l$ is below threshold, but that $l + k$ is above threshold. That means that module regularity for $\mathcal{M}_{\pm}$ of order $k$ gives above threshold regularity at the opposite radial set $\mathcal{R}_{\mp}$, which is the key to obtaining estimates such as (2.4). This statement can be rephrased as follows: for such $k \in \mathbb{N}$ and $l \in \mathbb{R}$, the operator $H - \lambda^2$ is an isomorphism from $\mathcal{X}_{\pm}^{s,l;k,k'}$ to $\mathcal{Y}_{\pm}^{s-2,l+1;k,k'}$, and its inverse is the resolvent $R(\lambda \pm i0)$.

The reason for introducing the small module $\mathcal{N}$ to treat the nonlinear problem is explained in detail in the introduction of [1]. We remark here only that for multiplication of distributions in $H_{+}^{s,l;\kappa,k}$, large module regularity produces loss in decay – see, e.g., [1, Corollary 2.10] from which one has $H_{+}^{s,l;\kappa,k} \cdot H_{+}^{s,l;\kappa,k} \subset H_{+}^{s,2l+(n/2)-\kappa;\kappa,k}$. Small module regularity is used to minimize this loss of $\kappa$ in the spatial order.

### 2.3. Microlocalization

It will be important to microlocalize distributions both near to and away from the radial set. To this end, we recall some features of the operator wavefront set $\mathrm{WF}'(A)$ of $A \in \Psi_{\mathrm{sc}}^{s,l}(M)$. We will work mostly with those $A$ which are compactly microlocalized in frequency. Concretely, this means that

$$A = \mathrm{Op}(a(z,\zeta)) + E, \quad \operatorname{supp} a(z,\zeta) \subset \{|\zeta| < C\},\ \kappa_E \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n),$$

for some $C > 0$ ($\kappa_E$ is the Schwartz kernel of $E$). The condition on $\kappa_E$ is equivalent to $E$ being a *residual* operator, meaning

$$E \in \Psi_{\mathrm{sc}}^{-\infty,-\infty} := \bigcap_{m,l \in \mathbb{R}\times\mathbb{R}} \Psi_{\mathrm{sc}}^{m,l}.$$

For such $A$, $\mathrm{WF}'(A) \subset \mathbb{S}_{\hat{z}}^{n-1} \times \mathbb{R}_{\zeta}^{n}$ is by definition the complement of the set $(\hat{z}, \zeta)$ such that $a$ is Schwartz in $\zeta$ in the $\hat{z}$ direction.

Clearly, $\mathrm{WF}'(A) = \varnothing \implies A \in \Psi_{\mathrm{sc}}^{-\infty,-\infty}$, and thus for any $S, L \in \mathbb{R}$,

$$\mathrm{WF}'(A) = \varnothing \implies A\colon H^{S,L}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

Also, wavefront sets have the expected algebraic property of supports, namely, for $A \in \Psi_{\mathrm{sc}}^{s,l}$, $B \in \Psi_{\mathrm{sc}}^{s',l'}$,

$$\mathrm{WF}'(AB) \subset \mathrm{WF}'(A) \cap \mathrm{WF}'(B).$$

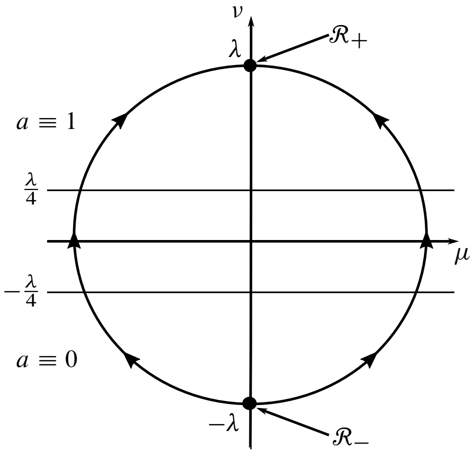


**Figure 1.** The bicharacteristic flow in the characteristic set $\Sigma(P) = \{x = 0,\ \nu^2 + |\mu|_h^2 = \lambda^2\}$ with $P = \Delta - \lambda^2$ and $a$ the principal symbol of the microlocalizer $A_+$.

Two particularly useful examples of such operators are those of the form $Q_\pm \in \Psi_{\mathrm{sc}}^{\mathbf{0},\mathbf{0}}$ which are microlocalized near the two components of the radial set $\mathcal{R}_\pm$. This can be done by defining $Q_\pm$ explicitly in a way similar to the definition of $Q$ above (2.3), specifically one can take

$$Q_\pm = \mathrm{Op}(q_\pm), \quad q_\pm(x, y, \nu, \mu) = \chi(x)\chi\Big(\frac{|\mu|_h^2 + \nu^2}{2}\lambda^2\Big)\chi\Big(\frac{\nu \mp \lambda}{\varepsilon}\Big), \tag{2.5}$$

where again $\chi(s) = 1$ for $s \le 1$ and $\chi(s) = 0$ for $s \ge 2$.

Alternatively, one can simply break up phase space into a region where $\nu$ is positive and its complement, excluding one radial set from each. We choose an operator $A_+ \in \Psi_{\mathrm{cl}}^{0,0}(M)$ such that $A_+$ is microlocally equal to the identity in a neighborhood of $\mathcal{R}_+$, and microlocally equal to 0 in a neighborhood of $\mathcal{R}_-$. We also let $A_- = \mathrm{Id} - A_+$; thus, $A_-$ is microlocally equal to the identity in a neighborhood of $\mathcal{R}_-$, and microlocally equal to 0 in a neighborhood of $\mathcal{R}_+$. It is convenient to choose $A_+$ such that its principal symbol $a$ is a function only of $\nu$ in a neighborhood of the characteristic variety of $H$, and is monotone. Indeed, we can take

$$A_+ = \mathrm{Op}(a), \quad a(x, \nu) = \chi(x)\Big(\frac{|\mu|_h^2 + \nu^2}{2}\lambda^2\Big)\tilde{\chi}(\nu), \tag{2.6}$$

where $\chi$ is as in the definition of $Q_\pm$ and $\tilde{\chi}(\nu) \equiv 1$ in $\nu > \lambda/4$ and $\tilde{\chi}(\nu) \equiv 0$ for $\nu < -\lambda/4$, and is monotone in between.

While microlocalization gives us a concrete mechanism for analyzing frequency-localized spatial decay, it will be useful to note that this is also possible using anisotropic Sobolev spaces, meaning Sobolev spaces $H^{s,\mathsf{l}}$ in which the parameters $s, \mathsf{l}$

are themselves functions on phase space. Since the operators under consideration here are elliptic outside a compact set in frequency, we will always take $s$ constant, but we shall employ variable spatial weights $\mathsf{l}_\pm \in S^{0,0}$ satisfying

$$\mathsf{l}_+ \text{ is equal to } -1/2 \text{ outside small neighborhoods of } \mathcal{R}_+ \text{ and } \mathcal{R}_-, \tag{2.7a}$$

$$\mathsf{l}_+ \text{ equals } -1/2 \mp \delta \text{ in smaller neighborhoods of } \mathcal{R}_\pm, \tag{2.7b}$$

$$\mathsf{l}_+ \text{ is nonincreasing along the Hamilton flow of } P \text{ within } \Sigma(P), \tag{2.7c}$$

and

$$\mathsf{l}_- = -1 - \mathsf{l}_+ \text{ has corresponding properties with } \mathcal{R}_- \text{ and } \mathcal{R}_+ \text{ switched.} \tag{2.7d}$$

For definiteness, we suppose that for some sufficiently small $\delta \in [0, 0.01]$, we have $\mathsf{l}_+$ equal to $-1/2$ outside the sets $\{x^2 + |\mu|_h^2 + (\nu \mp \lambda)^2 \leq 4\delta\lambda^2\}$, and is equal to $-1/2 \mp \delta$ within the sets $\{x^2 + |\mu|_h^2 + (\nu \mp \lambda)^2 \leq \delta\lambda^2\}$. Thus, distributions in $H^{s,\mathsf{l}_+}$ lie in $H^{s,-1/2-\varepsilon}$ and are microlocally $H^{s,-1/2+\varepsilon}$ near $\mathcal{R}_-$; that is, they are below threshold regularity at $\mathcal{R}_+$ and above threshold at $\mathcal{R}_-$, with a corresponding statement true with reversed signs for $H^{s,\mathsf{l}_-}$. Then, by [1, Theorem 3.2] we have the isomorphisms

$$\begin{gathered} H - \lambda^2 \colon \mathcal{X}^{s,\mathsf{l}_\pm} \to \mathcal{Y}^{s-2,\mathsf{l}_\pm+1}, \\ \mathcal{Y}^{s-2,\mathsf{l}_\pm+1} = H^{s-2,\mathsf{l}_\pm+1}, \quad \mathcal{X}^{s,\mathsf{l}_\pm} = \{u \in H^{s,\mathsf{l}_\pm} \colon (H - \lambda^2)u \in \mathcal{Y}^{s-2,\mathsf{l}_\pm+1}\}. \end{gathered} \tag{2.8}$$

The inverse maps are the incoming/outgoing resolvent $R(\lambda \pm i0)$. As with (2.4), we notice the sharp difference of one in the spatial regularity, and the avoidance of the threshold value at the radial sets.

Moreover, we can combine variable order spaces with module regularity, obtaining isomorphisms

$$\begin{aligned} H - \lambda^2 \colon \mathcal{X}_\pm^{s,\mathsf{l}_\pm;\kappa,k} &\to \mathcal{Y}_\pm^{s-2,\mathsf{l}_\pm+1;\kappa,k}, \\ \mathcal{Y}_\pm^{s',\mathsf{l}';\kappa,k} &= \{u \in \mathcal{Y}_\pm^{s',\mathsf{l}'} \colon \mathcal{N}^k \mathcal{M}_\pm^\kappa u \subset \mathcal{Y}_\pm^{s',\mathsf{l}'}\}, \\ \mathcal{X}_\pm^{s,\mathsf{l}_\pm;\kappa,k} &= \{u \in \mathcal{Y}_\pm^{s,\mathsf{l}_\pm;\kappa,k} \colon (H - \lambda^2)u \in \mathcal{Y}_\pm^{s-2,\mathsf{l}_\pm+1;\kappa,k}\}. \end{aligned} \tag{2.9}$$

We emphasize that in (2.9), we can allow $\kappa = 0$, that is, only consider small module regularity, in contrast to (2.4).

These variable order module regularity results do not appear explicitly in our previous work [1], but such results follow readily from the propagation estimates in Section 3 of that paper. (The reason for restricting the module regularity spaces to constant orders in [1] is that it is more convenient to analyze multiplicative properties of such spaces when the orders are constant.) We shall not review variable order spaces in detail here, referring the reader to [12]. Here we only mention one elementary property of such spaces. Namely, we can conclude containment in these types of variable order spaces using microlocalization. For example, it is straightforward to show that, given a tempered distribution $u$, for $\varepsilon > 0$ sufficiently small and $Q_+ \in \Psi_{sc}^{0,0}$ as in (2.5), if $Q_+ u \in H^{s,-1/2-\delta}$ and $(\mathrm{Id} - Q_+)u \in H^{s,-1/2+\delta}$, then $u \in H^{s,\mathsf{l}_+}$.

We note that in all these considerations, the value of $\mathsf{l}_\pm$ off $\Sigma$ is essentially irrelevant, since off $\Sigma$ one has elliptic estimates for $H - \lambda^2$.

## 3. Mapping properties of the linear Poisson operator

The Poisson operator furnishes a distorted or generalized Fourier transform for the operator $H$. Constructions in the Euclidean case go back a long way in scattering theory, see, e.g., [5, 6]. We use the results of Melrose and Zworski [8], who constructed the Poisson operator $P(\lambda)$, $\lambda > 0$, for the Helmholtz operator $H = \Delta - \lambda^2$ on asymptotically conic manifolds (scattering metrics). Their results extend readily to the case of a Schrödinger operator with potential in the conormal space $\mathcal{A}^\gamma(M)$, for $\gamma > 1$, which is by definition the space of smooth functions $V$ on $M^\circ$ satisfying estimates near $\partial M$ of the form

$$|(xD_x)^j D_y^\alpha V(x,y)| \le Cx^\gamma, \quad j \ge 0,\ \alpha \in \mathbb{N}^{n-1}.$$

These estimates on $V$ are equivalent to (1.5) in the case of Euclidean space.

The Poisson operator has the property that it maps a smooth function $\phi \in C^\infty(\partial M)$ to a function $u_0$ satisfying $Hu_0 = 0$, with the asymptotics

$$u_0 = u_- + u_+, \quad u_\pm = r^{-(n-1)/2} e^{\pm i\lambda r} f_\pm, \tag{3.1}$$

with $f_\pm \in C(M)$, $f_-|_{\partial M} = \phi$. In fact, the $f_\pm$ can be taken to lie in $C^\infty(M) + \mathcal{A}^{\gamma-1}(M)$ (which are continuous up to the boundary, as $\gamma - 1 > 0$); moreover, if $V$ is equal to $r^{-2}$ times a smooth function on $M$, then the $f_\pm$ can be taken in $C^\infty(M)$. We shall also define $P(-\lambda)$, again for $\lambda > 0$, to be the operator mapping $\phi \in C^\infty(\partial M)$ to a function $u_0$ satisfying $Hu_0 = 0$, with the asymptotics

$$u_0 = u_- + u_+, \quad u_\pm = r^{-(n-1)/2} e^{\pm i\lambda r} f_\pm,$$

with $f_\pm \in C^\infty(M) + \mathcal{A}^{\gamma-1}(M)$, $f_+|_{\partial M} = \phi$. Recall that $C^\infty$ functions on $M$, as opposed to $C^\infty$ functions on $M^\circ$, are (by definition) smooth functions of $x = 1/r$ and $y$ near $\partial M$.

The adjoint operator $P(\lambda)^*$ will also play a role in this article. To describe it, let $v$ be a Schwartz function[1] on $M$, and let $u_\pm = \pm R(\lambda \pm i0)v$. Then each $u_\pm$ has a

[1] By a Schwartz function on $M$ we mean a smooth function that vanishes, with all its derivatives, at the boundary. When $M$ is the radial compactification of $\mathbb{R}^n$ this corresponds to the usual meaning of Schwartz function.

similar expansion as in (3.1), but with only the outgoing (+)/incoming(−) oscillation [9, Proposition 12]:

$$u_\pm = r^{-(n-1)/2} e^{\pm i\lambda r} f_\pm, \quad f_\pm \in C^\infty(M) + \mathcal{A}^{\gamma-1}(M).$$

We claim that $P(\lambda)^* v = 2i\lambda f_-|_{\partial M}$. To see that this is true, we use a pairing identity that follows from Green's formula. Suppose that $u_1$ and $u_2$ are two functions of the form

$$u_i = r^{-(n-1)/2}(e^{-i\lambda r} f_{i,-} + e^{+i\lambda r} f_{i,+}), \quad f_{i,\pm} \in C^\infty(M) + \mathcal{A}^{\gamma-1}(M).$$

Suppose further that $(H-\lambda^2)u_i = O(r^{-(n+1)/2-\varepsilon})$ for some $\varepsilon > 0$. Then the following identity holds [9, equation (13.1)] (see also [3, Section 5]):

$$\int_M (u_1 \overline{(H-\lambda^2)u_2} - ((H-\lambda^2)u_1)\overline{u_2}) d\,\mathrm{vol}_g = 2i\lambda \int_{\partial M} (f_{1,+}\overline{f_{2,+}} - f_{1,-}\overline{f_{2,-}}) d\,\mathrm{vol}_h,$$

where we take the restrictions to $\partial M$ in the integral over the boundary. To prove this, one notes that the left-hand side is absolutely integrable, so it can be obtained as the limit, as $R \to \infty$, of the integral restricted to $\{r \le R\}$. Then one applies Green's formula and uses the asymptotic form of the functions $u_i$ to see that the limit as $R \to \infty$ is the right-hand side.

We apply this with $u_1 = P(\lambda)a$, $a \in C^\infty(\partial M)$ and $u_2 = u_-$ above. Then one has $(H-\lambda^2)u_1 = 0$, and $f_{2,+} = 0$, while $f_{1,-} = a$. We obtain

$$\int_M (P(\lambda)a)\bar{v} = 2i\lambda \int_{\partial M} a\overline{f_-}.$$

This immediately yields

$$P(\lambda)^* v = 2i\lambda f_-|_{\partial M}. \tag{3.2}$$

A similar argument with $u_+$ shows

$$P(-\lambda)^* v = 2i\lambda f_+|_{\partial M}. \tag{3.3}$$

Moreover, applying $P(\pm\lambda)$ to $f_\mp$, we obtain $u_+ + u_-$, since this is the unique eigenfunction with incoming/outgoing data equal to $f_\mp$. It follows that we have

$$P(\pm\lambda)P(\pm\lambda)^* = 2\lambda i(R(\lambda+i0) - R(\lambda-i0)), \tag{3.4}$$

acting on Schwartz functions. Notice that the quantity in (3.4) is equal to the spectral measure, up to a factor of $d\lambda/4\pi$. Note also the simple consequence of (3.2) and (3.3):

$$P(\pm\lambda)^* \text{ maps Schwartz functions on } M \text{ to smooth functions on } \partial M. \tag{3.5}$$

Our first purpose in this section is to analyze the range of $P(\pm\lambda)$ on the $L^2$-based Sobolev space $H^m(\partial M)$. We begin with a basic mapping property for the Poisson operator on $L^2(\partial M)$. For any $\delta > 0$, identity (3.4) implies that we have a continuous mapping

$$P(\pm\lambda)\colon L^2(\partial M) \to H^{s,-1/2-\delta}(M) \tag{3.6}$$

Indeed, first note that both $R(\lambda \pm i0)$ map $H^{-1,1/2+\delta}(M)$ to $H^{1,-1/2-\delta}(M)$. This result is well known but follows in particular from (2.8) taking $s = 1$ since, for either choice of $\pm$ and any $s$,

$$H^{s-2,1/2+\delta} \subset \mathcal{Y}_{\pm}^{s-2,\mathsf{l}_\pm+1}, \quad \mathcal{X}_{\pm}^{s,\mathsf{l}_\pm} \subset H^{s,-1/2-\delta}.$$

Formula (3.4) then shows that $P(\pm\lambda)$ extends from $C^\infty(\partial M)$ to a bounded map from $L^2(\partial M)$ to $H^{1,-1/2-\delta}(M)$, as this holds if and only if $P(\pm\lambda)P(\pm\lambda)^*$ maps $H^{-1,1/2+\delta}(M)$ to $H^{1,-1/2-\delta}(M)$ (since $H^{-1,1/2+\delta}(M)$ is the dual of $H^{1,-1/2-\delta}(M)$), whence (3.6) holds for $s = 1$. Finally, since the image of $P(\lambda)$ is contained in solutions to $(H - \lambda^2)u = 0$, the differential index 1 can be replaced by any $s$ using elliptic regularity.

Now, we note that (3.6) can be improved easily using the microlocalization discussed above. If we define

$$\mathsf{l}_{\min} = \min(\mathsf{l}_+, \mathsf{l}_-), \quad \mathsf{l}_{\max} = \max(\mathsf{l}_+, \mathsf{l}_-), \tag{3.7}$$

then these are smooth functions on phase space for $\delta' > 0$ sufficiently small, and (again for arbitrary $s$),

$$(H^{s,\mathsf{l}_{\min}})^* = H^{-s,-\mathsf{l}_{\min}} = H^{-s,\mathsf{l}_{\max}+1}$$

by the final property in (2.7). Repeating our argument to deduce mapping properties for $P(\lambda)$ from (3.4), we use

$$H^{s-2,\mathsf{l}_{\max}+1} \subset \mathcal{Y}_{\pm}^{s-2,\mathsf{l}_\pm+1}, \quad \mathcal{X}_{\pm}^{s,\mathsf{l}_\pm} \subset H^{s,\mathsf{l}_{\min}}$$

to conclude that for any $s \in \mathbb{R}$,

$$P(\pm\lambda)\colon L^2(\partial M) \to H^{s,\mathsf{l}_{\min}}(M), \quad P(\pm\lambda)^*\colon H^{s,\mathsf{l}_{\max}+1}(M) \to L^2(\partial M), \tag{3.8}$$

the latter following from adjunction of the former. This improves (3.6) since $\mathsf{l}_{\min} = -1/2 > -1/2 - \delta$ away from the radial sets.

Again, we can phrase this in terms of microlocalization, giving improved decay for $P(\pm\lambda)f$, $f \in L^2(\partial M)$ away from the radial sets. That is to say, if $Q \in \Psi_{\mathrm{sc}}^{0,0}$ has $\mathrm{WF}'(Q) \cap \mathcal{R}_+ \cup \mathcal{R}_- = \varnothing$, then

$$QP(\pm\lambda)\colon L^2(\partial M) \to H^{s,-1/2}(M), \tag{3.9}$$

as follows immediately from (3.8), provided $Q$ is microsupported where $\mathsf{l}_\pm = -1/2$, so that $Q\colon H^{s,\mathsf{l}_{\min}} \to H^{s,-1/2}$. Proposition 3.1, a crucial step in proving our main theorem, improves on this; it asserts that increased regularity of the boundary distribution $f$ corresponds to increased decay of $P(\pm\lambda)f$ away from the radial sets.

The Poisson operator extends to a map on distributions, or equivalently on negative order Sobolev spaces $H^k(\partial M)$ for all $k \leq 0$. The representation of the Poisson kernel as a distribution associated to an intersecting pair of Legendre submanifolds with conic points in [8] shows that there is a mapping property of the form

$$P(\pm\lambda)\colon H^k(\partial M) \to H^{s,k+c}(M) \quad \text{for all } k \leq 0 \tag{3.10}$$

for some $c \in \mathbb{R}$; this is obtained by passing derivatives from the distribution onto the Poisson kernel, thus reducing to the case $k \geq 0$. The next proposition, plus positive commutator estimates as in [1, Section 3], show that in fact $c = -1/2$ is sharp for (3.10), although we do not need this fact in the present article.

**Proposition 3.1.** *Let* $Q \in \Psi^{0,0}_{\mathrm{sc}}$ *satisfy* $\mathrm{WF}'(Q) \cap (\mathcal{R}_+ \cup \mathcal{R}_-) = \varnothing$. *Then for all* $s \in \mathbb{R}$, $k \in \mathbb{R}$,

$$QP(\pm\lambda)\colon H^k(\partial M) \to H^{s,k-1/2}(M). \tag{3.11}$$

Before we prove the proposition we note the straightforward corollary.

**Corollary 3.2.** *Let* $A_+$ *be chosen as in* (2.6), *and* $s$ *be any real number. Then for* $f \in H^k(\partial M)$,

$$[H, A_\pm]P(\lambda)f,\ [H, A_\pm]P(-\lambda)f \in H^{s,k+1/2}(M) \tag{3.12}$$

*and is microlocally trivial in a neighborhood of* $\mathcal{R}_+ \cup \mathcal{R}_-$. *Consequently, we also have*

$$[H, A_\pm]P(\lambda)f,\ [H, A_\pm]P(-\lambda)f \in \mathcal{Y}^{s,\mathsf{l}_{\max}+1;k,0}_\pm(M). \tag{3.13}$$

The statement about microlocal triviality here means that there is a neighborhood $U$ of $\mathcal{R}_- \cup \mathcal{R}_+$ so that for any $Q \in \Psi^{0,0}_{\mathrm{sc}}$ with $\mathrm{WF}'(Q) \subset U$, $Q[H, A_\pm]P(\pm\lambda)f \in H^{S,L}$ for any $S, L$, i.e., is rapidly decaying.

*Proof of Corollary.* Statement (3.12) follows immediately from Proposition 3.1 and the fact that $[H, A_\pm]$ are scattering pseudodifferential operators of order $(1, -1)$. To obtain (3.13), we first replace $s$ by $s + k$ in (3.12), and then note that the $k$-th power of the module $\mathcal{M}_\pm$ maps the functions $[H, A_\pm]P(\pm\lambda)f$ to $H^{s,1/2}(M)$ simply using the fact that module elements have order $(1, 1)$, whence we find that these functions are in $\mathcal{Y}^{s,1/2;k,0}_\pm(M)$. Finally, the microlocal triviality near the radial sets allows us to vary the spatial order $1/2$ arbitrarily near the radial sets, so we can replace the order $1/2$ with $\mathsf{l}_{\max} + 1$. ∎

We remark that when $H = H_0 := \Delta_{\mathbb{R}^n}$ is the flat structure and $k \in \mathbb{N}$, Proposition 3.1 follows without using the Poisson kernel used in the more general proof below. Let $P_0(\lambda)$ denote the Poisson operator in the flat case. Assuming, without loss of generality, that supp $Q \subset \{r \geq 1\}$, one can make use of the generators of rotation on $\mathbb{S}^{n-1}$, which form a family Rot $:= \{V\}$ of vector fields commuting with $\Delta_{\mathbb{S}^{n-1}}$ and generating the cotangent space at every point, in particular

$$f \in H^k(\mathbb{S}^{n-1}) \iff \mathrm{Rot}^j f \subset L^2(\mathbb{S}^{n-1}) \quad \text{for all } j \leq k.$$

Let $V$ denote an arbitrary such vector field. Using $[H_0, V] = 0$ in $r > 0$, we claim that for $f \in H^k$,

$$P_0(\lambda)Vf = VP_0(\lambda)f. \tag{3.14}$$

This follows from the explicit formula for the Poisson operator and integration by parts, but in using the results presented here one can argue as follows. For $f \in C^\infty(\mathbb{S}^{n-1})$, the expansion for $P_0(\lambda)f$ gives that $P_0(\lambda)Vf - VP_0(\lambda)f$ lies in $\mathcal{X}^{s,\mathrm{l}+}$, but we have $(H_0 - \lambda^2)(P_0(\lambda)Vf - VP_0(\lambda)f) = 0$, so since $H_0 - \lambda^2$ is injective on this domain, (3.14) follows. The condition on $\mathrm{WF}'(Q)$, means that $p := (\hat{z}, \nu, \mu) \in \mathrm{WF}'(Q)$ then $\mu \neq 0$, and thus we can find $V$ with $\sigma_{1,1}(V)(p) \neq 0$. Since $QV \in \Psi^{-\infty,1}$ is elliptic at $p$, if $f \in H^k$, since $QP_0(\lambda)Vf \in H^{s,-1/2}$ we have by scattering elliptic regularity that $QP_0(\lambda)f \in H^{s,-1/2+k}$. We will use the structure of the Schwartz kernel of the Poisson operator to deduce this same result for more general Hamiltonians.

*Proof of Proposition* 3.1. We prove (3.11) for $+\lambda$ only. This will follow by using the well-understood structure of the integral kernel of $P(\lambda)$, due in this generality to Melrose and Zworski [8]. The proof of (3.11) for $-\lambda$ follows analogously.

As we show below, using propagation of singularities, it will suffice to restrict the microsupport of $Q$ to punctured neighborhoods of $\mathcal{R}_-$ over small balls on the boundary, as follows. First, pick a small coordinate patch $V$ on $\partial M$ with coordinates $y$, and then a small neighborhood $V'$ of $\mathcal{R}_- \cap {}^{\mathrm{sc}}T^*_V M$; our $Q$ will be supported in the punctured neighborhood $U = V' \setminus \mathcal{R}_-$. Concretely, this can be parametrized as $\{(x, y, \nu, \mu) \colon x, \nu^2 + |\mu|_h^2 - \lambda^2 < \varepsilon, 0 < |\nu + \lambda| < \varepsilon\}$ with $y$ in the coordinate patch.

For such $Q$, the operator $QP(\lambda)$ takes the form [8]

$$e^{-i\lambda \cos d_h(y,y')/x}\tilde{a}(x, y, y') + e(x, y, y') \tag{3.15}$$

where $d_h$ is the distance function on $(\partial M, h)$, $\tilde{a}$ is smooth and is supported where $y$ is in a deleted neighborhood of $y'$ (with $y'$ varying in the same chosen coordinate patch as $y$) and we integrate with respect to the $h$-Riemannian measure on $\partial M$. Here $e$ is a smooth function which is rapidly decaying as $x \to 0$ which we subsequently ignore.

Since $\mathcal{F}(H^{s,k}(\mathbb{R}^n)) = H^{k,s}(\mathbb{R}^n)$, were we working in Euclidean space, it would now suffice to prove that $\mathcal{F} \circ QP(\lambda) \colon H^k \to H^{k-1/2,s}$. To use this approach, we identify the open set $(0,\varepsilon)_x \times V \subset M^\circ$ with a subset of Euclidean space by first choosing any local diffeomorphism $\psi\colon V \to \mathbb{S}^{n-1}$, and then writing $\hat{z} = \psi(y)$, $r = 1/x$, $z = r\hat{z}(y) \in \mathbb{R}^n$. Then $\mathcal{F} \circ QP(\lambda)$ has Schwartz kernel

$$\widetilde{K}(\zeta, y') := \int e^{-iz\cdot\zeta} e^{-i\lambda|z|\cos d_h(\hat{z},\psi(y'))} a(z, y')\, dz,$$

where $a(z, y) = \tilde{a}(x, y, y')$ is a classical symbol in the $z$-variable of order 0, smooth in $y$. We claim that for $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi(\zeta) = 1$ on $|\zeta| < 2\lambda$ and supp $\chi \subset \{|\zeta| < 4\lambda\}$, then

$$K(\zeta, y') := \int e^{-iz\cdot\zeta} e^{-i\lambda|z|\cos d_h(\hat{z},\psi(y'))} a(z, y')\chi(\zeta)\, dz, \quad a \in S^0$$

satisfies

$$\widetilde{K}(\bullet, y') - K(\bullet, y') \in \mathcal{S}(\mathbb{R}^n_\zeta), \tag{3.16}$$

smoothly in $y'$, i.e., it is rapidly decreasing as $\zeta \to \infty$, as follows from non-stationary phase.

We now view $K$ as a Fourier integral operator mapping from the sphere to the dual $\mathbb{R}^n$, parametrized by $\zeta$, with $z$ playing the role of a homogeneous phase variable. We use the mapping properties of FIOs found in [4, Chapter 25, in particular Theorem 25.3.8]. Recall throughout that $a$ is supported where $y$ is close to $y'$ but not equal to it. Here $K$ is an FIO of order $0 - ((n-1) + n - 2n)/4 = 1/4$ with phase function $\phi(\zeta, z, y') = -iz\cdot\zeta - i\lambda|z|\cos d_h(\hat{z}, y')$ where $z$ is the auxiliary variable. We compute that $\phi$ defines the canonical relation

$$C := \{(\zeta, z; y', \mu')\colon \zeta = \lambda\hat{z}\cos d_h(\hat{z}, y') + \lambda\theta_2 \sin d_h(\hat{z}, y'),\ \mu' = \lambda|z|\theta_1 \sin d_h(\hat{z}, y')\}$$

where $\theta_1$ and $\theta_2$ are the initial, resp. final, fiber coordinates of the unit length geodesic between $y'$ and $\hat{z}$ with respect to the Riemannian metric $h$ on $\partial M$, interpreted as being in $T^*_{y'}\partial M$ in the first case and $T^*_{y(\hat{z})}\partial M$ in the second. The dimension of the canonical relation is $2n-1$. It is parametrized by $z, y'$ since for $y, y'$ sufficiently close (recall that $y$ parametrizes $\hat{z}$), the $\theta_i$ are determined by $y, y'$ and therefore by $\hat{z}$ and $y$ and the other variables $\mu', \zeta$ are given in terms of these. On the other hand, the projection of $C$ to the “right” factor $T^*_{y'}\partial M$ is a surjective submersion since $\partial\mu'/\partial\hat{z}$ is full rank for $y, y'$ sufficiently close but not equal. Thus, the pullback of the symplectic form on $T^*\partial M_{y'}$ to $C$ is rank $2(n-1)$ or corank 1. Finally, at no point are either of the radial vector fields $\mu' D_{\mu'}$ in $T^*\partial M \setminus \{0\}$ or $zD_z$ in $T^*\mathbb{R}^n_\zeta \setminus \{0\}$ tangential to $C$.

Therefore, [4, Theorem 25.3.8] together with the standard argument (see, e.g., [4, Corollary 25.3.2]) in which one pre/post composes an FIO by invertible elliptic pseudodifferential operators, give that $K$ maps $H^k(\partial M)$ to $H^{k-1/2}(\mathbb{R}^n_\zeta)$ continuously. (Note that the theorem concludes a mapping to $H^{k-1/2}_{\mathrm{loc}}(\mathbb{R}^n_\zeta)$ but $K$ is compactly supported in $\zeta$.) Compact support of $\widetilde{K}$ in $\zeta$ and (3.16) then give that $\langle\zeta\rangle^s \widetilde{K}$ maps $H^k(\partial M)$ to $H^{k-1/2,s}(\mathbb{R}^n_\zeta)$ for any $s$, whence composing with the inverse Fourier transform gives (3.9) for $Q$ microlocalized near the radial sets and locally over the boundary.

To treat general $Q$ with $\mathrm{WF}'(Q) \cap (\mathcal{R}_+ \cup \mathcal{R}_-) = \varnothing$, we recall that the assertion of continuity in (3.11) can be microlocalized in the sense that it will follow if, for all $q \in \mathrm{WF}'(Q)$ there is $\widetilde{Q} \in \Psi^{0,0}_{\mathrm{sc}}$ such that $q \in \mathrm{Ell}_{0,0}(\widetilde{Q})$ and for some $C > 0$

$$\|\widetilde{Q}P(\lambda)f\|_{H^{s,k-1/2}(M)} \le C\|f\|_{H^k(\partial M)}.$$

for all $f \in H^m(\partial M)$. For $q \in \mathrm{Ell}_{2,0}(H - \lambda^2)$, this follows from the microlocal elliptic estimate (see, e.g., [1, Proposition 3.3])

$$\begin{aligned}\|\widetilde{Q}P(\lambda)f\|_{H^{s,k-1/2}(M)} &\lesssim \|(H-\lambda^2)P(\lambda)f\|_{H^{s-2,k-1/2}(M)} + \|P(\lambda)f\|_{H^{-S,-L}} \\ &= \|P(\lambda)f\|_{H^{-S,-L}(M)} \lesssim \|f\|_{H^k(\partial M)}\end{aligned}$$

where $S, L$ are chosen sufficiently large and the final bound comes from (3.10). (Here $\lesssim$ means there exists a $C > 0$ such that the left-hand side is bounded by $C$ times the right-hand side.) For $q \in \Sigma_{2,0}(H - \lambda^2) \setminus (\mathcal{R}_+ \cup \mathcal{R}_-)$, then there is a bicharacteristic ray $\gamma$ in $\Sigma$ such that $\gamma(0) = q$ and $\lim_{\sigma\to-\infty} \gamma(\sigma) \in \mathcal{R}_-$. The first part of our proof implies the existence of a $\sigma_0 \ll 0$ and $Q \in \Psi^{0,0}_{\mathrm{sc}}$ such that $\gamma(\sigma_0) \in \mathrm{Ell}_{0,0}(Q)$ and $Q$ satisfies (3.11). On the other hand, by the standard propagation of singularities estimate (see, e.g., [1, Proposition 3.4]), there exists $\widetilde{Q} \in \Psi^{0,0}_{\mathrm{sc}}$ with $q \in \mathrm{Ell}_{0,0}(\widetilde{Q})$ such that

$$\begin{aligned}\|\widetilde{Q}P(\lambda)f\|_{H^{s,k-1/2}(M)} &\lesssim \|QP(\lambda)f\|_{H^{s,k-1/2}(M)} \\ &\quad + \|(H-\lambda^2)P(\lambda)f\|_{H^{s-2,k+1/2}(M)} + \|P(\lambda)f\|_{H^{-S,-L}} \\ &= \|QP(\lambda)f\|_{H^{s,k-1/2}(M)} + \|P(\lambda)f\|_{H^{-S,-L}} \\ &\lesssim \|QP(\lambda)f\|_{H^{s,k-1/2}(M)} + \|f\|_{H^k},\end{aligned}$$

where we choose $S, L$ sufficiently large and use (3.10) to estimate $P(\lambda)f$ by $f$. Then, using $\|QP(\lambda)f\|_{H^{s,k-1/2}(M)} \lesssim \|f\|_{H^k}$ gives the result for $q$ on the characteristic set away from the radial sets, completing the proof. ■

The next proposition is known to experts, see, e.g., [10] and [2, Section 3] in a slightly different context. It is implicit in older works, e.g., [7]; no doubt many other references could be given. Notice that in (3.17), $R(\lambda + i0) - R(\lambda - i0)$ is, up to a constant, the "spectral projection" $dE(\lambda^2)$ for the operator $H$. This operator is not an actual projection, however; in fact, it is not bounded on any natural Hilbert space. The proposition shows that a modification of this operator leads to a genuine projection, that acts as the identity on the generalized eigenfunctions.

**Proposition 3.3.** *Let $u = P(\lambda)f$, for $f \in H^k(\partial M)$, $k = 0, 1, 2, \ldots$, and let $s \in \mathbb{R}$. Then we can express*

$$\begin{aligned} u &= u_+ + u_-, \\ u_\pm &= A_\pm u = R(\lambda \pm i0)[H, A_\pm]u \in \mathcal{X}_\pm^{s+2,\mathsf{l}_\pm;k,0}(M). \end{aligned} \tag{3.17}$$

*Conversely, let $w \in H^{s,k+1/2}(M)$ be such that $w$ is microlocally trivial in a neighborhood of $\mathcal{R}_- \cup \mathcal{R}_+$. Then, defining $u$ by*

$$u := (R(\lambda + i0) - R(\lambda - i0))w,$$

*$u$ can be obtained as $u = P(\lambda)f$ for a function $f \in H^k(\mathbb{S}^{n-1})$, namely the function $f = (i/2\lambda)P(\lambda)^* w$.*

*Proof.* First we show (3.17). Using the identity $(H - \lambda^2)u = 0$ we evaluate the right-hand side:

$$\begin{aligned} &(R(\lambda + i0) - R(\lambda - i0))[H, A_+]u \\ &\quad = R(\lambda + i0)[H - \lambda^2, A_+]u + R(\lambda - i0)[H - \lambda^2, A_-]u \\ &\quad = R(\lambda + i0)(H - \lambda^2)A_+u + R(\lambda - i0)(H - \lambda^2)A_-u. \end{aligned}$$

We claim that $A_+u$ is in the space $\mathcal{X}^{s,\mathsf{l}_+}$ (for any $s$). To see this, we use (3.6), which shows that $P(\lambda)f$ is in $H^{s,-1/2-\varepsilon}$ globally, for arbitrary $\varepsilon > 0$, and Proposition 3.1, which shows that it is in $H^{s,-1/2}$ microlocally away from $\mathcal{R}_+$, together with the definition of the operator $A_+$ which microlocalizes away from $\mathcal{R}_-$. Similarly, $A_-u$ is in the space $\mathcal{X}^{s,\mathsf{l}_-}$. On these spaces, $R(\lambda + i0)$, respectively $R(\lambda - i0)$ is a left inverse to $H - \lambda^2$ (see (2.8)). It follows that the right-hand side is equal to $A_+u + A_-u = u$, proving the identity (3.17) with $u_\pm = A_\pm u$. Moreover, by Corollary 3.2, $[H, A_\pm]u$ is in $\mathcal{Y}_\pm^{s,\mathsf{l}_{\max}+1;k,0} \subset \mathcal{Y}_\pm^{s,\mathsf{l}_\pm+1;k,0}$, so using the resolvent mapping property (2.9), we see that $u_\pm$ is in $\mathcal{X}_\pm^{s+2,\mathsf{l}_\pm;k,0}(M)$ as claimed.

We show the converse statement first for $k = 0$. This follows immediately from (3.4) and (3.8) since the assumption on $w$ implies that $w \in H^{s,\mathsf{l}_{\max}+1}$, provided that $\delta$ in the definition of $\mathsf{l}_\pm$ is sufficiently small, so that $w$ is microlocally trivial in the region where $\mathsf{l}_{\max} \neq -1/2$. For any positive integer $k$, we write

$$P(\lambda)^* w = P(\lambda)^* Q w + P(\lambda)^* (\mathrm{Id} - Q) w,$$

where $Q$ is microlocally equal to the identity on the microsupport of $w$, and microlocally trivial near the radial sets. Then $(\mathrm{Id} - Q)w$ is Schwartz, so using (3.5), $P(\lambda)^*(\mathrm{Id} - Q)w$ is $C^\infty$. For the other term, the adjoint of Proposition 3.1 shows that $P(\lambda)^* Q f$ is in $H^k(\partial M)$. ∎

We now begin our analysis of the incoming/outgoing data of distributions in the image of $R(\lambda \pm i0)$. The main improvement in the regularity of this incoming/outgoing data comes from a combination of the mapping properties of the Poisson operator and its adjoint, together with the reproducing formula (3.17).

**Proposition 3.4.** *Let $s \geq 0$, and assume $F$ is in $\mathcal{Y}_+^{s,\mathfrak{l}_{\max}+1;1,k-1}$, $k \geq 2$, where $\mathfrak{l}_{\max}$ is defined in (3.7). Assume that $\delta$ satisfies*

$$0 < 2\delta < \min(1, \gamma - 1), \tag{3.18}$$

*where $\gamma$ is as in (1.5). Then $u_+ := R(\lambda + i0)F$ is such that, in a collar neighborhood of the boundary $\partial M \times (R, \infty)_r$, the limit*

$$\mathcal{L}(\lambda)u_+ := \lim_{r\to\infty} r^{(n-1)/2} e^{-i\lambda r} u_+(r, \cdot)$$

*exists in $H^{k-2}(\partial M)$, with estimates*

$$\begin{gathered} \|\mathcal{L}(\lambda)u_+\|_{H^{k-2}(\partial M)} \leq C\|F\|_{\mathcal{Y}_+^{s,\mathfrak{l}_{\max}+1;1,k-1}}, \\ \|r^{(n-1)/2}e^{-i\lambda r}u_+(r,\cdot) - \mathcal{L}(\lambda)u_+\|_{H^{k-2}(\partial M)} = O(r^{-\varepsilon}) \end{gathered} \tag{3.19}$$

*for $0 < \varepsilon < \delta$. Similarly, if $F \in \mathcal{Y}_-^{s,\mathfrak{l}_{\max}+1;1,k-1}$, $k \geq 2$, then $u_- := R(\lambda - i0)F$ is such that the limit*

$$\mathcal{L}(-\lambda)u_- := \lim_{r\to\infty} r^{(n-1)/2} e^{+i\lambda r} u_-(r, \cdot)$$

*exists in $H^{k-2}(\partial M)$, with estimates*

$$\begin{gathered} \|\mathcal{L}(-\lambda)u_-\|_{H^{k-2}(\partial M)} \leq C\|F\|_{\mathcal{Y}_-^{s,\mathfrak{l}_{\max}+1;1,k-1}}, \\ \|r^{(n-1)/2}e^{+i\lambda r}u_-(r,\cdot) - \mathcal{L}(-\lambda)u_-\|_{H^{k-2}(\partial M)} = O(r^{-\varepsilon}) \end{gathered} \tag{3.20}$$

*for $0 < \varepsilon < \delta$.*

*Proof.* We follow the argument of [1, Section 4], which in turn is based on [9]. We do this only for $u_+$, as the argument is similar for $u_-$. From now on, we denote $u_+$ by $u$.

We use the microlocalizing operators $Q_\pm$ from Section 2.3, and complete these to a partition of unity, $\mathrm{Id} = Q_+ + Q_- + Q_3$. Thus, $Q_+$ is microsupported close to $\mathcal{R}_+$, $Q_-$ is microsupported close to $\mathcal{R}_-$, and $Q_3$ is microsupported away from $\mathcal{R}_+ \cup \mathcal{R}_-$. Assuming that the microsupports of $Q_\pm$ are sufficiently close to $\mathcal{R}_\pm$, the hypothesis on $F$ implies that $Q_\pm F$ is in the space $\mathcal{Y}_+^{s,1/2+\delta;1,k-1}$, while $Q_3 F$ is in $\mathcal{Y}_+^{s,1/2;1,k-1}$.

We notice that all of the commutators $[H, Q_\bullet]$ have boundary order 1 and microlocally vanish near $\mathcal{R}_+ \cup \mathcal{R}_-$, so $r^2[H, Q_\bullet]$ belongs in both modules, $\mathcal{M}_+$ and $\mathcal{M}_-$. We write $u_1 = Q_+ u$, $u_2 = Q_- u$, and $u_3 = Q_3 u$.

Write $\tilde{u}_1 = \chi(r) r^{(n-1)/2} e^{-i\lambda r} u_1$, where $\chi$ is supported in $r > R$ and identically 1 near $r \geq 2R$. Our first goal is to show that $\tilde{u}_1(r, y)$ has a limit $b(y)$ in $H^{k-2}(\partial M)$, and that $\tilde{u}_1(r, y) - b(y) = O_{H^{k-2}(\partial M)}(r^{-\varepsilon})$ as $r \to \infty$. To do this, we write the operator $H - \lambda^2$ in the form

$$H - \lambda^2 = D_r^2 - i(n-1)r^{-1}D_r + r^{-2}L + r^{-2}\tilde{L} + V - \lambda^2, \tag{3.21}$$

where $L$ is a second order differential operator involving only tangential $D_{y_j}$ derivatives, and $\tilde{L}$ is a scattering differential operator of order $(1, 0)$. Since $(H - \lambda^2)u = F$, we obtain

$$\begin{aligned}(D_r + \lambda)\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big)u_1 = Q_+F + [H, Q_+]u - r^{-2}\tilde{L}u_1 \\ + \frac{1}{2r^2}(i(n-1)r(D_r - \lambda) + (n-1) - 2L)u_1 \\ - Vu_1.\end{aligned} \tag{3.22}$$

We are going to show that the right-hand side lies in $\mathcal{Y}^{s,1/2+\delta;0,k-2}$, with the norm in this space bounded by

$$C\|F\|_{\mathcal{Y}_+^{s,\mathrm{l}_{\max}+1;1,k-1}}. \tag{3.23}$$

This has already been noted for the first term, $Q_+F$. For the remaining terms, it suffices to bound them by

$$C\|u\|_{\mathcal{Y}^{s+2,-1/2-\delta;1,k-1}} \tag{3.24}$$

since, as we have seen in (2.9), this is bounded by (3.23). For the second term, we use the fact that $[H, Q_+]$ is $r^{-2}$ times a small module element, so the $\mathcal{Y}^{s,1/2+\delta;0,k-2}$-norm of this term is bounded by the $\mathcal{Y}^{s+1,-3/2+\delta;0,k-1}$ norm of $u$, which is bounded by (3.24) since $\delta < 1/2$ according to (3.18). The next term is very similar: since $r^{-2}\tilde{L}$ has order $(1, -2)$, the $\mathcal{Y}^{s,1/2+\delta;0,k-2}$-norm of this term is bounded by the $\mathcal{Y}^{s+1,-3/2+\delta;0,k-2}$ norm of $u$, which is again bounded by (3.24). For the term with the $r^{-2}$ prefactor, we observe that the differential operator in large parentheses is contained in $\mathcal{M}_+ \cdot \mathcal{N}$, that is, one factor in the large module and one in the small module. It follows that the $\mathcal{Y}^{s,1/2+\delta;0,k-2}$-norm of this term is bounded by the $\mathcal{Y}^{s,-3/2+\delta;1,k-1}$ norm of $u_1$, which again is bounded by (3.24). Finally, for the $V$ term this follows from the fact that $V$ satisfies the conormal estimates (1.5), hence multiplication by $V$ is a scattering pseudodifferential operator of order $(0, -\gamma)$. Recalling (3.18), we have $-1/2 - \delta + \gamma > 1/2 + \delta$, and it follows that the $\mathcal{Y}^{s,1/2+\delta;0,k-2}$-norm of $Vu_1$ is bounded by the $\mathcal{Y}^{s,-1/2-\delta;0,k-2}$-norm of $u_1$, which is bounded by (3.24).

Now, observe the operator $D_r + \lambda$ is elliptic everywhere on $\mathrm{WF}'(Q_+)$, since $Q_+$ is microsupported near $\mathcal{R}_+$. Thus, we may write invert this operator microlocally; that is, we can write

$$\mathrm{Id} = J(D_r + \lambda) + R',$$

where $J \in \Psi_{\mathrm{sc}}^{-1,0}$ is a microlocal inverse, and the microsupport of the remainder $R'$ is disjoint from $Q_+$. Then for any scattering pseudodifferential operator $A$, $R'AQ_+u$

can be bounded by (a suitable multiple of) any Sobolev norm of $u$, for example $\|u\|_{s+2,-1/2-\delta}$, which in turn is bounded by (3.24). Applying this operator identity to $AQ_+u = Au_1$, where $A = (D_r - \lambda - i(n-1)/(2r))$, then applying (3.22) and the estimate (3.23) on the right-hand side of this equation, we find

$$\begin{gathered}\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big)u_1 \in \mathcal{Y}^{s,1/2+\delta;0,k-2}, \\ \Big\|\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big)u_1\Big\|_{\mathcal{Y}^{s,1/2+\delta;0,k-2}} \leq C\|F\|_{\mathcal{Y}_+^{s,\mathrm{l}_{\max}+1;1,k-1}}.\end{gathered} \tag{3.25}$$

Now, using $s \geq 0$ and observing that

$$D_r\tilde{u}_1 = \chi(r)r^{(n-1)/2}e^{-i\lambda r}\Big(D_r - \lambda - \frac{i(n-1)}{2r}\Big)u_1 + (D_r\chi)r^{(n-1)/2}e^{-i\lambda r}u_1,$$

we find that $D_r\tilde{u}_1 \in H^{0,1/2+\delta-(n-1)/2;0,k-2}$, or equivalently

$$D_r\tilde{u}_1 \in r^{n/2-1-\delta}L^2(([R,\infty), r^{n-1}dr); H^{k-2}(\partial M)),$$

with a corresponding norm estimate (where we used the support property of $D_r\chi$ for the inclusion in $H^{0,1/2+\delta-(n-1)/2;0,k}$ of the second term). Combining this estimate with the inclusions

$$\begin{aligned} &r^{n/2-1-\delta}L^2(([R,\infty), r^{n-1}dr); H^{k-2}(\partial M)) \\ &\quad\subseteq r^{-1/2-\delta}L^2(([R,\infty), dr); H^{k-2}(\partial M)) \\ &\quad\subseteq r^{-\varepsilon}L^1(([R,\infty), dr); H^{k-2}(\partial M)), \quad 0 < \varepsilon < \delta, \end{aligned}$$

we find

$$\|r^{\varepsilon}D_r\tilde{u}_1\|_{L^1(([R,\infty),dr);H^{k-2}(\partial M))} \leq C\|F\|_{\mathcal{Y}_+^{s,\mathrm{l}_{\max}+1;1,k-1}}. \tag{3.26}$$

We note that, since $\tilde{u}_1$ is locally $H^1$ in $r$ with values in $H^{k-2}(\partial M)$, it is in fact continuous in $r$ with values in $H^{k-2}(\partial M)$. By (3.26), we can integrate to infinity to find

$$b(y) = \int_R^\infty \partial_{r'}\tilde{u}_1(r',y)\,dr', \quad \|b\|_{H^{k-2}(\partial M)} \leq C\|F\|_{\mathcal{Y}_+^{s,\mathrm{l}_{\max}+1;1,k-1}},$$

is well defined as an element of $H^{k-2}(\partial M)$. Moreover,

$$\begin{aligned} \tilde{u}_1(r,y) - b(y) &= -\int_r^\infty \partial_{r'}\tilde{u}_1(r',y)\,dr', \\ \|\tilde{u}_1(r,y) - b(y)\|_{H^{k-2}(\partial M)} &\leq Cr^{-\varepsilon}\|r^{\varepsilon}D_r\tilde{u}_1\|_{L^1(([R,\infty),dr);H^{k-2}(\partial M))} \\ &\leq Cr^{-\varepsilon}\|F\|_{\mathcal{Y}_+^{s,\mathrm{l}_{\max}+1;1,k-1}}. \end{aligned}$$

To treat $u_2$, we apply a similar argument with the role of $\mathcal{R}_+$ and $\mathcal{R}_-$ interchanged. In this case we define $\tilde{u}_2 = \chi(r) r^{(n-1)/2} e^{+i\lambda r} u_2$ and write (3.22) in the form

$$(D_r - \lambda)\Big(D_r + \lambda - \frac{i(n-1)}{2r}\Big)u_2 = Q_- F + [H, Q_-]u_2 + \frac{i(n-1)}{2r^2}(r(D_r + \lambda))u_+ + \Big(\frac{n-1}{2r^2} - r^{-2}L - r^{-2}\tilde{L} - V\Big)u_+.$$

Following the same reasoning as above, we find that $\tilde{u}_2$ has a limit as $r \to \infty$, with the same $O(r^{-\varepsilon})$ of convergence. But here, the mapping property of the *outgoing* resolvent near the *incoming* radial set (the inverse mapping to (2.9), see [1, Theorem 3.1]) shows that $u_2$ is actually in $H^{s,-1/2+\delta}$, i.e., the spatial order is above threshold, since $\mathsf{l}_+ = -1/2 + \delta$ on the microsupport of $Q_-$. Therefore, since we know that $\tilde{u}_2$ has a limit in $H^{k-2}(\partial M)$ as $r \to \infty$, this limit must be zero (were the limit nonzero, then $\tilde{u}_2$ would fail to lie in the space $H^{s,-1/2+\delta-(n-1)/2}$ for $\delta > 0$). It then follows that $\chi(r) r^{(n-1)/2} e^{-i\lambda r} u_2 = e^{-2i\lambda r}\tilde{u}_2$ also has a zero limit in $H^{k-2}(\partial M)$ as $r \to \infty$, with the same rate of convergence as $\tilde{u}_2$.

It remains to discuss $u_3$. We claim that $u_3$ is an element of $\mathcal{Y}_+^{s+2,1/2-\delta;1,k-2}$. We argue separately in the microlocal regions (i) near the characteristic variety $\Sigma$, and (ii) away from $\Sigma$. The mapping property (2.9) shows that $u$ is in $\mathcal{Y}_+^{s+2,-1/2-\delta;1,k-1}$.

In region (i), since $u_3$ is microsupported away from the radial sets, the small module $\mathcal{N}$ is elliptic there, so we can trade one order of small module regularity for a gain of one spatial order. In region (ii), we already have $u \in \mathcal{Y}_+^{s+2,+1/2-\delta;1,k-1}$ since $H - \lambda^2$ is elliptic there and we incur no loss in the spatial regularity from applying the resolvent in this region.

Now, using the $\mathcal{M}_+$ module regularity, and replacing the differential order with zero, we see that

$$r\Big(D_r - \lambda - i\frac{n-1}{2r}\Big)u_3 \in H^{0,1/2-\delta;0,k-2},$$

that is,

$$\Big(D_r - \lambda - i\frac{n-1}{2r}\Big)u_3 \in H^{0,3/2-\delta;0,k-2},$$

which is stronger than (3.25) as $\delta < 1/2$. We can thus apply the same reasoning as for $u_1$ to obtain a limit for $\chi(r) r^{(n-1)/2} e^{\pm i\lambda r} u_3$ (which are necessarily zero, for the same reason as for $u_2$), with the same rate of convergence.

The estimates in (3.19) are obtained by adding the contributions from $u_1$, $u_2$, and $u_3$. The estimates (3.20) are obtained similarly. ∎

**Remark 3.5.** From the proof above, we see the following: if $Q \in \Psi_{\mathrm{sc}}^{0,0}$ is microsupported away from $\mathcal{R}_+$, then $\mathcal{L}(\lambda)(Qu_+) = 0$. Similarly, if $Q' \in \Psi_{\mathrm{sc}}^{0,0}$ is microsupported away from $\mathcal{R}_-$, then $\mathcal{L}(-\lambda)(Q'u_-) = 0$. Also note that $\mathcal{L}(\lambda)(Qu_+) = 0$

immediately implies $\mathcal{L}(-\lambda)(Qu_+) = 0$. In the same way, $\mathcal{L}(-\lambda)(Q'u_-) = 0$ implies $\mathcal{L}(\lambda)(Q'u_-) = 0$.

**Proposition 3.6.** *Let $f \in H^k(\partial M)$, with $k \in \mathbb{N}$, $k \geq 2$, and let $u = P(\lambda)f$. Then $A_- u$ is such that the limit*

$$\mathcal{L}(-\lambda)(A_- u) = \lim_{r\to\infty} r^{(n-1)/2} e^{i\lambda r}(A_- u)(r, \cdot) \tag{3.27}$$

*exists in $H^{k-2}(\partial M)$. Moreover, this limit is $f$. Similarly, the limit*

$$\mathcal{L}(\lambda)(A_+ u) = \lim_{r\to\infty} r^{(n-1)/2} e^{-i\lambda r}(A_+ u)(r, \cdot) \tag{3.28}$$

*exists in $H^{k-2}(\partial M)$ as $r \to \infty$. Moreover, this limit is in $H^k(\partial M)$. Both limits are achieved with an $O(r^{-\varepsilon})$ convergence rate, as in Proposition 3.4.*

*Proof.* As we have already seen in Proposition 3.3, $A_- u$ is the incoming resolvent applied to $[H, A_-]u$, and $A_+ u$ is the outgoing resolvent applied to $[H, A_+]u = -[H, A_-]u$. We have also seen in Corollary 3.2 that for $f \in H^k(\partial M)$, $[H, A_\pm]u$ is in the module regularity space $\mathcal{Y}_\pm^{s,\mathfrak{l}_{\max}+1;k,0}$. Therefore, the existence of the limits (3.27) and (3.28) in $H^{k-2}(\partial M)$ follows from Proposition 3.4.

We next note that, for $f \in C^\infty(\partial M)$, and $u = P(\lambda)f$, the limit $\mathcal{L}(-\lambda)(A_- u)$ is exactly $f$. This is a defining property of the Poisson kernel; see [8, Equations (0.2) and (0.4)]. However, we will elaborate on this point, as it is closely related to the form of (3.15). Suppose, without loss of generality, that $f$ is supported in a small neighborhood of $b \in \partial M$. Outside any microlocal neighborhood of $\mathcal{R}_-$, the contribution of $A_- u$ to this limit is zero, as it is in $H^{s,k}$ for every $k$ by Proposition 3.1. Therefore, given the canonical relation of $P(\lambda)$, see [8, Propositions 4 and 19], we can replace $A_- P(\lambda) f$ by $QP(\lambda) f$ where $Q$ is microsupported near $\mathcal{R}_-$ and its kernel is supported near $(b, b)$. The kernel of $QP(\lambda)$ is then as in (3.15), that is,

$$e^{-i\lambda \cos d_h(y,y')/x} \tilde{a}(x, y, y') + e(x, y, y'),$$

except that $\tilde{a}(x, y, y')$ is now supported close to $(0, b, b)$ but not in a *deleted* neighborhood, as was the case in (3.15). In fact, we have

$$\tilde{a}(0, y, y) = (\lambda/(2\pi))^{(n-1)/2} e^{-i(n-1)\pi/4} \text{ for } y \text{ near } b.$$

Then, in normal coordinates around $y$, we have

$$\cos d_h(y, y') = 1 - \frac{|y - y'|^2}{2} + O(|y - y'|^3),$$
$$dh(y') = dy'(1 + O(|y - y'|)),$$

the stationary phase lemma shows that indeed $\mathcal{L}(-\lambda)(QP(\lambda)f)(y) = f(y)$ in a neighborhood of $b$.

Now, for arbitrary $f \in H^k(\partial M)$, we choose a sequence of smooth $f_j$ converging to $f$ in $H^k(\partial M)$. Let $u_j = P(\lambda)f_j$; then $[H, A_-]u_j$ converges to $[H, A_-]u$ in $\mathcal{Y}_-^{s,\mathrm{l}_{\max}+1;1,k-1}$ using Corollary 3.2. As above, we have $A_-u_j = R(\lambda - i0)[H, A_-]u_j$. Using (3.19) in Proposition 3.4, $\lim_{j\to\infty} \mathcal{L}(-\lambda)A_-u_j$ exists in $H^{k-2}(\partial M)$ and is $\mathcal{L}(-\lambda)A_-u$. On the other hand, since the $f_j$ are smooth, $\mathcal{L}(-\lambda)A_-u_j$ is precisely $f_j$, which converge to $f$ in $H^k$, and so *a fortiori* in $H^{k-2}$, showing that $\mathcal{L}(-\lambda)A_-u = f$.

To obtain the result for $A_+u$, we could just appeal to the main result of [8] that says that the limit is the scattering matrix $S(\lambda)$ applied to $f$, and since $S(\lambda)$ is an FIO of order zero, then $S(\lambda)f$ is in $H^k$. However, we prefer a direct argument. Using the formula (3.4), we see that $u$ can be expressed as

$$u = \frac{1}{2\lambda i} P(-\lambda)P(-\lambda)^*[H, A_+]u.$$

That is, $u$ is equal to $P(-\lambda)f'$, where $f' = (2\lambda i)^{-1}P(-\lambda)^*[H, A_+]u$. Now, arguing as in the proof of the converse to Proposition 3.3, but for $P(-\lambda)$ instead of $P(\lambda)$, we see that $f'$ is in $H^k(\partial M)$. Now, applying the argument in the first half of this proof, with signs switched, we conclude that the limit (3.28) exists in $H^{k-2}(\partial M)$ and is equal to $f' \in H^k(\partial M)$.

The statement about the convergence rate is shown by applying Proposition 3.4 to $F = [H, A_\pm]u$. ■

**Remark 3.7.** The previous proof shows that the operators $\mathcal{L}(\pm\lambda) \circ R(\lambda \pm i0)$ in Proposition 3.4 coincide with $\pm(2i\lambda)^{-1}P(\mp\lambda)^*$. Also, we remark that $\mathcal{L}(\lambda)A_+P(\lambda)f$, the limit in (3.28), is precisely $S_{\mathrm{lin}}(\lambda)f$, the linear scattering matrix applied to $f$.

**Remark 3.8.** There is a subtlety here: the limiting functions in (3.27) and (3.28) are more regular than one would expect based on the topology of convergence. In fact, the convergence does *not* take place, in general, in the topology of $H^k(\partial M)$. To see this, consider the operator that maps $f$ to $r_0^{(n-1)/2}P(\lambda)f$ restricted to $\{r = r_0\}$. This is a semiclassical FIO of order zero on $\partial M$ (with $1/r_0$ playing the role of semiclassical parameter) but the canonical relation has fold singularities. This is best seen in the case of flat Euclidean space, where the phase function of this FIO is $\Phi(\hat{z}, \omega) = -\lambda\hat{z}\cdot\omega$, $z = r\hat{z}$, $\hat{z}, \omega \in \mathbb{S}^{n-1}$. Such an FIO cannot be expected to be (and is not) bounded on $H^k$ uniformly in $r_0$, and hence, convergence cannot be expected to take place, even weakly, in $H^k$ as $r_0 \to \infty$.

We combine the previous two propositions to obtain

**Proposition 3.9.** *The limits in Proposition 3.4 lie in $H^k(\partial M)$.*

*Proof.* In the notation of Proposition 3.4, we have $F \in \mathcal{Y}_{\pm}^{s,\mathsf{l}_{\max}+1;1,k-1}$, and therefore it lies in both of the variable order module regularity spaces $\mathcal{Y}^{s,\mathsf{l}_{\pm}+1;0,k}$. It follows that $u_{\pm} := R(\lambda \pm i0)F$ lies in $\mathcal{X}^{s+2,\mathsf{l}_{\pm};0,k}$. In particular, $u := u_+ - u_-$ is in $H^{s+2,k-1/2}$ microlocally away from small neighborhoods of $\mathcal{R}_+$ and $\mathcal{R}_-$, since $\mathsf{l}_{\pm} = -1/2$ there. (Notice that, on the characteristic set but away from the radial sets, the small module is elliptic, so the small module regularity of order $k$ affords $k$ orders of spatial regularity. Away from the characteristic set, $u$ is rapidly decaying by microlocal ellipticity, since $(H - \lambda^2)u = 0$.)

We next observe that, by (3.4), $u = P(\lambda)f'$ with $f' = (2\lambda i)^{-1}P(\lambda)^* F$. Using (3.8) we see that $f' \in L^2$. Applying Proposition 3.3, with $k = 0$, we have $u = (R(\lambda + i0) - R(\lambda - i0))[H, A_+]u$. But then, since $[H, A_+]u$ is in $H^{s+1,k+1/2}$, and microlocalized away from $\mathcal{R}_+$ and $\mathcal{R}_-$, Proposition 3.3 shows that $u = P(\lambda)f$ with $f \in H^k(\partial M)$.

Consider the limit $\mathcal{L}(\lambda)u_+$. We claim that is the same as $\mathcal{L}(\lambda)A_+u$. In fact, the difference is

$$\begin{aligned} \mathcal{L}(\lambda)(u_+ - A_+u) &= \mathcal{L}(\lambda)(u_+ - A_+u_+ - A_+u_-) \\ &= \mathcal{L}(\lambda)(A_-u_+ - A_+u_-). \end{aligned}$$

Using Remark 3.5, we see that $\mathcal{L}(\lambda)A_-u_+ = 0$ and $\mathcal{L}(-\lambda)A_+u_- = 0$, which implies that $\mathcal{L}(\lambda)A_+u_- = 0$. Thus, $\mathcal{L}(\lambda)u_+ = \mathcal{L}(\lambda)A_+u$. Similarly, $\mathcal{L}(-\lambda)u_- = \mathcal{L}(-\lambda)A_-u$. The conclusion then follows from applying Proposition 3.6 to $u$. ∎

## 4. Proof of the main theorem

We now elaborate on the construction and regularity of nonlinear Helmholtz eigenfunction $u$, whose asymptotic behavior is the subject of the main theorem. We begin by discussing (linear) generalized eigenfunctions.

Given $f \in H^k(\partial M)$, we let $u_0 = P(\lambda)f$ and decompose using Proposition 3.3 into

$$u_0 = u_- + u_+, \quad u_{\pm} = A_{\pm}u_0 \in \mathcal{X}_{\pm}^{s+2,-1/2-\delta;k,0}(M)$$

where $A_{\pm}$ are as in (2.6). According to Proposition 3.6, we have

$$u_0 = u_- + u_+ = r^{-(n-1)/2}(e^{-i\lambda r}f(y) + e^{+i\lambda r}b_0(y) + O_{H^{k-2}}(r^{-\varepsilon})), \qquad (4.1)$$

where $b_0$ is in $H^k(\partial M)$.

To address the nonlinear problem, following [1], we obtain a nonlinear Helmholtz eigenfunction $u$ satisfying

$$\begin{gathered} u = u_- + w, \quad u \text{ solves } (1.1), \\ u_- \in \mathcal{X}_-^{s+2,-1/2-\delta;1,k-1}, \quad w \in \mathcal{X}_+^{s+2,-1/2-\delta;1,k-1}, \quad s \in \mathbb{N}, \\ w = u_+ + R(\lambda + i0)N[u_- + w]. \end{gathered} \tag{4.2}$$

Moreover, if the nonlinearity $N$ involves products of degree $p$, then, as described in detail in [1, Section 4.2],

$$N[u_- + w] \in H_+^{s,\ell';1,k-1}$$

provided

$$\ell' \leq \frac{(p-1)(n-1)}{2} - \frac{3}{2} - p\delta. \tag{4.3}$$

The contraction mapping argument which produces this $w$ requires that $\ell' = 1/2 - \delta$ for the same $\delta$ appearing in (2.7) and (3.18), whence the bound for $p$ in (1.3), which in fact allows for $\ell' = 1/2 + \delta$ for $\delta > 0$ sufficiently small to satisfy all the above conditions. Thus, $F = N[u_- + w]$ satisfies the assumptions of Proposition 3.4. We conclude that

$$w - u_+ = r^{-(n-1)/2} e^{i\lambda r}(b_1(y) + O_{H^{k-2}}(r^{-\varepsilon})), \quad b_1 \in H^k(\partial M), \tag{4.4}$$

using Proposition 3.9 for the regularity of $b_1$. Combining $u = u_0 + (w - u_+)$ using (4.1) and (4.4) proves the asymptotic behavior stated in the main theorem, with $b = b_0 + b_1 \in H^k(\partial M)$.

Now, we assume we are given $k \in \mathbb{N}$, $k > (n-1)/2$, and $f \in H^k(\partial M)$ with $\|f\|_{H^k(\partial M)} < c$, as in the statement of Theorem 2. In addition, we suppose

$$(p-1)(n-1)/2 > 3$$

and that $N[u]$ only involves derivatives of $u$ and $\bar{u}$ up to order one. Then by (4.3) we can take $\ell' = 3/2 + \delta$, which is to say we obtain $w$ with

$$N[u_- + w] \in H_+^{s+1,3/2+\delta;1,k-1} \subset H_+^{s,1/2+\delta;2,k-1} \subset H_+^{s,1/2+\delta;1,k}.$$

The first containment is because we can exchange one order of scattering differentiability plus one order of spatial decay, and gain one order of module regularity. Then applying (2.4), we see that $w - u_+$ is in the better space $\mathcal{X}_+^{s+2,-1/2-\delta;1,k}$, that is, one additional order of small module regularity compared to (4.2). Applying Propositions 3.4 and 3.9, we have

$$w - u_+ = r^{-(n-1)/2} e^{i\lambda r}(b_1(y) + O_{H^{k-1}}(r^{-\varepsilon})), \quad b_1 \in H^{k+1}(\partial M),$$

Thus, under the stronger assumption on $p$, the decomposition $b = b_0 + b_1$ holds with $b_0 = S_{\text{lin}}(\lambda) f \in H^k$ and $b_1 \in H^{k+1}$. If $f \in H^{k+j}(\partial M)$ (together with the $H^k$ smallness assumption on $f$), then $u_\pm \in H_\pm^{s,1/2-\delta;k+j,0}$ and by a bootstrap argument we get $b_1 \in H^{k+j+1}(\partial M)$. In particular, since $S_{\text{lin}}(\lambda)$ is an FIO of order zero [8], if $f$ is in $C^\infty(\partial M)$, then $b = S_{\text{lin}}(\lambda) f + b_1$ is also in $C^\infty(\partial M)$.

Uniqueness follows from the same considerations as in [1]. Namely, given $u_- \in \mathcal{X}_-^{s+2,-1/2-\delta;1,k-1}$, as the function $w$ above is produced using a contraction mapping on $\mathcal{X}_-^{s+2,-1/2-\delta;1,k-1}$ , $w = u - u_-$ is uniquely determined in a small ball in this space.

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